A stabilized finite element method for the incompressible magnetohydrodynamic equations

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Abstract

We propose and analyze a stabilized finite element method for the incompressible magnetohydrodynamic equations. The numerical results that we present show a good behavior of our approximation in experiments which are relevant from an industrial viewpoint. We explain in particular in the proof of our convergence theorem why it may be interesting to stabilize the magnetic equation as soon as the hydrodynamic diffusion is small and even if the magnetic diffusion is large. This observation is confirmed by our numerical tests.

Keywords : magnetohydrodynamics, stabilized finite elements.
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1 Introduction

This work deals with the numerical resolution of the incompressible magnetohydrodynamic (MHD) equations by a stabilized finite element method. The system of partial differential equations that we consider here results from a coupling between the stationary incompressible Navier-Stokes equations and the stationary Maxwell equations. It governs the behavior of an incompressible fluid carrying an electrical current in presence of a magnetic field.

The unknowns of our problem are the velocity field u, the pressure p in the fluid and the magnetic field B (in fact the magnetic *induction*). They satisfy the following MHD equations (see for example R. Moreau [30] or

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W.F. Hughes and F.J. Young [25] for a physical viewpoint) :

$$\left\{\begin{array}{rcl} \rho u.\nabla u - \eta \triangle u + \nabla p + \dfrac{1}{\mu} B \times \operatorname{curl} B &=& \rho f, \\ \operatorname{div} u &=& 0, \\ \dfrac{1}{\mu\sigma} \operatorname{curl}(\operatorname{curl} B) - \operatorname{curl}(u \times B) &=& 0, \\ \operatorname{div} B &=& 0. \end{array}\right. \tag{1.1}$$

where f denotes an external force, ρ is the density of the fluid, η the viscosity, σ the electrical conductivity and μ the magnetic permeability. The parameters ρ, η, σ and μ are assumed to be constant over the domain. The system is set on a bounded simply-connected domain Ω and is completed by the boundary conditions

$$\left\{\begin{array}{rcl} u &=& 0, \\ B.n &=& q, \\ \operatorname{curl} B \times n &=& k \times n, \end{array}\right. \tag{1.2}$$

where n denotes the outward-pointing normal to Ω, and the quantities q and k are supposed to be known. Note that our analysis carries through for other cases (such as transient equations or non-homogeneous coefficients for example).

The resolution of the MHD equations may be useful in various physical or industrial contexts (such as aluminum electrolysis, electromagnetic pumping, MHD generator). In these situations, the hydrodynamic diffusivity η is generally very small whereas the magnetic diffusivity $1/\mu\sigma$ is high. We refer the interested reader to R. Berton [3] for more precise values of these parameters in the different cases. The small hydrodynamic diffusion may induce some well-known numerical instabilities. It is therefore natural to use some stabilizing techniques for the Navier-Stokes equations. One of the conclusion of this study is that it may be *also* useful to stabilize the magnetic equation in spite of the high values of the magnetic diffusivity.

The layout of this paper is as follows. In Section 2, we introduce some notations and we briefly present the stabilized finite element method for the advection-diffusion and the Stokes equations. Section 3 is devoted to the approximation of the MHD equations by stabilized finite elements together with the proof of our convergence result. We finally present various numerical results in Section 4 and we draw some conclusions in Section 5.

2 Preliminaries

2.1 The MHD equations

Many studies have already been devoted to the incompressible MHD equations. For theoretical results, let us just mention those by G. Duvaut and

J.-L. Lions [12], M. Sermange and R. Temam [36], J.-M. Domingez de la Rasilla [10], K. Kerieff [27], E. Sanchez-Palancia [34, 35], J. Rappaz and R. Touzani [33, 32]. J.-F. Gerbeau and C. Le Bris [19, 20]. An interesting alternative viewpoint which consists in considering the electrical current rather than the magnetic field as the main electromagnetic unknown is proposed by A.J. Meir and P.G. Schmidt [28, 29].

Numerical methods conserving the dissipative properties of the continuum transient system in 2D are presented in F. Armero, J.C. Simo [1]. M.D. Gunzburger, A.J. Meir, J.S. Peterson give in [22] a complete study of the Galerkin approximation of the stationary equations and a proof of existence of solutions.

Before presenting our approximation result, we find it convenient to rewrite the MHD system in a non-dimensional form. For this purpose we introduce a characteristic value B_0 of the magnetic field, a characteristic value U_0 for the velocity field and a characteristic length L, and we define the following non-dimensional numbers :

$$
\begin{aligned}
\text{Reynolds number :} \quad & Re = \frac{\rho U_0 L}{\eta}, \\
\text{Magnetic Reynolds number :} \quad & Rm = \mu\sigma U_0 L, \\
\text{Coupling number :} \quad & S = \frac{B_0^2}{\mu\rho U_0^2}.
\end{aligned}
$$

Denoting by $\tilde{u}$, $\tilde{B}$, $\tilde{p}$ and $\tilde{f}$ the physical quantities occurring in (1.1), we introduce the non-dimensional variables $u = \tilde{u}/U_0$, $B = \tilde{B}/B_0$, $p = \tilde{p}/\rho U_0^2$ and $f = \tilde{f}L/u_0^2$. Then the non-dimensional MHD system reads :

$$
\left\{
\begin{aligned}
u.\nabla u - \frac{1}{Re}\triangle u + \nabla p + S\, B \times \operatorname{curl} B &= f, \\
\operatorname{div} u &= 0, \\
\frac{1}{Rm}\operatorname{curl}(\operatorname{curl} B) - \operatorname{curl}(u \times B) &= 0, \\
\operatorname{div} B &= 0.
\end{aligned}
\right. \tag{2.3}
$$

In the industrial cases we have in mind, $Re \approx 10^5$, $Rm \approx 10^{-1}$ and $S \approx 1$. Let us just mention that the following numbers are often used in the MHD literature :

$$
\begin{aligned}
\text{Hartmann number :} \quad & Ha = \sqrt{Re\, Rm\, S} = \sqrt{\frac{\sigma}{\eta}} BL, \\
\text{Alfvén number :} \quad & A = \frac{1}{\sqrt{S}} = \sqrt{\mu\rho}\frac{U_0}{B_0}.
\end{aligned}
$$

In order to solve numerically this non-linear system, we consider here the following coupled Picard algorithm : assuming that (u^n, B^n, p^n) is given, we compute the approximated solution at step $n+1$ by solving the linear problem

$$\left\{\begin{array}{rcl} u^n\cdot\nabla u^{n+1}-\dfrac{1}{Re}\triangle u^{n+1}+\nabla p^{n+1}+S\,B^n\times\operatorname{curl}B^{n+1} & = & f,\\ \operatorname{div}u^{n+1} & = & 0,\\ \dfrac{1}{Rm}\operatorname{curl}(\operatorname{curl}B^{n+1})-\operatorname{curl}(u^{n+1}\times B^n) & = & 0,\\ \operatorname{div}B^{n+1} & = & 0.\end{array}\right. \tag{2.4}$$

Many other schemes could be used to tackle the MHD equations (see for example M.D. Gunzburger, A.J. Meir and J.S. Peterson [22] and J.-F. Gerbeau [18, 17]). For the sake of clarity, we only consider this Picard algorithm in the theoretical part of this paper. Therefore, in the following, we focus on the discretisation of the linear problem : find u, B and p such that

$$a\cdot\nabla u-\frac{1}{Re}\triangle u+\nabla p+S\,b\times\operatorname{curl}B = f, \tag{2.5}$$

$$\operatorname{div}u = 0, \tag{2.6}$$

$$\frac{1}{Rm}\operatorname{curl}(\operatorname{curl}B)-\operatorname{curl}(u\times b) = 0, \tag{2.7}$$

$$\operatorname{div}B = 0. \tag{2.8}$$

where a and b stand for u^n and B^n in the Picard iterations and are supposed to be known and regular. Again, for the sake of simplicity, in the theoretical part of this work, we suppose that $\operatorname{div}a=0$ (this is not necessary, see Remark 3.1), and we deal with homogeneous boundary conditions on Γ :

$$u = 0, \tag{2.9}$$

$$B.n = 0, \tag{2.10}$$

$$\operatorname{curl}B\times n = 0. \tag{2.11}$$

We leave to the reader the slight extensions of our work that are necessary to deal with other cases.

2.2 Notations

The domain Ω is partitioned in a quasi-uniform mesh $\mathcal{T}_h$ (see for example P.G. Ciarlet [9]), which consists of tetrahedral or hexahedral elements K. The diameter of an element K is denoted by h_K, and $h=\max\{h_K\}$. We consider the Lagrangian finite element space defined by

$$X_h^k=\{v_h\in C^0(\overline{\Omega}),v_h|_K\in P_k(K),\forall K\in\mathcal{T}_h\},$$

when the elements K are tetrahedra or by

$$X_h^k=\{v_h\in C^0(\overline{\Omega}),v_h|_K\in Q_k(K),\forall K\in\mathcal{T}_h\},$$

when the elements K are hexahedra. A classical result of the approximation theory gives an upper-bound of the difference between $u \in H^{k+1}(K)$ and its interpolate $\Pi_h u$ in X_h^k :

$$||u - \Pi_h u||_{L^2(K)} + h_K ||\nabla(u - \Pi_h u)||_{L^2(K)} + h_K^2 ||\triangle (u - \Pi_h u)||_{L^2(K)} \\ \leq C h_K^{k+1} |u|_{k+1,K}. \tag{2.12}$$

The velocity u is approximated in $V_h = (X_h^k \cap H_0^1(\Omega))^3$, the pressure p in $M_h = X_h^m \cap L_0^2(\Omega)$ (where $L_0^2(\Omega)$ is the space of $L^2(\Omega)$ functions with zero mean in Ω) and the magnetic field B in $W_h = (X_h^l)^3 \cap \mathbb{H}_n^1(\Omega)$ (where $\mathbb{H}_n^1(\Omega)$ is the space of vector fields B belonging to $H^1(\Omega)^3$ such that $B.n = 0$ on Γ). We shall use the inverse inequality (see e.g. P.G. Ciarlet [9])

$$\sum_{K \in \mathcal{T}_h} h_K^2 \int_K |\triangle v_h|^2 \, dx \leq d_0 \int_\Omega |\nabla v_h|^2 \, dx, \qquad \forall v_h \in V_h, \tag{2.13}$$

where d_0 is a non-negative constant independent of h.

With the assumption made on Ω, we have the following inequality for $B \in \mathbb{H}_n^1(\Omega)$ (see V. Girault and P.A. Raviart[21], Theorem 3.9) :

$$\int_\Omega |\nabla B|^2 \, dx \leq \int_\Omega |\mathrm{curl}\, B|^2 + |\mathrm{div}\, B|^2 \, dx \tag{2.14}$$

We shall use the formulae

$$\int_\Omega |\mathrm{div}\, u|^2 \, dx \leq 3 \int_\Omega |\nabla u|^2 \, dx, \tag{2.15}$$

and

$$\int_\Omega \mathrm{curl}\, B.C \, dx = \int_\Omega B.\mathrm{curl}\, C \, dx + \int_{\partial\Omega} n \times B.C \, dx. \tag{2.16}$$

2.3 A brief presentation of some stabilized methods

In this section we show how the advection-diffusion and the Stokes equations can be solved by the stabilized finite element methods. This is not an exhaustive presentation, our aim being just to introduce the basic ideas that we use in the following section for the MHD equations. For further details, the reader is referred to the extensive literature we now give a very brief overview of.

The “Streamline Upwind Petrov Galerkin” (SUPG) method (also named “Streamline Diffusion”) is generally presented as the first stabilized finite element method. It was introduced by A.N. Brooks and T.J.R. Hughes in [8] to cure the numerical oscillations due to a small diffusion in the advection-diffusion and Navier-Stokes equations. T.J.R. Hughes, M. Mallet and A. Mizukami propose in [24] to add a term to SUPG in order to avoid oscillations in boundary layers.

C. Johnson and J. Saranen analyze in [26] an extension of the Streamline Diffusion method to the transient Navier-Stokes and Euler equations. Other alternative methods are proposed by L.P. Franca and E.G. Dutra do Carmo [13], L.P. Franca, S.L. Frey and T.J.R Hughes [15] for the advection-diffusion equation.

T.J.R. Hughes, L.P. Franca and M. Balestra use stabilized finite element in [23] to solve the Stokes problem by circumventing the inf-sup condition. J. Douglas and J. Wang propose an unconditionally stable alternative in [11]. The fact that the same stabilizing tricks work with both advection-diffusion and Stokes equations is explained in L.P. Franca and T.J.R Hughes [16] where a symmetric advective-diffusive form of the Stokes equations is presented.

L.P. Franca and S.L. Frey study in [14] the linearized Navier-Stokes equations. In their approximation, the pressure and the velocity may be approximated in the same space, and, following their work on the advection-diffusion equation in [15], they use different stabilization parameters according to the regime of the flow (diffusion or advection-dominated) with a new definition of the local Reynolds number.

Some analysis of stabilization methods for the *non-linear* Navier-Stokes equations can be found in the work [39] by Tian-Xiao Zhou and Min-Fu Feng.

Streamline diffusion methods are related to the process of addition and elimination of suitable bubble functions to the finite element space (see C. Baiocchi, F .Brezzi and L.P. Franca [2], F. Brezzi, M. Bristeau, L.P. Franca, M. Mallet and G. Rogé [5], and a practical computation of scaled bubble functions in J.C. Simo, F Armero and C.A. Taylor in [37]). This establishes the existence of a link between the two families of stabilizing techniques. The problem of determining an optimal stabilization parameter – which is the difficulty generally considered as the major drawback of streamline diffusion methods – may be therefore replaced by the problem of an optimal choice of the bubble space. The optimal bubble space can be determined by solving a boundary value problem in each element, this is the so-called "residual-free bubbles" method. These very interesting issues will be not considered here, the interested reader is referred to F. Brezzi and A. Russo [7] and to F. Brezzi, D. Marini and A. Russo [6],

2.3.1 The advection-diffusion equation

We first consider the advection-diffusion equation :

$$-\eta \triangle u + a.\nabla u = f \text{ in } \Omega. \tag{2.17}$$

It is well-known that the finite element method is not well-suited to solve this problem when the diffusion is overtaken by the convection at the cell scale, in other words when $||a||_{\infty,K} h_K/\eta$ is large. The stabilization methods improve the convergence in that case. For the sake of simplicity, we suppose that $u = 0$ on Γ and that $\operatorname{div} a = 0$. For a more complete study we refer to A. Quarteroni and A. Valli [31] for example.

In this subsection, V_h denote the finite element space X_h^k. We define the bilinear form

$$\Phi(w,v) = \eta \int_\Omega \nabla w \nabla v \, dx + \int_\Omega a \cdot \nabla w \, v \, dx,$$

and

$$< F, v >= \int_\Omega f v \, dx.$$

In the sequel, the solution of the continuous problem (2.17) is denoted by u, the interpolate of u in V_h is denoted by $\tilde{u}_h$ and the solution of the discrete Galerkin problem is denoted by u_h. Thus we have :

$$\Phi(u_h, v_h) =< F, v_h >, \forall v_h \in V_h. \tag{2.18}$$

The interpolation error is denoted by $\pi_h = u - \tilde{u}_h$, the approximation error $e_h = \tilde{u}_h - u_h$ and the global error $\epsilon_h = u - u_h$. We have $\epsilon_h = \pi_h + e_h$.

By very standard estimates, one shows that the interpolation error is bounded as follows

$$||\nabla e_h||_{\mathbb{L}^2} \leq C \sqrt{1 + \frac{||a||_\infty^2 h^2}{\eta^2}} |u|_{k+1} h^k, \tag{2.19}$$

where C is a constant that does not depend on h, a, u and η. For advection-dominated flows, the right-hand side of this estimate is large and numerical results exhibit oscillations.

The way to estimate the Galerkin method's error is based on four properties : linearity, strong consistence, coercivity and continuity. The stabilized methods we consider here are some generalized Galerkin methods where the bilinear form Φ of the continuous formulation is replaced by a bilinear form Φ_h depending on the mesh and still satisfying the four above properties.

We define Φ_h on $V_h \times V_h$:

$$\Phi_h(w,v) = \Phi(w,v) + \sum_{K \in \mathcal{T}_h} \int_K \tau(-\eta \triangle w + a \cdot \nabla w)(\xi \eta \triangle v + a \cdot \nabla v) \, dx, \tag{2.20}$$

and

$$< F_h, v >= \int_\Omega f v \, dx + \sum_{K \in \mathcal{T}_h} \int_K \tau f (\xi \eta \triangle v + a \cdot \nabla v) \, dx. \tag{2.21}$$

The coefficient τ is the stabilization parameter. It is defined on Ω by :

$$\tau|_K = \tau_K \text{ with } \tau_K(x) = \frac{\lambda h_K}{|a(x)|}, \qquad \forall x \in K,$$

where λ is a nonnegative constant that will be fixed later to ensure the stability of the approximation.The value of ξ depends on the method : -1 (Douglas-Wang method, DWG), 0 ("Streamline Upwind Petrov Galerkin", SUPG) or 1 ("Galerkin Least Square", GLS).

It can be proved that the error estimate for the stabilization methods is

$$||\nabla e_h||_{\mathbb{L}^2} \leq C\sqrt{1+\frac{||a||_\infty h}{\eta}}\,|u|_{k+1}h^k, \tag{2.22}$$

where C is a constant independent of h, a, u and η.

Comparing (2.19) and (2.22), this shows that the stabilized methods may improve the results when $||a||_\infty h/\eta$ is large. In the case when $||a||_\infty h/\eta < 1$ (diffusion-dominated flow), the estimate (2.22) is worse than those obtained with the classical Galerkin approximation. To avoid this drawback we can choose $\tau_K = \lambda h_K^2/\eta$ in the cells K where diffusion dominates; thus we find a estimated like (2.19) and the stabilization parameter τ remains continuous when $||a||_\infty h/\eta$ takes the value 1.

2.3.2 The Stokes equations

The Galerkin mixed formulation of the Stokes problem reads : find $(u_h, p_h) \in V_h \times M_h$ such that $(v_h, q_h) \in V_h \times M_h$,

$$\Phi(u_h, p_h; v_h, q_h) = < F; v_h, q_h >,$$

with

$$\Phi(u_h, p_h; v_h, q_h) = \int_\Omega \eta \nabla u_h \cdot \nabla v_h\,dx - \int_\Omega p_h \mathrm{div}\,v_h\,dx + \int_\Omega q_h \mathrm{div}\,u_h\,dx,$$

and

$$< F; v_h, q_h > = \int_\Omega f \cdot v_h\,dx.$$

It is well-known that the spaces V_h and M_h must satisfy a compatibility condition. Following the same idea as for the advection-diffusion equation, the stabilized method consists in adding a "strongly consistent" term to the classical Galerkin formulation, in other words, Φ is replaced by Φ_h :

$$\Phi_h(u_h, p_h; v_h, q_h) = \Phi(u_h, p_h; v_h, q_h) + \sum_{K\in\mathcal{T}_h}\int_K \tau(-\eta\triangle u_h + \nabla p_h)\cdot(-\xi\eta\triangle v_h + \nabla q_h)\,dx,$$

and F by F_h :

$$< F_h; v_h, q_h > = < F; v_h, q_h > + \sum_{K\in\mathcal{T}_h}\int_K \tau f.(-\xi\eta\triangle v_h + \nabla q_h)\,dx.$$

where $\tau|_K = \lambda h_K^2$, λ is a constant, and ξ is equal to -1, 0 or 1 depending on the method.

The convergence of the stabilized method may then be proved without any inf-sup condition. This property is interesting for the practical implementation, specially in 3D. We refer to J. Douglas and J. Wang [11], T.J.R. Hughes, L.P. Franca and M. Balestra [23] for the original papers and to A. Quarteroni and A. Valli [31] for a pedagogical presentation.

3 Stabilized finite element methods for the MHD equations

In this section our aim is to extend the stabilized finite element methods introduced above to the MHD problem (2.5)-(2.11). As far as we know, this has never been presented in the literature before. We consider the non-dimensional form of the MHD equations and we denote $1/Re$ by η and $1/Rm$ by α. For the sake of simplicity, we deal with the globally advection-dominated case in the Navier-Stokes equations (but see Remark 3.3). In other words, we suppose that we have within each element K :

$$\frac{|a(x)|h_K}{\eta} > 1, \qquad \forall x \in K. \tag{3.1}$$

In view of the applications we have in mind, the magnetic equations is supposed to be diffusion-dominated. Let be λ_u and λ_B two positive constants. We define the stabilization coefficients by :

$$\tau_u|_K = \frac{\lambda_u h_K}{|a(x)|},$$

and

$$\tau_B|_K = \frac{\lambda_B h_K^2}{\alpha}.$$

Note that we have

$$\eta \tau_u|_K \leq \lambda_u h_K^2. \tag{3.2}$$

The resolution of the linearized MHD equations (2.5)-(2.11) by the classical Galerkin method consists in finding $u_h \in V_h$, $B_h \in W_h$ and $p_h \in M_h$ such that for all $(v_h, C_h, q_h) \in (V_h, W_h, M_h)$

$$\Phi_G(u_h, B_h, p_h; v_h, C_h, q_h) =< F_G; v_h, C_h, q_h >,$$

with

$$\begin{aligned}
\Phi_G(u,B,p;v,C,q) &= \int_\Omega (\eta\nabla u\cdot\nabla v + a\cdot\nabla u\cdot v - p\,\mathrm{div}\,v + S\,b\times \mathrm{curl}\,B\cdot v)\,dx \\
&\quad + \int_\Omega q\,\mathrm{div}\,u\,dx \\
&\quad + \int_\Omega (\alpha S\,\mathrm{curl}\,B\cdot\mathrm{curl}\,C + \alpha S\,\mathrm{div}\,B\mathrm{div}\,C - S\,u\times b\cdot\mathrm{curl}\,C)\,dx,
\end{aligned}$$

and

$$< F_G; v, C, q > \;=\; \int_\Omega f\cdot v\,dx.$$

Let us now define the stabilization terms :

$$\begin{aligned}
\Phi_S(u,B,p;v,C,q) &= \sum_{K\in\mathcal{T}_h}\int_K \tau_u \quad (a\cdot\nabla u - \eta\triangle u + \nabla p + S\,b\times\mathrm{curl}\,B)\cdot \\
&\qquad\qquad (a\cdot\nabla v + \xi\eta\triangle v + \nabla q + S\,b\times\mathrm{curl}\,C)\,dx \\
&\quad + \sum_{K\in\mathcal{T}_h}\int_K \tau_B \quad (-\alpha S\triangle B - S\mathrm{curl}\,(u\times b))\cdot \\
&\qquad\qquad (\alpha\xi\triangle C - \mathrm{curl}\,(v\times b))\,dx,
\end{aligned}$$

and

$$< F_S; v, C, q > \;=\; \sum_{K\in\mathcal{T}_h}\int_K \tau_u \quad f\cdot(a\cdot\nabla v + \xi\eta\triangle v + \nabla q + S\,b\times\mathrm{curl}\,C)\,dx.$$

where ξ is a constant equal to -1, 0 or 1 according to the stabilization method (namely the Douglas-Wang, the SUPG or the Galerkin Least Square methods in the context of the advection diffusion equation). Notice that the three methods coincide when linear or bilinear elements are used.

The stabilized problem that we now propose to analyze reads : find $u_h\in V_h$, $B_h\in W_h$ and $p_h\in M_h$ such that for all $(v_h,C_h,q_h)\in(V_h,W_h,M_h)$

$$\Phi(u_h,B_h,p_h;v_h,C_h,q_h) = < F; v_h, C_h, q_h > \tag{3.3}$$

with $\Phi=\Phi_G+\Phi_S$ and $F=F_G+F_S$. Our main result is :

Theorem 1
Let us recall that $V_h=(X_h^k\cap H_0^1(\Omega))^3$, $M_h=X_h^m\cap L_0^2(\Omega)$ and $W_h=(X_h^l)^3\cap\mathbb{H}_n^1(\Omega)$. We denote by (u,B,p) the exact solution of the MHD equations (2.5)-(2.11), by $(\tilde u_h,\tilde B_h,\tilde p_h)$ the interpolate of (u,B,p) in $V_h\times W_h\times M_h$ and by (u_h,B_h,p_h) the solution obtained by the stabilized finite element method (3.3).

Then, under hypothesis (3.1), the approximation error $(e_u,e_B,e_p)=$

$(\tilde{u}_h - u_h, \tilde{B}_h - B_h, \tilde{p}_h - p_h)$ *can be estimated as follows :*

$$\frac{1}{2}\int_\Omega (\eta|\nabla e_u|^2 + S\alpha|curl\, e_B|^2 + S\alpha|div\, e_B|^2)\, dx + \int_\Omega \tau_B S|curl\,(e_u \times b)|^2\, dx$$
$$+ \int_\Omega \tau_u |a.\nabla e_u + \nabla e_p + S\, b \times curl\, e_B|^2\, dx \leq$$
$$C_{\lambda_u,\lambda_B}\Bigg(\bigg(\lambda_u(||a||_\infty h + \eta) + \frac{S^2h^2}{\alpha}(||b||^2_\infty + h^2||\nabla b||^2_\infty)\bigg)\, h^{2k}|u|^2_{k+1}$$
$$+ \left(\frac{\lambda_u h}{||a||_\infty} + \frac{h^2}{\eta}\right) h^{2m}|p|^2_{m+1}$$
$$+ \left(\lambda_u \frac{S^2||b||^2_\infty}{||a||_\infty} h + \lambda_B \alpha\right) h^{2l}|B|^2_{l+1}\Bigg). \tag{3.4}$$

This shows the convergence of the approximation 3.3.$\diamondsuit$

It is useful to notice that the stabilization terms in the Navier-Stokes equations are necessary to prove the convergence of the pressure but the stabilization terms in the Maxwell equation are *not* necessary to prove the convergence of B_h (see Remark 3.2).

The following of this section is devoted to the proof of this theorem. We begin by proving the stability in the three cases $\xi = -1, 0, 1$, and then the convergence in the case $\xi = 1$ (the cases $\xi = -1, 0$ may be treated by the same arguments).

Remark 3.1 *It has already been mentioned in Section 2.1 that the fields a and b might be seen as the fields u^n and B^n occurring in the Picard iterations. Then $a \in V_h$ and $b \in W_h$. Let us see what should be changed in this case. First, the field a is no longer divergence free. Nevertheless, in the following proof, this assumption is only needed to ensure the antisymmetry of the trilinear advection term. Thus, it suffices to replace classically (see R. Temam [38])*

$$\int_\Omega a \cdot \nabla v \cdot w\, dx$$

by

$$\int_\Omega a \cdot \nabla v \cdot w\, dx + \frac{1}{2}\int_\Omega w \cdot v\; div\, a\, dx.$$

Let us now see what involve the assumption $b \in W_h$. We have the following inverse inequality (see P.G. Ciarlet [9], Theorem 17.2 for example) :

$$||\nabla b||_\infty \leq \frac{C}{h}||b||_\infty.$$

Thus, the term $(||b||^2_\infty + h^2||\nabla b||^2_\infty)$ could be replaced by $C||b||^2_\infty$ in estimate (3.4).

3.1 Stability

In this section, we prove the stability of the approximation (3.3). Let us start with $\Phi_G(u,B,p;u,B,p)$ corresponding to the classical Galerkin method. The assumption $\operatorname{div} a = 0$ yields the cancellation of the advection term. The term ∇p is compensated by $\operatorname{div} u$. Likewise the Lorentz force is compensated by the Maxwell advection term because of the relation

$$\int_\Omega Sb \times \operatorname{curl} B \cdot u \, dx = \int_\Omega Su \times b \cdot \operatorname{curl} B \, dx.$$

This relation makes useless the assumption $\operatorname{div} b = 0$, and this is to use this important cancellation property that we have chosen to linearize the MHD equations like that for the Picard iterations 2.4. Finally, we get :

$$\Phi_G(u,B,p;u,B,p) = \int_\Omega \left(\eta|\nabla u|^2 + \alpha S|\operatorname{curl} B|^2 + \alpha S|\operatorname{div} B|^2\right) \, dx. \tag{3.5}$$

Let us now consider the stabilization terms.

- **Case $\xi = 1$.**

Using (3.2) and the inverse inequality (2.13), we have :

$$\begin{aligned}
\Phi_S(u,B,p;u,B,p) &= \sum_{K\in\mathcal{T}_h} \int_K (\tau_u|a.\nabla u + \nabla p + S\, b \times \operatorname{curl} B|^2 - \tau_u \eta^2 |\triangle u|^2) \, dx \\
&\quad + \sum_{K\in\mathcal{T}_h} \int_K (\tau_B S|\operatorname{curl}(u \times b)|^2 - \tau_B S\alpha^2|\triangle B|^2) \, dx \\
&\geq \int_\Omega \tau_u|a.\nabla u + \nabla p + S\, b \times \operatorname{curl} B|^2 \, dx - d_0\lambda_u \int_\Omega \eta|\nabla u|^2 \, dx \\
&\quad + \int_\Omega \tau_B S|\operatorname{curl}(u \times b)|^2 \, dx - d_0\lambda_B \int_\Omega S\alpha|\nabla B|^2 \, dx.
\end{aligned}$$

Along with (2.14) and (3.5), we deduce

$$\begin{aligned}
\Phi(u,B,p;u,B,p) &\geq (1 - d_0\lambda_u) \int_\Omega \eta|\nabla u|^2 \, dx + (1 - d_0\lambda_B) \int_\Omega S\alpha|\operatorname{curl} B|^2 \, dx \\
&+ (1 - d_0\lambda_B) \int_\Omega S\alpha|\operatorname{div} B|^2 \, dx + \int_\Omega \tau_B S|\operatorname{curl}(u \times b)|^2 \, dx \\
&+ \int_\Omega \tau_u|a.\nabla u + \nabla p + S\, b \times \operatorname{curl} B|^2 \, dx.
\end{aligned}$$

The stability of the approximation is therefore achieved as soon as λ_u and λ_B are such that :

$$1 - d_0\lambda_B > 0 \qquad \text{and} \qquad 1 - d_0\lambda_u > 0. \tag{3.6}$$

- **Case $\xi = -1$.**

Let $\gamma > 1$ be an arbitrary constant. Using the obvious inequality

$$|A - B|^2 \geq |A|^2 - 2|A||B| + |B|^2 \geq (1 - 1/\gamma)|B|^2 - (\gamma - 1)|A|^2,$$

we have

$$\begin{aligned}
\Phi_S(u,B,p;u,B,p) &= \sum_{K\in\mathcal{T}_h} \int_K \tau_u |a.\nabla u - \eta \triangle u + \nabla p + S\, b \times \operatorname{curl} B|^2 \, dx \\
&\quad + \sum_{K\in\mathcal{T}_h} \int_K \tau_B S| - \operatorname{curl}(u \times b) - \alpha \triangle B|^2 \, dx \\
&\geq -(\gamma - 1) \sum_{K\in\mathcal{T}_h} \int_K \eta^2 \tau_u |\triangle u|^2 + \alpha^2 S \tau_B |\triangle B|^2 \, dx \\
&\quad +(1 - 1/\gamma) \int_\Omega \tau_u |a \cdot \nabla u + \nabla p + S\, b \times \operatorname{curl} B|^2 \, dx + \\
&\quad +(1 - 1/\gamma) \int_\Omega \tau_B S |\operatorname{curl}(u \times B)|^2 \, dx.
\end{aligned}$$

With (3.2) and (2.13), this yields

$$\begin{aligned}
\Phi_S(u,B,p;u,B,p) &\geq -(\gamma - 1)\lambda_u d_0 \int_\Omega \eta |\nabla u|^2 \, dx - (\gamma - 1)\lambda_B d_0 \int_\Omega \alpha S |\nabla B|^2 \, dx \\
&\quad +(1 - 1/\gamma) \int_\Omega \tau_u |a \cdot \nabla u + \nabla p + S\, b \times \operatorname{curl} B|^2 \, dx + \\
&\quad (1 - 1/\gamma) \int_\Omega \tau_B S |\operatorname{curl}(u \times B)|^2 \, dx.
\end{aligned}$$

Therefore, with $\gamma > 1$ such that $1 > (\gamma - 1)\lambda_u d_0$ and $1 > (\gamma - 1)\lambda_B d_0$, we deduce the stability just like in the previous case. Notice that we do not need any assumptions on λ_u and λ_B.

- **Case $\xi = 0$.**

Following the same steps as in the case $\xi = 1$:

$$\begin{aligned}
&\Phi_S(u,B,p;u,B,p) = \\
&\sum_{K\in\mathcal{T}_h} \int_K (\tau_u |a.\nabla u + \nabla p + S\, b \times \operatorname{curl} B|^2 - \tau_u \eta \triangle u \cdot (a \cdot \nabla u + \nabla p + S\, b \times \operatorname{curl} B))\, dx \\
&\quad + \sum_{K\in\mathcal{T}_h} \int_K (\tau_B S |\operatorname{curl}(u \times b)|^2 + \tau_B S \alpha \triangle B \cdot \operatorname{curl}(u \times b))\, dx \geq \\
&\geq \frac{1}{2} \int_\Omega \tau_u |a.\nabla u + \nabla p + S\, b \times \operatorname{curl} B|^2 \, dx - \frac{1}{2}\lambda_u d_0 \int_\Omega \eta |\nabla u|^2 \, dx \\
&\quad + \frac{1}{2} \int_\Omega \tau_B S |\operatorname{curl}(u \times b)|^2 - \frac{1}{2}\lambda_B d_0 \int_\Omega \alpha S |\nabla B|^2 \, dx.
\end{aligned}$$

Thus, we deduce the stability as soon as

$$1 - \frac{1}{2} d_0 \lambda_B > 0 \qquad \text{and} \qquad 1 - \frac{1}{2} d_0 \lambda_u > 0.$$

3.2 Convergence

We now proceed to establish the convergence in the case when $\xi = 1$ (cases $\xi = 0, -1$ can be treated with the same arguments). We choose λ_u and λ_B such that $1 - d_0\lambda_u \geq 1/2$ and $1 - d_0\lambda_B \geq 1/2$. In view of (3.6), this yields the stability of the approximation.

We denote by u, B, p the solution of the continuous problem (2.5)-(2.11), $\tilde{u}$ the interpolate of u in V_h, $\tilde{B}$ the interpolate of B in W_h, $\tilde{p}$ the interpolate of p in M_h. We denote by u_h, B_h and p_h the solution of the stabilized discrete problem. We denote the interpolation error for the velocity field $\pi_u = u - \tilde{u}_h$, the approximation error $e_u = \tilde{u}_h - u_h$, and the global error by $\epsilon_u = u - u_h$. We define in the same way π_B, e_B, ϵ_B for the magnetic field and π_p, e_p, ϵ_p for the pressure. Note that we have the relation $\epsilon_u = \pi_u + e_u$, $\epsilon_B = \pi_B + e_B$, $\epsilon_p = \pi_p + e_p$.

The strong consistence of the stabilized formulation implies :

$$\begin{aligned}\Phi(e_u, e_B, e_p; e_u, e_B, e_p) &= \Phi(\epsilon_u - \pi_u, \epsilon_B - \pi_B, \epsilon_p - \pi_p; e_u, e_B, e_p) \\ &= -\Phi(\pi_u, \pi_B, \pi_p; e_u, e_B, e_p).\end{aligned}$$

Therefore, the stability property proved in the previous section yields

$$\begin{aligned}&\int_\Omega (\eta|\nabla e_u|^2 + S\alpha|\mathrm{curl}\, e_B|^2 + S\alpha|\mathrm{div}\, e_B|^2)\, dx + 2\int_\Omega \tau_B S|\mathrm{curl}\,(e_u \times b)|^2\, dx \\ &+2\int_\Omega \tau_u|a.\nabla e_u + \nabla e_p + S\, b \times \mathrm{curl}\, e_B|^2\, dx \leq 2|\Phi(\pi_u, \pi_B, \pi_p; e_u, e_B, e_p)|.\end{aligned} \tag{3.7}$$

We now estimate the right-hand side of this inequality in order to "swallow" the terms e_u, e_B and e_p in the left-hand side. Let us recall Φ has been split into a classical Galerkin part Φ_G and a stabilization part Φ_S. As far as the Galerkin part is concerned, we have

$$\begin{aligned}2\Phi_G(\pi_u, \pi_B, \pi_p; e_u, e_B, e_p) = {}& 2\int_\Omega e_p\, \mathrm{div}\, \pi_u\, dx + \\ & 2\int_\Omega (\eta\nabla\pi_u \cdot \nabla e_u + a \cdot \nabla\pi_u \cdot e_u - \pi_p\, \mathrm{div}\, e_u + S\, b \times \mathrm{curl}\, \pi_B \cdot e_u)\, dx \\ & +2\int_\Omega (\alpha S\, \mathrm{curl}\, \pi_B \cdot \mathrm{curl}\, e_B + \alpha S\, \mathrm{div}\, \pi_B \mathrm{div}\, e_B - S\, \pi_u \times b \cdot \mathrm{curl}\, e_B)\, dx.\end{aligned}$$

The diffusion terms are treated straightforwardly :

$$2\eta\int_\Omega |\nabla\pi_u||\nabla e_u|\, dx \leq \gamma_1\eta\int_\Omega |\nabla\pi_u|^2\, dx + \frac{\eta}{\gamma_1}\int_\Omega |\nabla e_u|^2\, dx,$$

$$\begin{aligned}&2\alpha S\int_\Omega (|\mathrm{curl}\, \pi_B||\mathrm{curl}\, e_B| + |\mathrm{div}\, \pi_B||\mathrm{div}\, e_B|)\, dx \leq \\ &\qquad \gamma_2\alpha S\int_\Omega (|\mathrm{curl}\, \pi_B|^2 + |\mathrm{div}\, \pi_B|^2)\, dx + \frac{\alpha S}{\gamma_2}\int_\Omega (|\mathrm{curl}\, e_B|^2 + |\mathrm{div}\, e_B|^2)\, dx.\end{aligned}$$

We use the stabilization terms in order to control the convection and the incompressibility terms :

$$2\left|\int_\Omega a\cdot\nabla\pi_u\cdot e_u + e_p \mathrm{div}\,\pi_u - S\pi_u\times b\cdot\mathrm{curl}\,e_B\,dx\right| \leq$$
$$\frac{1}{\gamma_3}\int_\Omega \tau_u|a.\nabla e_u+\nabla e_p + S\,b\times\mathrm{curl}\,e_B|^2\,dx + \gamma_3\int_K \frac{1}{\tau_u}|\pi_u|^2\,dx.$$

Intuitively, it is worth noticing that there is a “swap” between the Navier-Stokes and the Maxwell equations : on the one hand, the term $\pi_u\times b\cdot\mathrm{curl}\,e_B$ comes from the Maxwell equations and it is “swallowed” by the stabilization term of the Navier-Stokes equations. On the other hand, $b\times\mathrm{curl}\,\pi_B\cdot e_u$ which comes from the Lorentz force of the Navier-Stokes equations is “swallowed” by the stabilization term of the Maxwell equations (see Remark 3.2). Using formula (2.16), we actually have :

$$2\left|\int_\Omega S\,b\times\mathrm{curl}\,\pi_B\cdot e_u\,dx\right| \leq \frac{S}{\gamma_4}\int_\Omega \tau_B|\mathrm{curl}\,(e_u\times b)|^2\,dx + S\gamma_4\int_\Omega\frac{1}{\tau_B}|\pi_B|^2\,dx.$$

Finally the pressure term is bounded by using (2.15) :

$$2\int_\Omega|\pi_p||\mathrm{div}\,e_u|\,dx \leq \frac{3\eta}{\gamma_5}\int_\Omega|\nabla e_u|^2\,dx + \frac{\gamma_5}{\eta}\int_\Omega|\pi_p|^2\,dx.$$

Note that this control would be slightly different if we introduce the term of Remark 3.4).

Let us now estimate the stabilization part :

$$\begin{aligned}\Phi_S(\pi_u,\pi_B,\pi_p;e_u,e_B,e_p) \;=\; & \sum_{K\in\mathcal{T}_h}\int_K \tau_u \quad (a\cdot\nabla\pi_u - \eta\triangle\,\pi_u + \nabla\pi_p + S\,b\times\mathrm{curl}\,\pi_B)\cdot \\ & \qquad\qquad (a\cdot\nabla e_u + \eta\triangle\,e_u + \nabla e_p + S\,b\times\mathrm{curl}\,e_B)\,dx \\ + & \sum_{K\in\mathcal{T}_h}\int_K \tau_B \quad (-\alpha S\triangle\,\pi_B - S\mathrm{curl}\,(\pi_u\times b))\cdot \\ & \qquad\qquad (\alpha\triangle\,e_B - \mathrm{curl}\,(e_u\times b))\,dx.\end{aligned}$$

The stabilization terms of the Navier-Stokes equations are bounded as follows :

$$2\sum_{K\in\mathcal{T}_h}\int_K\tau_u|a\cdot\nabla\pi_u||a\cdot\nabla e_u+\nabla e_p+S\,b\times\mathrm{curl}\,e_B|\,dx \leq \gamma_3\int_\Omega\tau_u|a\cdot\nabla\pi_u|^2\,dx +$$
$$\frac{1}{\gamma_3}\int_\Omega\tau_u|a\cdot\nabla e_u+\nabla e_p+S\,b\times\mathrm{curl}\,e_B|^2\,dx,$$

$$2\sum_{K\in\mathcal{T}_h}\int_K\tau_u|a\cdot\nabla\pi_u||\eta\triangle\,e_u|\,dx \quad\leq\quad \gamma_6\int_\Omega\tau_u|a\cdot\nabla\pi_u|^2\,dx + \frac{d_0\lambda_u}{\gamma_6}\int_\Omega\eta|\nabla e_u|^2\,dx,$$

$$2\sum_{K\in\mathcal{T}_h}\int_K \tau_u|\eta\triangle\pi_u||a\cdot\nabla e_u+\nabla e_p+S\,b\times\mathrm{curl}\,e_B|\,dx \le \gamma_3\sum_{K\in\mathcal{T}_h}\int_K \tau_u\eta^2|\triangle\pi_u|^2\,dx+$$
$$\frac{1}{\gamma_3}\int_\Omega \tau_u|a\cdot\nabla e_u+\nabla e_p+S\,b\times\mathrm{curl}\,e_B|^2\,dx,$$

$$2\sum_{K\in\mathcal{T}_h}\int_K \tau_u|\eta\triangle\pi_u||\eta\triangle e_u|\,dx \quad\le\quad \frac{d_0\lambda_u}{\gamma_6}\int_\Omega \eta|\nabla e_u|^2\,dx+\gamma_6\sum_{K\in\mathcal{T}_h}\int_K \tau_u\eta^2|\triangle\pi_u|^2\,dx,$$

$$2\sum_{K\in\mathcal{T}_h}\int_K \tau_u|\nabla\pi_p||a\cdot\nabla e_u+\nabla e_p+S\,b\times\mathrm{curl}\,e_B|\,dx \le \gamma_3\int_\Omega \tau_u|\nabla\pi_p|^2\,dx+$$
$$\frac{1}{\gamma_3}\int_\Omega \tau_u|a\cdot\nabla e_u+\nabla e_p+S\,b\times\mathrm{curl}\,e_B|^2\,dx,$$

$$2\sum_{K\in\mathcal{T}_h}\int_K \tau_u|\nabla\pi_p||\eta\triangle e_u|\,dx \quad\le\quad \frac{d_0\lambda_u}{\gamma_6}\int_\Omega \eta|\nabla e_u|^2\,dx+\gamma_6\int_\Omega \tau_u|\nabla\pi_p|^2\,dx,$$

$$2\sum_{K\in\mathcal{T}_h}\int_K \tau_u|S\,b\times\mathrm{curl}\,\pi_B||a\cdot\nabla e_u+\nabla e_p+S\,b\times\mathrm{curl}\,e_B|\,dx \le$$
$$\gamma_3\int_\Omega \tau_u S^2\,|b\times\mathrm{curl}\,\pi_B|^2\,dx+\frac{1}{\gamma_3}\int_\Omega \tau_u|a\cdot\nabla e_u+\nabla e_p+S\,b\times\mathrm{curl}\,e_B|^2\,dx,$$

$$2\sum_{K\in\mathcal{T}_h}\int_K \tau_u|S\,b\times\mathrm{curl}\,\pi_B||\eta\triangle e_u|\,dx \le$$
$$\frac{d_0\lambda_u}{\gamma_6}\int_\Omega \eta|\nabla e_u|^2\,dx+\gamma_6\int_\Omega \tau_u S^2|b\times\mathrm{curl}\,\pi_B|^2\,dx.$$

Finally, let us estimate the stabilization terms of the Maxwell equations :

$$2\sum_{K\in\mathcal{T}_h}\int_K \tau_B|\alpha S\triangle\pi_B||\alpha\triangle e_B|\,dx \le \frac{d_0\lambda_B}{\gamma_7}\int_\Omega \alpha|\mathrm{curl}\,e_B|^2+\alpha|\mathrm{div}\,e_B|^2\,dx+$$
$$\sum_{K\in\mathcal{T}_h}\gamma_7\int_K \tau_B\alpha^2S^2|\triangle\pi_B|^2\,dx,$$

$$2\sum_{K\in\mathcal{T}_h}\int_K \tau_B|\alpha S\triangle\pi_B||\mathrm{curl}\,(e_u\times b)|\,dx \le \frac{1}{\gamma_4}\int_\Omega \tau_B|\mathrm{curl}\,(e_u\times b)|^2\,dx+$$
$$\textstyle\sum_{K\in\mathcal{T}_h}\gamma_4\int_K \tau_B\alpha^2S^2|\triangle\pi_B|^2\,dx,$$

$$2\sum_{K\in\mathcal{T}_h}\int_K \tau_B|S\,\mathrm{curl}\,(\pi_u\times b)||\alpha\triangle e_B|\,dx \le \gamma_7\int_\Omega \tau_B S^2|\mathrm{curl}\,(\pi_u\times b)|^2\,dx+$$
$$\frac{d_0\lambda_B}{\gamma_7}\int_\Omega \alpha|\mathrm{curl}\,e_B|^2+\alpha|\mathrm{div}\,e_B|^2\,dx,$$

$$2\sum_{K\in\mathcal{T}_h}\int_K \tau_B|S\,\mathrm{curl}\,(\pi_u\times b)||\mathrm{curl}\,(e_u\times b)|\,dx \le \gamma_4\int_\Omega \tau_B S^2|\mathrm{curl}\,(\pi_u\times b)|^2\,dx+$$
$$\frac{1}{\gamma_4}\int_\Omega \tau_B|\mathrm{curl}\,(e_u\times b)|^2\,dx.$$

If we now insert the above inequality (with convenient γ_i) in (3.7), we obtain :

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega (\eta|\nabla e_u|^2 + S\alpha|\mathrm{curl}\, e_B|^2 + S\alpha|\mathrm{div}\, e_B|^2)\,dx + \int_\Omega \tau_B S|\mathrm{curl}\,(e_u\times b)|^2\,dx \\
&+\int_\Omega \tau_u|a.\nabla e_u + \nabla e_p + S\, b\times \mathrm{curl}\, e_B|^2\,dx \le \\
&\qquad c_1\int_\Omega \tau_u|a\cdot\nabla\pi_u|^2\,dx + c_2\sum_{K\in\mathcal{T}_h}\int_K \tau_u\eta^2|\triangle\,\pi_u|^2\,dx + \\
&\qquad c_3\int_\Omega \tau_u|\nabla\pi_p|^2\,dx + \frac{c_4}{\eta}\int_\Omega|\pi_p|^2\,dx + c_5\int_\Omega \tau_u S^2|b\times\mathrm{curl}\,\pi_B|^2 + \\
&\qquad c_6\sum_{K\in\mathcal{T}_h}\int_K \tau_B\alpha^2|\triangle\,\pi_B|^2\,dx + c_7\int_K S^2\tau_B|\mathrm{curl}\,(\pi_u\times b)|^2\,dx,
\end{aligned}
\tag{3.8}
$$

where the constants c_i does not depend on h_K, η, α, a and b. In order to achieve the proof, we establish the following interpolation estimates. Using $\tau_u\eta^2 \le \lambda_u\eta h_K^2$, we have :

$$
c_1\int_K \tau_u|a\cdot\nabla\pi_u|^2\,dx + c_2\int_K \tau_u\eta^2|\triangle\,\pi_u|^2\,dx \le C(||a||_\infty h_K + \eta)\lambda_u h_K^{2k}|u|^2_{k+1,K},
$$

$$
c_3\int_K \tau_u|\nabla\pi_p|^2\,dx + \frac{c_4}{\eta}\int_K|\pi_p|^2\,dx \le C(\frac{\lambda_u}{||a||_\infty} + \frac{h_K}{\eta})h_K^{2m+1}|p|^2_{m+1,K},
$$

$$
c_5\int_K \tau_u S^2|b\times\mathrm{curl}\,\pi_B|^2 + c_6\int_K \tau_B\alpha^2|\triangle\,\pi_B|^2\,dx \le C\left(\lambda_u\frac{S^2||b||^2_\infty}{||a||_\infty}h_K + \lambda_B\alpha\right)h_K^{2l}|B|^2_{l+1,K},
$$

$$
\begin{aligned}
c_7\int_K S^2\tau_B|\mathrm{curl}\,(\pi_u\times b)|^2\,dx &= c_7\int_K S^2\tau_B|\pi_u|^2(|\mathrm{div}\, b|^2 + |\nabla b|^2)\,dx \\
&\quad + c_7\int_K S^2\tau_B|b|^2(|\mathrm{div}\,\pi_u|^2 + |\nabla\pi_u|^2)\,dx \\
&\le C(||b||^2_\infty + h_K^2||\nabla b||^2_\infty)\frac{S^2h_K^2}{\alpha}h_K^{2k}|u|^2_{k+1,K}.
\end{aligned}
$$

Inserting these inequalities in (3.8), we obtain (3.4). This concludes the proof of Theorem 1. $\diamondsuit$

Remark 3.2 *It has been noticed in this proof that $\pi_u\times b\cdot curl\, e_B$ coming from the Maxwell equations was swallowed by the stabilization term of the Navier-Stokes equations, and conversely for $b\times curl\,\pi_B\cdot e_u$. That is why we have stabilized also the Maxwell equation. It is worth mentioning how one would control $b\times curl\,\pi_B\cdot e_u$ if the Maxwell equations were not stabilized (in other words if $\tau_B = 0$) :*

$$
\begin{aligned}
\left|\int_\Omega b\times curl\,\pi_B\cdot e_u\,dx\right| &= \left|\int_\Omega curl\,(e_u\times b)\cdot\pi_B\,dx\right| \\
&\le 2C(||b||_\infty + ||\nabla b||_\infty)||\nabla e_u||_{L^2}||\pi_B||_{L^2} \\
&\le \frac{\eta}{\gamma_1}\int_\Omega|\nabla e_u|^2\,dx + C\frac{\gamma_1||b||^2_{W^{1,\infty}}}{\eta}\int_\Omega|\pi_B|^2\,dx.
\end{aligned}
$$

Thus, the following term would occur in the error estimate on B :

$$\frac{h^2||b||^2_{W^{1,\infty}}}{\eta} h^{2l}|B|^2_{l+1}.$$

This yields the convergence of the approximation, but the control is less accurate than that of Theorem 1 as soon as $||a||_\infty h/\eta$ is large.

Remark 3.3 *For the sake of simplicity, it has been assumed in this proof that the regime was advection-dominated on the whole domain. Nevertheless, for the numerical simulations, we have used the more sophisticated definition of the stabilization parameters proposed by L.P. Franca and S.L. Frey in [14] in order to adapt the stabilization to the regime of the flow.*

Remark 3.4 *It is proposed in [14] to add the following term to Φ :*

$$\int_\Omega \delta\, div\, u\, div\, v\, dx,$$

where δ is a variable parameter. It is straightforward to modify the above proof taking into account this term by following the arguments of [14]. Our numerical tests have been done with this slight modification.

4 Numerical results

The numerical tests presented in this section have been performed with an academic code. The fluid dynamic code FIDAP has been used for the grids generation and the postprocessing.

As far as we know, the experiments presented here are new in the framework of the scientific computing literature on MHD. Other experiments can be found in F. Armero and J.C. Simo [1] (Hartmann flows in 2D and MHD flow past a cylinder) and in J.-F. Gerbeau [18] (Hartmann flows in 2D and 3D with perfectly conducting and insulating walls).

4.1 MHD flow over a step

In the first experiment, we consider the flow of a fluid over a step in presence of a transverse magnetic field $B_0(0, 1)$. The sides of the duct are assumed to be perfectly conducting. The inlet and the outlet values of the velocity are the Poiseuille profile (Hartmann profile might be better but this is not relevant for our purpose). See Figure 1 for the definition of the geometry and the boundary conditions.

We first compare the pressure contours obtained with the Q2/P1 classical elements and with the Q1/Q1 stabilized elements. Figure 2 show a good agreement between the two results.

On Figure 3, we show that the magnetic field acting on the fluid tends to decrease the recirculation after the step. This brings to the fore, in this particular configuration, a well-known property of a MHD flow (see e.g. R. Moreau [30]).

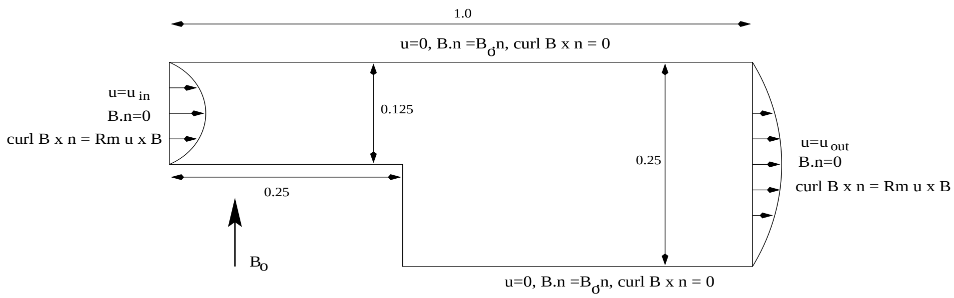
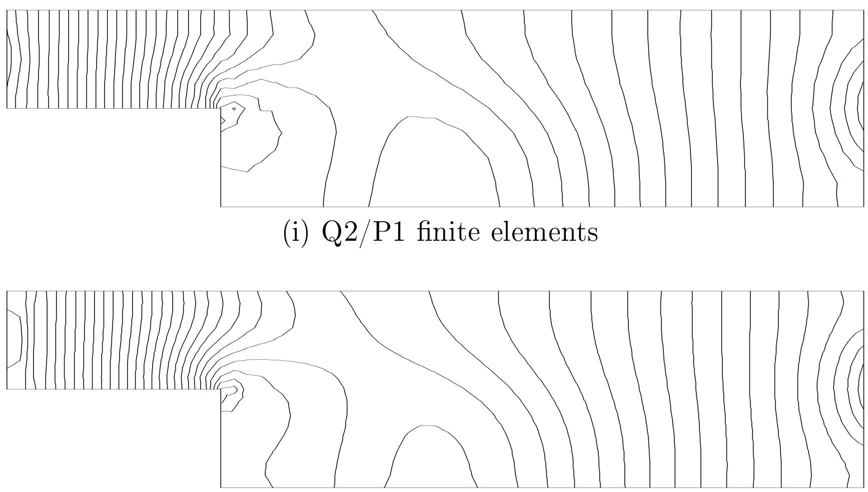


Figure 1: Flow over a step in presence of a transverse magnetic field.



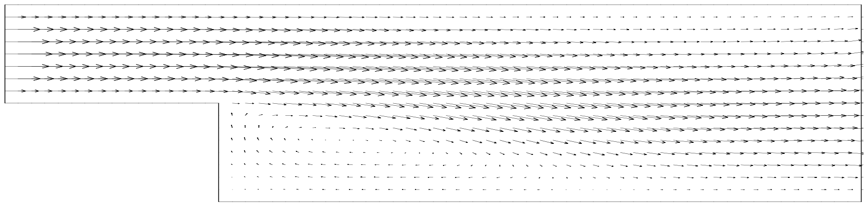
(i) Q2/P1 finite elements

(ii) stabilized Q1/Q1 finite elements

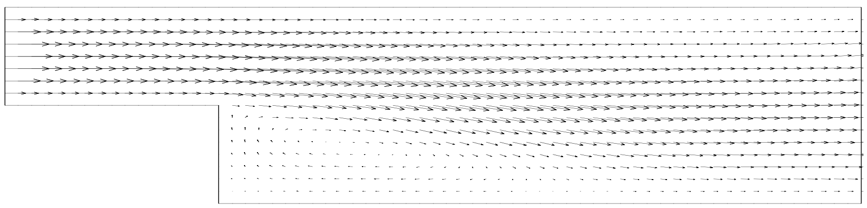
Figure 2: Pressure in a MHD flow over a step Re=100, Rm=0.125, S=1.

4.2 A fluid carrying current in presence of a magnetic field

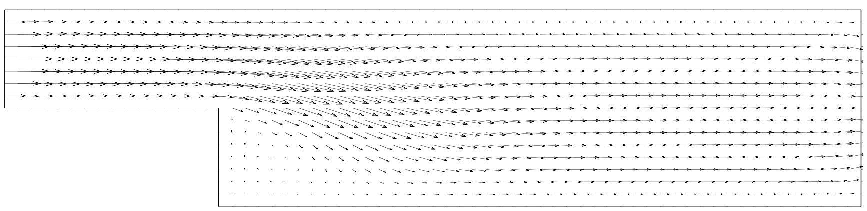
The following 3D computations model the behavior of a fluid crossed by an homogeneous electric current from the top to the bottom. The fluid is enclosed within a parallelepiped whose top and bottom are assumed to be perfectly conducting and whose side is perfectly insulating. Two linear conductors surround the parallelepiped and create a magnetic field (see Fig. 4). The grid we used is coarse (about 4000 nodes, Fig. 5). We emphasize that



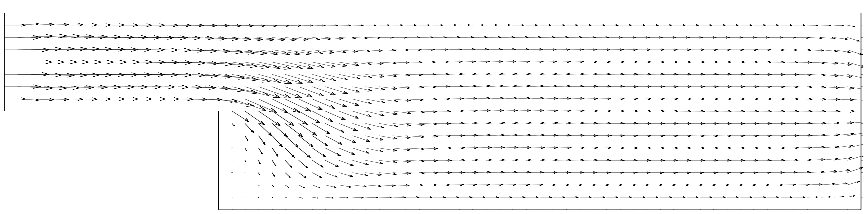
(i) Re=100, Ha=0 (no magnetic field).



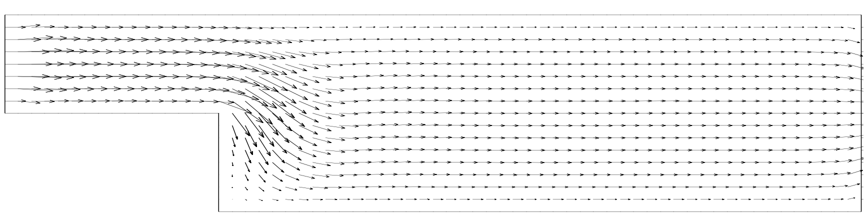
(i) Re=100, Ha=1.



(ii) Re=100, Ha=5.



(iii) Re=100, Ha=10.



(iv) Re=100, Ha=20.

Figure 3: Effect of a magnetic field on a flow over a step. Notice the diminution of the recirculating area when the Hartmann number increases.

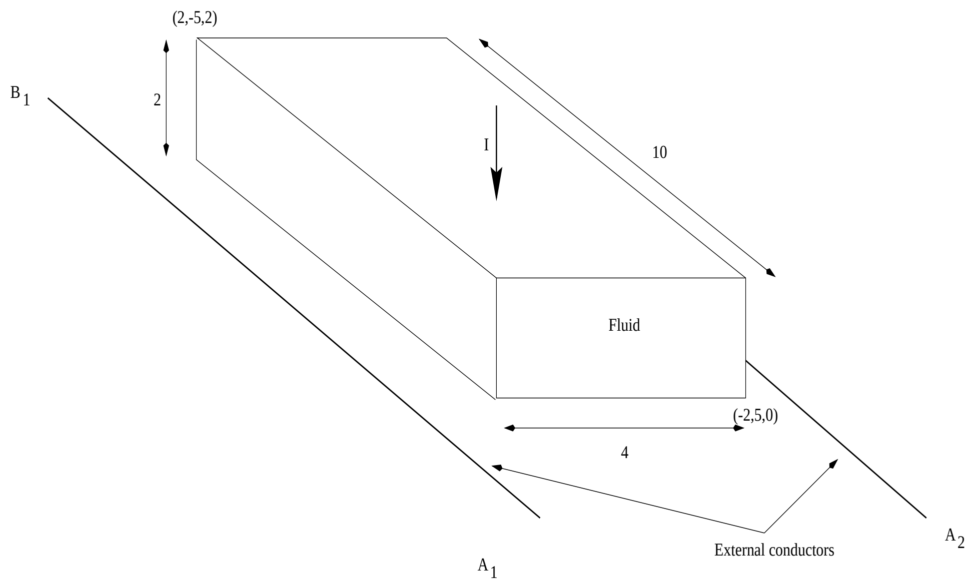


Figure 4: A fluid crossed by a current in presence of external conductors

such an experiment may be interesting from an industrial viewpoint, in the context of aluminum electrolysis cells for example (more sophisticated external conductors should of course be considered in this case).

The magnetic field created by the external conductors and the homogeneous current crossing the parallelepiped (the fluid being at rest) can be straightforwardly computed by the Biot and Savart formula. This field is used to enforce the boundary conditions on B in order to perform the MHD computation. This is of course an approximation of the reality since we do not take into account the influence of the hydrodynamic on the magnetic boundary values. Nevertheless, as will be shown, the results we obtain are qualitatively good.

Three configurations are considered for the external conductors. The tops of Figures 6, 7 and 8 show the different configurations and their effects on the fluid according to the physical intuition (see J.M. Blanc and P. Entner [4] for example). Our numerical results are in perfect agreement with the predictions.

The computations of Figures 6, 7 and 8 are performed with $Re = 100$. In this range of Reynolds number, similar results may be obtained with classical finite elements. But when the Reynolds number increases, oscillations appear with the classical approximation whereas the stabilization methods still exhibit good results. This is shown on Figures 9 and 10 (this case corresponds to the same experiment as on Figure 6 but with $Re = 300$).

Finally, for the experiment of Figure 6 (one vortex), we have compared

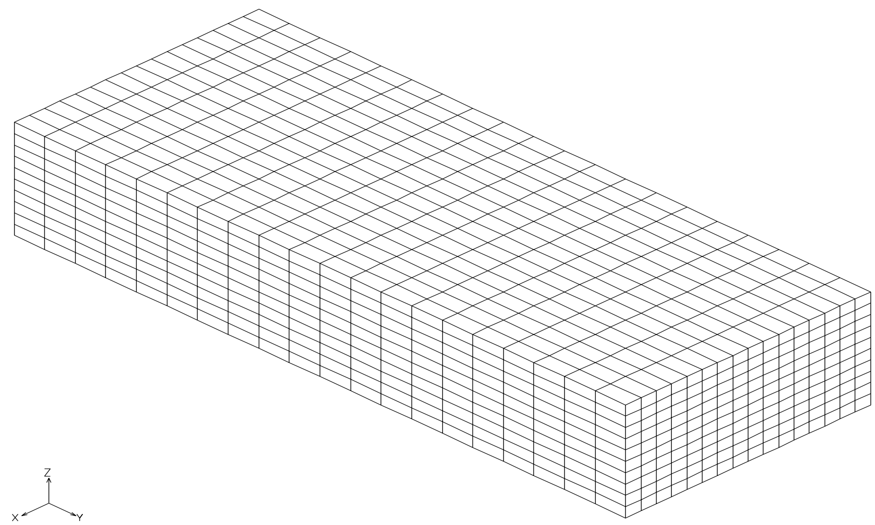


Figure 5: The mesh (about 4000 nodes).

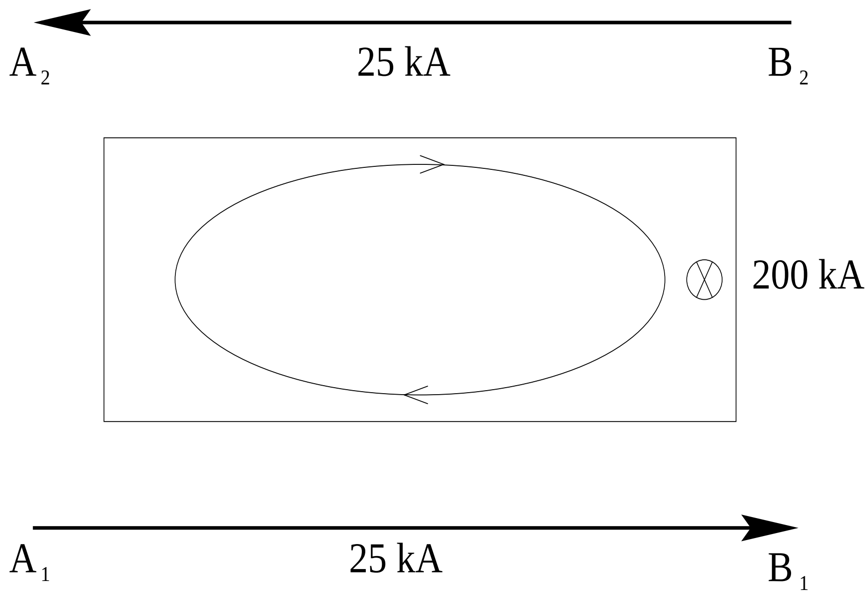
the three following kinds of approximation with various Reynolds numbers : Q1/P0 (non-stabilized), Q1/Q1 when the Navier-Stokes is stabilized ("Q1/Q1 Stab NS") and Q1/Q1 when the Navier-Stokes *and* the Maxwell equations are stabilized ("Q1/Q1 Stab NS+Max").

The computation is initialized by four Picard iterations and the convergence is achieved with a Newton algorithm. The relative non-linear residual is 10^{-3}. The linear systems are solved with an ILU preconditioned CGS algorithm, the relative linear residual is 10^{-9}. Tabular 1 shows the *total* number of CGS iterations required to solve the linear systems. This value gives a good idea of the CPU time since the total computational time is essentially dedicated to the resolution of the linear systems.

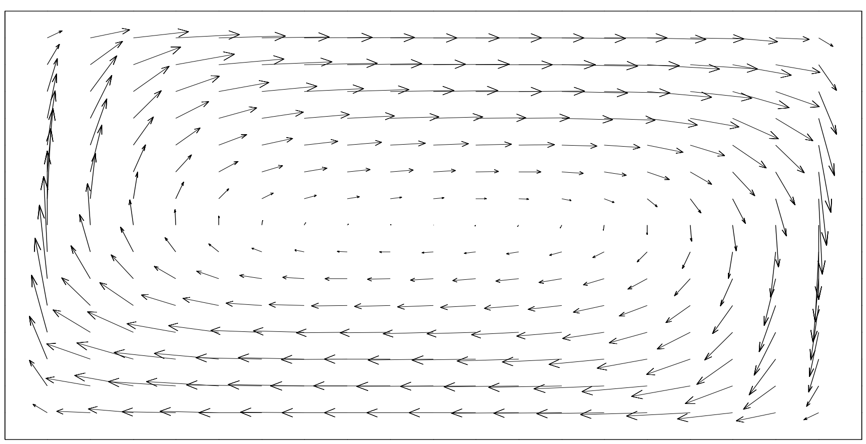
Re	Q1/P0	Q1/Q1 Stab NS	Q1/Q1 Stab NS+Max
100	309	146	141
200	1274	239	213
500	diverges	420	373
1000	diverges	diverges	560

Table 1: Total number of iterations required to solve the linear systems.

It is worth noticing that these results confirm Remark 3.2 following the proof of the convergence theorem : it is all the more interesting to stabilize the magnetic equation as the Reynolds number is large, even if the magnetic Reynolds number is small.

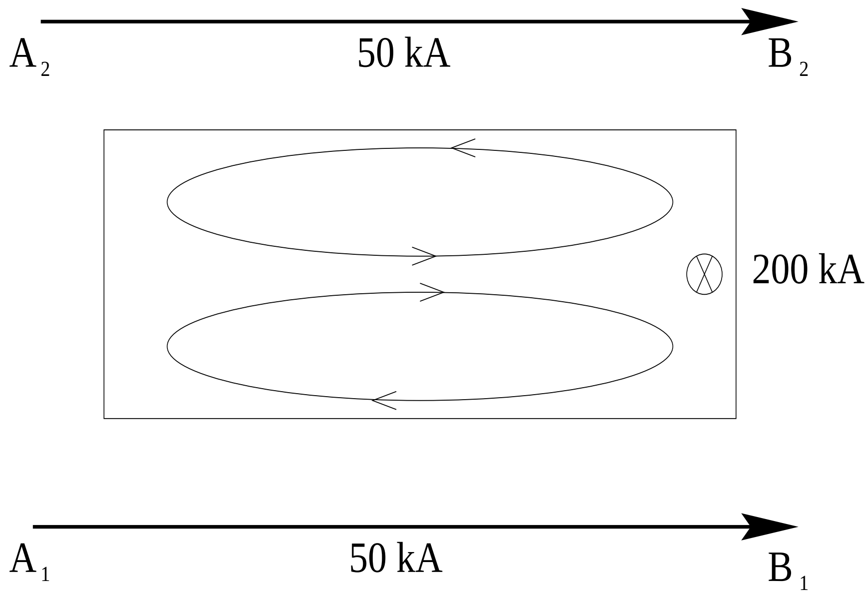


(i) Predicted result.

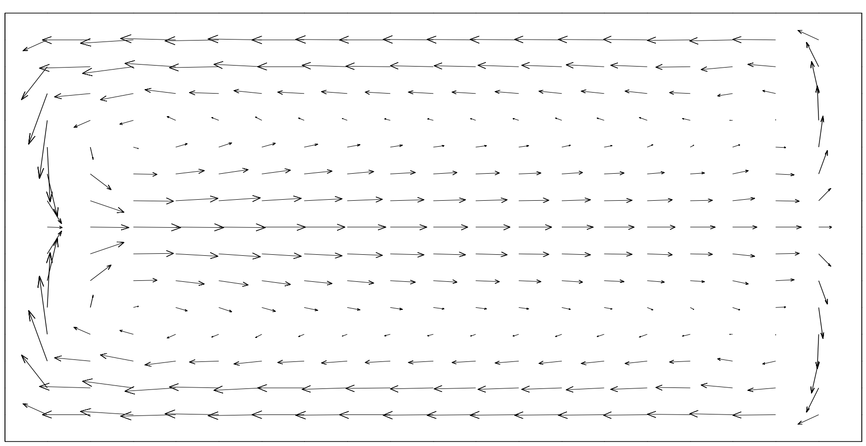


(ii) 3D computation, top view, plane z=1.

Figure 6: First configuration. Re=100, S=Rm=1.

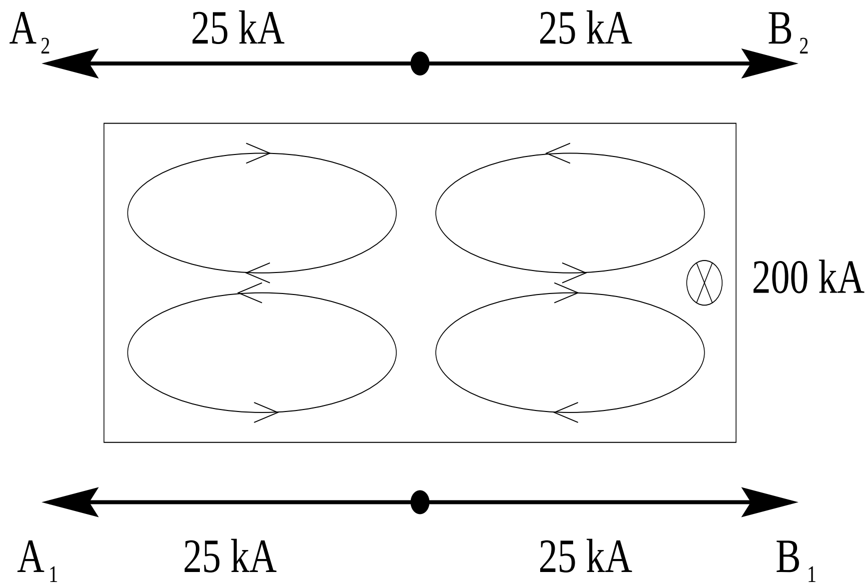


(i) Predicted result.

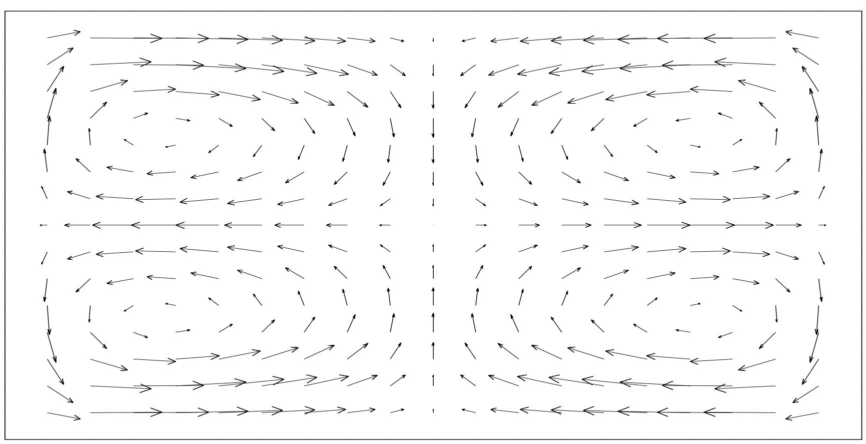


(ii) 3D computation, top view, plane z=1.

Figure 7: Second configuration. Re=100, S=Rm=1.

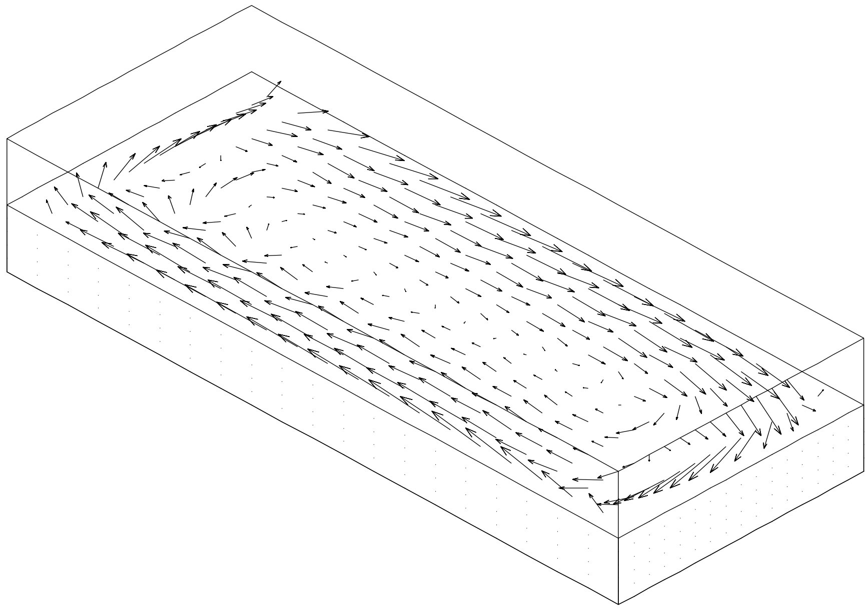


(i) Predicted result.

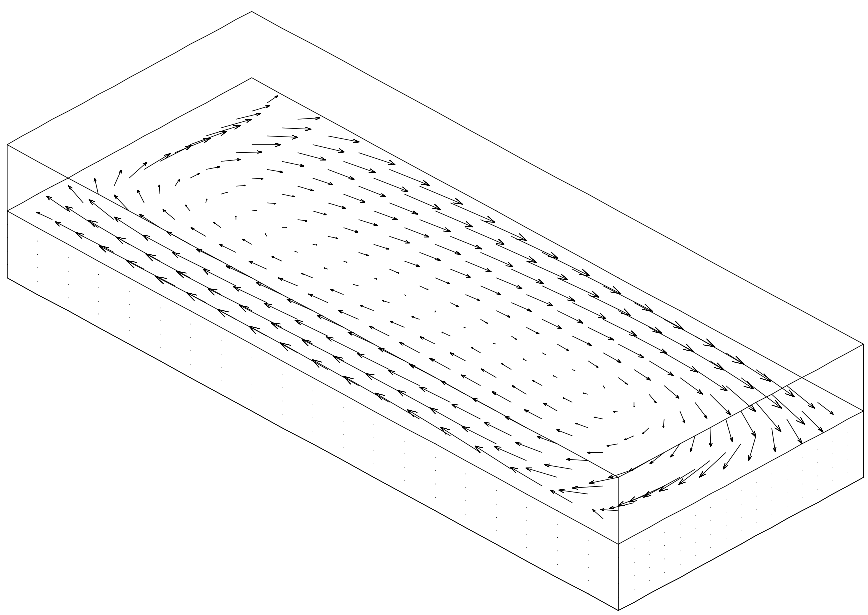


(ii) 3D computation, top view, plane z=1.

Figure 8: Third configuration. Re=100, S=Rm=1.



(i) Q1/P0 elements (notice the oscillations).



(ii) Stabilized Q1/Q1 elements.

Figure 9: Same experiment as on Figure 6 with $Re = 300$.

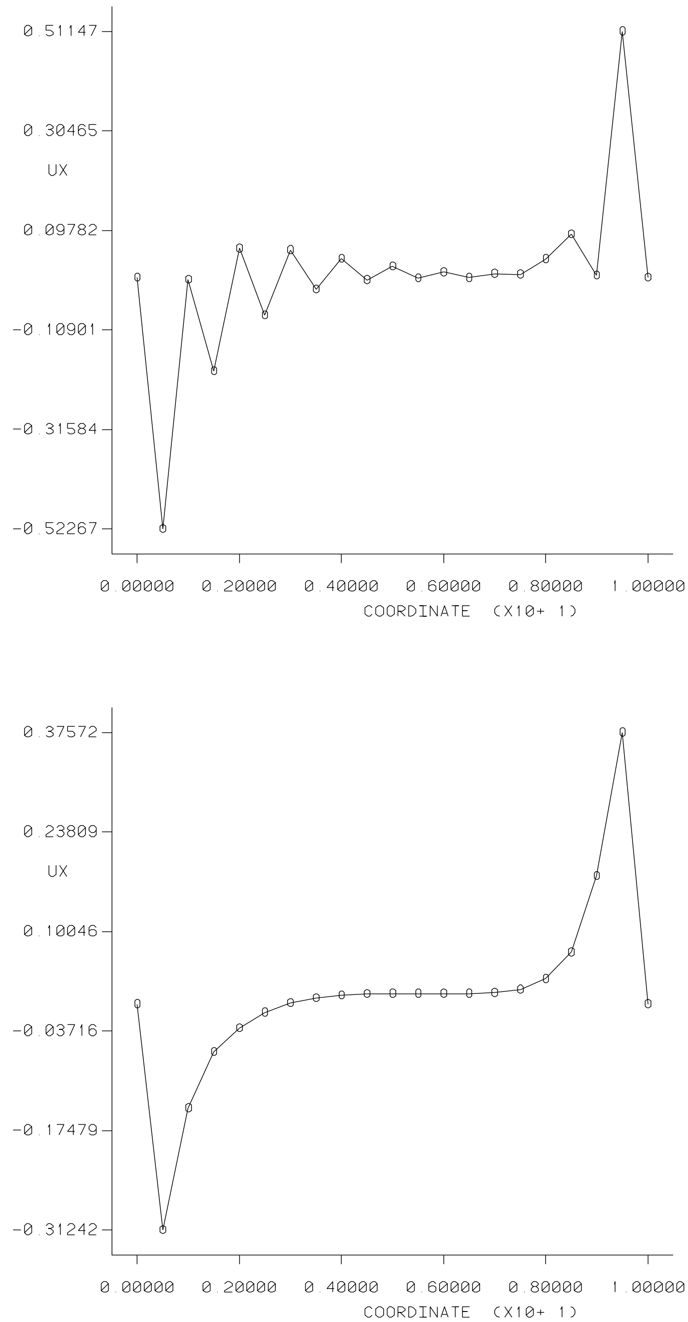


Figure 10: Computation of Figure 9 : first component of the velocity along a straight line with Q1/P0 (oscillations) and stabilized Q1/Q1 elements.

5 Conclusion

After a brief presentation of some well-known stabilized finite element methods for the advection-diffusion and the Stokes equations, we have proposed and analyzed an extension of such methods for the incompressible MHD equations. Three remarks may be drawn from our study. First, the pressure contours obtained with equal order stabilized finite element spaces are as good as those obtained with classical pairs of finite elements satisfying the inf-sup conditions (in our experiments at least). Second, non-stabilized methods exhibit oscillations when the Reynolds number tends to realistic physical values whereas stabilized methods do not. Third, even when the magnetic diffusion is high – this is generally the case in the physical applications we have in mind – it may be useful to stabilized both Navier-Stokes and Maxwell equations. These conclusions have of course to be confirmed in further tests.

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