Path properties of superprocesses with a general branching mechanism

Jean-François DELMAS *

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Abstract

We first consider a super Brownian motion X with a general branching mechanism. Using the Brownian snake representation with subordination, we get the Hausdorff dimension of supp X_t, the topological support of X_t, and more generally the Hausdorff dimension of $\cup_{t\in B}$supp X_t. We also provide estimations on the hitting probability of small balls for those random measures. We then deduce that the support is totally disconnected in high dimension. Eventually, considering a super α-stable process with a general branching mechanism, we prove that in low dimension, this random measure is absolutely continuous with respect to the Lebesgue measure.

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1 Introduction

Superprocesses $(X_t, t \geq 0)$ are measure valued branching processes whose distribution can be characterized by a pair (γ, Ψ), where γ is the underlying Markov process, playing the role of the spatial motion and Ψ is the branching mechanism function. We refer to Dynkin [10, 11] for basic facts about superprocesses and their construction as limits of branching particle systems. Some recent studies on super-Brownian motion (corresponding to the case when γ is a Brownian motion in $\mathbb{R}^d$ and $\Psi(\lambda) = \lambda^2$) give the exact Hausdorff measure of its support supp X_t, at fixed time $t > 0$, see Perkins [20, 21], Dawson, Iscoe, and Perkins [8], Tribe [25], see also Dawson [7], theorem 9.3.3.5 for the Hausdorff dimension of supp X_t with $\Psi(\lambda) = \lambda^{1+\rho}$, $\rho \in (0, 1)$. The proof relies on approximation of super-Brownian motion by branching particle systems. Another way to study this superprocess is to use the Brownian snake introduced by Le Gall [17, 18] which is a path valued Markov process. In [3], Bertoin, Le Gall and Le Jan succeeded through a subordination method to use the Brownian snake to represent superprocesses with a rather general branching mechanism. Their construction applies in particular to the stable case $\Psi(\lambda) = \lambda^{1+\rho}$ for $\rho \in (0, 1]$. In the present paper,

*ENPC-CERMICS, 6 et 8, avenue Blaise Pascal, Cité Descartes, Champs-sur-Marne, 77455 Marne La Vallée, France.

we shall use this path representation to derive some properties of the (γ, Ψ) superprocess when γ is a Brownian motion in $\mathbb{R}^d$ and Ψ is of the type considered in [3]. In particular we give the Hausdorff dimension of the closure of $\bigcup_{t\in B} \mathrm{supp}\ X_t$, when B is a closed subset of $(0,\infty)$ (theorem 2.1). We also provide sufficient conditions for the a.s. absolute continuity of the measure X_t (theorem 2.5), thus extending to a general branching mechanism a well-known result for super-Brownian motion (see Dawson [7]). The result can be generalized to α-stable superprocesses, extending results of Fleischmann [15] and of Dawson [7]. We then use exit measures to give precise lower and upper bounds for hitting probabilities of small balls (theorem 2.3). As an application, we can prove that if the dimension is large enough, the support of X_t is totally disconnected (theorem 2.4). This extends a result of Tribe [26] concerning super-Brownian motion.

Let us now describe more precisely the contents of the following sections. In section 2, we recall the definition of Hausdorff dimension and upper box-counting dimension. We introduce the special type of branching mechanism function Ψ that we will consider. We recall the definition of the (γ, Ψ) superprocess X, where γ is a Brownian motion in $\mathbb{R}^d$. The Laplace transform of X is related to the solution of an integral equation (1). We then state the main results of this paper. In particular, theorem 2.1 provides upper and lower bounds on the Hausdorff dimension of the closure of $\bigcup_{t\in B} \mathrm{supp}\ X_t$. Under suitable assumptions, the lower and upper bounds coincide and we get the exact value of the dimension.

With the branching mechanism Ψ, we can associate a subordinator S that plays a key role in the subordination method. Section 3 is devoted to some preliminary results on this subordinator. We give short proofs for the reader's convenience.

In section 4, we first recall the subordination method of [3] based on the Brownian snake. Precisely, we consider the path-valued process of [17] when the underlying (Markov) spatial motion is a triple (ξ_t, L_t, Γ_t) whose law can be described as follows. First ξ is the residual lifetime process associated with S: $\xi_t = \inf\{S_r - t; r \geq 0, S_r > t\}$. Second L_t is the right-continuous inverse of S (equivalently it is the local time at 0 of ξ). Finally $\Gamma_t = \gamma_{L_t}$, where γ is a Brownian motion in $\mathbb{R}^d$ independent of S. Using the Brownian snake with spatial motion (ξ, L, Γ), we can give an explicit formula for the (γ, Ψ)-superprocess. This formula is crucial for our investigation of path properties.

In section 5, we prove theorem 2.1. The proof of the lower bound on the Hausdorff dimension uses a "Palm measure formula" for the exit measure associated to the Brownian snake (proposition 4.2), classical results from Falconer [13] and technical results that are derived in the appendix. The upper bound is a bit more complex, and really relies on the path properties of the Brownian snake and its transition kernel. At this point, the Brownian snake approach is used in its full strength.

Section 6 is devoted to our bounds on hitting probabilities of small balls and the result about connected components of the support of super-Brownian motion. Lower bounds on hitting probabilities are quite easy to prove from the integral equation (1). The upper bounds use the special Markov property of the Brownian snake and the connection between exit measures and solutions of nonlinear partial differential equations (see Dynkin [11, 12], see also Le Gall [18] for the snake approach). The proof of the theorem on connected components then follows from a technique of Perkins (see [22] p.1041).

Finally in section 7, we discuss the absolute continuity of the measure $\int \mu(ds)\ X_s$. Assume that $\iint \mu(ds)\mu(dt)\ \ |s-t|^{-q} < \infty$, where $q \in [0,1)$. Then we prove that in the ρ-stable branching case ($\Psi(\lambda) = \lambda^{1+\rho}$), $\int \mu(ds)\ X_s$ is absolutely continuous if $d < 2(q + 1/\rho)$. If the underlying Brownian motion is replaced by an α-stable symmetric Lévy process in $\mathbb{R}^d$, $\alpha \in (0,2)$, then the measure $\int \mu(ds)\, X_s$ is absolutely continuous if $d < \alpha(q + 1/\rho)$.

2 Notation and results

First we introduce some notation. We denote by $(M_f, \mathcal{M}_f)$ the space of all finite nonnegative measures on $\mathbb{R}^d$, endowed with the topology of weak convergence. We denote by $\mathcal{B}(\mathbb{R}^p)$ the set of all measurable functions defined on $\mathbb{R}^p$ taking values in $\mathbb{R}$. With a slight abuse of notation, we also denote by $\mathcal{B}(\mathbb{R}^p)$ the Borel σ-field on $\mathbb{R}^p$. For every measure $\nu \in M_f$, and every nonnegative function $f \in \mathcal{B}(\mathbb{R}^d)$, we shall use both notations $\int f(y)\nu(dy) = (\nu, f)$. We also write $\nu(A) = (\nu, \mathbf{1}_A)$ for $A \in \mathcal{B}(\mathbb{R}^d)$. For $A \in \mathcal{B}(\mathbb{R}^p)$, let $\mathcal{C}l(A)$ be the closure of A. We recall briefly the definition of Hausdorff dimension and upper box-counting dimension (cf [13]). Let $A \in \mathcal{B}(\mathbb{R}^p)$ bounded. Let $C_\varepsilon(A)$ denote the set of all coverings $C = \{B_i, i \in I\}$ of A with balls B_i of radius $|B_i| \le \varepsilon$. Then for every $r > 0$, we consider

$$H^r_\varepsilon(A) = \inf_{C \in C_\varepsilon(A)} \sum_{i \in I} |B_i|^r .$$

Clearly $H^r_\varepsilon(A)$ increases to $H^r(A) \in [0,\infty]$, as ε decreases to $0+$. The mapping $r \mapsto H^r(A)$ is decreasing. Moreover we see that if $H^r(A) < \infty$, then $H^{r'}(A) = 0$ for every $r' > r$; and if $H^r(A) > 0$, then $H^{r'}(A) = \infty$ for every $r' < r$. The critical value

$$\dim A = \sup\{r > 0, H^r(A) = \infty\} = \inf\{r > 0, H^r(A) = 0\},$$

with the convention $\sup \emptyset = 0$, is called the Hausdorff dimension of A. Then consider $N_\varepsilon(A)$ the minimal number of balls of radius ε necessary to cover A. Define the upper box-counting dimension of A by

$$\overline{\dim}\, A = \limsup_{\varepsilon \to 0+} \frac{\log N_\varepsilon(A)}{\log 1/\varepsilon}.$$

Plainly we have $\overline{\dim}\, A \ge \dim A$.

We consider the increasing function Ψ defined on $\mathbb{R}^+$ by

$$\Psi(\lambda) = 2b\lambda^2 + \int_{(0,\infty)} \frac{2h\lambda^2}{1 + 2h\lambda} \Pi(dh),$$

where $b \ge 0$ and Π is a Radon measure on $(0,\infty)$ such that $\int_{(0,\infty)} (1 \wedge h)\, \Pi(dh) < \infty$. To avoid trivial cases, we assume either $b > 0$ or $\Pi((0,\infty)) = \infty$. Note that $\Psi(\lambda) \le c\lambda$ for $\lambda \in [0,1]$. The function Ψ can be expressed in the usual form for branching mechanism functions:

$$\Psi(\lambda) = 2b\lambda^2 + \int_{(0,\infty)} \Pi'(du) \left[\mathrm{e}^{-u\lambda} - 1 + u\lambda \right],$$

where $\Pi'(du) = \left[\int_{(0,\infty)} \Pi(dh)\, \mathrm{e}^{-u/2h} (4h^2)^{-1} \right] du$ satisfies $\int_{(0,\infty)} (u \wedge u^2) \Pi'(du) < \infty$. Notice that if we take $b = 0$ and $\Pi(dh) = c' h^{-1-\rho} dh$ then we get the stable case $\Psi(\lambda) = c\lambda^{1+\rho}$.

Let γ be a Brownian motion in $\mathbb{R}^d$, and $(P_s, s \ge 0)$ its transition kernel.

We then consider $X := \big((X_t, t\geq 0), (\mathbb{P}^X_\nu, \nu \in M_f)\big)$ the canonical realization of the (γ,Ψ)-superprocess defined on $\mathbb{D} := \mathbb{D}([0,\infty), M_f)$, the set of all càdlàg functions defined on $[0,\infty)$ with values in M_f. We refer to [9, 10, 12, 14] for its construction and general properties. We recall that the superprocess X is a càdlàg strong Markov process with values in M_f characterized by $X_0 = \nu$ $\mathbb{P}^X_\nu$-a.s. and for every nonnegative bounded function $f \in \mathcal{B}(\mathbb{R}^d)$, $t\geq s\geq 0$,

$$\mathbb{E}^X_\nu \left[\mathrm{e}^{-(X_t,f)} \mid \sigma(X_u, 0\leq u\leq s)\right] = \mathrm{e}^{-(X_s, v(t-s,\cdot))},$$

where v is the unique nonnegative measurable solution of the integral equation

$$v(t,x) + \int_0^t ds\; P_{t-s}[\Psi(v(s,\cdot))](x) = P_t f(x), \quad t\geq 0, x\in\mathbb{R}^d. \tag{1}$$

We define the constants $\underline{\rho}$ and $\overline{\rho}$ by:

$$\underline{\rho} = -1 + \liminf_{\lambda\to\infty} \frac{\log\Psi(\lambda)}{\log\lambda}, \quad \text{and} \quad \overline{\rho} = -1 + \limsup_{\lambda\to\infty} \frac{\log\Psi(\lambda)}{\log\lambda}.$$

Since $\int_{(0,\infty)} (1\wedge h)\Pi(dh) < \infty$, we easily get $0\leq \underline{\rho}\leq\overline{\rho}\leq 1$. From the definition of $\overline{\rho}$, $\underline{\rho}$, for every $\delta\in(0,1)$, there exists $\lambda_\delta\in(0,\infty)$ such that for every $\lambda>\lambda_\delta$

$$\lambda^{1+\underline{\rho}-\delta} \leq \Psi(\lambda) \leq \lambda^{1+\overline{\rho}+\delta}. \tag{2}$$

We will consider the following two assumptions:

(H1): *We have* $0<\underline{\rho}$*.*

(H2): *The function* Ψ *is regularly varying at* ∞ *with index* $1+\rho$ *where* $\rho\in(0,1]$*, that is to say:*

$$\lim_{\lambda\to\infty} \frac{\Psi(t\lambda)}{\Psi(\lambda)} = t^{1+\rho} \quad \textit{for every} \quad t>0.$$

Notice that (H2) implies (H1) and $\underline{\rho} = \overline{\rho} = \rho$. The stable case $\Psi(\lambda) = c\lambda^{1+\rho}$ satisfies (H2).

We can now give our first result about the Hausdorff dimension of the topological support of the measure X_t. Let supp ν denote the topological support of a measure $\nu\in M_f$. Set $\sigma_X = \inf\{s>0; X_s = 0\}$.

Theorem 2.1. *Assume (H1). Then for every* $\nu\in M_f$*, for every nonempty compact set* $B\subset(0,\infty)$*, we have* $\mathbb{P}^X_\nu$*-a.s. on* $\{B\subset(0,\sigma_X)\}$*,*

$$\left(\frac{2}{\overline{\rho}} + 2\dim B\right)\wedge d \leq \dim \mathcal{C}l\left(\bigcup_{t\in B} \operatorname{supp} X_t\right) \leq \left(\frac{2}{\underline{\rho}} + 2\,\overline{\dim}\, B\right)\wedge d.$$

Moreover, if (H2) holds, then $\mathbb{P}^X_\nu$*-a.s. on* $\{B\subset(0,\sigma_X)\}$*,*

$$\dim \mathcal{C}l\left(\bigcup_{t\in B} \operatorname{supp} X_t\right) = \left(\frac{2}{\rho} + 2\dim B\right)\wedge d.$$

Let $\mathcal{R} = \bigcup_{\varepsilon>0} \mathcal{C}l\left(\bigcup_{t\geq\varepsilon} \text{supp } X_t\right)$ be the range of the superprocess X. We deduce then

Corollary 2.2. *Assume (H1). Then a.s. we have*

$$\left(\frac{2}{\overline{\rho}}+2\right)\wedge d \leq \dim \mathcal{R} \leq \left(\frac{2}{\underline{\rho}}+2\right)\wedge d.$$

Moreover, if (H2) holds, then a.s. we have

$$\dim \mathcal{R} = \left(\frac{2}{\rho}+2\right)\wedge d.$$

In the special case $\Psi(\lambda) = \lambda^2$, Tribe [25] (theorem 2.13) proved a stronger form of theorem 2.1. Precisely, Tribe showed that the last assertion of the theorem holds simultaneously for all sets B outside a set of zero probability. Our next result is about the hitting probabilities of small balls. We denote by $B_\varepsilon(0)$ the ball centered at 0 with radius ε, and by p the Brownian transition density on $\mathbb{R}^d$

$$p(t,x) = \frac{1}{(2\pi t)^{d/2}} \exp -\frac{|x|^2}{2t}, \quad (t,x) \in (0,\infty)\times\mathbb{R}^d.$$

We say a positive function l, defined on $(0,\infty)$ is slowly varying at $0+$, if for every $t>0$, $\lim_{\lambda\downarrow 0} l(\lambda t)/l(\lambda) = 1$. Let δ_x be the Dirac mass at point $x\in\mathbb{R}^d$.

Theorem 2.3. *Assume (H2) and $\rho d > 2$. There exists a positive function l_1, which is slowly varying at $0+$, such that for every $t>0$, $\varepsilon>0$:*

$$\mathbb{P}^X_{\delta_x}\left[X_t(B_\varepsilon(0))>0\right] \leq t^{-d/2}\varepsilon^{d-2/\rho} l_1(\sqrt{t}\wedge\varepsilon).$$

Moreover if $\limsup_{\lambda\to 0+}\lambda^{-1-\rho}\Psi(\lambda) < \infty$, then for every $M>0$, there exists a positive increasing function l_2, which is slowly varying at $0+$, such that for every $M\sqrt{t}>\varepsilon>0$, we have

$$\mathbb{P}^X_{\delta_x}\left[X_t(B_\varepsilon(0))>0\right] \geq \frac{1}{2}\wedge\left[\varepsilon^{d-2/\rho} p\left(\frac{\rho t}{1+\rho}, x\right) l_2(\varepsilon)\right].$$

Our next result is about the connected components of X_t.

Theorem 2.4. *Assume (H2) and $d>4/\rho$. Let $\nu\in M_f$, $t>0$. Then $\mathbb{P}^X_\nu$-a.s. the support of X_t is totally disconnected.*

The last result deals with the absolute continuity of superprocesses in the case where the underlying process is not only a Brownian motion but also a symmetric α-stable process. We first introduce the α-stable superprocess.

Let γ^α be a symmetric α-stable process on $\mathbb{R}^d$ of index $\alpha\in(0,2)$ started at x under $\mathbb{P}_x$. For every $y\in\mathbb{R}^d$, for every $t\geq 0$, we have

$$\mathbb{E}_x\, \mathrm{e}^{-i\langle y,\gamma^\alpha_t - x\rangle} = \mathrm{e}^{-t\int_{|z|=1}|\langle y,z\rangle|^\alpha\,\chi(dz)}$$

where $\langle .,. \rangle$ denotes the usual scalar product on $\mathbb{R}^d$ and χ is a finite symmetric measure on the sphere $\{z \in \mathbb{R}^d, |z| = 1\}$. In order to avoid degenerate cases we assume that

$$\inf_{|y|=1} \int_{|z|=1} |\langle y, z \rangle|^\alpha \chi(dz) > 0.$$

In particular the transition density is continuous on $(0, \infty) \times \mathbb{R}^d$ (see [16] theorem 10.1). For $\alpha = 2$ we consider $\gamma^2 = \gamma$, the Brownian motion in $\mathbb{R}^d$ started at x under $\mathbb{P}_x$. We consider $X^\alpha = (X_t^\alpha, t \geq 0)$ the canonical realization of the (γ^α, Ψ)-superprocess defined on $\mathbb{D}$. We refer again to [9, 10, 12, 14] for its construction and general properties.

Theorem 2.5. *Assume (H1). Let $\alpha \in (0, 2]$. Let μ be a finite positive measure with support in $(0, \infty)$ and $q \in [0, 1)$ such that*

$$\iint \mu(dt)\mu(ds) \, |t - s|^{-q} < \infty.$$

If $\dfrac{\alpha}{\overline{\rho}} + \alpha q > d$, then for every $\nu \in M_f$, $\mathbb{P}_\nu^X$-a.s. $\int \mu(dt) X_t^\alpha$ is absolutely continuous with respect to Lebesgue measure.

As a particular case, taking $\mu = \delta_t$ for $t > 0$, and $q = 0$, we get that if $\alpha/\overline{\rho} > d$, then for $t > 0$, $\mathbb{P}_\nu^X$-a.s. X_t^α is absolutely continuous with respect to Lebesgue measure.

Hypothesis (H1) will be in force from now on.

3 Preliminary estimates

Notice that the function defined on $\mathbb{R}^+$ by $\eta(\lambda) = b\lambda + \int_0^\infty [1 - \exp(-\lambda h)] \, \Pi(dh)$ is the Laplace exponent of a subordinator. By comparing the functions $2u/(1 + 2u)$ and $1 - \exp(-u)$, it is easy to obtain the following bounds:

$$\frac{2}{3} \lambda \eta(\lambda) \leq \Psi(\lambda) \leq 2\lambda\eta(\lambda), \quad \lambda \geq 0. \tag{3}$$

The constants $\underline{\rho}$ and $\overline{\rho}$ thus correspond to the lower index and upper index of the subordinator associated to η (cf [6]). We give an elementary result about η.

Lemma 3.1. *If (H2) is satisfied and if $\rho < 1$, then the function η is regularly varying at ∞ with index ρ.*

We shall need the usual notation $\bar{\Pi}(h) = \Pi([h, \infty))$

Proof. Assume (H2) and $\rho < 1$. The latter condition implies $b = 0$. Fubini's theorem gives

$$\Psi(\lambda) = \int_0^\infty 2\lambda^2 (1 + 2\lambda h)^{-2} \bar{\Pi}(h) dh = 2\lambda^2 \int_0^\infty (h + 2\lambda)^{-2} \bar{\Pi}(1/h) dh.$$

Thanks to theorems 1.7.4 and 1.7.2 of [4], we deduce that the function $\bar{\Pi}$ is regularly varying with index $-\rho$ at $0+$. Then theorem 1.7.1' of [4] implies that the function η is regularly varying with index ρ at ∞. $\square$

We now give some simple results about the subordinator with Laplace exponent η. We refer to [2] for definitions and properties of subordinators. Let $S = (S_t, t \geq 0)$ be a subordinator with Laplace exponent η. We denote by $L = (L_t, t \geq 0)$ the right continuous inverse of S, that is $L_t = \inf \{u \geq 0; S_u > t\}$.

Lemma 3.2. *1. For every $\delta > 0$, there exists $h_\delta > 0$ such that for every $h \in [0, h_\delta]$,*

$$\mathbb{E}L_h \leq h^{\underline{\rho}-\delta}.$$

Furthermore there exists a constant C_δ, such that for every $h \geq 0$, $\mathbb{E}L_h \leq C_\delta(h \vee h^{\underline{\rho}-\delta})$.

2. The process L is locally Hölder with exponent α, for every $\alpha \in [0, \underline{\rho})$.

3. For every $\alpha \in [0, 1/\overline{\rho})$, $s > 0$, a.s. there exists $\varepsilon \in (0, s)$, depending on $(S_t, 0 \leq t < s)$ and α, such that for every $u \in [s - \varepsilon, s)$, we have

$$S_{s-} - S_u \leq (s-u)^\alpha.$$

4. For every $\delta > 0$, there exists a sequence $(R_n, n \geq 1)$ of positive real numbers, decreasing to zero, such that for every $M \in (0, \infty)$, we have

$$\lim_{n \to \infty} \mathbb{P}\left[\inf_{i \geq n} R_i^{-\underline{\rho}(1+\delta)} L_{R_i} > M\right] = 1.$$

5. If (H2) holds, then for every $(\delta, M) \in (0, \infty)^2$, we have

$$\lim_{r \to 0} \mathbb{P}\left[\inf_{h \in (0,r]} h^{-\rho(1+\delta)} L_h > M\right] = 1.$$

Proof. 1. Using the links between S and L, we have for $\lambda > 0$,

$$\begin{aligned}\eta(\lambda)^{-1} &= \int_0^\infty dt\ \mathbb{E}\mathrm{e}^{-\lambda S_t} = \mathbb{E}\int_0^\infty dL_h\ \mathrm{e}^{-\lambda h} = \lambda \int_0^\infty dh\ \mathrm{e}^{-\lambda h}\,\mathbb{E}[L_h] \\ &\geq \lambda \int_{1/\lambda}^{2/\lambda} dh\ \mathrm{e}^{-2}\,\mathbb{E}[L_{1/\lambda}] = \mathrm{e}^{-2}\,\mathbb{E}[L_{1/\lambda}].\end{aligned}$$

The first part of the lemma follows from (2) and (3). The second one is then trivial.

2. The variable $L_{t+h} - L_t$ is bounded from above in distribution by L_h. By a standard argument for additive functionals, we have also $\mathbb{E}[(L_h)^p] \leq p!\,(\mathbb{E}[L_h])^p$. Thus for every $t \geq 0$, $\delta > 0$, if h_δ is defined as in 1., and $h \in [0, h_\delta]$, we have

$$\mathbb{E}[(L_{t+h} - L_t)^p] \leq \mathbb{E}[(L_h)^p] \leq p!\,(\mathbb{E}[L_h])^p \leq p! h^{p(\underline{\rho}-\delta)}.$$

From the classical Kolmogorov lemma, we obtain that L is locally Hölder with exponent α, for any $\alpha \in [0, \underline{\rho})$.

3. Let $s > 0$. The two processes $(S_{s-} - S_u, 0 \leq u < s)$ and $\big(S_{(s-u)-}, 0 \leq u < s\big)$ have the same law. So it is sufficient to prove the analogous result for $V_u = S_{(s-u)-}$.

- If $\overline{\rho} = 1$, the result is a consequence of proposition 8 p.84 of [2].
- If $\overline{\rho} < 1$, then $b = 0$, and we have

$$\Psi(\lambda) \geq \lambda \int_{1/\lambda}^\infty 2h\lambda(1 + 2h\lambda)^{-1}\Pi(dh) \geq \frac{2}{3}\lambda\bar{\Pi}(1/\lambda).$$

Then the upper bound (2) implies that the integral $\int_{0+} \bar{\Pi}(t^\alpha)dt$ is convergent for every $\alpha \in (0, 1/\overline{\rho})$. Thanks to theorem 9 p.85 of [2], we have for every $\alpha \in (1, 1/\overline{\rho})$ a.s.

$$\lim_{u \to s,\, u<s} V_u/(s-u)^\alpha = 0.$$

The desired result follows.

4. Fix $\delta \in (0, \infty)$. Note that (2) and (3) imply $\liminf_{\lambda\to\infty} \lambda^{-\underline{\rho}(1+\delta/2)}\eta(\lambda) = 0$. We can find a sequence $(R_n, n \geq 1)$ of positive reals decreasing to zero such that

$$\text{for every} \quad n \geq 1, \quad \eta(1/R_n) \leq R_n^{-\underline{\rho}(1+\delta/2)} \quad \text{and} \quad \sum_{n\geq 1} R_n^{\delta/2} < \infty.$$

In order to bound for every $M \in (0, \infty)$,

$$\mathbb{P}\left[L_{R_n} < MR_n^{\underline{\rho}(1+\delta)}\right] = \mathbb{P}\left[R_n^{-1} S_{MR_n^{\underline{\rho}(1+\delta)}} > 1\right],$$

we consider the Laplace transform of S:

$$\mathbb{E}\exp\left[-R_n^{-1} S_{MR_n^{\underline{\rho}(1+\delta)}}\right] = \exp\left[-MR_n^{\underline{\rho}(1+\delta)}\eta(1/R_n)\right] \geq \exp\left[-MR_n^{\underline{\rho}\delta/2}\right].$$

An easy calculation shows that

$$\mathbb{P}\left[R_n^{-1} S_{MR_n^{\underline{\rho}(1+\delta)}} > 1\right] \leq (1-1/\mathrm{e})^{-1}\left[1 - \mathbb{E}\exp\left[-R_n^{-1} S_{MR_n^{\underline{\rho}(1+\delta)}}\right]\right] \leq (1-1/\mathrm{e})^{-1} MR_n^{\underline{\rho}\delta/2}.$$

Since the series $\sum_{n\geq 1} R_n^{\underline{\rho}\delta/2}$ converges, we get

$$\sum_{n\geq 1} \mathbb{P}\left[L_{R_n} < MR_n^{\underline{\rho}(1+\delta)}\right] < \infty.$$

The desired result then follows from the Borel-Cantelli lemma.

5. If (H2) holds, we have $\rho = \overline{\rho}$, and we deduce from the proof of 3., that for every $m > 0$,

$$\lim_{r\to 0} \mathbb{P}\left[\sup_{u\in(0,r]} u^{-1/\rho(1+\delta)} S_u < m\right] = 1.$$

The desired result follows since L is the inverse of S.

4 The subordination approach to superprocesses

4.1 The Brownian snake

Our main goal in this section is to explain how superprocesses with a general branching mechanism can be constructed using the Brownian snake and a subordination method taken from [3]. We start from a subordinator $S = (S_t, t \geq 0)$ as in section 3. We denote by ξ the associated residual lifetime process defined by $\xi_t = \inf\{S_s - t; S_s > t\}$, and by L the right continuous inverse of S, $L_t = \inf\{s; S_s > t\}$. We also consider an independent Brownian motion in $\mathbb{R}^d$ denoted by $\gamma = (\gamma_t, t \geq 0)$. We shall be interested in the process $\bar{\xi}_t = (\xi_t, L_t, \gamma_{L_t})$, which is a Markov process with values in $E=\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^d$. Let $\bar{\mathbb{P}}_z$ be the law of $\bar{\xi}$ started at $z\in E$. For simplicity we write $\Gamma_t=\gamma_{L_t}$, and $\bar{\mathbb{P}}_x=\bar{\mathbb{P}}_z$ when $z=(0,0,x)$.

We then introduce the Brownian snake with spatial motion $\bar{\xi}$ (cf [17], our construction is slightly different here because the first coordinate of $\bar{\xi}$ is not a continuous process). The Brownian snake is a Markov process taking values in the set of all killed paths in E. By definition a killed path in E is a càdlàg mapping $w:[0,\zeta)\to E$ where $\zeta=\zeta_w>0$ is called the lifetime of the path. By convention we also agree that every point $z\in E$ is a killed path with lifetime 0. The set $\mathcal{W}$ of all killed paths is a Polish space when equipped with the metric

$$d(w,w'):=\left|\zeta-\zeta'\right|+\left|w(0)-w'(0)\right|+\int_0^{\zeta\wedge\zeta'}\left[d_u\left(w_{\leq u},w'_{\leq u}\right)\wedge 1\right]\,du,$$

where $w_{\leq u}$ denotes the restriction of w to $[0,u]$, and d_u is the Skorokhod distance on the space of all càdlàg functions from $[0,u]$ into E.

Let us fix $z\in E$ and denote by $\mathcal{W}_z$ the subset of $\mathcal{W}$ of all killed paths with initial point $w(0)=z$ (in particular $z\in\mathcal{W}_z$). Let $w\in\mathcal{W}_z$ with lifetime $\zeta>0$. If $0\leq a<\zeta$, and $b\geq a$, we let $Q_{a,b}(w,dw')$ be the unique probability measure on $\mathcal{W}_z$ such that:

- $\zeta'=b$, $Q_{a,b}(w,dw')$-a.s.,
- $w'(t)=w(t)$, $\forall t\in[0,a]$, $Q_{a,b}(w,dw')$-a.s.,
- the law under $Q_{a,b}(w,dw')$ of $(w'(a+t),0\leq t<b-a)$ is the law of $(\bar{\xi},0\leq t<b-a)$ under $\bar{\mathbb{P}}_{w(a)}$.

By convention we set $Q_{0,b}(z,dw')$ for the law of $(\bar{\xi},0\leq t<b)$ under $\bar{\mathbb{P}}_z$.

Denote by $\theta_s^\zeta(dadb)$ the joint distribution of $(\inf_{[0,s]}B_r,B_s)$ where B is a one dimensional reflecting Brownian motion in $\mathbb{R}^+$ with initial value $B_0=\zeta\geq 0$.

$$\begin{aligned}\theta_s^\zeta(dadb)=\frac{2(\zeta+b-2a)}{\sqrt{2\pi s^3}}\exp\left(-\frac{(\zeta+b-2a)^2}{2s}\right)\mathbf{1}_{\{0<a<\zeta\wedge b\}}dadb\\ +\sqrt{\frac{2}{\pi s}}\exp\left(-\frac{(\zeta+b)^2}{2s}\right)\mathbf{1}_{\{0<b\}}\delta_0(da)db.\end{aligned}$$

We recall proposition 5 of [3].

Proposition 4.1. *There exists a continuous strong Markov process in $\mathcal{W}_z$, denoted by $W=(W_s,s\geq 0)$, whose transition kernels are given by the formula*

$$Q_s(w,dw')=\int_{[0,\infty)^2}\theta_s^\zeta(dadb)Q_{a,b}(w,dw').$$

If ζ_s denotes the lifetime of W_s, the process $(\zeta_s,s\geq 0)$ is a reflecting Brownian motion in $\mathbb{R}_+$.

Intuitively the path W_s is erased from its tip when the lifetime ζ_s decreases, and it is extended, independently of the past, when ζ_s increases, according to the law of the underlying spatial motion $\bar{\xi}$. It is easy to check that a.s. for every $s<s'$, the two killed paths W_s and $W_{s'}$

coincide for $t < m(s,s') := \inf_{r\in[s,s']} \zeta_r$. They also coincide at $t = m(s,s')$ if $m(s,s') < \zeta_s \wedge \zeta_{s'}$. In the sequel, we shall refer to this property as the "snake property" of W.

Denote by $\mathcal{E}_w$ the probability measure under which W starts at w, and by $\mathcal{E}^*_w$ the probability under which W starts at w and is killed when ζ reaches zero.

Here thanks to the properties of the process $\bar{\xi}$ (and in particular assumption (H1)), we can get stronger continuity properties for the process W. First introduce an obvious notation for the coordinates of a path $w \in \mathcal{W}$:

$$w(t) = (\xi_t(w), L_t(w), \Gamma_t(w)) \quad \text{for} \quad 0 \le t < \zeta_w.$$

We also set $\hat{w} = \lim_{t\uparrow\zeta_w} \Gamma_t(w)$ if the limit exists, $\hat{w} = \partial$ otherwise, where ∂ is a cemetery point added to $\mathbb{R}^d$. Fix $w_0 \in \mathcal{W}_z$, such that the functions $t \mapsto L_t(w_0)$ and $t \mapsto \Gamma_t(w_0)$ are continuous on $[0, \zeta_{w_0})$ and have a continuous extension on $[0, \zeta_{w_0}]$. By using the Hölder properties of the processes L (cf lemma 3.2) and Γ one can prove that $\mathcal{E}_{w_0}$-a.s. for every $s \ge 0$, the functions $t \mapsto L_t(W_s)$ and $t \mapsto \Gamma_t(W_s)$ which are a priori defined on $[0, \zeta_s)$ are continuous and have a continuous extension to $[0, \zeta_s]$ (cf lemma 10 and its proof in [3], see also the proof of lemma 5.3 below). Furthermore the mappings $s \mapsto (L_{t\wedge\zeta_s}(W_s), t \ge 0)$ and $s \mapsto (\Gamma_{t\wedge\zeta_s}(W_s), t \ge 0)$ are continuous with respect to the uniform topology. The processes $L_{\zeta_s}(W_s)$ and $\hat{W}_s$ are continuous $\mathcal{E}_{w_0}$-a.s.

It is clear that the trivial path $z \in \mathcal{W}_z$ is a regular recurrent point for W. We denote by $\mathbb{N}_z$ the associated excursion measure (see [5]). The law under $\mathbb{N}_z$ of $(\zeta_s, s \ge 0)$ is the Itô measure of positive excursions of linear Brownian motion. We assume that $\mathbb{N}_z$ is normalized so that

$$\mathbb{N}_z \left[\sup_{s\ge 0} \zeta_s > \varepsilon \right] = \frac{1}{2\varepsilon}.$$

We also set $\sigma = \inf\{s > 0, \zeta_s = 0\}$, which represents the duration of the excursion. Then for any nonnegative measurable function G on $\mathcal{W}_z$, we have:

$$\mathbb{N}_z \int_0^\sigma G(W_s)\, ds = \int_0^\infty ds\, \bar{\mathbb{E}}_z \left[G\left((\bar{\xi}_t, 0 \le t < s) \right) \right]. \tag{4}$$

For simplicity we write $\mathbb{N}_x = \mathbb{N}_z$ when $z = (0,0,x)$. The continuity properties mentioned above under $\mathcal{E}_{w_0}$ also hold under $\mathbb{N}_z$. In particular the two processes $(L_{\zeta_s}(W_s), s \ge 0)$ and $\left(\hat{W}_s, s \ge 0\right)$ are well defined and continuous under $\mathbb{N}_z$.

Remark. We set $\mathcal{G} := \left\{ \left(L_{\zeta_s}(W_s), \hat{W}_s \right), s \ge 0 \right\}$. Since $L_{\zeta_s}(W_s)$ and $\hat{W}_s$ are continuous under $\mathcal{E}_x$, we deduce that for any open set $\Delta \subset \mathbb{R}^+ \times \mathbb{R}^d$ such that $(0,x) \in \Delta$, we have $\mathbb{N}_x [\mathcal{G} \cap \Delta^c \neq \emptyset] < \infty$.

4.2 Exit measures

Let D be an open subset of E with $z \in D$ (or $w_0(0) \in D$). As in [3], we can define the exit local time from D, denoted by $\left(L^D_s, s \ge 0\right)$. $\mathbb{N}_z$-a.e. (or $\mathcal{E}_{w_0}$-a.s.), the exit local time L^D is a continuous increasing process given by the approximation: for every $s \ge 0$,

$$L^D_s = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \int_0^s \mathbf{1}_{\{\tau_D(W_u) < \zeta_u < \tau_D(W_u) + \varepsilon\}} du,$$

where $\tau_D(w) = \inf\{r; w(r) \notin D\}$ with the convention $\inf \emptyset = +\infty$. We then define under the excursion measure $\mathbb{N}_z$ a random measure Y^D on $\mathbb{R}^d$ by the formula: for every bounded nonnegative function $\varphi \in \mathcal{B}(\mathbb{R}^d)$,

$$(Y_D, \varphi) = \int_0^\sigma \varphi(\hat{W}_s) dL_s^D.$$

The first moment of the random measure can be derived from the following fact. By passing to the limit in (4) (see [18] proposition 3.3 for details), we have for every nonnegative measurable function G on $\mathcal{W}_z$

$$\mathbb{N}_z \int_0^\sigma G(W_s)\, dL_s^D = \bar{\mathbb{E}}_z^D\, [G], \tag{5}$$

where $\bar{\mathbb{P}}_z^D$ is the sub-probability on $\mathcal{W}_z$ defined as the law of $\bar{\xi}$ stopped at time τ_D under $\bar{\mathbb{P}}_z(\cdot \cap \{\tau_D < \infty\})$.

We apply this construction with $D = D_t = \mathbb{R}^+ \times [0, t) \times \mathbb{R}^d$. For convenience we write $\tau_t(w) = \tau_{D_t}(w)$, $L_s^t = L_s^{D_t}$, $Y_t = Y_{D_t}$, and $\bar{\mathbb{P}}_z^t = \bar{\mathbb{P}}_z^{D_t}$. We also will write $\bar{\mathbb{P}}_x^t = \bar{\mathbb{P}}_z^t$ when $z = (0, 0, x)$. When $z \notin D_t$, we then take $L_s^t = 0$ for all $s \geq 0$ and $Y_t = 0$. Using (5), we get in particular the first moment for the process $(Y_t, t \geq 0)$: for every bounded nonnegative function $\varphi \in \mathcal{B}(\mathbb{R}^d)$,

$$\mathbb{N}_x\, [(Y_t, \varphi)] = P_t\varphi(x).$$

To get a measurable version of $(Y_t, t \geq 0)$, we take a measurable version of $\left(L_s^t, t \geq 0, s \geq 0\right)$: for t such that $z \in D_t$,

$$L_s^t = \liminf_{p \to \infty} L_s^{t, 2^{-p}},$$

where for $\varepsilon > 0$, $\quad L_s^{t,\varepsilon} = \dfrac{1}{\varepsilon} \displaystyle\int_0^s \mathbf{1}_{\{\tau_t(W_u) < \zeta_u < \tau_t(W_u) + \varepsilon\}} du.$

Remark. As a simple consequence of (4), we have for $t > 0$,

$$\mathbb{N}_x\, \left[L_\sigma^{t,\varepsilon}\right] = \frac{1}{\varepsilon}\, \bar{\mathbb{E}}_x \left[\int_0^\infty \mathbf{1}_{\{\tau_t < u < \tau_t + \varepsilon\}} du\right] = 1.$$

If μ is a finite Radon measure on $[0, \infty)$, then $\mu(dt)$-a.e. $\mathbb{N}_z$-a.e. the function $s \mapsto L_s^t$ is increasing and continuous. Similar observations hold under $\mathcal{E}_{w_0}$. We shall be interested in the random measure $\int \mu(ds)\, Y_s$. By arguing as in [18], theorem 4.1, we easily get a "Palm measure formula" for this random measure.

Proposition 4.2. *For every nonnegative measurable function F on $\mathbb{R}^d \times M_f$, for every $t > 0$ and $z \in D_t$, we have*

$$\begin{aligned} \mathbb{N}_z & \left[\int Y_t(dy) F\left(y, \int_0^\infty \mu(ds)\, Y_s\right)\right] \\ &= \int \bar{\mathbb{P}}_z^t(dw) \mathbb{E}\left[F\left(\hat{w}, \int \mathcal{N}_w(du, d\mathrm{W}) \int_0^\infty \mu(ds)\, Y_s(\mathrm{W}) \mathbf{1}_{\{u < \tau_s(w)\}}\right)\right], \end{aligned}$$

where for every $w \in \mathcal{W}_z$, $\mathcal{N}_w(du, d\mathrm{W})$ denotes a Poisson measure on $\mathbb{R}^+ \times C\left(\mathbb{R}^+, \mathcal{W}\right)$ with intensity

$$4\, \mathbf{1}_{[0, \zeta_w]}(u)\, du\, \mathbb{N}_{w(u)}[d\mathrm{W}].$$

4.3 The subordinate superprocess

We introduced the process Y because its distribution under the excursion measure $\mathbb{N}_x$ is the canonical measure of the (γ,Ψ)-superprocess started at δ_x. More precisely, we have the following result.

Proposition 4.3. *Let $\nu \in M_f$ and let $\sum_{i\in I}\delta_{W^i}$ be a Poisson measure on $C(\mathbb{R}^+,\mathcal{W})$ with intensity $\int \nu(dx)\mathbb{N}_x[\cdot]$. The process*

$$X_0=\nu, \quad X_t=\sum_{i\in I} Y_t(W^i), \quad \textit{for} \quad t>0,$$

is a (Ψ,γ)-superprocess. Moreover, a.s. for every $t>0$, the collection $\left(\left(Y_s(W^i), s\geq t\right), i\in I\right)$ has only a finite number of non zero terms.

The proposition is proved in [3], except for the last assertion. For this it is enough to check that $\mathbb{N}_x\left[Y_t\neq 0\right]<\infty$ for $t>0$. We know from [3], that

$$\mathbb{N}_x\left[1-\mathrm{e}^{-n(Y_t,\mathbf{1})}\right]=v_n(t,x), \quad t\geq 0,\ \ x\in\mathbb{R}^d,$$

is the only nonnegative measurable solution of (1) with $f=n$. By a uniqueness argument, we have $v_n(t,x)=v_n(t)$. Then (1) implies $v_n(0)=n$, $\frac{d}{dt}v_n(t)=-\Psi(v_n(t))$, from which we easily get:

$$\int_{v_n(t)}^{n}\Psi(u)^{-1}du=t, \quad \text{for} \quad t\geq 0.$$

By (2) and (H1), we have $\int^\infty \Psi(u)^{-1}du<\infty$. Thus if $v(t)=\lim_{n\to\infty}v_n(t)=\mathbb{N}_x\left[Y_t\neq 0\right]$, we get from the previous equation that $v(t)<\infty$ and more precisely

$$\int_{v(t)}^{\infty}\Psi(u)^{-1}du=t, \quad \text{for} \quad t\geq 0. \tag{6}$$

Remark. We can use the continuity of the mapping $t\mapsto v(t)$ to derive a fact that will be useful later. For $t>0$ fixed, observe that $\mathbb{N}_x$-a.e.

$$\left\{\sup_{s\geq 0} L_{\zeta_s}(W_s)>t\right\}\subset\{Y_t\neq 0\}\subset\left\{\sup_{s\geq 0} L_{\zeta_s}(W_s)\geq t\right\}.$$

The second inclusion follows from the construction of L^t and the first one is easily deduced from the special Markov property (cf [3] proposition 7). It follows that

$$\mathbb{N}_x\left[\sup_{s\geq 0} L_{\zeta_s}(W_s)\geq t\right]=\mathbb{N}_x\left[Y_t\neq 0\right]=v(t),$$

and so

$$\mathbb{N}_x\left[\sup_{s\geq 0} L_{\zeta_s}(W_s)=t\right]=0.$$

We shall also need the following result, which is a consequence of (6) and theorem 1.5.12 of [4]:

Corollary 4.4. *Under (H2), the function $v(t)=\mathbb{N}_x\left[Y_t\neq 0\right]$ is regularly varying at $0+$ with index $-1/\rho$.*

4.4 The support of the exit measure

In this section, we give a technical result about the support of the exit measure L^t, which is crucial for the proof of theorem 2.1. Recall that we defined $\tau_t(W_s) = \inf\{r < \zeta_s; L_r(W_s) \geq t\}$. However we know that $\mathbb{N}_z$-a.e. (or $\mathcal{E}_{w_0}$-a.s.), for every $s \geq 0$ the mapping $r \mapsto L_r(W_s)$, $r \in [0, \zeta_s)$ has a continuous extension to $[0, \zeta_s]$. Thanks to this fact, we slightly modify the previous definition of τ_t by taking $\tau_t(W_s) = \zeta_s$ when $L_{\zeta_s}(W_s) = t$ and $L_r(W_s) < t$ for $r < \zeta_s$. For $t > 0$, we introduce under $\mathbb{N}_x$ the set

$$\mathcal{H}_t = \{s \in [0, \sigma]; \zeta_s = \tau_t(W_s)\}.$$

Recall that supp ν denotes the closed topological support of a measure ν.

Lemma 4.5. *$\mathbb{N}_x$-a.e. for every $t > 0$, the set $\mathcal{H}_t$ is closed. Furthermore for every fixed $t > 0$, $\mathbb{N}_x$-a.e., we have* $\operatorname{supp} dL^t \subset \mathcal{H}_t$.

Proof. We prove the first part of the lemma. From the "snake property", it is easy to see that $\{s; \tau_t(W_s) < \zeta_s - \varepsilon\}$ is open. Note also that the set $\{s; \tau_t(W_s) \leq \zeta_s\} = \{s; L_{\zeta_s}(W_s) \geq t\}$ is closed since the function $s \mapsto L_{\zeta_s}(W_s)$ is continuous. Thus $A_\varepsilon = \{s; \zeta_s - \varepsilon \leq \tau_t(W_s) \leq \zeta_s\}$ is closed. We deduce the set $\mathcal{H}_t = \cap_{n \geq 1} A_{1/n}$ is closed.

For the second part of the lemma, fix $t > 0$. By the definition of $L_s^{t,\varepsilon}$, we have

$$\operatorname{supp} dL^{t,\varepsilon} \subset \mathcal{C}l\left(\{s; \tau_t(W_s) < \zeta_s < \tau_t(W_s) + \varepsilon\}\right) \subset \{s; \zeta_s - \varepsilon \leq \tau_t(W_s) \leq \zeta_s\}.$$

Since $L_s^t = \lim_{\varepsilon \to 0} L_s^{t,\varepsilon}$, we deduce that supp $L^t \subset \mathcal{H}_t$ $\mathbb{N}_x$-a.e. □

5 Proof of theorem 2.1

We prove theorem 2.1 in three steps. In the first one we reduce the proof to proposition 5.1. The second and third steps deal respectively with the proof of the lower bound and the proof of the upper bound of proposition 5.1.

5.1 Preliminary reduction

Let $q \in [0, 1)$, and μ a measure on $\mathbb{R}^+$, such that supp $\mu \subset (0, \infty)$ and

$$0 < \iint \mu(dt)\, \mu(ds)\, |t - s|^{-q} < \infty. \tag{7}$$

Let B a compact subset of $(0, \infty)$. We set $\sigma_Y = \sup_{s \in [0,\sigma]} L_{\zeta_s}(W_s)$ and $\mathcal{H}_B = \cup_{t \in B} \mathcal{H}_t$.

Proposition 5.1. *Let $x \in \mathbb{R}^d$. $\mathbb{N}_x$-a.e., on $\{\operatorname{supp} \mu \subset (0, \sigma_Y)\}$, we have the lower bound*

$$\dim \operatorname{supp} \int \mu(ds)\, Y_s \geq \left(\frac{2}{\rho} + 2q\right) \wedge d.$$

$\mathbb{N}_x$-a.e., on $\{B \subset (0, \sigma_Y)\}$, we have the upper bound

$$\dim \left\{\hat{W}_s; s \in \mathcal{H}_B\right\} \leq \left(\frac{2}{\underline{\rho}} + 2\,\overline{\dim}\, B\right) \wedge d.$$

Moreover if (H2) holds, then we have the stronger upper bound: $\mathbb{N}_x$*-a.e., on* $\{B \subset (0, \sigma_Y)\}$,

$$\dim \left\{ \hat{W}_s; s \in \mathcal{H}_B \right\} \leq \left(\frac{2}{\rho} + 2 \dim B \right) \wedge d.$$

We first show how theorem 2.1 follows from proposition 5.1. For every $q \in (0, \dim B)$ (take $q = 0$ if $\dim B = 0$), there exists a Radon measure μ, supported on B, such that (7) holds (cf theorem 4.13 of [13]). We deduce from proposition 4.3 and the first part of proposition 5.1 that $\mathbb{P}_\nu^X$-a.s., on $\{B \subset (0, \sigma_X)\}$,

$$\dim \operatorname{supp} \int \mu(ds)\ X_s \geq \left(\frac{2}{\rho} + 2q \right) \wedge d.$$

Since $\operatorname{supp} \int \mu(ds)\ X_s \subset \mathcal{C}l \left(\bigcup_{t \in B} \operatorname{supp} X_t \right)$ and since q can be chosen arbitrarily close to $\dim B$, we get the lower bound of theorem 2.1.

Let B' be a countable subset of B such that every point of B is the limit of a decreasing sequence of points of B'. The proof of the following lemma is postponed until the end of this subsection.

Lemma 5.2. *We have* $\mathbb{N}_x$*-a.e.*

$$\mathcal{C}l \left(\bigcup_{t \in B'} \operatorname{supp} Y_t \right) \subset \left\{ \hat{W}_s; s \in \mathcal{H}_B \right\}.$$

Since the process X is càdlàg, and all points of B are limits of decreasing sequences of points of B', it is clear that on $\{B \subset (0, \sigma_X)\}$,

$$\mathcal{C}l \left(\bigcup_{t \in B} \operatorname{supp} X_t \right) = \mathcal{C}l \left(\bigcup_{t \in B'} \operatorname{supp} X_t \right).$$

It is then easy to deduce the upper bounds in theorem 2.1 from the upper bounds in proposition 5.1, proposition 4.3 and lemma 5.2. □

Proof of lemma 5.2. Using the properties of the Brownian snake (in particular the "snake property"), $\mathbb{N}_x$-a.e. for every $t > 0$, we have $\left\{ \hat{W}_s; s \in \mathcal{H}_t \right\} = \left\{ \hat{W}_s; s \in [0, \sigma], L_{\zeta_s}(W_s) = t \right\}$. Thus, we have $\left\{ \hat{W}_s; s \in \mathcal{H}_B \right\} = \left\{ \hat{W}_s; s \in [0, \sigma], L_{\zeta_s}(W_s) \in B \right\}$. Since the mappings $s \mapsto L_{\zeta_s}(W_s)$ and $s \mapsto \hat{W}_s$ are continuous, we deduce that the set $\left\{ \hat{W}_s; s \in [0, \sigma], L_{\zeta_s}(W_s) \in B \right\}$ is compact, and thus closed. Finally we deduce from lemma 4.5 that $\mathbb{N}_x$-a.e., for every $t \in B'$,

$$\operatorname{supp} Y_t = \left\{ \hat{W}_s; s \in \operatorname{supp} dL^t \right\} \subset \left\{ \hat{W}_s; s \in \mathcal{H}_t \right\} \subset \left\{ \hat{W}_s; s \in \mathcal{H}_B \right\}.$$

The desired result follows. □

5.2 The lower bound of proposition 5.1

We introduce the set $K = \{s \in \operatorname{supp} \mu; \int \mu(dt)\, |t-s|^{-q} < \infty\}$. Notice that $\mu(K^c) = 0$. In a first step we show that for every $\kappa \in (0, (2q + 2/\overline{\rho}) \wedge d)$, $\delta \in (0, \kappa/2)$, $s_0 \in K$,

$$\mathbb{N}_x \left[\int Y_{s_0}(dz)\ F_{\kappa - 2\delta} \left(z, \int \mu(dt)\ Y_t \right) \right] = 0,$$

where if $\theta > 0$, F_θ is the measurable function on $\mathbb{R}^d \times M_f$ defined by

$$F_\theta(y, \nu) = \mathbf{1}_{\left\{ \limsup_{n \to \infty} \nu(B_{2^{-n}}(y)) 2^{n\theta} > 0 \right\}},$$

where $B_r(y)$ is the ball centered at y with radius r. By proposition 4.2, we have

$$\begin{aligned} &\mathbb{N}_x \left[\int Y_{s_0}(dy) F_\theta \left(y, \int \mu(dt)\ Y_t \right) \right] \\ &\qquad = \int \bar{\mathbb{P}}_x^{s_0}(dw) \mathbb{E} \left[F_\theta \left(\hat{w}, \int \mathcal{N}_w(du, d\mathrm{W}) \int \mu(dt)\ \mathbf{1}_{\{u < \tau_t(w)\}}\ Y_t(\mathrm{W}) \right) \right]. \qquad (8) \end{aligned}$$

In order to use the Borel-Cantelli lemma, we first bound $\iint \bar{\mathbb{P}}_x^{s_0}(dw) \mathbb{P}(d\omega) \mathbf{1}_{A_n}(w, \omega)$, where

$$A_n := \left\{ (w, \omega);\ 2^{n(\kappa - 2\delta)} \int \mathcal{N}_w(\omega)(du, d\mathrm{W}) \int \mu(dt)\ \mathbf{1}_{\{u < \tau_t(w)\}}\ Y_t(\mathrm{W})\, (B_{2^{-n}}(\hat{w})) \geq C_\kappa 2^{-n\delta} \right\}$$

and $C_\kappa = C_\kappa(w)$ is a finite positive constant that does not depend on n and ω, and depends on w only through $(S_v(w), 0 \leq v < s_0)$ (the choice of this constant will be made precise later). Conditioning on $\mathcal{S}_0 = \sigma(S_v(w), 0 \leq v < s_0)$, and using the Markov inequality, we obtain

$$\begin{aligned} &\bar{\mathbb{E}}_x^{s_0} \left[\mathbb{E}[\mathbf{1}_{A_n}] \right] \\ &\quad \leq \bar{\mathbb{E}}_x^{s_0} \left[\bar{\mathbb{E}}_x^{s_0} \left[\mathbb{E} \left[C_\kappa^{-1} 2^{n(\kappa - \delta)} \int \mathcal{N}_w(du, d\mathrm{W}) \int \mu(dt) \mathbf{1}_{\{u < \tau_t(w)\}} Y_t(\mathrm{W})\, (B_{2^{-n}}(\hat{w})) \right] \middle| \mathcal{S}_0 \right] \right] \\ &\quad = 2^{n(\kappa - \delta)} \bar{\mathbb{E}}_x^{s_0} \left[C_\kappa^{-1} \left[\bar{\mathbb{E}}_x^{s_0} \left[\int \mu(dt) 4 \int_0^{\zeta_w} du\ \mathbf{1}_{\{u < \tau_t(w)\}} \mathbb{N}_{w(u)} \left[Y_t \left(B_{2^{-n}}(y) \right) \right]_{y = \hat{w}} \middle| \mathcal{S}_0 \right] \right] \right] \\ &\quad = 4\ 2^{n(\kappa - \delta)} \bar{\mathbb{E}}_x^{s_0} \left[C_\kappa^{-1} \left[\bar{\mathbb{E}}_x^{s_0} \left[\int \mu(dt) \int_0^{\tau_{s_0} \wedge \tau_t} du\ \bar{\mathbb{P}}_{w(u)}^t \left[\hat{w} \in B_{2^{-n}}(y) \right]_{y = \hat{w}} \middle| \mathcal{S}_0 \right] \right] \right] \\ &\quad = 4\ 2^{n(\kappa - \delta)} \bar{\mathbb{E}}_x^{s_0} \left[C_\kappa^{-1} \int \mu(dt) \int_{[0, s_0 \wedge t)} dS_{u'}\ \mathbb{E}_x \left[\mathbb{P}_{\gamma_{u'}} \left[\gamma_{t - u'} \in B_{2^{-n}}(y) \right]_{y = \gamma_{s_0}} \right] \right], \end{aligned}$$

where γ is under $\mathbb{P}_x$ a Brownian motion in $\mathbb{R}^d$ started at x. In the first equality we used the form of the intensity of the Poisson measure $\mathcal{N}_w$. In the second one, we applied (5) with $D = D_t$. In the third one, we made the formal change of variable $u = S_{u'}$, using the specific properties of the process ξ, and in particular the fact that Γ is constant over each interval (S_{u-}, S_u). We have

$$\mathbb{E}_x \left[\mathbb{P}_{\gamma_{u'}} \left[\gamma_{t - u'} \in B_{2^{-n}}(y) \right]_{y = \gamma_{s_0}} \right] = g_2(2^{-n}, s_0 + t - 2u'),$$

where $g_2(r,t) = \mathbb{P}_0[|\gamma_t| \leq r]$. We prove in the appendix (lemma 8.1) that under the assumption $s_0 \in K$, we can choose a finite constant C_κ depending only on $(S_v(w), 0 \leq v < s_0)$ such that for $r \in (0,1]$,

$$\int \mu(dt) \int_{[0,s_0\wedge t)} g_2(r, s_0 + t - 2u) dS_u \leq C_\kappa r^\kappa.$$

As a consequence, we have for every $n \geq 1$,

$$\bar{\mathbb{E}}_x^{s_0} \left[\mathbb{E}[\mathbf{1}_{A_n}]\right] \leq 4\ 2^{-n\delta}.$$

Applying the Borel-Cantelli lemma to the sequence $(A_n, n \geq 1)$, we get $\bar{\mathbb{P}}_x^{s_0}$-a.s., $\mathbb{P}$-a.s.

$$\limsup_{n\to\infty} 2^{n(\kappa-2\delta)} \int \mathcal{N}_w(du, d\mathrm{W}) \int \mu(dt)\ \mathbf{1}_{\{u<\tau_t(w)\}}\ Y_t(\mathrm{W})\left(B_{2^{-n}}(\hat{w})\right) = 0.$$

Hence by the definition of F_θ and (8), we get for every $s_0 \in K$,

$$\mathbb{N}_x\left[\int Y_{s_0}(dy) F_{\kappa-2\delta}\left(y, \int \mu(dt)\ Y_t\right)\right] = 0.$$

Since $\mu(K^c) = 0$, integrating with respect to $\mu(ds_0)$ gives $\mathbb{N}_x$-a.e.

$$\int \mu(ds) \int Y_s(dy) F_{\kappa-2\delta}\left(y, \int \mu(dt)\ Y_t\right) = 0. \tag{9}$$

We deduce from theorem 4.9 of [13], that for every $\kappa \in (0, (2q + 2/\overline{\rho}) \wedge d)$, $\delta \in (0, \kappa/2)$, $\mathbb{N}_x$-a.e. on $\left\{\int \mu(dt)\ Y_t \neq 0\right\}$,

$$\dim\ \mathrm{supp} \int \mu(ds) Y_s \geq \kappa - 2\delta.$$

The lower bound of proposition 5.1 follows. □

5.3 The upper bounds of proposition 5.1

First of all we give a technical result about the Brownian snake.

Lemma 5.3. *1. $\mathbb{N}_x$-a.e. the function $s \mapsto L_{\zeta_s}(W_s)$, respectively $s \mapsto \hat{W}_s = \Gamma_{\zeta_s}(W_s)$, is locally Hölder with index $\underline{\rho}/2 - \delta$, respectively $\underline{\rho}/4 - \delta$, for every $\delta \in (0, \underline{\rho}/4)$.*

2. The adapted increasing process $(M_t, t > 0)$, defined by:

$$M_t := \sup_{s\in(0,t]} \ \sup_{u\neq v, (u,v)\in[0,\zeta_s]^2} \frac{|L_u(W_s) - L_v(W_s)|}{|u-v|^{\underline{\rho}(1-\delta/2)}},$$

is $\mathbb{N}_x$-a.e. finite for every $\delta \in (0,1)$.

Proof. 1. Recall that $\mathcal{E}_x$-a.s. the mapping $s \mapsto \left(L_{\zeta_s}(W_s), \hat{W}_s\right)$ is continuous. Thanks to the Kolmogorov lemma it is sufficient to prove that for every integer $k \geq 1$, and $\delta \in (0, \underline{\varrho})$, $N > 0$, there exists a constant c'_N such that for every $0 \leq s, s' \leq N$,

$$\mathcal{E}_x \left[\left| L_{\zeta_s}(W_s) - L_{\zeta_{s'}}(W_{s'}) \right|^{2k} \right] \leq c'_N \left| s - s' \right|^{\left(\underline{\varrho} - \delta\right) k}, \tag{10}$$

$$\mathcal{E}_x \left[\left| \hat{W}_s - \hat{W}_{s'} \right|^{2k} \right] \leq c'_N \left| s - s' \right|^{\left(\underline{\varrho} - \delta\right) k/2}. \tag{11}$$

First, we prove (11). Since $\mathcal{E}_x$-a.s. $\Gamma_{m(s,s')}(W_s) = \Gamma_{m(s,s')}(W_{s'})$, we have by symmetry

$$\mathcal{E}_x \left[\left| \hat{W}_s - \hat{W}_{s'} \right|^{2k} \right] \leq 2.2^{2k-1} \mathcal{E}_x \left[\left| \Gamma_{\zeta_s}(W_s) - \Gamma_{m(s,s')}(W_s) \right|^{2k} \right].$$

Conditionally on ζ the distribution of $\Gamma_{\zeta_s}(W_s) - \Gamma_{m(s,s')}(W_s)$ is the same as that of $\Gamma_{\zeta_s} - \Gamma_{m(s,s')}$ under $\bar{\mathbb{P}}_x$. Thus we get

$$\mathcal{E}_x \left[\left| \Gamma_{\zeta_s}(W_s) - \Gamma_{m(s,s')}(W_s) \right|^{2k} \right] = \mathcal{E}_x \left[\bar{\mathbb{E}}_x \left[\left| \Gamma_u - \Gamma_v \right|^{2k} \right]_{u=\zeta_s, v=m(s,s')} \right].$$

By scaling and using the same arguments as in the proof of lemma 3.2, we get

$$\begin{aligned} \bar{\mathbb{E}}_x \left[\left| \Gamma_u - \Gamma_v \right|^{2k} \right] = \mathbb{E}_0 \left[\left| \gamma_1 \right|^{2k} \right] \bar{\mathbb{E}}_0 \left[L_u - L_v \right]^k &\leq \mathbb{E}_0 \left[\left| \gamma_1 \right|^{2k} \right] k! \left[\mathbb{E}\left[L_{u-v} \right] \right]^k \\ &\leq c_1 \left[\left| u - v \right| \vee \left| u - v \right|^{\underline{\varrho} - \delta} \right]^k, \end{aligned}$$

by lemma 3.2,1. From this inequality and standard bounds on the moments of the increments of ζ, we easily get

$$\mathcal{E}_x \left[\left| \hat{W}_s - \hat{W}_{s'} \right|^{2k} \right] \leq c_2 \left[\left| s - s' \right| \vee \left| s - s' \right|^{\underline{\varrho} - \delta} \right]^{k/2},$$

where the constant c_2 is independent of s and s'. Since s and s' are bounded, (11) follows. The proof of (10) is similar.

2. Thanks to lemma 3.2,1. for every integer $k \geq 1$, and $1/2 > \delta > 0$, $A > 0$, $T > 0$, there exists a constant c_1 such that for every $(u, v) \in [0, A]^2$,

$$\mathbb{E} \left[\left| L_u - L_v \right|^k \right] \leq c_1 \left| u - v \right|^{k \underline{\varrho}(1-\delta)}.$$

Furthermore, there exists a constant c_2, such that for every $(s, t) \in [0, T]^2$,

$$\mathcal{E}_x \left[\sup_{r, q \in [s,t]} \left| \zeta_r - \zeta_q \right|^{k \underline{\varrho}(1-\delta)} \right] \leq c_2 \left| s - t \right|^{k \underline{\varrho}(1-\delta)/2}.$$

For convenience, we put $L_u(W_s) = L_{\zeta_s}(W_s)$ when $u > \zeta_s$. Using the above inequalities and the snake properties, we then bound for every integer $k \geq 2\underline{\varrho}^{-1}$, and $u \geq v$, $(u, v) \in [0, A]^2$,

$(s,t) \in [0,T]^2$,

$$
\begin{aligned}
&\mathcal{E}_x\left[|L_u(W_s) - L_v(W_s) - L_u(W_t) + L_v(W_t)|^k\right] \\
&\qquad \le \mathcal{E}_x\left[\mathbf{1}_{m(s,t)\le v \le u}\,[(L_u(W_s) - L_v(W_s)) + (L_u(W_t) - L_v(W_t))]^k\right] \\
&\qquad\quad + \mathcal{E}_x\left[\mathbf{1}_{v< m(s,t)\le u}\,\left[\left(L_u(W_s) - L_{m(s,t)}(W_s)\right) + \left(L_u(W_t) - L_{m(s,t)}(W_t)\right)\right]^k\right] \\
&\qquad \le c_3\mathcal{E}_x\left[\mathbf{1}_{m(s,t)\le v \le u}\,[|\zeta_s \wedge u - \zeta_s \wedge v + \zeta_t \wedge u - \zeta_t \wedge v|]^{k\underline{\varrho}(1-\delta)}\right] \\
&\qquad\quad + c_3\mathcal{E}_x\left[\mathbf{1}_{v< m(s,t)\le u}\,[|\zeta_s \wedge u + \zeta_t \wedge u - 2m(s,t)|]^{k\underline{\varrho}(1-\delta)}\right] \\
&\qquad \le c_4\mathcal{E}_x\left[(u-v)^{k\underline{\varrho}(1-\delta)} \wedge \sup_{r,q\in[s,t]} |\zeta_r - \zeta_q|^{k\underline{\varrho}(1-\delta)}\right] \\
&\qquad \le c_5\left[|u-v| \wedge |s-t|^{1/2}\right]^{k\underline{\varrho}(1-\delta)},
\end{aligned}
$$

where the constant c_5 is independent of u, v, s and t. For s, t fixed consider the continuous random process $Z_u^{s,t} = L_u(W_s) - L_u(W_t)$. Fix $\eta \in (1/2, 1)$. The previous inequality gives:

$$
\mathcal{E}_x\left[\left|Z_u^{s,t} - Z_v^{s,t}\right|^k\right] \le c_5\,|u-v|^{k\eta\underline{\varrho}(1-\delta)}\,|t-s|^{k(1-\eta)\underline{\varrho}(1-\delta)/2}.
$$

The Kolmogorov lemma (theorem 1.2.1 of [23]) implies that the process $Z^{s,t}$ is locally Hölder with exponent $\theta_0 = \eta\underline{\varrho}(1-3\delta/2)$, and moreover a close look at the arguments of the proof shows that

$$
\mathcal{E}_x\left[\left(\sup_{u\neq v;(u,v)\in[0,A]^2} \frac{\left|Z_u^{s,t} - Z_v^{s,t}\right|}{|u-v|^{\theta_0}}\right)^k\right] \le c_6\,|t-s|^{k(1-\eta)\underline{\varrho}(1-\delta)/2},
$$

where the constant c_6 is independent of t, s. Now consider the norm on the Banach space of all real functions f on $[0, A]$ that are Hölder with exponent θ_0 and such that $f(0) = 0$, defined by

$$
\|\,f\,\|_{\theta_0} := \sup_{u\neq v;(u,v)\in[0,A]^2} |f(u) - f(v)|\,|u-v|^{-\theta_0}.
$$

The above inequality can be written as

$$
\mathcal{E}_x\left[\|\,L(W_s) - L(W_t)\,\|_{\theta_0}^k\right] \le c_6\,|t-s|^{k(1-\eta)\underline{\varrho}(1-\delta)/2}.
$$

We can again use the Kolmogorov lemma to get $\mathcal{E}_x$-a.s.

$$
\sup_{s\in[0,T]} \|\,L(W_s)\,\|_{\theta_0} < \infty.
$$

Note that $\mathcal{E}_x$-a.s. $\sup_{s\in[0,T]} \zeta_s$ is finite, and so $\sup_{s\in[0,T]} \zeta_s \le A$ if A is large enough. The fact that $M_t < \infty$ follows from the last bound by taking η sufficiently close to 1 and δ small enough. This completes the proof. $\square$

Since we have proved the function $s \mapsto \hat{W}_s$ is $\mathbb{N}_x$-a.e. locally Hölder with index $\frac{\underline{\rho}}{4} - \delta$, for any $\delta \in (0, \underline{\rho}/4)$, proposition 2.2 from [13] implies that $\mathbb{N}_x$-a.e.,

$$\dim \left\{ \hat{W}_s, s \in \mathcal{H}_B \right\} \leq \frac{4}{\underline{\rho}} \dim \mathcal{H}_B.$$

We will now prove in three steps that for every $[\alpha, \beta] \in (0, \infty)$, $\delta' \in (0, 1/2)$, $\mathbb{N}_x$-a.e.,

$$\dim \left(\mathcal{H}_B \cap [\alpha, \beta] \right) \leq \frac{1}{2} (1 + \underline{\rho} \, \overline{\dim} \, B) + \delta'. \tag{12}$$

Moreover if (H2) is satisfied then $\mathbb{N}_x$-a.e.,

$$\dim \left(\mathcal{H}_B \cap [\alpha, \beta] \right) \leq \frac{1}{2} (1 + \rho \, \dim B) + \delta'. \tag{13}$$

This will be sufficient to prove the upper bounds of proposition 5.1.

Proof of (12) and (13). In a first step we start from a covering of B by open sets and construct an associated covering of $\mathcal{H}_B \cap [\alpha, \beta]$. In a second step, lemma 5.4 gives us an upper bound on the cardinality of this covering. In the last step, we prove (12) and (13) by letting the maximal diameter of the open sets in the covering of B tend to 0.

Let $[\alpha, \beta] \subset (0, \infty)$, with $\alpha < 1$, and let $\delta > 0$ small enough. We set $\theta = \underline{\rho}(1 + \delta)/2$.

First step. Let i be an integer and $\varepsilon > 0$, $h > \varepsilon^\theta$. We define the stopping times:

$$T_i^\varepsilon = \inf \left\{ u \in [i\varepsilon, (i+1)\varepsilon] \, ; u \in \mathcal{H}_{[h - \varepsilon^\theta, h]} \cap [\alpha, \beta + 1] \right\},$$

with the convention $\inf \emptyset = \sigma$. If $[t]$ is the unique integer k such that $k \leq t < k+1$, we define

$$N_{\varepsilon, h, \delta} = \sum_{i = [\alpha/\varepsilon]}^{[\beta/\varepsilon]} \mathbf{1}_{\left\{ T_i^\varepsilon < \sigma \right\}}.$$

The random variable $N_{\varepsilon, h, \delta}$ represents an upper bound on the total number of intervals of the form $[i\varepsilon, (i+1)\varepsilon)$ which intersect $\mathcal{H}_{[h - \varepsilon^\theta, h]} \cap [\alpha, \beta]$. Let $\varepsilon_0 \in (0, \alpha/2)$, and $((h_n, r_n), n \geq 1)$ a possibly finite sequence in $(0, \infty) \times (0, \varepsilon^\theta]$ such that $h_n > r_n > 0$ for all $n \geq 1$, and the family of open sets $(h_n - r_n, h_n)$ covers the compact set B. It is clear that

$$\mathcal{H}_B \cap [\alpha, \beta] \subset \bigcup_{n \geq 1} \mathcal{H}_{[h_n - r_n, h_n]} \cap [\alpha, \beta].$$

We finally denote by Δ the collection of all pairs $(i, \varepsilon) \in \mathbb{N}^* \times (0, \varepsilon_0]$ such that there exists $n \geq 1$ for which

$$\varepsilon = r_n^{1/\theta}, \quad \text{and} \quad [i\varepsilon, (i+1)\varepsilon) \cap \mathcal{H}_{[h_n - r_n, h_n]} \cap [\alpha, \beta] \neq \emptyset.$$

The collection of balls $([i\varepsilon, (i+1)\varepsilon] \, ; (i, \varepsilon) \in \Delta)$ covers the set $\mathcal{H}_B \cap [\alpha, \beta]$. Moreover for ε fixed of the form $\varepsilon = r_n^{1/\theta}$,

$$\text{Card } \left\{ i \in \left\{ [\alpha/\varepsilon], \ldots, [\beta/\varepsilon] \right\} ; (i, \varepsilon) \in \Delta \right\} \leq N_{\varepsilon, h_n, \delta}.$$

Second step. We are mainly concerned by a control on the expectation of $N_{\varepsilon,h,\delta}$. Recall the notation of 4.2, and observe that for $\varepsilon \in (0,\alpha/2)$,

$$
\begin{aligned}
1 \geq \mathbb{N}_x L_{\beta+1}^{h,\sqrt{\varepsilon}} &\geq \frac{1}{\sqrt{\varepsilon}}\mathbb{N}_x \int_{\alpha/2}^{\beta+1} \mathbf{1}_{\{\tau_h(W_u)<\zeta_u<\tau_h(W_u)+\sqrt{\varepsilon}\}} du \\
&\geq \frac{1}{2\sqrt{\varepsilon}} \sum_{i=[\alpha/\varepsilon]}^{[\beta/\varepsilon]} \mathbb{N}_x \left[T_i^\varepsilon < \sigma; \int_{i\varepsilon}^{(i+2)\varepsilon} \mathbf{1}_{\{\tau_h(W_u)<\zeta_u<\tau_h(W_u)+\sqrt{\varepsilon}\}} du \right] \\
&\geq \frac{1}{2\sqrt{\varepsilon}} \sum_{i=[\alpha/\varepsilon]}^{[\beta/\varepsilon]} \mathbb{N}_x \left[T_i^\varepsilon < \sigma; \mathcal{E}^*_{W_{T_\varepsilon^i}} \left[\int_0^\varepsilon \mathbf{1}_{\{\tau_h(W_u)<\zeta_u<\tau_h(W_u)+\sqrt{\varepsilon}\}} du \right]\right],
\end{aligned}
$$

where we used the strong Markov property at time T_ε^i for the last inequality. To go further, let us introduce some notation and state a technical lemma. Recall the notation of lemma 3.2 to define, for every real number $u > 0$ of the form $u = (4R_n)^2$,

$$
Z_u^1(y) := \bar{\mathbb{P}}_0 \left[\inf_{i\geq n} R_i^{-\underline{\rho}(1+\delta/2)} L_{R_i} > (y+1) \right].
$$

Note that the sequence (R_n) depends on δ. If (H2) holds, then consider

$$
Z_u^2(y) := \bar{\mathbb{P}}_0 \left[\inf_{v\in(0,\sqrt{u}/4]} v^{-\rho(1+\delta/2)} L_v > (y+1) \right].
$$

We will use the same notation $Z_u(y)$, for both functions $Z_u^1(y)$ and $Z_u^2(y)$. This function is defined for $(u,y) \in \mathbb{F} \times \mathbb{R}^+$, where $\mathbb{F} = \{(4R_n)^2; n \geq 1\}$ in the first case and $\mathbb{F} = (0,\infty)$ in the second one. Clearly the function Z is positive and bounded above by 1, and is decreasing in both variables u and y. Moreover thanks to lemma 3.2, we have for every $y > 0$, $\lim_{u\in\mathbb{F},u\to 0+} Z_u(y) = 1$. Recall the process M_t was defined in lemma 5.3. The proof of the following lemma is postponed to the end of this section.

Lemma 5.4. *There exists a universal constant C_0, such that for every $\delta \in (0,1/2)$, $h > 0$, $\varepsilon \in \mathbb{F}\cap(0,1/2)$, $\mathbb{N}_x$-a.e. for every stopping time T taking values in $\left(\mathcal{H}_{[h-\varepsilon^\theta,h]} \cap [\alpha,\beta]\right) \cup \{\sigma\}$, we have*

$$
\mathcal{E}^*_{W_T} \left[\int_0^\varepsilon \mathbf{1}_{\{\tau_h(W_u)<\zeta_u<\tau_h(W_u)+\sqrt{\varepsilon}\}} du \right] \geq C_0\, \varepsilon^{1+\delta} Z_\varepsilon(M_T) \mathbf{1}_{T<\sigma}\, \mathcal{E}^*_{W_T}[\sigma > \varepsilon].
$$

Using this lemma with $T = T_i^\varepsilon$, we get for $\varepsilon \in \mathbb{F}\cap(0,1/2)$

$$
\begin{aligned}
&\mathbb{N}_x \left[T_i^\varepsilon < \sigma; \mathcal{E}^*_{W_{T_\varepsilon^i}} \left[\int_0^\varepsilon \mathbf{1}_{\{\tau_h(W_u)<\zeta_u<\tau_h(W_u)+\sqrt{\varepsilon}\}} du \right]\right] \\
&\qquad\geq C_0 \varepsilon^{1+\delta} \mathbb{N}_x \left[T_i^\varepsilon < \sigma; Z_\varepsilon(M_{T_i^\varepsilon})\, \mathcal{E}^*_{W_{T_i^\varepsilon}}[\sigma > \varepsilon] \right] \\
&\qquad\geq C_0 \varepsilon^{1+\delta} \mathbb{N}_x \left[T_i^\varepsilon < \sigma; \sigma > (i+2)\varepsilon; Z_\varepsilon(M_{T_i^\varepsilon}) \right] \\
&\qquad\geq C_0 \varepsilon^{1+\delta} \mathbb{N}_x \left[T_i^\varepsilon < \sigma; Z_\varepsilon(M_{T_i^\varepsilon}) \right] - C_0 \varepsilon^{1+\delta} \mathbb{N}_x \left[\sigma \in [i\varepsilon, (i+2)\varepsilon] \right].
\end{aligned}
$$

We then sum over $i \in \{[\alpha/\varepsilon], \dots, [\beta/\varepsilon]\}$, and use the monotonicity of the mapping $y \mapsto Z_\varepsilon(y)$ to get

$$\begin{aligned} 1 \geq \mathbb{N}_x L^{h,\sqrt{\varepsilon}}_{\beta+1} &\geq \frac{1}{2\sqrt{\varepsilon}} C_0 \varepsilon^{1+\delta} \sum_{i=[\alpha/\varepsilon]}^{[\beta/\varepsilon]} \mathbb{N}_x \left[T^\varepsilon_i < \sigma; Z_\varepsilon(M_{\beta+1})\right] - \frac{1}{\sqrt{\varepsilon}} C_0 \varepsilon^{1+\delta} \mathbb{N}_x \left[\sigma > \alpha/2\right]] \\ &\geq 2^{-1} C_0 \varepsilon^{1/2+\delta} \mathbb{N}_x \left[Z_\varepsilon(M_{\beta+1}) N_{\varepsilon,h,\delta}\right] - 2C_0 \varepsilon^{1/2+\delta} [\alpha\pi]^{-1/2}. \end{aligned}$$

In the last bound we also used the definition of $N_{\varepsilon,h,\delta}$ and the well-known formula $\mathbb{N}_x\left[\sigma > a\right] = (2/\pi a)^{1/2}$. From the monotonicity of the mapping $\varepsilon \mapsto Z_\varepsilon(y)$, we get for $\varepsilon_0 \in \mathbb{F}$ small enough and $\varepsilon_0 \geq \varepsilon \in \mathbb{F}$

$$\mathbb{N}_x \left[Z_{\varepsilon_0}(M_{\beta+1}) N_{\varepsilon,h,\delta}\right] \leq 4 C_0^{-1} \varepsilon^{-1/2-\delta}.$$

Third step. Let κ be such that $2(\kappa - 1/2)/\underline{\rho} > d_1$ where $d_1 = \dim B$ if (H2) is satisfied, $d_1 = \overline{\dim} B$ otherwise. Let $\delta > 0$ be so small that $(\kappa - 1/2 - \delta)/\theta \geq d_1 + \delta$ with $\theta = \underline{\rho}(1+\delta)/2$. By the definition of upper box-counting dimension, and Hausdorff dimension, for every integer p there exists a sequence $((h^p_n, r^p_n), n \geq 1)$, where $h^p_n > r^p_n > 0$, such that the family of open sets $(h^p_n - r^p_n, h^p_n)$ covers B and such that $(r^p_n)^{1/\theta} \in \mathbb{F} \cap (0, 2^{-p} \wedge \alpha/2]$ for all $n \geq 1$ and

$$\sum_{n\geq 1} (r^p_n)^{d_1+\delta} \leq 2^{-2p}.$$

For each p consider the set Δ_p associated to the sequence $((h^p_n, r^p_n), n \geq 1)$ as in the first step of the proof. For p big enough we deduce from the last inequality of the second step that, if $\varepsilon_0(p) = \sup \mathbb{F} \cap (0, (\alpha/2) \wedge 2^{-p}]$,

$$\begin{aligned} \mathbb{N}_x \left[Z_{\varepsilon_0(p)}(M_{\beta+1}) \sum_{(i,\varepsilon)\in\Delta_p} \varepsilon^\kappa \right] &\leq \sum_{n\geq 1} (r^p_n)^{\kappa/\theta} \mathbb{N}_x \left[Z_{\varepsilon_0(p)}(M_{\beta+1}) N_{(r^p_n)^{1/\theta}, h^p_n, \delta} \right] \\ &\leq 4 C_0^{-1} \sum_{n\geq 1} (r^p_n)^{(\kappa - 1/2 - \delta)/\theta} \\ &\leq 4 C_0^{-1} 2^{-2p}. \end{aligned}$$

By the Borel-Cantelli lemma we get the existence of p' such that for every integer $p \geq p'$,

$$Z_{\varepsilon_0(p)}(M_{\beta+1}) \sum_{(i,\varepsilon)\in\Delta_p} \varepsilon^\kappa \leq 2^{-p}.$$

We have $\lim_{p\to\infty} \varepsilon_0(p) = 0$. Thanks to the properties of Z, we get $\lim_{p\to\infty} Z_{\varepsilon_0(p)}(M_{\beta+1}) = 1$. We have thus proved that $\mathbb{N}_x$-a.e.,

$$\lim_{p\to\infty} \sum_{(i,\varepsilon)\in\Delta_p} \varepsilon^\kappa = 0.$$

Since the collection $([i\varepsilon, (i+1)\varepsilon]; (i,\varepsilon) \in \Delta_p)$ covers $\mathcal{H}_B \cap [\alpha, \beta]$, we obtain that $\mathbb{N}_x$-a.e.

$$\dim \mathcal{H}_B \cap [\alpha, \beta] \leq \kappa.$$

Since this bound holds for every κ such that $2(\kappa-1/2)/\underline{\rho} > \overline{\dim} B$ (and $2(\kappa-1/2)/\underline{\rho} > \dim B$, if (H2) is satisfied), we obtain (12) and (13), which completes the proof of proposition 5.1. □

Proof of lemma 5.4. Let $\delta \in (0,1/2)$, $h>0$, $\varepsilon \in \mathbb{F} \cap (0,1/2)$. We set $\theta = \underline{\rho}(1+\delta)/2$. Let T be a stopping time with values in $\left(\mathcal{H}_{[h-\varepsilon^\theta,h]} \cap [\alpha,\beta]\right) \cup \{\sigma\}$. Note that, on $\{T<\sigma\}$, $L_r(W_T) < L_{\zeta_T}(W_T)$ for every $r \in [0,\zeta_T)$. We introduce the following three sets, where $m(s,s') = \inf_{r\in[s,s']} \zeta_r$ and $b_\varepsilon := \left[\frac{1}{16}\,\varepsilon^{1+2\delta}\right] \wedge (\zeta_T/2)^2$,

$$A'_\varepsilon := \Big\{ m(T,T+b_\varepsilon) \in \left[\zeta_T - \sqrt{b_\varepsilon}, \zeta_T - \sqrt{b_\varepsilon}/2\right]; m(T+b_\varepsilon, T+\varepsilon/2) \geq \zeta_T; \\ \zeta_{T+\varepsilon/2} \in \left[\zeta_T + 3\sqrt{\varepsilon}/8, \zeta_T + 5\sqrt{\varepsilon}/8\right] \Big\},$$

$$A_\varepsilon := A'_\varepsilon \cap \left\{\forall s \in [T+\varepsilon/2, T+\varepsilon], \zeta_s \in \left(\zeta_T + \sqrt{\varepsilon}/4, \zeta_T + 3\sqrt{\varepsilon}/4\right]\right\}$$

and

$$B_\varepsilon := \left\{ L_{y_\varepsilon+\sqrt{\varepsilon}/4}(W_{T+\varepsilon/2}) - L_{y_\varepsilon}(W_{T+\varepsilon/2}) \geq \varepsilon^\theta (M_T+1) \right\},$$

where $y_\varepsilon := \inf\left\{r > m(T,T+\varepsilon); L_r(W_T) > L_{m(T,T+\varepsilon)}(W_T)\right\}$. The lemma is then a simple consequence of the following two results:

A. $\mathbb{N}_x$-a.e. on $A_\varepsilon \cap B_\varepsilon \cap \{T<\sigma\}$, we have

$$\int_T^{T+\varepsilon} \mathbf{1}_{\left\{\tau_h(W_s) < \zeta_s < \tau_h(W_s)+\sqrt{\varepsilon}\right\}} ds \geq \varepsilon/2,$$

B. There exists a universal constant C_0 such that

$$\mathcal{E}^*_{W_T}\left[A_\varepsilon \cap B_\varepsilon\right] \geq 2C_0 \varepsilon^\delta \mathbf{1}_{T<\sigma} Z_\varepsilon(M_T) \mathcal{E}^*_{W_T}\left[\sigma > \varepsilon\right].$$

□

Proof of A. The proof is based on the properties of the Brownian snake. Let us first show that on $A_\varepsilon \cap B_\varepsilon \cap \{T<\sigma\}$, for every $s \in [T+\varepsilon/2, T+\varepsilon]$, $\tau_h(W_s) < \zeta_s$. Notice that we have $y_\varepsilon \leq \zeta_T$ on $\{T<\sigma\} \cap \{m(T,T+\varepsilon) < \zeta_T\} \subset \{T<\sigma\} \cap A_\varepsilon$. On $\{T<\sigma\} \cap A_\varepsilon$ we get $m(T+\varepsilon/2, T+\varepsilon) > \zeta_T + \sqrt{\varepsilon}/4 \geq y_\varepsilon + \sqrt{\varepsilon}/4$. Thus for every $s \in [T+\varepsilon/2, T+\varepsilon]$, the paths $t \mapsto L_t(W_s)$ coincide for $t \in [0, y_\varepsilon + \sqrt{\varepsilon}/4]$. Thus we have for every $s \in [T+\varepsilon/2, T+\varepsilon]$

$$L_{y_\varepsilon+\sqrt{\varepsilon}/4}(W_{T+\varepsilon/2}) = L_{y_\varepsilon+\sqrt{\varepsilon}/4}(W_s) \leq L_{\zeta_s}(W_s).$$

Furthermore, the paths $t \mapsto L_t(W_T)$ and $t \mapsto L_t(W_{T+\varepsilon/2})$ coincide over $[0, m(T,T+\varepsilon/2)]$. Since $m(T,T+\varepsilon/2) \geq \zeta_T - \sqrt{b_\varepsilon}$, we get

$$\begin{aligned} L_{y_\varepsilon}(W_{T+\varepsilon/2}) &\geq L_{m(T,T+\varepsilon/2)}(W_{T+\varepsilon/2}) \\ &= L_{m(T,T+\varepsilon/2)}(W_T) \geq L_{\zeta_T - \sqrt{b_\varepsilon}}(W_T). \end{aligned}$$

Using the definition of M_t (cf lemma 5.3), we see that

$$L_{\zeta_T - \sqrt{b_\varepsilon}}(W_T) \geq L_{\zeta_T}(W_T) - M_T b_\varepsilon^{\rho(1-\delta/2)/2} > L_{\zeta_T}(W_T) - M_T \varepsilon^\theta.$$

Then we get that on the event $\{T < \sigma\} \cap A_\varepsilon$, for every $s \in [T + \varepsilon/2, T + \varepsilon]$,

$$\begin{aligned} L_{\zeta_s}(W_s) &\geq L_{y_\varepsilon + \sqrt{\varepsilon}/4}(W_{T+\varepsilon/2}) - L_{y_\varepsilon}(W_{T+\varepsilon/2}) + L_{y_\varepsilon}(W_{T+\varepsilon/2}) \\ &> L_{y_\varepsilon + \sqrt{\varepsilon}/4}(W_{T+\varepsilon/2}) - L_{y_\varepsilon}(W_{T+\varepsilon/2}) + L_{\zeta_T}(W_T) - M_T \varepsilon^\theta. \end{aligned}$$

It is then clear that on $A_\varepsilon \cap B_\varepsilon \cap \{T < \sigma\}$, we have for every $s \in [T + \varepsilon/2, T + \varepsilon]$,

$$L_{\zeta_s}(W_s) > (M_T + 1)\,\varepsilon^\theta + L_{\zeta_T}(W_T) - M_T \varepsilon^\theta = L_{\zeta_T}(W_T) + \varepsilon^\theta.$$

Since, on $\{T < \sigma\}$, $T \in \mathcal{H}_{[h-\varepsilon^\theta, h]}$, we have $L_{\zeta_T}(W_T) \geq h - \varepsilon^\theta$. It follows that $L_{\zeta_s}(W_s) > h$ for $s \in [T + \varepsilon/2, T + \varepsilon]$. Thus we have also $\tau_h(W_s) < \zeta_s$ for $s \in [T + \varepsilon/2, T + \varepsilon]$.

Finally let us prove that on $\{T < \sigma\} \cap A_\varepsilon$, for every $s \in [T + \varepsilon/2, T + \varepsilon]$, $\tau_h(W_s) > \zeta_s - \sqrt{\varepsilon}$. For every $s \in [T, T + \varepsilon]$, the paths $t \mapsto L_t(W_s)$ coincide over $[0, m(T, T + \varepsilon)]$. The inequality

$$L_{m(T,T+\varepsilon)}(W_s) = L_{m(T,T+\varepsilon)}(W_T) < L_{\zeta_T}(W_T) \leq h,$$

implies $\tau_h(W_s) > m(T, T + \varepsilon)$ for every $s \in [T + \varepsilon/2, T + \varepsilon]$. Recall that on $\{T < \sigma\} \cap A_\varepsilon$, for every $s \in [T + \varepsilon/2, T + \varepsilon]$,

$$m(T, T + \varepsilon) \geq \zeta_T - \sqrt{b_\varepsilon} \geq \zeta_s - \sqrt{b_\varepsilon} - 3\sqrt{\varepsilon}/4 > \zeta_s - \sqrt{\varepsilon}.$$

Then we have for every $s \in [T + \varepsilon/2, T + \varepsilon]$, $\tau_h(W_s) > \zeta_s - \sqrt{\varepsilon}$. In a nutshell we have obtained that $\mathbb{N}_x$-a.e. on $A_\varepsilon \cap B_\varepsilon \cap \{T < \sigma\}$, for every $s \in [T + \varepsilon/2, T + \varepsilon]$,

$$\tau_h(W_s) < \zeta_s < \tau_h(W_s) + \sqrt{\varepsilon}.$$

This completes the proof of A. □

Proof of B. Let $\varepsilon \in \mathbb{F} \cap (0, 1/2)$. By conditioning on $\sigma(W_s, 0 \leq s \leq T + \varepsilon/2)$ and using a scaling argument we get

$$\mathcal{E}^*_{W_T}[A_\varepsilon] \geq \mathcal{E}^*_{W_T}\left[A'_\varepsilon\right] P_0 \left[\forall s \in [0, 1/2], |B_s| < 1/8\right],$$

where B is under P_y a linear Brownian motion started at $y \in \mathbb{R}$. We set $\tilde{m}(s,t) = \inf_{r \in [s,t]} B_r$. Using the Markov property at time b_ε for B, we get:

$$\begin{aligned} \mathcal{E}^*_{W_T}\left[A'_\varepsilon\right] =& P_{\zeta_T}\Bigg[\tilde{m}(0, b_\varepsilon) \in \left[\zeta_T - \sqrt{b_\varepsilon}, \zeta_T - \sqrt{b_\varepsilon}/2\right]; \\ & \qquad \tilde{m}(b_\varepsilon, \varepsilon/2) \geq \zeta_T; B_{\varepsilon/2} \in \left[\zeta_T + 3\sqrt{\varepsilon}/8, \zeta_T + 5\sqrt{\varepsilon}/8\right]\Bigg] \\ \geq& P_0\Bigg[\tilde{m}(0, b_\varepsilon) \in \left[-\sqrt{b_\varepsilon}, -\sqrt{b_\varepsilon}/2\right]; \sqrt{b_\varepsilon}/2 \leq B_{b_\varepsilon} \leq \sqrt{b_\varepsilon}; \\ & \qquad P_{B_{b_\varepsilon}}\left[\tilde{m}(0, \frac{\varepsilon}{2} - b_\varepsilon) \geq 0, B_{\frac{\varepsilon}{2} - b_\varepsilon} \in \left[3\sqrt{\varepsilon}/8, 5\sqrt{\varepsilon}/8\right]\right]\Bigg]. \end{aligned}$$

Using standard properties of linear Brownian motion, we easily see that the above expression is bounded below by a universal constant times $\sqrt{b_\varepsilon/\varepsilon}$. So there exists a universal constant C_0 such that

$$\mathcal{E}^*_{W_T}[A_\varepsilon] > 8\, C_0 \sqrt{b_\varepsilon/\varepsilon}.$$

We finally get a lower bound on $\mathcal{E}^*_{W_T}[A_\varepsilon \cap B_\varepsilon]$. We denote by $\mathcal{S}_T$ the σ-field $\sigma(W_s, s \leq T) \vee \sigma(\zeta_s, s \geq 0)$. Recall that the two paths W_T and $W_{T+\varepsilon/2}$ coincide over $[0, m(T, T+\varepsilon/2)]$. Conditionally on $\mathcal{S}_T$, the distribution of

$$\left(\left(\xi_{m(T,T+\varepsilon/2)+u}(W_{T+\varepsilon/2}), L_{m(T,T+\varepsilon/2)+u}(W_{T+\varepsilon/2})\right), 0 \leq u < \zeta_{T+\varepsilon/2} - m(T, T+\varepsilon/2)\right)$$

is the law of (ξ_u, L_u), started at $\left(\xi_{m(T,T+\varepsilon/2)}(W_T), L_{m(T,T+\varepsilon/2)}(W_T)\right)$ and killed at time $\zeta_{T+\varepsilon/2} - m(T, T+\varepsilon/2)$. Notice $y_\varepsilon = \inf\left\{r > m(T, T+\varepsilon); L_r(W_T) > L_{m(T,T+\varepsilon)}(W_T)\right\}$ is $\mathcal{S}_T$-measurable by construction. Moreover on $\{T < \sigma\} \cap A_\varepsilon$, we have $y_\varepsilon = m(T, T+\varepsilon/2) + \xi_{m(T,T+\varepsilon/2)}(W_T)$, as a consequence of the behavior of the process ξ. Thus conditionally on $\mathcal{S}_T$, on $\{T < \sigma\} \cap A_\varepsilon$, we obtain that $\left(\left(\xi_{y_\varepsilon+u}(W_{T+\varepsilon/2}), L_{y_\varepsilon+u}(W_{T+\varepsilon/2})\right), 0 \leq u < \zeta_{T+\varepsilon/2} - y_\varepsilon\right)$ is distributed as (ξ_u, L_u), started at $\left(0, L_{m(T,T+\varepsilon/2)}(W_T)\right)$ and killed at time $\zeta_{T+\varepsilon/2} - y_\varepsilon$. Notice also that on $\{T < \sigma\} \cap A_\varepsilon$, we have

$$y_\varepsilon + \sqrt{\varepsilon}/4 \leq \zeta_T + \sqrt{\varepsilon}/4 < \zeta_{T+\varepsilon/2}.$$

Thus, conditionally on $\mathcal{S}_T$, on $\{T < \sigma\} \cap A_\varepsilon$, $\left(L_{y_\varepsilon+u}(W_{T+\varepsilon/2}) - L_{y_\varepsilon}(W_{T+\varepsilon/2}), 0 \leq u \leq \sqrt{\varepsilon}/4\right)$ is distributed as $(L_u, 0 \leq u \leq \sqrt{\varepsilon}/4)$, under $\bar{\mathbb{P}}_0$. Hence we get

$$\begin{aligned}
\mathbf{1}_{T<\sigma}\, \mathcal{E}^*_{W_T}[A_\varepsilon \cap B_\varepsilon] &= \mathbf{1}_{T<\sigma}\, \mathcal{E}^*_{W_T}\left[A_\varepsilon, \mathcal{E}^*_{W_T}[B_\varepsilon \mid \mathcal{S}_T]\right] \\
&= \mathbf{1}_{T<\sigma}\, \mathcal{E}^*_{W_T}\left[A_\varepsilon, \bar{\mathbb{P}}_0\left[L_{\sqrt{\varepsilon}/4} > \varepsilon^\theta (M+1)\right]\right]_{M=M_T}.
\end{aligned}$$

Since $\varepsilon \in \mathbb{F}$, by the definition of Z_ε, we have

$$\bar{\mathbb{P}}_0\left[L_{\sqrt{\varepsilon}/4} > \varepsilon^\theta (M+1)\right] \geq Z_\varepsilon(M).$$

Then we have

$$\mathbf{1}_{T<\sigma}\, \mathcal{E}^*_{W_T}[A_\varepsilon \cap B_\varepsilon] \geq \mathbf{1}_{T<\sigma}\, \mathcal{E}^*_{W_T}[A_\varepsilon]\, Z_\varepsilon(M_T) \geq \mathbf{1}_{T<\sigma}\, 8\, C_0 \sqrt{b_\varepsilon/\varepsilon}\, Z_\varepsilon(M_T). \tag{14}$$

To conclude note that the law of σ under $\mathcal{E}^*_{W_T}$ is the law of $\zeta_T^2 N^{-2}$ where N is a standard normal variable. Thus we have

$$\mathcal{E}^*_{W_T}[\sigma > \varepsilon] \leq 1 \wedge (\zeta_T \varepsilon^{-1/2}) \leq 4\varepsilon^{-(\frac{1}{2}+\delta)} \sqrt{b_\varepsilon}.$$

Combine this with the inequality (14) to complete the proof of B. $\square$

6 Hitting probability of small balls and proofs of theorems 2.3 and 2.4

From now on we assume (H2) holds. In the next two sections, we state and prove upper and lower bound for the hitting probability of small balls for the process Y_t (cf [1] for $\Psi(\lambda) = \lambda^2$). Then we derive theorem 2.3. In the fourth section, we prove theorem 2.4.

6.1 Upper bound for the hitting probability of small balls

The next proposition gives an upper bound for the hitting probability of small balls.

Proposition 6.1. *Assume $\rho d > 2$. There exists a positive function l_1, which is slowly varying at $0+$, such that for every $t > 0$, $\varepsilon > 0$:*

$$\mathbb{N}_x \left[Y_t(B_\varepsilon(0)) > 0\right] \le t^{-d/2}\varepsilon^{d-2/\rho}\, l_1(\sqrt{t} \wedge \varepsilon).$$

Proof. We are following the proof of proposition 8 from [19]. We first consider the case $0 < 2\varepsilon < \sqrt{t}$. We introduce the open set

$$\Delta := \left\{(r,y) \in \mathbb{R}^+ \times \mathbb{R}^d,\ r < t, |y| > 2\varepsilon\right\} \bigcup \left\{(r,y) \in \mathbb{R}^+ \times \mathbb{R}^d,\ r < t-\varepsilon^2, |y| \le 2\varepsilon\right\}.$$

Formula (5), with $D = \mathbb{R}^+ \times \Delta$ implies the measure $Y_{\mathbb{R}^+ \times \Delta}$ defined in section 4.2 is supported on $\{0\} \times \partial\Delta$. For convenience, let us denote by $\tilde{Y}_\Delta$ its restriction to $\partial\Delta \subset \mathbb{R}^+ \times \mathbb{R}^d$, that is $\delta_0 \otimes \tilde{Y}_\Delta = Y_{\mathbb{R}^+\times\Delta}$. By the special Markov property (cf [3] proposition 7), if N is the number of excursions of the Brownian snake outside $\mathbb{R}^+ \times \Delta$, that reach $\mathbb{R}^+ \times \{t\} \times B_\varepsilon(0)$, then we have:

$$\mathbb{N}_x \left[Y_t\left(B_\varepsilon(0)\right) > 0\right] \le \mathbb{N}_x\left[N\right] = \mathbb{N}_x \left[\int \tilde{Y}_\Delta(dr,dy)\mathbb{N}_{(0,r,y)}\left[\mathcal{G} \cap (\{t\} \times B_\varepsilon(0)) \neq \emptyset\right]\right], \tag{15}$$

where the set $\mathcal{G}$ has been defined in section 4.1. Since the measure $\tilde{Y}_\Delta$ is supported by $\partial\Delta$, it is sufficient to bound the integrand for $(r,y) \in \partial\Delta$:

- if $r = t$, $|y| > \varepsilon$, then $\mathcal{G} \cap (\{t\} \times B_\varepsilon(0)) = \emptyset$, $\mathbb{N}_{(0,r,y)}$-a.e.
- if $r = t - \varepsilon^2$ and $|y| \le 2\varepsilon$, then using the function v defined in section 4.3, we get

$$\mathbb{N}_{(0,r,y)}\left[\mathcal{G} \cap (\{t\} \times B_\varepsilon(0)) \neq \emptyset\right] \le \mathbb{N}_{(0,0,y)}\left[\sup_{s\ge 0} L_{\zeta_s} > \varepsilon^2\right] = v(\varepsilon^2). \tag{16}$$

- if $t - \varepsilon^2 < r < t$ and $|y| = 2\varepsilon$, then by time translation and symmetry we get

$$\begin{aligned}\mathbb{N}_{(0,r,y)}\left[\mathcal{G} \cap (\{t\} \times B_\varepsilon(0)) \neq \emptyset\right] &\le \mathbb{N}_{(0,0,y)}\left[\mathcal{G} \cap \left(\mathbb{R}^+ \times B_\varepsilon(0)\right) \neq \emptyset\right] \\ &\le \mathbb{N}_{(0,0,y')}\left[\mathcal{G} \cap \left(\mathbb{R}^+ \times \left((-\infty,0] \times \mathbb{R}^{d-1}\right)\right) \neq \emptyset\right],\end{aligned}$$

where $y' = (\varepsilon, 0, \ldots, 0) \in \mathbb{R}^d$. Let $u(y')$ denotes the right-hand side of the previous formula. It can be deduced from the remark in section 4.1 that the function u is bounded on every compact set of $(0,\infty) \times \mathbb{R}^{d-1}$. The arguments of propositions 6 to 8 from [3] and propositions 4.3 to 5.3 from [18] can be adapted to prove that u solves

$$\frac{1}{2}\Delta u = \Psi(u),$$

on $(0,\infty) \times \mathbb{R}^{d-1}$, with the boundary condition

$$\lim_{y_1 > 0,\ y_1 \to 0} u(y) = \infty,$$

where we write $y = (y_1, \dots, y_d)$. Obviously, by space homogeneity, the function u depends only on y_1. For simplicity we write $u(y_1)$ for $u((y_1, \dots, y_d))$. Therefore $u : (0, \infty) \to \mathbb{R}^+$ solves

$$u''(s) = 2\Psi(u(s)), \quad s > 0 \quad \text{and} \quad \lim_{s \to 0} u(s) = \infty.$$

Using the fact that u is decreasing, we get for $r > 1$

$$u'(r) = -\left[u'(1)^2 + 4\int_{u(1)}^{u(r)} \Psi(h)dh\right]^{1/2}.$$

Integrating over $(0, s]$ and making the change of variable $t = u(r)$, we get for $s \in (0, 1)$:

$$\int_{u(s)}^{\infty} \left[u'(1)^2 + 4\int_{u(1)}^{t} \Psi(h)dh\right]^{-1/2} dt = s.$$

Notice that the integrand is regularly varying at ∞ with index $-1-\rho/2$. Thanks to theorems 1.5.10 and 1.5.12 of [4], we deduce that u is regularly varying at $0+$ with index $-2/\rho$. Recall that

$$\mathbb{N}_{(0,r,y)}\left[\mathcal{G} \cap (\{t\} \times B_\varepsilon(0)) \neq \emptyset\right] \leq u(\varepsilon). \tag{17}$$

Recall that the function v is regularly varying at $0+$ with index $-1/\rho$. Since the functions u and v are positive, there exists a positive function, l', which is slowly varying at $0+$ such that $u(\varepsilon) + v(\varepsilon^2) \leq \varepsilon^{-2/\rho} l'(\varepsilon)$. We can then substitute (16) and (17) into inequality (15) to obtain

$$\mathbb{N}_x\left[Y_t(B_\varepsilon(0)) > 0\right] \leq \varepsilon^{-2/\rho} l'(\varepsilon) \mathbb{N}_x\left[\left(\tilde{Y}_\Delta, \mathbf{1}_{(0,t)\times\mathbb{R}^d}\right)\right].$$

Then formula (5) gives

$$\mathbb{N}_x\left[\left(\tilde{Y}_\Delta, \mathbf{1}_{(0,t)\times\mathbb{R}^d}\right)\right] = \mathbb{P}_x\left[T_\Delta < t\right],$$

where $T_\Delta = \inf\{s > 0, (s, \gamma_s) \notin \Delta\}$ (recall that γ_s is a Brownian motion in $\mathbb{R}^d$ started at x under $\mathbb{P}_x$). Then we easily get the existence of constants c_1 depending only on d such that:

$$\mathbb{P}_x\left[T_D < t\right] \leq c_1 t^{-d/2} \varepsilon^d.$$

Thus we have

$$\mathbb{N}_x\left[Y_t(B_\varepsilon(0)) > 0\right] \leq c_1 t^{-d/2} \varepsilon^{d-2/\rho} l'(\varepsilon).$$

Now if $0 < \sqrt{t} < \varepsilon$, we have the elementary upper bound:

$$\mathbb{N}_x\left[Y_t(B_\varepsilon(0)) > 0\right] \leq v(t) \leq t^{-d/2} \varepsilon^{d-2/\rho}\, l'(\sqrt{t}).$$

Taking $l_1 = (c_1 + 1)l'$ gives the desired inequality. □

Notice that in the stable case, a scaling argument shows that we can replace l by a constant.

6.2 Lower bound for the hitting probability of small balls

We assume only in this section that

$$\limsup_{\lambda\to 0+} \lambda^{-1-\rho}\Psi(\lambda) < \infty. \tag{18}$$

Proposition 6.2. *Assume that $\rho d > 2$. For every $M > 0$, there exists a positive increasing function l_2, which is slowly varying at $0+$, such that for every $M\sqrt{t} > \varepsilon > 0$, we have*

$$\mathbb{N}_x \left[Y_t \left(B_\varepsilon(0)\right) > 0\right] \geq \varepsilon^{d-2/\rho} p\left(\frac{\rho t}{1+\rho}, x\right) l_2(\varepsilon).$$

Moreover, if $\limsup_{\lambda\to\infty} \lambda^{-1-\rho}\Psi(\lambda) < \infty$, we can replace l_2 by a positive constant.

Notice that all the assumptions on Ψ are satisfied in the stable case.
Proof. Let $A \geq \kappa > 0$. We have (cf [3]):

$$\mathbb{N}_x \left[Y_t \left(B_\varepsilon(0)\right) > 0\right] \geq v_\varepsilon(t,x) := \mathbb{N}_x \left[1 - \exp\left[-\kappa\varepsilon^{-2/\rho} Y_t\left(B_\varepsilon(0)\right)\right]\right],$$

where the function v_ε is the only nonnegative solution of (1) with $f = \kappa\varepsilon^{-2/\rho}\mathbf{1}_{B_\varepsilon(0)}$. As

$$v_\varepsilon(t,x) \leq \kappa\varepsilon^{-2/\rho} P_t \mathbf{1}_{B_\varepsilon(0)}(x),$$

we deduce from (1) and the monotonicity of Ψ, that

$$v_\varepsilon(t,x) \geq \kappa\varepsilon^{-2/\rho} P_t \mathbf{1}_{B_\varepsilon(0)}(x) - \int_0^t du\ P_u \left[\Psi\left(\kappa\varepsilon^{-2/\rho} P_{t-u}\mathbf{1}_{B_\varepsilon(0)}\right)\right](x). \tag{19}$$

We now bound the second term of the right-hand side, which we denote by I_t. Thanks to (18) and [4] (ex:4 p.58), we know that the function l_A defined on $(0,\infty)$ by

$$l_A(r) := \sup_{\lambda\in(0,Ar^{-2/\rho}]} \lambda^{-1-\rho}\Psi(\lambda)$$

is decreasing and slowly varying at $0+$. Using the monotonicity of l_A, it follows that

$$\begin{aligned} I_t &\leq \left(\kappa\varepsilon^{-2/\rho}\right)^{1+\rho} \int_0^t du\ P_u\left[\left(P_{t-u}\mathbf{1}_{B_\varepsilon(0)}\right)^{1+\rho}\right](x) l_A(\varepsilon) \\ &\leq \kappa^{1+\rho}\varepsilon^{-2(1+1/\rho)} t \int_0^1 du\ P_u\left[\left(P_{1-u}\mathbf{1}_{B_{\varepsilon/\sqrt{t}}(0)}\right)^{1+\rho}\right](x/\sqrt{t}) l_A(\varepsilon). \end{aligned}$$

Let $\lambda \in (0, M)$. We now give an upper bound on

$$\int_0^1 du \int dz\ p(u, z-x)\left[\left[\int_{B_\lambda(0)} dy\ p(1-u, y-z)\right]^{1+\rho}\right].$$

We decompose the above integral in two terms by considering the integral du on the sets $\{u < 1/2\}$ (integral J_1), $\{u \geq 1/2\}$ (integral J_2). Using $\rho d > 2$, the integral J_1 is bounded above by

$$\int_0^{1/2} du \int dz\ p(u, z-x)\left[c\lambda^d\right]^{1+\rho} \leq c_1\lambda^{d+2},$$

where c_1 depends only on M and d. Now by scaling we get

$$\begin{aligned} J_2 &\leq \int_0^{1/2} du \int dz \, \left[\int_{B_\lambda(0)} dy \; p(u, y-z) \right]^{1+\rho} \\ &\leq \lambda^{d+2} \int_0^{\infty} du \int dz \, \left[\int_{B_1(0)} dy \; p(u, y-z) \right]^{1+\rho} \\ &= c_2 \lambda^{d+2}. \end{aligned}$$

We use $\rho d > 2$ to get $c_2 < \infty$. Combining those results together with $\lambda = \varepsilon/\sqrt{t}$, we get that there exists a constant c'_M depending only on M and d such that

$$I_t \leq c'_M \kappa^{1+\rho} \varepsilon^{-2(1+1/\rho)} t^{1-(d+2)/2} \varepsilon^{d+2} l_A(\varepsilon). \tag{20}$$

On the other hand, there exists a constant c_d depending only on d such that:

$$P_t \mathbf{1}_{B_\varepsilon(0)}(x) \geq c_d \left[1 \wedge \left(\left(\varepsilon/\sqrt{t} \right)^d \mathrm{e}^{-|x|^2/2t} \right) \right]$$

Thus for $M\sqrt{t} > \varepsilon > 0$, we have

$$P_t \mathbf{1}_{B_\varepsilon(0)}(x) \geq c_d M^{-d} t^{-d/2} \varepsilon^d \, \mathrm{e}^{-|x|^2/2t}.$$

Plugging the previous inequality and (20) into (19), we get

$$v_\varepsilon(t,x) \geq \kappa t^{-d/2} \varepsilon^{d-2/\rho} \left[c_d M^{-d} \, \mathrm{e}^{-|x|^2/2t} - c'_M \kappa^\rho l_A(\varepsilon) \right].$$

Since the constants A and κ are arbitrary, we can take $A = (c_d M^{-d} {c'}_M^{-1})^{1/\rho}$ and $\kappa = A(\mathrm{e}^{-|x|^2/2t} [1 + l_A(\varepsilon)]^{-1})^{1/\rho}$ to get

$$\mathbb{N}_x \left[Y_t \left(B_\varepsilon(0) \right) > 0 \right] \geq v_\varepsilon(t,x) \geq c_M \varepsilon^{d-2/\rho} p\left(\rho t/(1+\rho), x \right) l(\varepsilon),$$

where $l(\varepsilon) = [1 + l_A(\varepsilon)]^{-1-1/\rho}$ is increasing and slowly varying at $0+$, and the constant c_M is independent of x, t and ε. Moreover, if $\limsup_{\lambda\to\infty} \lambda^{-1-\rho}\Psi(\lambda) < \infty$, then l_A is bounded above by a positive constant independent of A, and we can let l be a constant. $\square$

6.3 Proof of theorem 2.3

We deduce from proposition 4.3 that for every $\lambda > 0$,

$$\mathbb{P}^X_{\delta_x} \left[\mathrm{e}^{-\lambda X_t(B_\varepsilon(0))} \right] = \mathrm{e}^{-\mathbb{N}_x [1 - \exp -\lambda Y_t(B_\varepsilon(0))]}.$$

Letting $\lambda \to \infty$, we get

$$\mathbb{P}^X_{\delta_x} \left[X_t(B_\varepsilon(0)) > 0 \right] = 1 - \mathrm{e}^{-\mathbb{N}_x [Y_t(B_\varepsilon(0)) > 0]}.$$

Then theorem 2.3 is a consequence of proposition 6.1, proposition 6.2 and the inequality $(1 \wedge u)/2 \leq 1 - \mathrm{e}^{-u} \leq u$.

6.4 Proof of theorem 2.4

Before proving the theorem, we give a result on the intersection of the support of two independent copies of Y. In the next lemma, we consider the product measure $\mathbb{N}_{x_1} \otimes \mathbb{N}_{x_2}$ on the space $C(\mathbb{R}^+, \mathcal{W})^2$. The canonical process on this space is denoted by (W^1, W^2), and we write Y^1, respectively Y^2, for the measure-valued process associated with W^1, respectively W^2.

Lemma 6.3. *Assume $\rho d > 4$. Then for every $t > 0$, $s > 0$, we have $\mathbb{N}_{x_1} \otimes \mathbb{N}_{x_2}$-a.e.*

$$\text{supp } Y_t^1 \cap \text{supp } Y_s^2 = \emptyset.$$

Proof. Fix $t > 0$ and $s > 0$, and let $\delta \in (0, 1 \wedge \sqrt{t} \wedge \sqrt{s})$, $y \in \mathbb{R}^d$. We can cover the ball $B_1(y)$ with less than $\left[4\sqrt{d}\delta^{-1}\right]^d$ balls $(B_\delta(y_i), i \in J)$ with radius δ and centers y_i belonging to $y + \delta d^{-1/2}\mathbb{Z}^d$. Use proposition 6.1 to write

$$\begin{aligned}
\mathbb{N}_{x_1} \otimes \mathbb{N}_{x_2} &\left[\text{supp } Y_t^1 \cap \text{supp } Y_s^2 \cap B_1(y) \neq \emptyset\right] \\
&\leq \sum_{i \in J} \mathbb{N}_{x_1} \left[\text{supp } Y_t \cap B_\delta(y_i) \neq \emptyset\right] \mathbb{N}_{x_2} \left[\text{supp } Y_s \cap B_\delta(y_i) \neq \emptyset\right] \\
&\leq \sum_{i \in J} t^{-d/2} s^{-d/2} \left[\delta^{d-2/\rho}\; l(\delta)\right]^2 \\
&\leq (ts)^{-d/2} (4\sqrt{d})^d \delta^{d-4/\rho}\; l(\delta)^2.
\end{aligned}$$

Since $\rho d > 4$, let δ go to 0 to see that the left-hand side is 0. As this is true for every $y \in \mathbb{R}^d$, the desired result follows. $\square$

Recall from section 4.1 the definition of the set $\mathcal{G}$.

Lemma 6.4. *For $\varepsilon > 0$, $t > 0$ set*

$$h_\varepsilon(t) = \mathbb{N}_0 \left[\mathcal{G} \cap ([0, t] \times B_\varepsilon(0)^c) \neq \emptyset\right].$$

Then for every $\varepsilon > 0$, $\lim_{t \downarrow 0} h_\varepsilon(t) = 0$.

Proof. We start by making the simple observation that $\mathbb{N}_0$-a.e. for every $s \geq 0$ such that $L_{\zeta_s}(W_s) = 0$ we have $\zeta_s = 0$, and thus $\hat{W}_s = 0$. Indeed, if there would exist s such that $L_{\zeta_s}(W_s) = 0$ and $\zeta_s > 0$, then the snake property would yield a rational s' "close" to s such that $L_t(W_{s'}) = 0$ for $t \in [0, \theta)$, for some $\theta > 0$. This is impossible since under $\mathbb{N}_0$, conditionally on $\zeta_{s'}$, $W_{s'}$ is distributed as $\bar{\xi}$ started at $(0, 0, 0)$ and killed at time $\zeta_{s'}$.

Then let (t_n) be a sequence decreasing to 0, and let $A_n = \{\mathcal{G} \cap ([0, t_n] \times B_\varepsilon(0)^c) \neq \emptyset\}$. Thanks to the remark in section 4.1, we have $\mathbb{N}_0 [A_n] < \infty$. We claim that $\mathbb{N}_0 \left[\bigcap_{n \geq 1} A_n\right] = 0$. In fact, on the event $\bigcap_{n \geq 1} A_n \neq \emptyset$, the definition of $\mathcal{G}$ yields a sequence (s_n) in $[0, \sigma]$ such that

$$L_{\zeta_{s_n}}(W_{s_n}) \leq t_n \qquad \text{and} \quad \hat{W}_{s_n} \in B_\varepsilon(0)^c.$$

We can extract from the sequence (s_n) a subsequence converging to s_∞. By the continuity of the mappings $s \mapsto L_{\zeta_s}(W_s)$ and $s \mapsto \hat{W}_s$, we get that $L_{\zeta_{s_\infty}}(W_{s_\infty}) = 0$ and $\hat{W}_{s_\infty} \in B_\varepsilon(0)^c$, which contradicts the beginning of the proof.

Since the function h_ε is monotone increasing and $h_\varepsilon(t_n) = \mathbb{N}_0[A_n]$, the statement of lemma 6.4 follows from the fact that $\mathbb{N}_0\left[\bigcap_{n\geq 1} A_n\right] = 0$. $\square$

Proof of theorem 2.4. We adapt an argument of Perkins ([22], p.1041). Let us fix $t > 0$ and $\delta \in (0,t)$. By combining the Markov property of X at time $t-\delta$ and proposition 4.3, we obtain that the distribution of X_t under $\mathbb{P}_\nu^X$ is the same as the law of $\sum_{i\in I} Y_\delta(W^i)$, where conditionally on $X_{t-\delta}$, $\sum_{i\in I}\delta_{W^i}$ is a Poisson measure on $C(\mathbb{R}^+, \mathcal{W})$ with intensity $\int X_{t-\delta}(dy)\mathbb{N}_y[\cdot]$. With a slight abuse of notation, we may assume that the point measure $\sum_{i\in I} Y_\delta(W^i)$ is also defined under $\mathbb{P}_\nu^X$. It follows from lemma 6.3 and properties of Poisson measures that a.s. for every $i \neq j$,

$$\text{supp } Y_\delta(W^i) \cap \text{supp } Y_\delta(W^j) = \emptyset.$$

For $\varepsilon > 0$, let U_ε denote the event "supp X_t is contained in a finite union of disjoint compact sets with diameter less than ε". It is easy to check that U_ε is measurable. Furthermore, by the previous observations, and denoting by y_i the common starting point of the paths W_s^i,

$$\begin{aligned}
\mathbb{P}_\nu^X[U_\varepsilon] &\geq \mathbb{P}_\nu^X\left[\forall i \in I, \text{diam }(\text{supp } Y_\delta(W^i)) \leq \varepsilon\right] \\
&\geq \mathbb{P}_\nu^X\left[\forall i \in I, \text{supp } Y_\delta(W^i) \subset B_{\varepsilon/2}(y_i)\right] \\
&= \mathbb{E}_\nu^X\left[\exp - \int X_{t-\delta}(dy)\mathbb{N}_y[\text{supp } Y_\delta \cap B_{\varepsilon/2}(y)^c \neq \emptyset]\right] \\
&= \mathbb{E}_\nu^X\left[\exp -(X_{t-\delta}, \mathbf{1})\mathbb{N}_0[\text{supp } Y_\delta \cap B_{\varepsilon/2}(0)^c \neq \emptyset]\right] \\
&\geq \mathbb{E}_\nu^X\left[\exp -h_{\varepsilon/2}(\delta)(X_{t-\delta}, \mathbf{1})\right].
\end{aligned}$$

We can now let δ go to 0, using lemma 6.4, to conclude that $\mathbb{P}_\nu^X[U_\varepsilon] = 1$. Since this holds for every $\varepsilon > 0$, we conclude that supp X_t is totally disconnected $\mathbb{P}_\nu^X$-a.s. $\square$

7 Absolute continuity of the superprocess in the Brownian case and in the symmetric α-stable case

In this section we prove theorem 2.5. In fact, it is enough to prove the theorem for a finite measure μ with support in $[\underline{m}, \overline{m}] \subset (0, \infty)$. The construction of the Brownian snake W associated with the process $\bar{\xi}_t = \left(\xi_t, L_t, \gamma_{L_t}^\alpha\right)$ is performed as in section 4, following the general results of [3] (see section 4 and hypothesis (H) therein). In fact only the spatial motion Γ has to be modified. However the processes $t \mapsto \Gamma_t(W_s)$ and $s \mapsto \hat{W}_s$ are no longer continuous. The construction of the measure L^t in section 4.2 remain valid and we can still define the exit measure by the formula

$$(Y_t, \varphi) = \int_0^\sigma \varphi(\hat{W}_s)dL_s^t.$$

Proposition 4.2 remains also valid.

Let $\nu \in M_f$. Let $\sum_{i\in I}\delta_{W^i}$ be a Poisson measure on $C(\mathbb{R}^+, \mathcal{W})$ with intensity $\int \nu(dx)\mathbb{N}_x[\cdot]$. The process

$$X_0^\alpha = \nu, \quad X_t^\alpha = \sum_{i\in I} Y_t(W^i), \quad \text{for} \quad t > 0,$$

is a (γ^α, Ψ)-superprocess (see [3]). Moreover, a.s. the collection $\left(\left(Y_s(W^i), s \geq \underline{m}\right), i \in I\right)$ has finitely many non zero terms. Then theorem 2.5 is a consequence of the next proposition.

Proposition 7.1. *Let μ a finite measure on $[\underline{m}, \overline{m}] \subset (0, \infty)$ and $q \in [0,1)$ such that*

$$\iint \mu(dt)\mu(ds) \, |t-s|^{-q} < \infty.$$

If $\dfrac{\alpha}{\rho} + \alpha q > d$, then for every $x \in \mathbb{R}^d$, $\mathbb{N}_x$-a.e. the measure $\int \mu(dt) Y_t$ is absolutely continuous with respect to Lebesgue measure.

We shall now give a proof of this proposition. The arguments are very similar to section 5.2.

Proof. Thanks to theorem 7.15 from [24], it is sufficient to prove that $\mathbb{N}_x$-a.e. $\int \mu(dt) Y_t(dy)$-a.e.

$$F\left(y, \int \mu(dt)\ Y_t\right) := \mathbf{1}_{\left\{\liminf\limits_{n\to\infty} 2^{nd} \int \mu(dt) Y_t(B_{2^{-n}}(y)) = \infty\right\}} = 0,$$

Let $K = \left\{s \in \operatorname{supp} \mu; \int \mu(dt) \, |t-s|^{-q} < \infty\right\}$. Notice that $\mu(K^c) = 0$. Therefore it is enough to verify that for $s_0 \in K$,

$$\mathbb{N}_x \left[\int Y_{s_0}(dz)\ F\left(z, \int \mu(dt)\ Y_t\right)\right] = 0, \tag{21}$$

Thanks to proposition 4.2, we get

$$\begin{aligned} &\mathbb{N}_x \left[\int Y_{s_0}(dy) F\left(y, \int \mu(dt)\ Y_t\right)\right] \\ &\qquad = \int \bar{\mathbb{P}}_x^{s_0}(dw) \mathbb{E}\left[F\left(\hat{w}, \int \mathcal{N}_w(du, d\mathrm{W}) \int \mu(dt)\ \mathbf{1}_{\{u<\tau_t(w)\}}\ Y_t(\mathrm{W})\right)\right]. \end{aligned} \tag{22}$$

Conditioning on $\mathcal{S}_0 = \sigma\left(S_v(w), 0 \leq v < s_0\right)$, we shall prove that $\bar{\mathbb{P}}_x^{s_0}$-a.s.,

$$U = \bar{\mathbb{E}}_x^{s_0}\left[\mathbb{E}\left[\liminf_{n\to\infty} 2^{nd} \int \mathcal{N}_w(du, d\mathrm{W}) \int \mu(dt) \mathbf{1}_{\{u<\tau_t(w)\}} Y_t(\mathrm{W})\left(B_{2^{-n}}(\hat{w})\right)\right] \middle| \mathcal{S}_0\right] < \infty.$$

To this end, we use Fatou's lemma to get

$$\begin{aligned} U &\leq \liminf_{n\to\infty} 2^{nd}\ \bar{\mathbb{E}}_x^{s_0}\left[\mathbb{E}\left[\iint \mathcal{N}_w(du, d\mathrm{W}) \int \mu(dt) \mathbf{1}_{\{u<\tau_t(w)\}} Y_t(\mathrm{W})\left(B_{2^{-n}}(\hat{w})\right)\right] \middle| \mathcal{S}_0\right] \\ &= \liminf_{n\to\infty} 2^{nd}\ \bar{\mathbb{E}}_x^{s_0}\left[\int \mu(dt) 4 \int_0^{\zeta_w} du\ \mathbf{1}_{\{u<\tau_t(w)\}} \mathbb{N}_{w(u)}\left[Y_t\left(B_{2^{-n}}(y)\right)\right]_{y=\hat{w}} \middle| \mathcal{S}_0\right] \\ &= 4 \liminf_{n\to\infty} 2^{nd}\ \bar{\mathbb{E}}_x^{s_0}\left[\int \mu(dt) \int_0^{S_{s_0-} \wedge S_{t-}} du\ \bar{\mathbb{P}}^t_{w(u)}\left[\hat{w} \in B_{2^{-n}}(y)\right]_{y=\hat{w}} \middle| \mathcal{S}_0\right] \\ &= 4 \liminf_{n\to\infty} 2^{nd} \int \mu(dt) \int_{[0, s_0 \wedge t)} dS_{u'}\ \mathbb{E}_x\left[\mathbb{P}_{\gamma^\alpha_{u'}}\left[\gamma^\alpha_{t-u'} \in B_{2^{-n}}(y)\right]_{y=\gamma^\alpha_{s_0}}\right]. \end{aligned}$$

We used the formula for the intensity of $\mathcal{N}_w$ in the first equality, then formula (5) in the second one, and finally the change of variables $u = S_{u'}$ in the last one. We have

$$\mathbb{E}_x \left[\mathbb{P}_{\gamma^\alpha_{u'}} \left[\gamma^\alpha_{t-u'} \in B_{2^{-n}}(y) \right]_{y=\gamma^\alpha_{s_0}} \right] = g_\alpha(2^{-n}, s_0 + t - 2u'),$$

where $g_\alpha(r,t) = \mathbb{P}_0 \left[|\gamma^\alpha_t| \le r \right]$. Since $s_0 \in K$ and $\frac{\alpha}{\bar{\rho}} + \alpha q > d$, we can apply lemma 8.1 below with $\kappa = d$, and we get $U \le 4C_\kappa < \infty$ $\mathbb{P}^{s_0}_x$-a.s. Formula (22) then gives (21), which completes the proof of the proposition. □

8 Appendix

Let γ^α be a symmetric α-stable process in $\mathbb{R}^d$ as in section 2. Let the function g_α be defined on $\mathbb{R}^+ \times (0, \infty)$ by

$$g_\alpha(r,t) = \mathbb{P}_0 \left[|\gamma^\alpha_t| \le r \right] = \mathbb{P}_0 \left[|\gamma^\alpha_1| \le r t^{-1/\alpha} \right]$$

Since the law of the random variable γ^α_1 has a continuous density with respect to Lebesgue measure on $\mathbb{R}^d$, there exists a constant c_α, such that $g_\alpha(r,t) \le c_\alpha \left[1 \wedge r^d t^{-d/\alpha} \right]$ on $(r,t) \in \mathbb{R}^+ \times (0,\infty)$. Hence we have also $g_\alpha(r,t) \le c_\alpha r^\theta t^{-\theta/\alpha}$ for every $\theta \in [0,d]$. Let μ be a non zero finite measure with support in $[\underline{m}, \overline{m}] \subset (0,\infty)$. Let $s_0 \in \operatorname{supp} \mu$, and $q \in [0,1)$ such that

$$\int \mu(dt) \ |s_0 - t|^{-q} < \infty.$$

Let S be a subordinator as in section 3.

Lemma 8.1. *Let $\kappa \in [0,d]$, such that $\kappa < \alpha(q + 1/\bar{\rho})$. Then $\bar{\mathbb{P}}^{s_0}_x(dw)$-a.s. there exists a finite constant C_κ depending on w only through $(S_v(w), 0 \le v < s_0)$, such that for every $r \le 1$:*

$$\int \mu(dt) \int_{[0, s_0 \wedge t)} g_\alpha(r, s_0 + t - 2u) \ dS_u \le C_\kappa r^\kappa.$$

Proof. Let $\kappa \in [0,d]$, such that $\kappa < \alpha(q + 1/\bar{\rho})$. Let $\delta \in (0,1)$ small enough such that $\kappa < \alpha(q + \frac{1}{\bar{\rho}} - \delta)$ and $\kappa \ne \alpha(\frac{1}{\bar{\rho}} - \delta)$. Recall the upper bound $g_\alpha(r,t) \le c_\alpha r^\kappa t^{-\kappa/\alpha}$. The lemma will be proved as soon as we can verify that $\bar{\mathbb{P}}^{s_0}_x$-a.s.

$$\int \mu(dt) \int_{[0, s_0 \wedge t)} [s_0 + t - 2u]^{-\kappa/\alpha} \ dS_u < \infty. \tag{23}$$

By lemma 3.2 3., we can find $\bar{\mathbb{P}}^{s_0}_x$-a.s. a (random) constant $\varepsilon \in (0, \underline{m}/2)$, such that for every $u \in [s_0 - \varepsilon, s_0)$,

$$S_{s_0-} - S_u \le [s_0 - u]^{\frac{1}{\bar{\rho}} - \delta}.$$

In order to bound the left-hand side of (23), we first observe that

$$\int \mu(dt) \int_0^{(s_0 - \varepsilon) \wedge t} [s_0 + t - 2u]^{-\kappa/\alpha} \ dS_u \le (\mu, \mathbf{1}) \varepsilon^{-\kappa/\alpha} S_{\overline{m}}.$$

Consider the case $u \in [(s_0 - \varepsilon) \wedge t, s_0 \wedge t)$. If $t \neq s_0$ or $q = 0$, an integration by parts gives

$$\begin{aligned}\int_{[(s_0-\varepsilon)\wedge t, s_0\wedge t)} [s_0 + t - 2u]^{-\kappa/\alpha} \, dS_u \\ = \left[S_{(s_0\wedge t)-} - S_{((s_0-\varepsilon)\wedge t)}\right] [s_0 + t - 2((s_0 - \varepsilon) \wedge t)]^{-\kappa/\alpha} \\ + \frac{2\kappa}{\alpha} \int_{(s_0-\varepsilon)\wedge t}^{s_0\wedge t} [s_0 + t - 2u]^{-1-\kappa/\alpha} \left[S_{(s_0\wedge t)-} - S_u\right] du.\end{aligned}$$

Now for $u \in [(s_0 - \varepsilon) \wedge t, s_0 \wedge t)$, we have

$$S_{(s_0\wedge t)-} - S_u \leq S_{s_0-} - S_u \leq (s_0 - u)^{\frac{1}{\rho} - \delta}.$$

Thus the integral $\int_{(s_0-\varepsilon)\wedge t}^{s_0\wedge t} [s_0 + t - 2u]^{-1-\kappa/\alpha}[S_{(s_0\wedge t)-} - S_u]du$ is bounded above by

$$\int_{(s_0-\varepsilon)\wedge t}^{s_0\wedge t} [s_0 + t - 2u]^{-1-\frac{\kappa}{\alpha}+\frac{1}{\rho}-\delta} du \leq \begin{cases} C\,|s_0 - t|^{-\frac{\kappa}{\alpha}+\frac{1}{\rho}-\delta} & \text{if } \kappa > \frac{\alpha}{\rho} - \alpha\delta, \\ C & \text{if } \kappa < \frac{\alpha}{\rho} - \alpha\delta, \end{cases}$$

where the constant C is independent of $t \in [\underline{m}, \overline{m}]$. Notice that $\mu(\{s_0\}) > 0$ implies $q = 0$. Thus by combining the previous estimates, we see that the left-hand side of (23) is bounded above by

$$2(\mu, \mathbf{1})\varepsilon^{-\kappa/\alpha} S_{\overline{m}} + \begin{cases} \frac{2\kappa}{\alpha} C \int \mu(dt)\ |s_0 - t|^{-\frac{\kappa}{\alpha}+\frac{1}{\rho}-\delta} & \text{if } \kappa > \frac{\alpha}{\rho} - \alpha\delta, \\ \frac{2\kappa}{\alpha} C(\mu, \mathbf{1}) & \text{if } \kappa < \frac{\alpha}{\rho} - \alpha\delta. \end{cases}$$

This quantity is finite by the assumption $\int \mu(dt)\ |s_0 - t|^{-q} < \infty$ and the choice of δ. The lemma follows. $\square$

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