Probabilistic approximation for a porous medium equation

B.Jourdain*

November 6, 1998

Abstract

In this paper, we are interested in the one-dimensional porous medium equation when the initial condition is the distribution function of a probability measure. We associate a nonlinear martingale problem with it. After proving uniqueness for the martingale problem, we show existence thanks to a propagation of chaos result for a system of weakly interacting diffusion processes. The particle system obtained by increasing reordering from these diffusions is proved to solve a stochastic differential equation with normal reflection. Last, we obtain propagation of chaos for the reordered particles to a probability measure which does not solve the martingale problem but is also linked to the porous medium equation.

Introduction

Let $q > 1$. We are interested in the the porous medium equation :

$$\frac{\partial u}{\partial t} = \frac{\partial^2(u^q)}{\partial x^2},\ (t,x) \in [0,+\infty) \times \mathbb{R} \tag{0.1}$$

Inoue [6] [7] and Benachour Chassaing Roynette and Vallois [3] have given probabilistic interpretations of this equation in terms of nonlinear diffusion processes when the initial condition is a probability measure on $\mathbb{R}$. We are interested in another class of initial conditions : the cumulative distribution functions of probability measures on $\mathbb{R}$. We follow the approach developped by Bossy and Talay [5] for the viscous Burgers equation and write the equation satisfied by $v = \partial_x u$

$$\frac{\partial v}{\partial t} = \frac{\partial^2}{\partial x^2}(qu^{q-1}v) = \frac{\partial^2}{\partial x^2}(q(H * v(t,.))^{q-1}v(t,.))$$

where $H(x) = 1_{\{x\geq 0\}}$ denotes the Heaviside function. From a probabilistic point of view, this equation can be interpreted as a nonlinear Fokker-Planck equation. That way, we associate with it the following martingale problem :

Definition 0.1 *Let X denote the canonical process on $C([0,+\infty),\mathbb{R})$. A probability measure $Q \in \mathcal{P}(C([0,+\infty),\mathbb{R}))$ with time marginals $(Q_t)_{t\geq 0}$ solves the martingale problem (MP) starting at $m \in \mathcal{P}(\mathbb{R})$ if*

1. $Q_0 = m$

*ENPC-CERMICS, 6-8 av Blaise Pascal, Cité Descartes, Champs sur Marne, 77455 Marne la Vallée Cedex 2, France - e-mail : jourdain@cermics.enpc.fr

2. $\forall \phi \in C_b^2(\mathbb{R}),\ M_t^\phi = \phi(X_t) - \phi(X_0) - q\int_0^t (H * P_s(X_s))^{q-1}\phi''(X_s)ds$ *is a Q-martingale*

3. $\forall t > 0,\ Q_t$ *does not weight points.*

We first show that if Q solves problem (MP) starting at m then $(s,x) \to H * Q_s(x)$ is a weak solution of the porous medium equation (0.1) for the initial condition $H * m(x)$.
Then we prove uniqueness for problem (MP) thanks to the following results concerning equation (0.1) given by [4] [9] and [1] : uniqueness of weak solutions for the initial condition $H * m(x)$ and existence of a Hölder continuous weak solution.
We introduce the interacting diffusion processes

$$X_t^{i,n} = X_0^i + \int_0^t \sqrt{2q}(H * \mu_s^n(X_s^{i,n}))^{(q-1)/2}dB_s^i,\ \ 1 \leq i \leq n,\ \text{ with }\ \mu^n = \frac{1}{n}\sum_{j=1}^n \delta_{X^{j,n}} \qquad (0.2)$$

where $B^i,\ 1 \leq i \leq n$ are independent Brownian motions and $X_0^i,\ 1 \leq i \leq n$ are initial variables I.I.D. with distribution m independent of the Brownian motions. We prove that the particle systems $(X^{1,n}, \ldots, X^{n,n})$ are P chaotic where P denotes the unique solution of problem (MP) starting at m.

Let $(Y_t^{1,n}, \ldots, Y_t^{n,n})$ denote the increasing reordering of $(X_t^{1,n}, \ldots, X_t^{n,n})$ i.e.

$$Y_t^{i,n} = \sup_{|A|=n-i+1}\ \inf_{j\in A} X_t^{j,n} \quad \text{where} \quad |A| \quad \text{denotes the cardinality of } A \subset \{1,\ldots,n\}.$$

As $\frac{1}{n}\sum_{j=1}^n H(X_s^{i,n} - X_s^{j,n}) = |\{j : X_s^{j,n} \leq X_s^{i,n}\}|/n$, we remark that $(Y^{1,n}, \ldots, Y^{n,n})$ is a diffusion with constant diagonal diffusion matrix

$$\mathrm{diag}(2q(1/n)^{q-1}, 2q(2/n)^{q-1}, \ldots, 2q(n/n)^{q-1})$$

normally reflected at the boundary of the convex set

$$D_n = \{y = (y_1, \ldots, y_n) \in \mathbb{R}^n,\ y_1 \leq y_2 \leq \ldots \leq y_n\}.$$

Of course, the driving Brownian motion is not $(B_t^1, \ldots, B_t^n)$. Let $\tilde{\mu}^n = \frac{1}{n}\sum_{i=1}^n \delta_{Y^{i,n}}$ denote the empirical measure of the reordered system. Since for any $t \geq 0$, the increasing reordering preserves the empirical measure at time t, the variables $\tilde{\mu}_t^n \in \mathcal{P}(\mathbb{R})$ converge in probability to P_t. But as the increasing reordering does not preserve the sample-paths, the asymptotic behaviour of $\tilde{\mu}^n$ is different from the one of $\mu^n = \frac{1}{n}\sum_{i=1}^n \delta_{X^{i,n}}$ which converges in probability to P (convergence equivalent to the propagation of chaos to P for $(X^{1,n}, \ldots, X^{n,n})$). The sequence $\tilde{\mu}^n$ converges in probability to $\tilde{P}$ which does not solve problem (MP) starting at m. For any $i \leq n$, $H * \tilde{\mu}_s^n(Y_s^{i,n})$ is constant and equal to i/n for $s \notin \{t,\ (Y_t^{1,n}, \ldots, Y_t^{n,n}) \in \partial D_n\}$. When $n \to +\infty$, this property yields that $\tilde{P}$ a.s., the function $s \in (0, +\infty) \to H * \tilde{P}_s(X_s)$ is constant. The probability measure $\tilde{P}$ is characterized by this assertion combined with the following property : $(s,x) \to H * \tilde{P}_s(x)$ is a weak solution of the porous medium equation (0.1) for the initial condition $H * m(x)$.
Our study can be seen as an extension of [12] p.187-190 : Sznitman deals with the simpler case (corresponding to $q = 1$) of reordered independent Brownian motions with initial distribution m atomless. He proves the convergence in probability of the empirical measures to the constant $Q \in \mathcal{P}(C([0,+\infty),\mathbb{R}))$ characterized by : Q a.s., $\forall s \geq 0,\ X_s = F_s^{-1}(F_0(X_0))$

$$\text{where} \quad F_s = H * \left(\frac{1}{\sqrt{2\pi s}}\exp(-x^2/2)\right) * m.$$

Note that $(s,x) \to F_s(x) = H * Q_s(x)$ is the unique weak solution of the heat equation for the initial condition $H * m(x)$.

Acknowledgement : It is a pleasure to thank Claude Martini for numerous fruitful discussions.

1 A first propagation of chaos result

1.1 Uniqueness for the martingale problem (MP)

This section is dedicated to the proof of the following proposition :

Proposition 1.1 *The martingale problem (MP) starting at m has no more than one solution.*

This uniqueness result is a consequence of results concerning the porous medium equation. Let us first prove the link between the martingale problem and this equation.

Lemma 1.2 *If P solves the martingale problem (MP) starting at m then the function $(s,x) \in [0,+\infty) \times \mathbb{R} \to H * P_s(x)$ is a weak solution of the porous medium equation (0.1) for the initial condition $H * m(x)$.*

Proof : To prove that $u(t,x) = H*P_t(x)$ satisfies (0.1), we first give the Fokker-Planck equation satisfied by $t \to P_t$ in $\mathcal{D}'((0,+\infty) \times \mathbb{R})$ (i.e. in the sense of distributions on $(0,+\infty) \times \mathbb{R}$) :

$$\partial_t P_t = \partial_{xx}(qu^{q-1}(t,.)P_t).$$

Clearly $P_t = \partial_x u(t,.)$ in $\mathcal{D}'((0,+\infty) \times \mathbb{R})$.
Let $t > 0$. By condition 3. of Definition 0.1, P_t does not weight points. Thus $x \to u(t,x)$ is a continuous function with bounded variation and

$$\forall x \leq y \in \mathbb{R},\ u^q(t,y) - u^q(t,x) = \int_x^y qu^{q-1}(t,z)du(t,z) = \int_x^y qu^{q-1}(t,z)P_t(dz).$$

Hence $qu^{q-1}(t,.)P_t = \partial_x u^q(t,.)$ in $\mathcal{D}'((0,+\infty) \times \mathbb{R})$. The Fokker-Planck equation writes

$$\partial_x\Big(\partial_t u - \partial_{xx}u^q\Big) = 0.$$

As a consequence, the distribution $\partial_t u - \partial_{xx}u^q$ is invariant by spatial translation and for $\phi \in C_K^\infty((0,+\infty) \times \mathbb{R})$ (the space of C^∞ functions with compact support on $(0,+\infty) \times \mathbb{R}$) and $z \in \mathbb{R}$,

$$\int_{(0,+\infty)\times\mathbb{R}} \left(u(t,x)\frac{\partial\phi}{\partial t}(t,x) + u^q(t,x)\frac{\partial^2\phi}{\partial x^2}(t,x)\right)dxdt$$

is equal to

$$\int_{(0,+\infty)\times\mathbb{R}} \left(u(t,x-z)\frac{\partial\phi}{\partial t}(t,x) + u^q(t,x-z)\frac{\partial^2\phi}{\partial x^2}(t,x)\right)dxdt.$$

By Lebesgue theorem, the last integral converges to 0 as $z \to +\infty$. Hence

$$\int_{(0,+\infty)\times\mathbb{R}} \left(u(t,x)\frac{\partial\phi}{\partial t}(t,x) + u^q(t,x)\frac{\partial^2\phi}{\partial x^2}(t,x)\right)dxdt = 0$$

and $u(t,x) = H * P_t(x)$ is a weak solution of (0.1).
As the map $t \to P_t$ is weakly continuous, dx a.e., $H * P_t(x)$ converges to $H * m(x)$ as $t \to 0$ and the initial condition is $H * m(x)$. ∎

The next lemma is dedicated to properties of the porous medium equation (0.1).

Lemma 1.3 *There is no more than one weak solution of the porous medium equation (0.1) with initial condition $H * m(x)$ of the form $H * P_s(x)$ where $P \in \mathcal{P}(C([0,+\infty),\mathbb{R}))$. Moreover any such solution satisfies :*

$$\forall s > 0,\ \forall x, y \in \mathbb{R},\ |(H * P_s)^{q-1}(x) - (H * P_s)^{q-1}(y)| \leq \frac{q-1}{q}\left(\frac{2}{(q+1)s}\right)^{\frac{1}{2}} |x-y|$$

Proof : Let $P \in \mathcal{P}(C([0,+\infty)\mathbb{R}))$ be such that $H * P_s(x)$ is a weak solution of the porous medium equation (0.1) for the initial condition $u_0(x) = H * m(x)$. Following the notations of Bénilan Crandall and Pierre [4], we have

$$l(u_0) = \lim_{r \to +\infty} \sup_{R \geq r} R^{-(1+2/(q-1))} \int_{-R}^{R} |u_0(x)| dx = 0 \quad \text{and} \quad T(u_0) = +\infty.$$

By the weak continuity of $s \to P_s$, $s \to H * P_s(.) \in C([0,+\infty), L^1_{loc}(\mathbb{R}))$. Moreover $\forall s \geq 0$, $H * P_s(.)$ is bounded by 1. Hence by Theorem U p.75 [4], $\forall s \geq 0$, $H * P_s(.) = U(s, u_0)$ where $U(s, u_0)$ denotes the weak solution of the porous medium equation constructed up to time $T(u_0) = +\infty$ in Theorem E p.54.

For $n \in \mathbb{N}^*$, let $u_{0,n} : \mathbb{R} \to [0,1]$ be a C^∞ function with compact support included in $[-n, n]$ such that $\int_{-n}^{n} |u_{0,n}(x) - u_0(x)| dx \leq 1/n$.
As the function $u_{0,n}$ is continuous and bounded by 1 and the function $(u_{0,n})^q$ is Lipschitz continuous, by Oleinik [9] Theorem 2 p.359, there exists a function $u_n(s,x)$ continuous on $[0,+\infty) \times \mathbb{R}$ and bounded on $[0,T] \times \mathbb{R}$ for any $T > 0$ which solves weakly the porous medium equation (0.1) and satisfies $u_n(0,x) = u_{0,n}(x)$.
Applying Theorem U p.75[4], we deduce that $\forall s \geq 0$, $u_n(s,.) = U(s, u_{0,n})$. By the ordering principle p.55, as $\forall x \in \mathbb{R}$, $u_{0,n}(x) \leq 1$, the function u_n is bounded by 1. According to [1], for any $\tau > 0$, the functions $(s,x) \to u_n(s,x)$ are Hölder continuous with exponent $\min(1, 1/(q-1))$ on $[\tau, +\infty) \times \mathbb{R}$ uniformly in n and the following estimate holds for the space variable :

$$\forall s > 0,\ \forall x, y \in \mathbb{R},\ |(u_n(s,x))^{q-1} - (u_n(s,y))^{q-1}| \leq \frac{q-1}{q}\left(\frac{2}{(q+1)s}\right)^{\frac{1}{2}} |x-y|. \tag{1.1}$$

By a diagonal extraction procedure, we obtain a subsequence $(u_{n'})_{n'}$ such that $u_{n'}$ converges uniformly on compact subsets of $(0,+\infty) \times \mathbb{R}$ to a function u. Clearly (1.1) still holds for u.
As $\lim_{n \to +\infty} \int_{\mathbb{R}} |u_{0,n}(x) - u_0(x)|/(1+x^2) dx = 0$, by the dependence on data result of Theorem E p.54 [4], $\forall s \geq 0$, $U(s, u_{0,n'}) = u_{n'}(s,.)$ converges to $U(s, u_0) = H * P_s(.)$ in $L^1([-p,p])$ for any $p \in \mathbb{N}^*$. Hence $H * P_s(.) = u(s,.)$. As $x \to H * P_s(x)$ is càd, $\forall s > 0, \forall x \in \mathbb{R},\ H * P_s(x) = u(s,x)$ and the conclusion holds. ∎

We are now ready to prove uniqueness for the martingale problem (MP) starting at m.
Proof of Proposition 1.1 : Let P and Q be solutions of problem (MP) starting at m. By Lemmas 1.2 and 1.3,

$$\forall (s,x) \in [0,+\infty) \times \mathbb{R},\ H * P_s(x) = H * Q_s(x) = u(s,x)$$

which implies that the time marginals of P and Q are identical. Moreover, the functions $x \to u^{q-1}(s,x)$ are Lipschitz continuous uniformly for $s \geq \tau > 0$.
Let P^τ and Q^τ denote respectively the image of P and Q by the mapping $x(.) \in C([0,+\infty), \mathbb{R}) \to$

$x(\tau+.) \in C([0,+\infty),\mathbb{R})$. Both P^τ and Q^τ solve the martingale problem : $R \in \mathcal{P}(C([0,+\infty),\mathbb{R}))$ is a solution if $R_0 = P_\tau$ and $\forall \phi \in C_b^2(\mathbb{R})$,

$$\phi(X_t) - \phi(X_0) - \int_0^t q u^{q-1}(\tau+s, X_s)\phi''(X_s)ds \quad \text{is a } R\text{-martingale.}$$

As the diffusion coefficient $2qu^{q-1}(\tau+s,x)$ is lipschitz continuous in x uniformly for $s \geq 0$, uniqueness holds for this martingale problem (see for instance [11] Theorem 8.2.1 p.204) and $P^\tau = Q^\tau$. Taking the limit $\tau \to 0$, we conclude that $P = Q$. ∎

1.2 Convergence of the particle systems (0.2)

We are interested in the n-dimensional stochastic differential equation (0.2) :

$$X_t^{i,n} = X_0^i + \int_0^t \sqrt{2q}\Big(\frac{1}{n}\sum_{j=1}^n 1_{\{X_s^{j,n} \leq X_s^{i,n}\}}\Big)^{(q-1)/2} dB_s^i \quad 1 \leq i \leq n$$

where the initial variables X_0^i, $1 \leq i \leq n$ are IID with distribution m and $(B^i)_{i\leq n}$ are independent Brownian motions independent of the initial variables. The corresponding diffusion matrix is diagonal

$$a(x) = \mathrm{Diag}\Bigg(2q\Big(\sum_{j=1}^n 1_{\{x_j \leq x_1\}}/n\Big)^{q-1}, \ldots, 2q\Big(\sum_{j=1}^n 1_{\{x_j \leq x_n\}}/n\Big)^{q-1}\Bigg).$$

and uniformly elliptic : $\forall x, y \in \mathbb{R}^n$, $y^* a(x) y \geq 2q|y|^2/n^{q-1}$. For $\tau \in S_n$ the group of permutations on $\{1,\ldots,n\}$, let A_τ denote the polyhedron $\{x \in \mathbb{R}^n,\ x_{\tau(1)} \leq x_{\tau(2)} \leq \ldots \leq x_{\tau(n)}\}$. The interiors of the polyhedra $(A_\tau)_{\tau\in S_n}$ are pairwise disjoint and $\mathbb{R}^d = \bigcup_{\tau\in S_n} A_\tau$. Moreover, on the interior of A_τ, the diffusion matrix is equal to $\mathrm{Diag}(2q(\tau^{-1}(1)/n)^{q-1}, \ldots, 2q(\tau^{-1}(n)/n)^{q-1})$ and therefore constant. Hence, by Bass and Pardoux [2] Theorem 2.1 p.559, the stochastic differential equation (0.2) admits a weak solution. Moreover, weak uniqueness holds for this equation.

Theorem 1.4 *The particle systems $(X^{1,n},\ldots,X^{n,n})$ are P chaotic where P denotes the unique solution of the martingale problem (MP) starting at m.*

As the particles $X^{i,n}$, $1 \leq i \leq n$ are exchangeable, this result result is equivalent to the convergence of the distributions π^n of the empirical measures $\mu^n = \frac{1}{n}\sum_{i=1}^n \delta_{X^{i,n}}$ to a probability measure concentrated on solutions of the martingale problem (MP) starting at m (see [12] and the references cited in it). Again by exchangeability, the tightness of the sequence $(\pi^n)_n$ is equivalent to the tightness of the distributions of the variables $(X^{1,n})_n$. As

$$\forall n \in \mathbb{N}^*,\ \forall x \in \mathbb{R}^n,\ \sqrt{2q}\Big(\frac{1}{n}\sum_{j=1}^n 1_{\{x_j \leq x_1\}}\Big)^{(q-1)/2} \leq \sqrt{2q},$$

the coefficient before dB_s^1 in (0.2) is bounded and both sequences are tight.
Let π^∞ be the limit of a converging subsequence that we still index by n for simplicity. We conclude the proof by the two next lemmas.

Lemma 1.5 *Let Q denote the canonical variable on $\mathcal{P}(C([0,+\infty),\mathbb{R}))$. π^∞ a.s., the function $(s,x) \to H * Q_s(x)$ is a weak solution of the porous medium equation (0.1) for the initial condition $H * m(x)$.*

Proof : Let $g \in C_K^\infty([0,+\infty)\times\mathbb{R})$ and $\phi(s,x) = \int_{-\infty}^x g(s,y)dy$. By Itô's formula, we get

$$< \mu_t^n, \phi(t,.) >=< \mu_0^n, \phi(0,.) > + \int_0^t < \mu_s^n, \frac{\partial \phi}{\partial s}(s,.) + q(H*\mu_s^n(.))^{q-1}\frac{\partial^2\phi}{\partial x^2}(s,.) > ds$$
$$+ \frac{1}{n}\sum_{i=1}^n \int_0^t \sqrt{2q}(H*\mu_s^n(X_s^{i,n}))^{(q-1)/2}\frac{\partial\phi}{\partial x}(s,X_s^{i,n})\, dB_s^i$$

Hence

$$\mathbb{E}\bigg(\bigg(< \mu_t^n, \phi(t,.) > - < \mu_0^n, \phi(0,.) > - \int_0^t < \mu_s^n, \frac{\partial \phi}{\partial s}(s,.) + q(H*\mu_s^n(.))^{q-1}\frac{\partial^2\phi}{\partial x^2}(s,.) > ds\bigg)^2\bigg) \leq \frac{2q\|g\|^2_{L^\infty}t}{n} \tag{1.2}$$

By the integration by parts formula,

$$\begin{aligned}
&< \mu_t^n, \phi(t,.) > - < \mu_0^n, \phi(0,.) > - \int_0^t < \mu_s^n, \frac{\partial\phi}{\partial s}(s,.) > ds \\
&= \int_{\mathbb{R}} g(t,y)dy - \int_{\mathbb{R}} g(t,y)H*\mu_t^n(y)dy - \int_{\mathbb{R}} g(0,y)dy + \int_{\mathbb{R}} g(0,y)H*\mu_0^n(y)dy \\
&\quad - \int_0^t \bigg(\int_{\mathbb{R}}\frac{\partial g}{\partial s}(s,y)dy - \int_{\mathbb{R}}\frac{\partial g}{\partial s}(s,y)H*\mu_s^n(y)dy\bigg)ds \\
&= -\int_{\mathbb{R}} g(t,y)H*\mu_t^n(y)dy + \int_{\mathbb{R}} g(0,y)H*\mu_0^n(y)dy + \int_0^t\int_{\mathbb{R}}\frac{\partial g}{\partial s}(s,y)H*\mu_s^n(y)dyds
\end{aligned} \tag{1.3}$$

As the diffusion matrix corresponding to the stochastic differential equation (0.2) is uniformly elliptic, applying the occupation times formula (see for instance Revuz Yor [10] p.209) to the semimartingales $X^{i,n} - X^{j,n}$ for $1 \leq i < j \leq n$, we obtain :

$$a.s.,\ \forall 1 \leq i < j \leq n,\ \forall t \geq 0,\ \int_0^t 1_{\{X_s^{i,n} - X_s^{j,n} = 0\}} ds = 0. \tag{1.4}$$

Hence a.s., ds a.e., the variables $X_s^{i,n}$, $1 \leq i \leq n$ are distinct. Therefore a.s., ds a.e.,

$$< \mu_s^n, q(H*\mu_s^n(.))^{q-1}\frac{\partial^2\phi}{\partial x^2}(s,.) >= -\frac{q}{n}\int_{\mathbb{R}}\frac{\partial^2 g}{\partial x^2}(s,y)\sum_{k=1}^{nH*\mu_s^n(y)}\left(\frac{k}{n}\right)^{q-1}dy$$

We deduce that a.s.,

$$\begin{aligned}
&\bigg|\int_0^t < \mu_s^n, q(H*\mu_s^n(.))^{q-1}\frac{\partial^2\phi}{\partial x^2}(s,.) > ds + \int_0^t\int_{\mathbb{R}}\frac{\partial^2 g}{\partial x^2}(s,y)(H*\mu_s^n(y))^q dyds\bigg| \\
&\qquad = \bigg|\int_0^t\int_{\mathbb{R}}\frac{\partial^2 g}{\partial x^2}(s,y)\bigg(-\frac{q}{n}\sum_{k=1}^{nH*\mu_s^n(y)}\left(\frac{k}{n}\right)^{q-1} + (H*\mu_s^n(y))^q\bigg)dyds\bigg| \\
&\qquad \leq K_g \sup_{l\leq n}\bigg|\left(\frac{l}{n}\right)^q - \frac{1}{n}\sum_{k=1}^l q\left(\frac{k}{n}\right)^{q-1}\bigg| \leq K_g \sup_{\substack{x,y\in[0,1]\\ |x-y|\leq\frac{1}{n}}} |qx^{q-1} - qy^{q-1}|
\end{aligned}$$

As the function $x \to x^{q-1}$ is uniformly continuous on $[0,1]$, the left hand side has a limit equal to 0 when $n \to +\infty$.
Combining this convergence with (1.2) and (1.3), we obtain $\lim_{n\to+\infty} \mathbb{E}(G^2(\mu^n)) = 0$ where

$$G(Q) = \int_{\mathbb{R}} g(t,y)H * Q_t(y)dy - \int_{\mathbb{R}} g(0,y)H * Q_0(y)dy \\ - \int_0^t \int_{\mathbb{R}} \left(\frac{\partial g}{\partial s}(s,y)H * Q_s(y) + \frac{\partial^2 g}{\partial x^2}(s,y)(H * Q_s(y))^q \right) dyds. \tag{1.5}$$

Since the function $G : \mathcal{P}(C([0,+\infty),\mathbb{R})) \to \mathbb{R}$ is continuous and bounded, the weak convergence of $(\pi^n)_n$ to π^∞ implies that $\mathbb{E}^{\pi^\infty}(G^2(Q)) = 0$. As the variables X_0^i are I.I.D. with distribution m, π^∞ a.s. $Q_0 = m$. Hence $\forall g \in C_K^\infty([0,+\infty) \times \mathbb{R})$, $\forall t \geq 0$, π^∞ a.s.

$$\int_{\mathbb{R}} g(t,y)H * Q_t(y)dy = \int_{\mathbb{R}} g(0,y)H * m(y)dy \\ + \int_0^t \int_{\mathbb{R}} \left(\frac{\partial g}{\partial s}(s,y)H * Q_s(y) + \frac{\partial^2 g}{\partial x^2}(s,y)(H * Q_s(y))^q \right) dyds \tag{1.6}$$

Choosing t, g in denumerate dense subsets and then taking limits, we obtain that π^∞ a.s., $\forall g \in C_K^\infty([0,+\infty) \times \mathbb{R})$, $\forall t \geq 0$, (1.6) holds. We conclude that π^∞ a.s. the function $(s,x) \to H * Q_s(x)$ is a weak solution of the porous medium equation for the initial condition $H * m(x)$. ∎

Lemma 1.6 *π^∞ a.s., Q solves the martingale problem (MP) starting at m.*

Proof : As the variables X_0^i are I.I.D. with distribution m, π^∞ a.s., $Q_0 = m$ i.e. π^∞ a.s., Q satisfies condition 1. of Definition 0.1.
Combining lemmas 1.5 and 1.3, we obtain that π^∞ a.s., $\forall s > 0$, Q_s does not weight points i.e. Q satisfies condition 3. of the definition.
To prove that π^∞ a.s., Q satisfies condition 2., we set $0 \leq s_1 \leq \ldots \leq s_p \leq s \leq t$, $g : \mathbb{R}^p \to \mathbb{R}$ continuous and bounded, $\phi \in C_b^2(\mathbb{R})$ and define $F : \mathcal{P}(C([0,+\infty),\mathbb{R})) \to \mathbb{R}$ by

$$F(Q) = < Q, \left(\phi(X_t) - \phi(X_s) - \int_s^t q(H * Q_r(X_r))^{q-1}\phi''(X_r)dr \right) g(X_{s_1}, \ldots, X_{s_p}) > .$$

By Itô's formula,

$$F(\mu^n) = \frac{1}{n} \sum_{i=1}^n \left(\int_s^t \sqrt{2q}(H * \mu_s^n(X_s^{i,n}))^{(q-1)/2} dB_s^i \right) g(X_{s_1}^{i,n}, \ldots, X_{s_p}^{i,n}).$$

Hence $\mathbb{E}(F^2(\mu^n)) \leq K/n$ and $\lim_{n\to+\infty} \mathbb{E}^{\pi^n}(F^2(Q)) = 0$.
For $Q, Q' \in \mathcal{P}(C([0,+\infty),\mathbb{R}))$,

$$|F(Q) - F(Q')| \leq K \int_s^t \sup_{x\in\mathbb{R}} |((H * Q_r)(x))^{q-1} - ((H * Q'_r)(x))^{q-1}| \, dr \\ + \left| < Q - Q', \left(\phi(X_t) - \phi(X_s) - \int_s^t q(H * Q_r(X_r))^{q-1}\phi''(X_r)dr \right) g(X_{s_1}, \ldots, X_{s_p}) > \right|$$

The functions $\nu \in \mathcal{P}(\mathbb{R}) \to H * \nu(x)$, $x \in \mathbb{R}$ are equicontinuous at any probability measure on $\mathbb{R}$ that does not weight points. Using the uniform continuity of $y \to y^{q-1}$ on $[0,1]$ and applying Lebesgue theorem, we deduce that F is continuous at any $Q \in \mathcal{P}(C([0,+\infty),\mathbb{R}))$ such that

$\forall s > 0$, Q_s does not weight points. Hence the continuity points of the bounded mapping F have full π^∞ measure. We conclude that

$$\mathbb{E}^{\pi^\infty}(F^2(Q)) = \lim_{n\to+\infty} \mathbb{E}^{\pi^n}(F^2(Q)) = 0$$

which puts an end to the proof. ∎

2 Propagation of chaos for the reordered particle systems

2.1 The reordered particle systems

Let $Y_t = (Y_t^{1,n}, \ldots, Y_t^{n,n})$ denote the order statistics of $X_t = (X_t^{1,n}, \ldots, X_t^{n,n})$ ((X, B) is a weak solution of (0.2)) i.e. $Y_t^{i,n} = \Phi_i(X_t)$ for

$\Phi_i : x = (x_1, \ldots, x_n) \in \mathbb{R}^n \to \sup_{|A|=n+1-i} \inf_{j\in A} x_j$ where $|A|$ denotes the cardinality of $A \subset \{1, \ldots, n\}$.

We first prove that Y is a diffusion with constant and diagonal diffusion matrix normally reflected at the boundary of the convex set $D_n = \{y = (y_1, \ldots, y_n) \in \mathbb{R}^n : y_1 \le \ldots \le y_n\}$. More precisely, let $\sigma^i = \sqrt{2q}(i/n)^{(q-1)/2}$ and $\beta_t = (\beta_t^1, \ldots, \beta_t^n)$ satisfy

$$\beta_t = \sum_{\tau\in S_n} \int_0^t 1_{\{\forall 1\le k\le n,\ \Phi^k(X_s)=X_s^{\tau(k),n}\}} dB_s^\tau \;\text{ where }\; B_t^\tau = (B_t^{\tau(1)}, \ldots, B_t^{\tau(n)}).$$

As by (1.4), ds a.e., the variables $X_s^{i,n}$, $1 \le i \le n$ are distinct, we check that $< \beta^i \beta^j >_t = 1_{\{i=j\}} t$ which implies that the martingale β_t is a n-dimensional Brownian motion.

Lemma 2.1

$$\forall\, 1 \le i \le n,\ Y_t^{i,n} = Y_0^{i,n} + \sigma^i \beta_t^i + V_t^i$$

for $V = (V^1, \ldots, V^n)$ a continuous process with bounded variation satisfying

$$|V|_t = \int_0^t 1_{\{(Y_s^{1,n},\ldots,Y_s^{n,n})\in\partial D_n\}} d|V|_s \qquad V_t = \int_0^t \nu_s d|V|_s$$

where $d|V|_s$ a.e., ν_s is a unit vector in the cone of inward normals to D_n ($|V|_t$ is the total variation of V defined as $\sup \sum_{k=1}^n |V_{t_k} - V_{t_{k-1}}|$ where the supremum is taken over all partitions $t_0 = 0 < t_1 < \ldots < t_n = t$.).

Remark 2.2 *For a given Brownian motion β and given initial variables $Y_0^{i,n}$, $1 \le i \le n$ independent of β, by Tanaka [13], there exists a unique couple $((Y^{1,n}, \ldots, Y^{n,n}), (V^1, \ldots, V^n)) \in C([0, +\infty), D_n) \times C([0, +\infty), \mathbb{R}^n)$ satisfying the properties stated in Lemma 2.1.*

Remark 2.3 *If $x \in \partial D_n$, then there exists $I \subset \{2, \ldots, n\}$ such that*

$$x \in \{\bigcap_{i\in I}\{y_{i-1} = y_i\}\} \bigcap \{\bigcap_{j\notin I}\{y_{j-1} < y_j\}\}.$$

Let e_i be the canonical basis on $\mathbb{R}^n$. It is easy to check that the cone of inward normals to D_n at x is $\{\sum_{i\in I} \lambda_i(e_i - e_{i-1}) \ :\ \forall i \in I,\ \lambda_i \ge 0\}$.

Proof of Lemma 2.1 : By Tanaka formula, when Z_t and Z'_t are continuous $\mathbb{R}$-valued semimartingales, so are $\sup(Z_t, Z'_t)$ and $\inf(Z_t, Z'_t)$. Hence

$$\forall i \leq n,\ Y_t^{i,n} = \Phi_i(X_t) = \sup_{|A|=n+1-i} \inf_{j\in A} X_t^{j,n}$$

is a continuous semimartingale. Let M_t^i and V_t^i denote respectively the martingale component and the finite variation component of its decomposition.
The function Φ_i is globally Lipschitz continuous and C^∞ on the opened set

$$O = \{x \in \mathbb{R}^n : \forall 1 \leq i < j \leq n,\ x_i \neq x_j\} \text{ with derivatives } \frac{\partial \Phi_i}{\partial x_j} = 1_{\{\Phi_i(x)=x_j\}},\quad \frac{\partial^2 \Phi_i}{\partial x_j^2} = 0.$$

Let ρ be a C^∞ probability density with compact support on $\mathbb{R}^n$ and $\rho^k(x) = k^n \rho(kx)$. We set $\Phi_i^k = \rho^k * \Phi^i$. Let $q \in \mathbb{Q}_+$ and $T_q = \inf\{t \geq q,\ X_t \in \partial O\}$. Suppose $T_q > q$ and set $t \in (q, T_q)$. By Itô's formula,

$$\Phi_i^k(X_t) = \Phi_i^k(X_q) + \sum_{j=1}^n \int_q^t \frac{\partial \Phi_i^k}{\partial x_j}(X_s) dX_s^{j,n} + \frac{1}{2}\sum_{j=1}^n \int_q^t \frac{\partial^2 \Phi_i^k}{\partial x_j^2}(X_s) d < X^{j,n} >_s . \tag{2.1}$$

By continuity of the sample-path $s \to X_s$, $\inf\{d(X_s, \partial O),\ s \in [q,t]\} > 0$. Hence for k big enough,

$$\forall s \in [q,t],\ \forall j \leq n,\ \frac{\partial \Phi_i^k}{\partial x_j}(X_s) = 1_{\{\Phi_i(X_s)=X_s^{j,n}\}} \quad \text{and} \quad \frac{\partial^2 \Phi_i^k}{\partial x_j^2}(X_s) = 0.$$

Taking the limit $k \to +\infty$ in (2.1), we get

$$Y_t^{i,n} = Y_q^{i,n} + \sum_{j=1}^n \int_q^t 1_{\{\Phi_i(X_s)=X_s^{j,n}\}} \sqrt{2q}\left(\frac{1}{n}\sum_{k=1}^n 1_{\{X_s^{k,n} \leq X_s^{j,n}\}}\right)^{(q-1)/2} dB_s^j.$$

If $X_s \in O$ and $\Phi_i(X_s) = X_s^{j,n}$ then $\sum_{k=1}^n 1_{\{X_s^{k,n} \leq X_s^{j,n}\}} = i$ and $\sqrt{2q}(\frac{1}{n}\sum_{k=1}^n 1_{\{X_s^{k,n} \leq X_s^{j,n}\}})^{(q-1)/2} = \sigma^i$. Moreover, it is easy to check that $\sum_{j=1}^n 1_{\{\Phi_i(X_s)=X_s^{j,n}\}} dB_s^j = d\beta_s^i$.
Hence $Y_t^{i,n} = Y_q^{i,n} + \int_q^t \sigma^i d\beta_s^i = Y_q^{i,n} + \sigma^i(\beta_t^i - \beta_q^i)$. By continuity of $Y^{i,n}$ and β^i,

$$a.s.,\ \forall q \in \mathbb{Q}_+,\ \forall t \in [q, T_q],\ Y_t^{i,n} - Y_q^{i,n} = \sigma^i(\beta_t^i - \beta_q^i).$$

If we write the open set $\{s > 0 : X_s \in O\}$ as a denumerate union of pairwise disjoint opened intervals (a_l, b_l), $l \in \mathbb{N}$, we deduce that

$$a.s.,\ \forall l \in \mathbb{N},\ \forall r \leq s \in [a_l, b_l],\ Y_s^{i,n} - Y_r^{i,n} = \sigma^i(\beta_s^i - \beta_r^i). \tag{2.2}$$

Hence a.s., $\forall l \in \mathbb{N}$, the quadratic variation of $Y^{i,n} - \sigma^i\beta^i$ is constant on $[a_l, b_l]$ and

$$a.s.,\ \forall t \geq 0,\ \int_0^t 1_{\{X_s \in O\}} d < Y^{i,n} - \sigma^i\beta^i >_s = \sum_{l\in\mathbb{N}} \int_{(a_l,b_l)\cap[0,t]} 1_{\{X_s \in O\}} d < Y^{i,n} - \sigma^i\beta^i >_s = 0. \tag{2.3}$$

As $\forall x, x' \in \mathbb{R}^n$, $\sum_{i=1}^n (\Phi_i(x) - \Phi_i(x'))^2 \leq \sum_{i=1}^n (x_i - x'_i)^2$ (for dimension $n = 2$ we check this inequality by an easy computation and for $n > 2$ we prove it by induction using the two-dimensional inequality), we easily prove that a.s. the measure $\sum_{i=1}^n d < Y^{i,n} >_s$ is absolutely continuous with respect to $\sum_{i=1}^n d < X^{i,n} >_s$.

As a consequence a.s. $d < Y^{i,n} >_s$ is absolutely continuous with respect to Lebesgue measure. So is $d < \beta^i >_s$.
Since by (1.4), a.s., ds a.e., $X_s \in O$, we deduce that $a.s.$, $\forall t \geq 0$, $\int_0^t 1_{\{X_s \in \partial O\}} d < Y^{i,n} - \sigma^i \beta^i >_s = 0$. Taking (2.3) into account we get that $a.s.$, $\forall t \geq 0$, $< Y^{i,n} - \sigma^i \beta^i >_t = 0$ which ensures that $M_t^i = \sigma^i \beta_t^i$.
Recalling the decomposition $Y_t^{i,n} = M_t^i + V_t^i$, we obtain from (2.2) that $a.s.$, $t \to |V|_t$ is constant on $[a_l, b_l]$, $\forall l \in \mathbb{N}$. As $X_s \in \partial O$ if and only if $Y_s \in \partial D_n$, we conclude that $|V|_t = \int_0^t 1_{\{Y_s \in \partial D_n\}} d|V|_s$.

Let $\nu_s = (\nu_s^1, \ldots, \nu_s^n) = (\frac{dV_s^1}{d|V|_s}, \ldots, \frac{dV_s^n}{d|V|_s})$. Clearly, $d|V|_s$ a.e., ν_s is a unit vector. We are now going to prove that $d|V|_s$ a.e., this vector belongs to the cone of inward normals to D_n. To do so, we introduce $\Psi_i(x) = \sum_{j=i}^n \Phi_j(x)$. This function is C^∞ in the open set $\{\Phi_{i-1}(x) < \Phi_i(x)\}$ (with the convention $\Phi_0 \equiv -\infty$). By a reasoning similar to the one made for Φ_i, we prove that

$$a.s., \ \forall 1 \leq i \leq n, \ \forall t \geq 0, \ \int_0^t 1_{\{Y_s^{i-1,n} < Y_s^{i,n}\}} d(V^i + \ldots + V^n)_s = 0.$$

(with the convention $Y_s^{0,n} \equiv -\infty$). We deduce that

$$d|V|_s \ a.e., \ \forall 1 \leq i \leq n, \ (Y_s^{i-1,n} < Y_s^{i,n}) \implies (\nu_s^i + \ldots + \nu_s^n = 0) \tag{2.4}$$

Therefore $d|V|_s$ $a.e.$, $\nu_s^1 = -(\nu_s^2 + \ldots + \nu_s^n)$ and $\nu_s = \sum_{i=2}^n (\nu_s^i + \ldots + \nu_s^n)(e_i - e_{i-1})$ where e_i, $1 \leq i \leq n$ denotes the canonical basis on $\mathbb{R}^n$. According to (2.4) and Remark 2.3, the proof is completed if we show that

$$d|V|_s \ a.e., \ \forall 2 \leq i \leq n, \ \nu_s^i + \ldots + \nu_s^n \geq 0. \tag{2.5}$$

Let $2 \leq i \leq n$. As $\forall s \geq 0$, $Y_s^{i-1,n} \leq Y_s^{i,n}$, applying Tanaka formula to compute $(Y_t^{i,n} - Y_t^{i-1,n})^-$, we obtain

$$\forall t \geq 0, \ \int_0^t 1_{\{Y_s^{i-1,n} = Y_s^{i,n}\}} d(Y^{i,n} - Y^{i-1,n})_s = \frac{1}{2} L_t^0 (Y^{i,n} - Y^{i-1,n}).$$

where $L_t^0(Y^{i,n} - Y^{i-1,n})$ denotes the local time in 0 of $Y^{i,n} - Y^{i-1,n}$. Since $\{Y_s^{i-1,n} = Y_s^{i,n}\} \subset \{X_s \in \partial O\}$, $\int_0^t 1_{\{Y_s^{i-1,n} = Y_s^{i,n}\}} d(M^i - M^{i-1})_s = 0$. Hence $\int_0^t 1_{\{Y_s^{i-1,n} = Y_s^{i,n}\}} d(V^i - V^{i-1})_s = \frac{1}{2} L_t^0(Y^{i,n} - Y^{i-1,n})$. As the local time is increasing, we deduce that

$$d|V|_s \ a.e., \ \forall 2 \leq i \leq n, \ (Y_s^{i-1,n} = Y_s^{i,n}) \implies (\nu_s^i \geq \nu_s^{i-1}).$$

Combining this property with (2.4), we easily obtain (2.5). ∎

Following Sznitman [12] p.187-190, we symmetrize $(Y^{1,n}, \ldots, Y^{n,n})$ by a random permutation in order to obtain tightness. Let Θ be a random variable uniformly distributed on S_n (independent of the processes $(X^{i,n}, B^i)_{1 \leq i \leq n}$). We set

$$(Z^{1,n}, \ldots, Z^{n,n}) = (Y^{\Theta(1),n}, \ldots, Y^{\Theta(n),n}).$$

Although the two systems are different, their empirical measures are identical.

Theorem 2.4 *Let X denote the canonical process on $C([0, +\infty), \mathbb{R})$.*
The particle systems $(Z^{1,n}, \ldots, Z^{n,n})$ are $\tilde{P}$-chaotic where $\tilde{P}$ denotes the unique probability measure in $\mathcal{P}(C([0, +\infty), \mathbb{R}))$ such that :

(i) *the function $(s, x) \to H * \tilde{P}_s(x)$ is a weak solution of the porous medium equation (0.1) for the initial condition $H * m(x)$,*

(ii) *$\tilde{P}$ a.s., $s \in (0, +\infty) \to H * \tilde{P}_s(X_s)$ is constant.*

2.2 Proof of Theorem 2.4

As the variables $Z^{i,n}$, $1 \leq i \leq n$ are exchangeable, it is enough to check that there is no more than one $Q \in \mathcal{P}(C([0,+\infty),\mathbb{R}))$ satisfying **(i)** and **(ii)** and that the distributions $\tilde{\pi}^n$ of the empirical measures $\tilde{\mu}^n = \frac{1}{n}\sum_{i=1}^n \delta_{Z^{i,n}} = \frac{1}{n}\sum_{i=1}^n \delta_{Y^{i,n}}$ converge weakly to a probability measure giving full measure to $\{Q$ satisfying **(i)** and **(ii)**$\}$. We are going to realize this program thanks to four lemmas. The first one is dedicated to the tightness of the sequence $(\tilde{\pi}^n)_n$.

Lemma 2.5 *The sequence $(\tilde{\pi}^n)_n$ is tight.*

Proof : By exchangeability, the conclusion is equivalent to the tightness of the distributions of the processes $Z^{1,n}$.
We easily check that for any $n \geq 1$, the variables $Z_0^{1,n}, \ldots, Z_0^{n,n}$ are I.I.D. with distribution m. Hence the sequence $(Z_0^{1,n})_n$ is constant in distribution. Let $t \geq s \geq 0$.

$$\mathbb{E}((Z_t^{1,n} - Z_s^{1,n})^4) = \frac{1}{n}\sum_{i=1}^n \mathbb{E}((Y_t^{i,n} - Y_s^{i,n})^4).$$

As $\forall x, x' \in \mathbb{R}^n$, $\sum_{i=1}^n (\Phi_i(x) - \Phi_i(x'))^4 \leq \sum_{i=1}^n ((x_i - x'_i)^4)$ (again, for dimension $n = 2$ we check this inequality by an easy computation and for $n > 2$ we prove it by induction using the two-dimensional inequality),

$$\mathbb{E}((Z_t^{1,n} - Z_s^{1,n})^4) \leq \frac{1}{n}\sum_{i=1}^n \mathbb{E}((X_t^{i,n} - X_s^{i,n})^4).$$

Since $\forall n \geq 1$, $\forall x \in \mathbb{R}^n$, $\forall 1 \leq i \leq n$, $\sqrt{2q}(\sum_{j=1}^n 1_{\{x_j \leq x_i\}}/n)^{(q-1)/2} \leq \sqrt{2q}$, we deduce that $\exists K < +\infty$, $\forall n \geq 1$, $\forall t \geq s \geq 0$, $\mathbb{E}((Z_t^{1,n} - Z_s^{1,n})^4) \leq K(t-s)^2$. By Kolmogorov criterion, we conclude that the distributions of the processes $Z^{1,n}$ are tight. ∎

Let $\tilde{\pi}^\infty$ be the limit of a converging subsequence of $(\tilde{\pi}^n)_n$ that we still index by n for notational simplicity.
As $\forall s \geq 0, \tilde{\mu}_s^n = \mu_s^n$, for G defined in (1.5), $G(\tilde{\mu}^n) = G(\mu^n)$. Therefore, by a reasoning similar to the end of the proof of Lemma 1.5, we obtain :

Lemma 2.6 *$\tilde{\pi}^\infty$ a.s. the function $(s,x) \to H * Q_s(x)$ is a weak solution of the porous medium equation for the initial condition $H * m(x)$.*

Hence $\tilde{\pi}^\infty$ a.s., Q satisfies condition **(i)** of Theorem 2.4. Let us now deal with condition **(ii)**.

Lemma 2.7 *$\tilde{\pi}^\infty$ a.s., Q a.s. the function $s \in (0,+\infty) \to H * Q_s(X_s)$ is constant.*

Proof : Let $i \leq n-1$. By (1.4), ds a.e., a.s., the variables $Y_s^{i,n}$ $1 \leq i \leq n$ are distinct. Hence there is a Borel set $\mathcal{N} \subset (0,+\infty)$ with Lebesgue measure 0 such that

$$\forall s \in \mathcal{N}^c,\ \forall n \geq 2,\ \text{a.s. } Y_s^{1,n} < Y_s^{2,n} < \ldots < Y_s^{n,n}$$

$$\text{which implies that } \forall 1 \leq i \leq n,\ H * \tilde{\mu}_s^n(Y_s^{i,n}) = \frac{i}{n}$$

Let $0 < s < t$ with $s, t \in \mathcal{N}^c$,

$$\mathbb{E}(< \tilde{\mu}^n, |H * \tilde{\mu}^n_s(X_s) - H * \tilde{\mu}^n_t(X_t)| >) = \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} |H * \tilde{\mu}^n_s(Y^{i,n}_s) - H * \tilde{\mu}^n_t(Y^{i,n}_t)|\right) = 0. \quad (2.6)$$

The functions $\nu \in \mathcal{P}(\mathbb{R}) \to H * \nu(x)$, $x \in \mathbb{R}$ are equicontinuous at any probability measure on $\mathbb{R}$ which does not weight points. Moreover, combining Lemmas 2.6 and 1.3, we obtain that $\tilde{\pi}^\infty$ a.s., $\forall s > 0$, Q_s does not weight points. We deduce that $\tilde{\pi}^\infty$ is concentrated on continuity points of the bounded mapping $Q \to < Q, |H * Q_s(X_s) - H * Q_t(X_t)| >$. Hence

$$\mathbb{E}^{\tilde{\pi}^\infty}(< Q, |H * Q_s(X_s) - H * Q_t(X_t)| >) = \lim_{n\to+\infty} \mathbb{E}(< \tilde{\mu}^n, |H * \tilde{\mu}^n_s(X_s) - H * \tilde{\mu}^n_t(X_t)| >) = 0.$$

Therefore $\forall s, t \in (0, +\infty) \cap \mathcal{N}^c$, $\tilde{\pi}^\infty$ a.s., Q a.s., $H * Q_s(X_s) = H * Q_t(X_t)$.
The condition $\forall s > 0$, Q_s does not weight points is equivalent to the continuity of $(s, x) \to H * Q_s(x)$ on $(0, +\infty) \times \mathbb{R}$. Hence $\tilde{\pi}^\infty$ a.s., Q a.s., $s \in (0, +\infty) \to H * Q_s(X_s)$ is continuous and the conclusion of the Lemma holds. ∎

To conclude the proof of the propagation of chaos result, it is enough to show that there is no more than one probability measure P satisfying conditions **(i)** and **(ii)** of Theorem 2.4.

Lemma 2.8 *There exists a unique probability measure $\tilde{P} \in \mathcal{P}(C[0, +\infty), \mathbb{R}))$ such that :*

(i) *the function $(s, x) \to H * \tilde{P}_s(x)$ is a weak solution of the porous medium equation (0.1) for the initial condition $H * m(x)$,*

(ii) *$\tilde{P}$ a.s., $s \in (0, +\infty) \to H * \tilde{P}_s(X_s)$ is constant.*

Proof : Existence is ensured by Lemmas 2.6 and 2.7.
To prove uniqueness, we consider two probability measures P and Q both satisfying **(i)** and **(ii)**. By Lemma 1.3, condition **(i)** implies that $\forall (t, x) \in [0, +\infty) \times \mathbb{R}$, $H * P_t(x) = H * Q_t(x) = u(t, x)$ i.e. $\forall t \geq 0$, $P_t = Q_t$.

Let $s \geq 0$ and $\tilde{u}(s, y) = \inf\{x : u(s, x) \geq y\}$ for $0 \leq y \leq 1$. We are going to prove that $P(X_s = \tilde{u}(s, u(s, X_s))) = 1$. We have $\{\tilde{u}(s, y) \leq x\} = \{y \leq u(s, x)\}$. Therefore

$$P(\tilde{u}(s, u(s, X_s)) \leq x) = P(u(s, X_s) \leq u(s, x)) = P(X_s \leq x) + P(X_s \in \{y > x : u(s, y) = u(s, x)\})$$

As $u(s, x) = P(X_s \leq x)$ the second term of right-hand-side is nil and $P(\tilde{u}(s, u(s, X_s)) \leq x) = u(s, x)$. Hence $P \circ \tilde{u}(s, u(s, X_s))^{-1} = P \circ X_s^{-1}$. Moreover, clearly $\forall x \in \mathbb{R}$, $x \geq \tilde{u}(s, u(s, x))$ which implies $P(X_s \geq \tilde{u}(s, u(s, X_s))) = 1$. Thus $P(X_s = \tilde{u}(s, u(s, X_s))) = 1$.
If $t, s > 0$, as by **(ii)** $P(u(s, X_s) = u(t, X_t)) = 1$, $P(X_s = \tilde{u}(s, u(t, X_t))) = 1$. More generally, for $0 < t_1 < t_2 < \ldots < t_k$,

$$P(X_{t_2} = \tilde{u}(t_2, u(t_1, X_{t_1})), \ldots, X_{t_n} = \tilde{u}(t_n, u(t_1, X_{t_1}))) = 1.$$

Hence the finite dimensional marginal $P_{t_1,\ldots,t_n}$ is the image of P_{t_1} by the mapping

$$x \in \mathbb{R} \to (x, \tilde{u}(t_2, u(t_1, x)), \ldots, \tilde{u}(t_n, u(t_1, x))) \in \mathbb{R}^n.$$

The same is true for Q. As $P_{t_1} = Q_{t_1}$, we deduce that

$$\forall 0 < t_1 < \ldots < t_n,\ P_{t_1,\ldots,t_n} = Q_{t_1,\ldots,t_n}.$$

By weak continuity, this equality still holds for $t_1 = 0$ and the finite dimensional marginals of P and Q are equal which implies $P = Q$. ∎

3 A possible generalization

Let $a, b : \mathbb{R} \to \mathbb{R}$ be C^1 functions with $\forall x > 0, a'(x) > 0$, $(\beta^i)_{i\in\mathbb{N}^*}$ a sequence of independent Brownian motions and $Y_0^{1,n} \leq Y_0^{2,n} \leq \ldots \leq Y_0^{n,n}$ the order statistics of n variables I.I.D. with law m independent of the Brownian motions. If we are interested in the more general partial differential equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2(a(u))}{\partial x^2} - \frac{\partial(b(u))}{\partial x}, \quad (t,x) \in [0,+\infty) \times \mathbb{R} \tag{3.1}$$

we can consider the unique couple $((Y^{1,n}, \ldots, Y^{n,n}), (V^1, \ldots, V^n)) \in C([0,+\infty), D_n \times \mathbb{R}^n)$ such that

$$\forall i \leq n,\ Y_t^{i,n} = Y_0^{i,n} + \sqrt{a'(i/n)}\ \beta_t^i + b'(i/n)t + V_t^i,$$

$V = (V^1, \ldots, V^n)$ is of bounded variation and satisfies

$$|V|_t = \int_0^t 1_{\{(Y_s^{1,n}, \ldots, Y_s^{n,n}) \in \partial D_n\}} d|V|_s \qquad V_t = \int_0^t \nu_s d|V|_s$$

where $d|V|_s$ a.e., ν_s is a unit vector in the cone of inward normals to D_n (see Tanaka [13]).
We can also introduce a weak solution of the stochastic differential equation (see [2])

$$X_t^{i,n} = X_0^i + \int_0^t \sqrt{a'}(H * \mu_s^n(X_s^{i,n}))dB_s^i + \int_0^t b'(H * \mu_s^n(X_s^{i,n}))ds,\ 1 \leq i \leq n,\ \ \mu^n = \frac{1}{n}\sum_{i=1}^n \delta_{X^{i,n}}$$

where the variables X_0^i are I.I.D. with law m and independent of the n-dimensional Brownian motion $(B^1, \ldots, B^n)$. Adapting the proof of Lemma 2.1, we easily obtain that the particle system obtained by increasing reordering from $(X_t^{1,n}, \ldots, X_t^{n,n})$ is a weak solution of the previous stochastic differential equation with normal reflection.
We still denote $\mu^n = \frac{1}{n}\sum_{i=1}^n \delta_{X^{i,n}}$ and $\tilde{\mu}^n = \frac{1}{n}\sum_{i=1}^n \delta_{Y^{i,n}}$ the empirical measures. We say that $Q \in \mathcal{P}(C([0,+\infty), \mathbb{R}))$ is a solution of the martingale problem (MP) starting at m if $Q_0 = m$, $\forall \phi \in C_b^2(\mathbb{R})$,

$$\phi(X_t) - \phi(X_0) - \int_0^t \frac{1}{2}a'(H * P_s(X_s))\phi''(X_s) + b'(H * P_s(X_s))\phi'(X_s)ds \ \text{ is a } Q\text{-martingale},$$

and $\forall t > 0$, Q_t does not weight points.

A key point in the approach developped for the porous medium equation is Lemma 1.3. Indeed, if we show that there is no more than one weak solution of (3.1) of the form $H * P_s(x)$ where $P \in \mathcal{P}(C([0,+\infty), \mathbb{R}))$ for the initial condition $H * m(x)$ and that any such solution is continuous on $(0,+\infty) \times \mathbb{R}$, then every result but uniqueness for problem (MP) can be adapted. In particular, the sequence $\tilde{\mu}_n$ converges in probability to the unique $\tilde{P} \in \mathcal{P}(C([0,+\infty), \mathbb{R}))$ such that $(s,x) \to H * \tilde{P}_s(x)$ is a weak solution of (3.1) for the initial condition $H * m(x)$ and $\tilde{P}$ a.s., the function $s \in (0,+\infty) \to H * \tilde{P}_s(x)$ is constant. If we also prove uniqueness for the martingale problem (MP) starting at m, then this problem admits a unique solution P and the sequence μ^n converges in probability to the constant P.
For instance, in the particular case $a(u) = u$, both these convergence results hold since :

Lemma 3.1 *Suppose that $a(u) = u$. Then there is no more than one weak solution of (3.1) of the form $H * P_s(x)$ where $P \in \mathcal{P}(C([0,+\infty),\mathbb{R}))$ for the initial condition $H * m(x)$. Any such solution is continuous on $(0,+\infty) \times \mathbb{R}$. Moreover, uniqueness holds for the martingale problem (MP) starting at m.*

Proof : Let $P, Q \in \mathcal{P}(C([0,+\infty),\mathbb{R}))$ be such that $u(t,x) = H * P_t(x)$ and $v(t,x) = H * Q_t(x)$ are weak solutions of (3.1) for the initial condition $H * m(x)$. For a good choice of test functions, we obtain that

$$\forall t > 0,\ dx\ a.e.,\ u(t,x) = G_t * H * m(x) - \int_0^t \frac{\partial G_{t-s}}{\partial x} * (b(u(s,.))(x)ds \tag{3.2}$$

where $G_t(x) = \exp(-x^2/2t)/\sqrt{2\pi t}$ denotes the heat kernel. The same equation holds for v. Writing the equation satisfied by $v - u$ and taking $\|\frac{\partial G_t}{\partial x}\|_{L^1} = \sqrt{2/\pi t}$ into account, we get

$$\begin{aligned}\forall t > 0,\ \|u(t,.) - v(t,.)\|_{L^\infty} &\leq \int_0^t \left\| \frac{\partial G_{t-s}}{\partial x} \right\|_{L^1} \|b(u(s,.)) - b(v(s,.))\|_{L^\infty} ds \\ &\leq \sqrt{\frac{2}{\pi}} \sup_{[0,1]} |b'| \int_0^t \frac{\|u(s,.) - v(s,.)\|_{L^\infty}}{\sqrt{t-s}} ds\end{aligned}$$

Iterating this equation, we conclude by Gronwall's lemma that $\forall t > 0,\ \|u(t,.) - v(t,.)\|_{L^\infty} = 0$. Hence $\forall (t,x) \in [0,+\infty) \times \mathbb{R},\ H * P_t(x) = H * Q_t(x)$.
Let us now prove that $(t,x) \to u(t,x) = H * P_t(x)$ is continuous on $(0,+\infty) \times \mathbb{R}$. As $t \to P_t$ is weakly continuous, it is enough to show that $\forall t > 0$, P_t does not weight points i.e. $x \to u(t,x) = H * P_t(x)$ is continuous. Let $0 < \alpha < t$,

$$\begin{aligned}&\left| G_t * H * m(x) - \int_0^t \frac{\partial G_{t-s}}{\partial x} * (b(u(s,.))(x)ds - G_t * H * m(y) + \int_0^t \frac{\partial G_{t-s}}{\partial x} * (b(u(s,.))(y)ds \right| \\ &\qquad \leq |G_t * H * m(x) - G_t * H * m(y)| + 2 \left\| \int_\alpha^t \frac{\partial G_{t-s}}{\partial x} * (b(u(s,.))ds \right\|_{L^\infty} \\ &\qquad + \left| \int_0^\alpha \int_{\mathbb{R}} \left(\frac{\partial G_{t-s}}{\partial x}(x-z) - \frac{\partial G_{t-s}}{\partial x}(y-z) \right) b(u(s,z))dzds \right|\end{aligned}$$

The first term of the right-hand-side converges to 0 as $y \to x$. The second term is arbitrarily small for α close to t. Last, for fixed α, by Lebesgue theorem, the third term converges to 0 when $y \to x$. Hence the function $x \to G_t * H * m(x) - \int_0^t \frac{\partial G_{t-s}}{\partial x} * (b(u(s,.))(x)ds$ is continuous. As $x \to u(t,x)$ is right-continuous, we deduce that equality (3.2) holds $\forall x \in \mathbb{R}$ and that $x \to u(t,x)$ is continuous.

Let now P and Q solve the martingale problem (MP) starting at m. By an easy adaptation of the proof of Lemma 1.2, we get that $H * P_s(x)$ and $H * Q_s(x)$ are weak solutions of (3.1) for he initial condition $H * m(x)$. Hence, by the first step of the proof, $\forall (t,x) \in [0,+\infty) \times \mathbb{R},\ H * P_t(x) = H * Q_t(x)$. Therefore both P and Q solve the linear martingale problem with diffusion coefficient equal to 1 and bounded drift coefficient $b'(H * P_t(x))$. By Girsanov theorem, uniqueness holds for this problem and $P = Q$. ∎

Remark 3.2 *For different proofs of the propagation of chaos result to the unique solution of (MP) for the diffusing particles $(X^{1,n}, \ldots, X^{n,n})$ see [5] which deals with the case $b(u) = u^2/2$ (viscous Burgers equation) and [8] in which b is supposed to be a C^2 function.*

References

[1] D.G. Aronson and L.A. Caffarelli. Optimal regularity for one-dimensional porous medium flow. *Revista Matematica Iberoamericana*, 2(4):357–366, 1986.

[2] R.F. Bass and E. Pardoux. Uniqueness for diffusions with piecewise constant coefficients. *Probability Theory and Related Fields*, 76:557–572, 1987.

[3] S. Benachour, P. Chassaing, B. Roynette, and P. Vallois. Processus associés à l'équation des milieux poreux. *Ann. Sc. Norm. Super. Pisa, Cl. Sci.*, IV. Ser. 23(4):793–832, 1996.

[4] P. Bénilan, M.G. Crandall, and M. Pierre. Solutions of the porous medium equation in $\mathbb{R}^n$ under optimal conditions on initial values. *Indiana University Mathematics Journal*, 33(1):51–87, 1984.

[5] M. Bossy and D. Talay. Convergence rate for the approximation of the limit law of weakly interacting particles: Application to the Burgers equation. *Annals of Applied Prob.*, 6(3):818–861, 1996.

[6] M. Inoue. Construction of diffusion processes associated with a porous medium equation. *Hiroshima Math. J.*, 19:281–297, 1989.

[7] M. Inoue. Derivation of a porous medium equation from many markovian particles and propagation of chaos. *Hiroshima Math. J.*, 21:85–110, 1991.

[8] B. Jourdain. *Sur l'interprétation probabiliste de quelques équations aux dérivées partielles non linéaires.* PhD thesis, Ecole Nationale des Ponts et Chaussées, 1998.

[9] O.A. Oleinik. On some degenerate quasilinear parabolic equations. In *Seminari dell'Instituto Nazionale di Alta Mathematica 1962-63*, pages 355–371. Odesiri,Gubbio, 1964.

[10] D. Revuz and M. Yor. *Continuous martingales and Brownian motion.* Springer-Verlag, 1991.

[11] D.W. Stroock and S.R.S. Varadhan. *Multidimensional Diffusion Processes.* Springer, 1997.

[12] A.S. Sznitman. Topics in propagation of chaos. In *Ecole d'été de probabilités de Saint-Flour XIX - 1989, Lect. Notes in Math. 1464*. Springer-Verlag, 1991.

[13] H. Tanaka. Stochastic differential equations with reflecting boundary condition in convex regions. *Hiroshima Math. J.*, 9:163–177, 1979.