Semilinear elliptic system arising in a threedimensional type-II superconductor for infinite κ.

R. Monneau
Ecole Nationale des Ponts et Chaussées

November 9, 1998

Abstract

We study a semilinear elliptic system arising in a threedimensional superconductor Ω. This model is formally derived from the Ginzburg-Landau energy at $\kappa = +\infty$ for a Meissner solution. If the tangential trace of the magnetic field $\mathbf{H}$ is given on $\partial\Omega$, we prove the existence of a unique solution for a small data, and nonexistence for large data. On another hand we prove that the current $\mathbf{J} = \mathbf{curl\ H}$ is such that $\mathbf{J}^2$ is maximum on the boundary $\partial\Omega$.

AMS Classification: 35B20, 35B30, 35J55, 35Q40.

Keywords: Nonlinear elliptic systems; Maximum principle; Ginzburg-Landau energy; Meissner solution; Inverse function theorem.

1 Introduction

Let $\Omega \subset \mathbf{R}^3$ be an open set homeomorphic to a ball. We study the following model, which describes the magnetic field $\mathbf{H}$ in a threedimensional type-II superconductor Ω for a Meissner solution at $\kappa = +\infty$. We search the solutions $\mathbf{H}$ such that $\mathbf{curl\ H} \cdot n = 0$ on $\partial\Omega$ and

$$-\mathbf{curl}\ (F(|\mathbf{curl\ H}|^2)\mathbf{curl\ H}) - \mathbf{H} = 0 \text{ on } \Omega \tag{1.1}$$

$$\mathbf{H}_T = \mathbf{H}_T^{ext} \text{ on } \partial\Omega \tag{1.2}$$

where

$$h = (1 - u^2)u \Longleftrightarrow u = F(h^2)h \tag{1.3}$$

and F is uniquely defined for $u^2 < \frac{1}{3}$ or equivalently for $h^2 < \frac{4}{27}$ and $F(0) = 1$. Here $\mathbf{H}_T$ denotes the tangential trace of the magnetic field $\mathbf{H}$: $\mathbf{H}_T = \mathbf{H} - (\mathbf{H} \cdot n)n$, where n is the exterior unit normal to Ω. Let us assume that $\partial\Omega \in C^{2,\alpha}$. Then we have:

PROPOSITION 1.1 *Let $\mathbf{H}_T^{ext} \in C^{2,\alpha}(\partial\Omega)$.*
i) Then $\exists \mathbf{H}^1 \in C^{2,\alpha}(\overline{\Omega})$ and $\mathbf{H}_T^1 = \mathbf{H}_T^{ext}$ on $\partial\Omega$.
ii)If $\exists \mathbf{H}^i \in C^{2,\alpha}(\overline{\Omega})$ and $\mathbf{H}_T^i = \mathbf{H}_T^{ext}$ on $\partial\Omega$, for $i = 1, 2$, then $\mathbf{curl}\ \mathbf{H}^1 \cdot n = \mathbf{curl}\ \mathbf{H}^2 \cdot n$.

Then we have the

DEFINITION 1.2 *If $\mathbf{H}_T^{ext} \in C^{2,\alpha}(\partial\Omega)$, such that for some $\mathbf{H}^1 \in C^{2,\alpha}(\overline{\Omega})$, $\mathbf{H}_T^1 = \mathbf{H}_T^{ext}$, we have $\mathbf{curl}\ \mathbf{H}^1 \cdot n = 0$ on $\partial\Omega$, then we simply write that $\mathbf{curl}\ \mathbf{H}_T^{ext} \cdot n = 0$ on $\partial\Omega$.*

Then we prove the

THEOREM 1.3 *i)$\exists \epsilon > 0$, $\forall \mathbf{H}_T^{ext} \in C^{2,\alpha}(\partial\Omega)$ with $\mathbf{curl}\ \mathbf{H}_T^{ext} \cdot n = 0$ on $\partial\Omega$, if $|\mathbf{H}_T^{ext}|_{C^{2,\alpha}(\partial\Omega)} < \epsilon$, then there exists a unique $\mathbf{H} \in C^{2,\alpha}(\Omega)$ solution of (1.1)-(1.2).*
ii) $\exists C = C(\Omega) > 0$, such that if $\mathbf{H}$ is solution of (1.1) and $\mathbf{J} = \mathbf{curl}\ \mathbf{H}$, then

$$|\mathbf{H}_T|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq C |F(\mathbf{J}_T^2)\mathbf{J}_T|_{H^{\frac{1}{2}}(\partial\Omega)} \tag{1.4}$$

Moreover $\forall \mathbf{g} \in C^{2+\alpha}(\partial\Omega), \exists \mu_0 > 0, \forall \mu > \mu_0$ there is no solution to (1.1)-(1.2) for $\mathbf{H}_T = \mu\mathbf{g}$.
iii)Let the current $\mathbf{J} = \mathbf{curl}\ \mathbf{H}$ for some $\mathbf{H} \in C^{2,\alpha}(\overline{\Omega})$ solution of (1.1) with $\mathbf{J}^2 < \frac{4}{27}$ on $\overline{\Omega}$, then $\mathbf{J}^2$ is maximal on $\partial\Omega$.
iv) If $\mathbf{J}_T$ is the tangential trace of the current $\mathbf{J}$, then there is at most one magnetic field $\mathbf{H}$ solution of (1.1) (and then at most one current $\mathbf{J} = \mathbf{curl}\ \mathbf{H}$) such that $(\mathbf{curl}\ \mathbf{H})_T = \mathbf{J}_T$ on $\partial\Omega$.
Moreover $\mathbf{J}_T$ cannot be chosen arbitrarily : $\forall \epsilon > 0$, $\exists \mathbf{J}_T \in C^{2,\alpha}(\partial\Omega)$, $|\mathbf{J}_T|_{C^{2,\alpha}(\partial\Omega)} < \epsilon$ such that there is no solution $\mathbf{H}$ (and $\mathbf{J}$).

REMARK 1.4 *Let us remark that on $\partial\Omega$, $\mathbf{J} \cdot n = 0$ and $|\mathbf{J}_T|^2$ is smaller than the critical value $\frac{4}{27}$. In particular for $\mathbf{J} = (1 - \mathbf{Q}^2)\mathbf{Q}$, we have $F(\mathbf{J}_T^2)\mathbf{J}_T = \mathbf{Q}_T$, i.e. $|\mathbf{Q}_T|^2 < \frac{1}{3}$.*

2 Derivation of the model

The superconductor material is represented by Ω, and in $\mathbf{R}^3\backslash\Omega$ we would impose (in a realistic case) an exterior magnetic field which is given at infinity. In fact this exterior magnetic field will create an interior magnetic field inside the superconductor. And this interior magnetic field will make a perturbation (essentially local in space) of the exterior magnetic field. This is the reason why we can only impose the exterior magnetic field at infinity, in a real experimentation.

Let us recall that if $\mathbf{A}$ is the interior magnetic potential vector, and Ψ the complex wave function which describes the superconducting state, then the Ginzburg-Landau energy (without exterior forces) is

$$\mathcal{E}_\kappa(\mathbf{A},\psi) = \int_\Omega |(\frac{1}{\kappa}\nabla - i\mathbf{A})\Psi|^2 + \frac{1}{2}(|\Psi|^2-1)^2 + |\mathbf{curl}\ \mathbf{A}|^2 \tag{2.1}$$

In the case where there is no vortex (the Meissner solution), we have $|\Psi| > 0$, and we can take (see [3]) the new gauge $\Psi = fe^{i\chi}$, $\mathbf{Q} = A - \frac{1}{\kappa}\nabla\chi$, where $f > 0$ and $\chi \in \mathbf{R}$. Then for

$$\mathcal{E}_\kappa(f,\mathbf{Q}) = \int_\Omega \frac{|\nabla f|^2}{\kappa^2} + |\mathbf{curl}\ \mathbf{Q}|^2 + \frac{(f^2-1)^2}{2} + f^2\mathbf{Q}^2$$

we obtain the Euler-Lagrange equations in Ω:

$$\left\{\begin{array}{l} \frac{1}{\kappa^2}\Delta f = f(f^2+\mathbf{Q}^2-1) \\ -(\mathbf{curl}\ )^2\mathbf{Q} = f^2\mathbf{Q} \end{array}\right. \tag{2.2}$$

and the interior magnetic field is $\mathbf{H}^{int} = \mathbf{curl}\ \mathbf{Q}$. At the exterior of the superconductor material we have the Maxwell equations for the exterior magnetic field $\mathbf{H}^{ext}$ in $\mathbf{R}^3\backslash\overline{\Omega}$:

$$\left\{\begin{array}{l} \mathrm{div}\ \mathbf{H}^{ext} = 0 \\ \mathbf{curl}\ \mathbf{H}^{ext} = 0 \end{array}\right. \tag{2.3}$$

Moreover at the interface $\partial\Omega$, we have the continuity of the magnetic field:

$$\mathbf{H}^{int} = \mathbf{H}^{ext} \tag{2.4}$$

and formally from the variation of the Ginzburg-Landau energy we find the boundary conditions:

$$\frac{\partial_n f}{\kappa^2} = 0 \tag{2.5}$$

$$\mathbf{Q} \cdot n = 0 \tag{2.6}$$

Moreover we impose in some sense the exterior magnetic field at infinity:

$$\mathbf{H}^{ext} = \mathbf{H}_0 \tag{2.7}$$

where $\mathbf{H}_0$ is the (constant) magnetic field that we would have if there was no superconducting material in the space.

REMARK 2.1 *Let us remark that the tangential trace of the interior magnetic field satisfies* $\mathbf{H}_T^{int} = \mathbf{H}_T^{ext}$ *and then is not arbitrar. From proposition 1.1,* $\mathbf{curl}\ \mathbf{H}^{int} \cdot n$ *only depends on* $\mathbf{H}_T^{int}$*, but on the other hand* $\mathbf{curl}\ \mathbf{H}^{ext} = 0$*, then* $\mathbf{curl}\ \mathbf{H}^{int} \cdot n = 0$*. It can also be seen from (2.2), (2.6) and the fact that* $\mathbf{H}^{int} = \mathbf{curl}\ \mathbf{Q}$*. In particular it means that the current* $\mathbf{J} = \mathbf{curl}\ \mathbf{H}$ *is tangential to* $\partial\Omega$*.*

Previous models

The case $\mathbf{H}_0 = H_0 e_3$ along the axis of the cylinder $\Omega = \omega \times \mathbf{R}$ with $\omega \subset \mathbf{R}^2$ was studied in [3] in the case $\kappa = +\infty$, and in [4] for finite κ. See also [1], [2].

A model in $\mathbf{R}^3$

We study the case where Ω is a bounded open set such that $\mathbf{R}^3 \backslash \Omega$ is connected, for $\kappa = +\infty$. For infinite κ, the boundary condition (2.5) has no sense, and then we do not consider it. In this case the first equation of (2.2) gives $f^2 = 1 - \mathbf{Q}^2$, and the second $-\mathbf{curl}\ \mathbf{H} = (1 - \mathbf{Q}^2)\mathbf{Q}$, where $\mathbf{curl}\ \mathbf{Q} = \mathbf{H}$. Now we get with the function F defined in (1.3): $\mathbf{Q} = -F(|\mathbf{curl}\ \mathbf{H}|^2)\mathbf{curl}\ \mathbf{H}$, and if we take the $\mathbf{curl}$, we get the new system:

$$\begin{cases} -\mathbf{curl}\ (F(|\mathbf{curl}\ \mathbf{H}|^2)\mathbf{curl}\ \mathbf{H}) - H = 0 \text{ in } \Omega \\ \mathbf{curl}\ \mathbf{H} = 0 \text{ in } \mathbf{R}^3 \backslash \overline{\Omega} \\ \mathbf{H} \text{ continuous on } \partial\Omega \\ \text{div } \mathbf{H} = 0 \text{ in } \mathbf{R}^3 \\ \mathbf{H} = \mathbf{H}_0 \text{ at infinity} \end{cases} \tag{2.8}$$

The model that we consider in this paper.

In fact we are only interested in the magnetic field in the superconductor, so we will consider that the tangential trace of the exterior magnetic field is given on the surface of the superconductor. Let us remark that such a model permits us to describe more general situations, for example the case of coexistence of many connected components of superconductor material, like grains. Then we consider the model:

$$\begin{cases} -\mathbf{curl}\ (F(|\mathbf{curl}\ \mathbf{H}|^2)\mathbf{curl}\ \mathbf{H}) - \mathbf{H} = 0 \text{ in } \Omega \\ \mathbf{H}_T = \mathbf{H}_T^{ext} \text{ on } \partial\Omega \end{cases} \tag{2.9}$$

where $\mathbf{H}_T^{ext}$ is chosen such that $\mathbf{curl}\ \mathbf{H}_T^{ext} \cdot n = 0$.

How can we obtain a more simple model?

Let us recall that the Ginzburg-Landau energy (2.1) is valuable for scale in space such that $\lambda = 1$. Now if we reintroduce the wave lenght of London λ, we get for Ω denoted by Ω_λ: $\Omega_1 = \lambda\Omega_\lambda$, $\tilde{x} \in \Omega_1$, $x \in \Omega_\lambda$, $\tilde{x} = \lambda x$, $\tilde{H}(\tilde{x}) = H(x)$:

$$\begin{cases} -\lambda^2\mathbf{curl}\ (F(\lambda^2|\mathbf{curl}\ \tilde{\mathbf{H}}|^2)\mathbf{curl}\ \tilde{\mathbf{H}}) - \tilde{\mathbf{H}} = 0 \text{ in } \Omega_1 \\ \tilde{\mathbf{H}}_T = \tilde{\mathbf{H}}_T^{ext} \text{ on } \partial\Omega_1 \end{cases} \tag{2.10}$$

At the limit $\lambda \to 0$, we get formally

$$\tilde{\mathbf{H}} = 0 \text{ on } \Omega_1$$

with some currents on $\partial\Omega$ (see subsection 4.2).

3 Proof of theorem 1.3.

3.1 Preliminaries

Proof of proposition 1.1

The i) is obvious. The fact that $\mathbf{curl}\ \mathbf{H}^{int} \cdot n$ only depends on $\mathbf{H}_T$ comes from

$$\int_{\partial\Omega} \phi\ \mathbf{curl}\ \mathbf{H} \cdot n = \int_\Omega \mathbf{curl}\ \mathbf{H} \cdot \nabla\phi = \int_{\partial\Omega} \mathbf{H} \wedge \nabla\phi \cdot n \tag{3.1}$$

where we have used div $\mathbf{curl}\ = 0$, $\mathbf{curl}\ \nabla = 0$. This proves the ii). Then proposition 1.1 is proved.

Now we define the primitive $G(s) = \int^s f$ for $s \in [0, \frac{4}{27} - \delta]$ for some arbitrarily small $\delta > 0$, which is extended on $\mathbf{R}^+$ as a C^∞-convex function, which is affine on $[\frac{4}{27}, +\infty[$.

In all what follows we consider the new system:

$$-\mathbf{curl}\ (G'(|\mathbf{curl}\ \mathbf{H}|^2)\mathbf{curl}\ \mathbf{H}) - \mathbf{H} = 0 \text{ on } \Omega \tag{3.2}$$

$$\mathbf{H}_T = \mathbf{H}_T^{ext} \text{ on } \partial\Omega \tag{3.3}$$

IMPORTANT REMARK

From now on we will prove results on the system (3.2)-(3.3), which will imply similar results for the system (1.1)-(1.2).

3.2 An approach by the inverse function theorem

We assume that $\partial\Omega \in C^{2+\alpha}$, let us define the spaces:

$$X_{0T,div}^{2+\alpha} = \{\mathbf{H} \in C^{2+\alpha}(\overline{\Omega}), \mathbf{H}_T = 0, \text{div } \mathbf{H} = 0\}$$

$$X_{div}^{m+\alpha} = \{\mathbf{H} \in C^{m+\alpha}(\overline{\Omega}), \text{div } \mathbf{H} = 0\},\ m = 0, 1, 2$$

We give us $\mathbf{H}_T^{ext} \in C^{2+\alpha}(\partial\Omega)$, such that $\mathbf{curl}\ \mathbf{H}_T^{ext} \cdot n = 0$.

LEMMA 3.1 *If $\mathbf{H}_T^{ext} \in C^{2+\alpha}(\partial\Omega)$, then there exists $\mathbf{H}^p \in C^{2+\alpha}(\overline{\Omega})$ such that div $\mathbf{H}^p = 0$ and $\mathbf{H}_T^p = \mathbf{H}_T^{ext}$.*

Proof of lemma 3.1

It is easy to construct a particular interior magnetic field $\mathbf{H}^p \in C^{2+\alpha}(\overline{\Omega})$ such that div $\mathbf{H}^p = 0$ and $\mathbf{H}_T^p = \mathbf{H}_T^{ext}$. To do this, take a field $\mathbf{H}^1 \in C^{2+\alpha}(\overline{\Omega})$, such that $\mathbf{H}_T^1 = \mathbf{H}_T^{ext}$. Then search the function $\phi \in C^{2+\alpha}(\overline{\Omega})$ such that $\Delta\phi = \text{div } \mathbf{H}^1$, $\phi = 1$ on $\partial\Omega$, and take $\mathbf{H}^p = \mathbf{H}^1 - \nabla\phi \in C^{1+\alpha}(\overline{\Omega})$. Then

$$\begin{cases} \Delta\mathbf{H}^p = -(\mathbf{curl}\ )^2\mathbf{H}^1 \in C^\alpha(\Omega) \\ \mathbf{H}_T^p = \mathbf{H}_T^{ext} \text{ in } C^{2+\alpha}(\partial\Omega) \\ \text{div } \mathbf{H}^p = 0 \text{ in } C^{1+\alpha}(\partial\Omega) \end{cases}$$

Then from the Schauder theory for elliptic system (see [9]) we get $\mathbf{H}^p \in C^{2+\alpha}(\overline{\Omega})$, which proves the lemma 3.1.

From lemma 3.1, there exists $\mathbf{H}^p \in X_{div}^{2+\alpha}$ such that $\mathbf{curl}\ \mathbf{H}^p \cdot n = 0$, and $\mathbf{H}_T^p = \mathbf{H}_T^{ext}$. Let

$$X = \mathbf{H}^p + X_{0T,div}^{2+\alpha}$$

and

$$\begin{array}{rccc} \mathcal{A}: & X & \longrightarrow & X_{div}^\alpha \\ & \mathbf{H} & \longmapsto & \mathcal{A}(\mathbf{H}) = -\mathbf{curl}\ (G'(|\mathbf{curl}\ \mathbf{H}|^2)\mathbf{curl}\ \mathbf{H}) - \mathbf{H} \end{array}$$

Then we have the

PROPOSITION 3.2 *For all $\mathbf{H}_T^{ext} \in C^{2+\alpha}(\partial\Omega)$, there exists a unique solution $\mathbf{H} \in X_{div}^{2+\alpha}$ solution of (3.2)-(3.3).*

This proposition proves in particular the i) of theorem 1.3.

Proof of proposition 3.2

Let us calulate $\mathcal{A}(\mathbf{H}) = G'(\mathbf{J}^2)(-(\mathbf{curl}\ )^2\mathbf{H}) - \nabla(G'(\mathbf{J}^2)) \wedge \mathbf{curl}\ \mathbf{H} - \mathbf{H}$, where $\mathbf{J} = \mathbf{curl}\ \mathbf{H}$ is the current. But $-(\mathbf{curl}\ )^2\mathbf{H} = \Delta\mathbf{H} - \nabla(\text{div } \mathbf{H})$ and $\text{div } \mathbf{H} = 0$, $\mathbf{J}_i = \epsilon_{ijk}\nabla_j\mathbf{H}_k$, and ϵ_{ijk} is the complelty antisymetric tensor such that $\epsilon_{123} = 1$. Then we get

$$(\mathcal{A}(\mathbf{H}))_i = A_{ijmn}(\mathbf{J}^2)D_{jm}^2\mathbf{H}_n - \mathbf{H}_i$$

with

$$A_{ijmn}(\mathbf{J}^2) = G'(\mathbf{J}^2)\delta_{jm}\delta_{in} - 2G''(\mathbf{J}^2)\mathbf{J}_k\epsilon_{kij}\mathbf{J}_l\epsilon_{lmn}$$

where we have used for the intermediate calculus: $(\nabla(G'(\mathbf{J}^2)) \wedge \mathbf{J})_i = \epsilon_{ijk}\nabla_j(G'(\mathbf{J}^2))\mathbf{J}_k$, $\nabla_j(G'(\mathbf{J}^2)) = 2G''\mathbf{J}_l\epsilon_{jmn}\nabla j\nabla_m\mathbf{H}_m$.

Now we can consider

$$\begin{array}{rccc} D_{\mathbf{H}}\mathcal{A}(\mathbf{H}): & X_{0T,div}^{2+\alpha} & \longrightarrow & X_{div}^{\alpha} \\ & \mathbf{h} & \longmapsto & D_{\mathbf{H}}\mathcal{A}(\mathbf{H})\cdot\mathbf{h} \end{array}$$

and

$$(D_{\mathbf{H}}\mathcal{A}(\mathbf{H})\cdot\mathbf{h})_i = A_{ijmn}(\mathbf{J}^2)D_{jm}^2\mathbf{h}_n + 2\frac{dA_{ilmn}}{d(\mathbf{J}^2)}(D_{jm}^2\mathbf{H}_n)\mathbf{J}_r\epsilon_{rpq}\nabla_p\mathbf{h}_q - \mathbf{h}_i \tag{3.4}$$

It is easy to calculate $det(L_{in}(x,\lambda))$ where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, and $L_{in}(x,\lambda) = \lambda_j\lambda_m A_{ijmn}(\mathbf{J}^2) = G'\lambda^2\delta_{in} + 2G''(\lambda\wedge\mathbf{J})_i(\lambda\wedge\mathbf{J})_n$. Thus, with $\mathbf{J} = \mathbf{J}(x)$:

$$det(L_{in}(x,\lambda)) = G'^2(\mathbf{J}^2)\lambda^4(G'(\mathbf{J}^2)\lambda^2 + 2G''(\mathbf{J}^2)|\lambda\wedge\mathbf{J}|^2) \neq 0 \text{ if } \lambda \neq 0$$

Then $D_{\mathbf{H}}\mathcal{A}(\mathbf{H})$ is elliptic on $X_{0T,div}^{2+\alpha}$, and we want to prove that it satisfies a Schauder estimate, which will be used later. To do this we first extend the operator $D_{\mathbf{H}}\mathcal{A}(\mathbf{H})$, in an elliptic operator $\mathcal{L}(\mathbf{H})$ which has the expression (3.4), and is defined from $X_{0T,0div}^{2+\alpha} = \{\mathbf{H} \in C^{2+\alpha}, \mathbf{H}_T = 0, (\text{div } \mathbf{H})_{|\partial\Omega} = 0\}$ into $X^\alpha = \{\mathbf{H} \in C^\alpha\}$. Now using the results in [9], it is easy to verify that the conditions $\mathbf{H}_T = 0$, $(\text{div } \mathbf{H})_{|\partial\Omega} = 0$ are complementar conditions, which in particular permit us to get the Schauder estimate:

$$|\mathbf{h}|_{X^{2+\alpha}} \leq C(\mathbf{H})\{|\mathcal{L}(\mathbf{H})\cdot\mathbf{h}|_{X^\alpha} + |\mathbf{h}_T|_{C^{2+\alpha}(\partial\Omega)} + |\text{div } \mathbf{h}|_{C^{1+\alpha}(\partial\Omega)} + |\mathbf{h}|_{L^1(\Omega)}\} \tag{3.5}$$

injectivity

Let us first prove that $Ker D_{\mathbf{H}}\mathcal{A}(\mathbf{H})=\{0\}$. To do this we remark that for the energy $\mathcal{E}(\mathbf{H})=\int_\Omega \frac{G(|\mathbf{curl\ H}|^2)}{2}+\frac{\mathbf{H}^2}{2}$, we have $\mathcal{E}'(\mathbf{H})\cdot\mathbf{h}=\int_\Omega G'(\mathbf{J}^2)\mathbf{curl\ H}\cdot\mathbf{curl\ h}+\mathbf{H}\cdot\mathbf{h}=\int_\Omega -\mathcal{A}(\mathbf{H})\cdot\mathbf{h}$, where we have used the general equality $\int_\Omega \mathbf{curl\ A}\cdot\mathbf{B}=\int_\Omega \mathbf{curl\ B}\cdot\mathbf{A}+\int_{\partial\Omega}(\mathbf{A}\wedge\mathbf{B})\cdot n$. Consequently

$$\mathcal{E}''(\mathbf{H})\cdot(\mathbf{h},\mathbf{h})=-\int_\Omega (D_{\mathbf{H}}\mathcal{A}(\mathbf{H})\cdot\mathbf{h})\cdot\mathbf{h} \tag{3.6}$$

But on another hand we calculate explicitly

$$\mathcal{E}''(\mathbf{H})\cdot(\mathbf{h},\mathbf{h})=\int_\Omega G'(\mathbf{J}^2)|\mathbf{curl\ h}|^2+2G''(\mathbf{J}^2)(\mathbf{curl\ H}\cdot\mathbf{curl\ h})^2+\mathbf{h}^2 \tag{3.7}$$

then $\mathbf{h}=0$ if $D_{\mathbf{H}}\mathcal{A}(\mathbf{H})\cdot\mathbf{h}=0$, which proves the injectivity of $D_{\mathbf{H}}\mathcal{A}(\mathbf{H})$.

surjectivity

One way to prove the surjectivity of $D_{\mathbf{H}}\mathcal{A}(\mathbf{H})$, is to prove it for a weak version using the Lax-Milgram theorem. We introduce the following symetric bilinear form defined on $Y^1_{0T,div}=\{\mathbf{H}\in\mathcal{H}^1(\Omega),\ \mathbf{H}_T=0,\ \text{div}\ \mathbf{H}=0\}$:

$$a(\mathbf{h},\mathbf{v})=\mathcal{E}''(\mathbf{H})\cdot(\mathbf{h},\mathbf{v}),\ \forall\mathbf{h},\mathbf{v}\in Y^1_{0T,div}$$

and we want to solve

$$a(\mathbf{h},\mathbf{v})=<-\mathbf{f},\mathbf{v}>,\ \forall\mathbf{v}\in Y^1_{0T,div}$$

where $\mathbf{f}\in(Y^1_{0T,div})'$.

Let us recall here the classical result (see [7] page 247): For Ω at least of class C^2, there exists a constant $C=C(\Omega)>0$ such that

$$\forall v\in\mathcal{H}^1(\Omega),|v|_{\mathcal{H}^1}\le C\{|v|_{L^2}+|\mathbf{curl}\ v|_{L^2}+|\nabla\cdot v|_{L^2}+|v_T|_{\mathcal{H}^{\frac{1}{2}}(\partial\Omega)}\} \tag{3.8}$$

From (3.7) and (3.8) we see that there exists a constant $C>0$ such that $a(\mathbf{h},\mathbf{h})\ge C|\mathbf{h}|_{\mathcal{H}^1(\Omega)}$, and then the Lax-Milgram theorem applies. In particular for all $\mathbf{f}\in X^\alpha_{div}\subset(Y^1_{0T,div})'$, we deduce from (3.6) that

$$\int_\Omega(\mathbf{curl}\ (\frac{d}{d\mathbf{H}}(G'(|\mathbf{curl\ H}|^2)\mathbf{curl\ H})\cdot\mathbf{h})+\mathbf{h}+\mathbf{f})\cdot\mathbf{v}=0$$

Now we use the following lemma (see theorem 3.4 in [8]):

LEMMA 3.3 *If $v\in L^2(\Omega)$, div $v=0$, then $\exists\phi\in\mathcal{H}^1(\Omega)$ such that $v=\mathbf{curl}\ \phi$.*

We apply this lemma to $\mathbf{h}$ and $\mathbf{f}$ such that $\exists \mathbf{k}, \mathbf{g} \in \mathcal{H}^1(\Omega)$ such that $\mathbf{h} = \mathbf{curl}\ \mathbf{k}$, $\mathbf{f} = \mathbf{curl}\ \mathbf{g}$. Then using the general equality $\int_\Omega \mathbf{v} \cdot \mathbf{curl}\ \mathbf{B} = \int_\Omega \mathbf{curl}\ \mathbf{v} \cdot \mathbf{B} + \int_{\partial\Omega} \mathbf{B} \cdot (\mathbf{v} \wedge n)$, and $\mathbf{v} \wedge n = 0$ on $\partial\Omega$ because $\mathbf{v} \in Y^1_{0T,div}$, we get

$$\int_\Omega \mathbf{curl}\ \mathbf{v} \cdot \mathbf{A} = 0$$

where $\mathbf{A} = \frac{d}{d\mathbf{H}}(G'(|\mathbf{curl}\ \mathbf{H}|^2)\mathbf{curl}\ \mathbf{H}) \cdot \mathbf{h} + \mathbf{k} + \mathbf{g} \in L^2(\Omega)$. But from the proof of proposition 4 page 260 in [7], we get $\mathbf{curl}\ Y^1_{0T,div} = \mathbf{curl}\ \{\mathbf{H} \in \mathcal{H}^1(\Omega), \mathbf{H}_T = 0\}$, and the orthogonal of this space for the scalar product in $L^2(\Omega)$ is the space $\{\mathbf{V} \in L^2(\Omega), \mathbf{curl}\ \mathbf{V} = 0\}$. Consequently, $\mathbf{curl}\ \mathbf{A} = 0$, which is the Euler-Lagrange equation : $D_\mathbf{H}\mathcal{A}(\mathbf{H}) \cdot \mathbf{h} = \mathbf{f}$. Then the Schauder estimate (3.5) applies and proves that $\mathbf{h} \in X^{2+\alpha}_{0T,div}$, which proves that $D_\mathbf{H}\mathcal{A}(\mathbf{H})$ is surjectif. Then $D_\mathbf{H}\mathcal{A}(\mathbf{H})$ is an isomorphism, and the inverse function theorem applies and proves the proposition 3.2.

From the results of Morrey [9], we deduce that $\mathbf{H} \in C^\infty(\Omega)$ and on $\{|\mathbf{curl}\ \mathbf{H}|^2 < \frac{4}{27} - \delta\}$, $\mathbf{H}$ is analytic.

3.3 Further results

We have the

PROPOSITION 3.4 *If $\mathbf{H}$ is solution of (3.2)-(3.3), then the current $\mathbf{J} = \mathbf{curl}\ \mathbf{H}$ is such that $\mathbf{J}^2$ is maximal on $\partial\Omega$.*

It implies the iii) of theorem 1.3.

Proof of proposition 3.4

We have $-\mathbf{curl}\ (G'(|\mathbf{curl}\ \mathbf{H}|^2)\mathbf{curl}\ \mathbf{H}) - \mathbf{H} = 0$ in Ω then taking the $\mathbf{curl}$ we get the equation satisfied by the current $\mathbf{J}$:

$$-(\mathbf{curl}\ )^2(G'(\mathbf{J}^2)\mathbf{J}) - \mathbf{J} = 0 \text{ in } \Omega \tag{3.9}$$

But $-(\mathbf{curl}\ )^2\mathbf{A} = \Delta\mathbf{A} - \nabla(\nabla \cdot \mathbf{A})$, it gives

$$\Delta(G'\mathbf{J}) - \nabla(G''\mathbf{J} \cdot \nabla(\mathbf{J}^2)) - \mathbf{J} = 0$$

where we have used div $\mathbf{J}=0$. The calcul of the first term gives with the notation $u=\mathbf{J}^2$: $\Delta(G'\mathbf{J}) = G'(u)\Delta\mathbf{J} + G''(u)(\Delta u)\mathbf{J} + b_1(\mathbf{J},\nabla\mathbf{J})\nabla u$. The second term gives $\nabla_i(G''\mathbf{J}\cdot\nabla(\mathbf{J}^2)) = G''(u)\mathbf{J}_j D^2_{ij}u + (b_2(\mathbf{J},\nabla\mathbf{J})\nabla u)_i = 0$. Then multiplying by $\mathbf{J}$ and using the equality $\Delta\frac{\mathbf{J}^2}{2} = \mathbf{J}\Delta\mathbf{J} + |\nabla\mathbf{J}|^2$, we get:

$$\frac{G'(u)}{2}\Delta u + G''(u)u\sum_{j\neq\frac{\mathbf{J}}{|\mathbf{J}|}} D^2_{jj}u + b(\mathbf{J},\nabla\mathbf{J})\nabla u - u = G'(u)|\nabla\mathbf{J}|^2 \geq 0$$

Then from the maximum principle we deduce that u can not be maximum inside Ω, except in the trivial case where $\mathbf{J}\equiv 0$. This proves the proposition.

Moreover, we have

PROPOSITION 3.5 *If $\mathbf{J}_T$ is the tangential trace of $\mathbf{J}$ on $\partial\Omega$, then there is at most one current $\mathbf{J} = \mathbf{curl}\ \mathbf{H}$ and one magnetic field $\mathbf{H}$ solution of (3.2) such that $(\mathbf{curl}\ \mathbf{H})_T = \mathbf{J}_T$.*

REMARK 3.6 *In fact we can prove this result by an inverse function theorem (as previously) applied to the equation*

$$\begin{cases} -(\mathbf{curl}\ )^2(G'(\mathbf{J}^2)\mathbf{J}) - \mathbf{J} = 0 \\ \mathbf{J}_T = \mathbf{J}_T^{ext} \end{cases} \tag{3.10}$$

We introduce the operator $\mathcal{B}(\mathbf{J}) = -(\mathbf{curl}\ )^2(G'(\mathbf{J}^2)\mathbf{J}) - \mathbf{J}$ from $\mathbf{J}^p + X^{2+\alpha}_{0T,div}$ into X^α_{div} where $\mathbf{J}^p \in X^{2+\alpha}_{div}$ et $\mathbf{J}^p_T = \mathbf{J}^{ext}_T$, div $\mathbf{J}^p = 0$. We can prove that $\mathcal{B}(\mathbf{J}) = \nabla_i(G'(\mathbf{J}^2)\nabla_i\mathbf{J}) - \mathbf{J}$ and that $D_{\mathbf{J}}\mathcal{B}(\mathbf{J})$ is inversible from $X^{2+\alpha}_{0T,div} \to X^\alpha_{div}$. Then there exists a unique solution $\mathbf{J}$ to (3.10). But $\mathbf{J}_T$ is not arbitrar, because it must be chosen such that moreover the solution $\mathbf{J}$ of (3.10) verifies $\mathbf{J}\cdot n = 0$ on $\partial\Omega$.

REMARK 3.7 *The current $\mathbf{J}$ is so not arbitrar that when the London wave length λ tends to 0, we get formally (see remark 4.1) particular currents on the surface $\partial\Omega$: these currents are of free divergence on the surface $\partial\Omega$.*

To build a current $\mathbf{J}_T$ such that (3.10) has no solutions, i.e. $\mathbf{J}\cdot n \not\equiv 0$, it is sufficient to build a magnetic field $\mathbf{H}_T \in C^{2+\alpha}(\partial\Omega)$ such that $\mathbf{curl}\ \mathbf{H}_T \cdot n \not\equiv 0$. Then the proposition 3.2 gives a $\mathbf{H} \in C^{2+\alpha}(\overline{\Omega})$ solution of (3.2)-(3.3). Then $\mathbf{J} = \mathbf{curl}\ \mathbf{H} \in C^{1+\alpha}(\overline{\Omega})$ is solution of (3.10) with $\mathbf{J} \cdot n \neq 0$. To finish we can mollify $\mathbf{J}_T$ such that $\mathbf{J}_T \in C^{2+\alpha}(\overline{\Omega})$ and we keep $\mathbf{J} \cdot n \not\equiv 0$.

For example we can take $\mathbf{H}_T = e_z \wedge n$, and from (3.1) we get with $\phi(x, y, z) = z$:

$$\int_{\partial\Omega} \phi \ \mathbf{curl} \ \mathbf{H} \cdot n = \int_{\partial\Omega} \mathbf{H}_T \cdot (\nabla \wedge n) = \int_{\partial\Omega} (e_z \wedge n)^2 > 0$$

then $\mathbf{curl} \ \mathbf{H}_T \cdot n \not\equiv 0$.

With the proposition 3.5, it proves the iv) of theorem 1.3.

Proof of proposition 3.5

An easy proof consists to write the difference of the equation (3.9) satisfied by two currents $\mathbf{J}^{(1)}$ and $\mathbf{J}^{(2)}$ and to multiply it by $\mathbf{v} = G'((\mathbf{J}^{(2)})^2)\mathbf{J}^{(2)} - G'((\mathbf{J}^{(1)})^2)\mathbf{J}^{(1)}$ and to integrate by part. The boundary term is nul because $\mathbf{J}_T^{(1)} = \mathbf{J}_T^{(2)}$, and we obtain:

$$\int_{\Omega} |\mathbf{curl} \ \mathbf{v}|^2 + (\mathbf{J}^{(2)} - \mathbf{J}^{(1)}) \cdot \mathbf{v} = 0$$

and by strict convexity of the map $\mathbf{J} \longmapsto G(\mathbf{J}^2)$, there exists $c = c(\delta) > 0$, such that $(\mathbf{J}^{(2)} - \mathbf{J}^{(1)}) \cdot \mathbf{v} \geq c(\mathbf{J}^{(2)} - \mathbf{J}^{(1)})^2$, which implies that $\mathbf{J}^{(2)} - \mathbf{J}^{(1)} = 0$, and proves the uniqueness of the current. Consequently the magnetic field is unique, because $\mathbf{H} = -\mathbf{curl} \ (G'(\mathbf{J}^2)\mathbf{J})$. This ends the proof.

We have

PROPOSITION 3.8 *There exists a constant $C > 0$, such that if $\mathbf{H}$ is a solution of (3.2), then $|\mathbf{H}_T|_{\mathcal{H}^{-\frac{1}{2}}(\partial\Omega)} \leq C|G'(\mathbf{J}_T^2)\mathbf{J}_T|_{\mathcal{H}^{\frac{1}{2}}(\partial\Omega)}$, with $\mathbf{J} = \mathbf{curl} \ \mathbf{H}$.*

Proof of proposition 3.8

Multiplying (3.9) by $G'(\mathbf{J}^2)\mathbf{J}$, and integrating by part, we obtain:

$$\int_{\Omega} \mathbf{H}^2 + \mathbf{J}^2 G'(\mathbf{J}^2) = \int_{\partial\Omega} G'(\mathbf{J}^2)(\mathbf{J} \wedge \mathbf{H}) \cdot n \tag{3.11}$$

Now, from the continuity of the trace map (see theorem 2 page 240 in [7]), we get:

$$|\mathbf{H}_T|^2_{\mathcal{H}^{-\frac{1}{2}}(\partial\Omega)} \leq C \int_{\Omega} \mathbf{H}^2 + |\mathbf{curl} \ \mathbf{H}|^2$$

But **curl H** $=$ **J**, then:

$$\begin{array}{rcl} |\mathbf{H}_T|^2_{\mathcal{H}^{-\frac{1}{2}}(\partial\Omega)} & \le & C\int_\Omega \mathbf{H}^2 + \mathbf{J}^2 \\ & \le & C|\mathbf{H}_T|_{\mathcal{H}^{-\frac{1}{2}}(\partial\Omega)} |G'(\mathbf{J}_T^2)\mathbf{J}_T|_{\mathcal{H}^{\frac{1}{2}}(\partial\Omega)} \end{array}$$

where we have used equality (3.11), and the fact that there exists $C_0 > 0$, such that $C_0 \le G'(\mathbf{J}^2) \le \frac{1}{C_0}$. Then we deduce the result.

Proof of iv) of theorem 1.3

Proposition 3.8 implies the inequality (1.4).
From lemma 3.1, for a given $\mathbf{g} \in C^{2+\alpha}(\partial\Omega)$, $\exists \mathbf{H}^p \in C^{2+\alpha}(\overline{\Omega})$ such that div $\mathbf{H}^p = 0$, $\mathbf{H}^p_T = \mathbf{g}$. Now we can search to minimize

$$\min_{\mathbf{H}\in Y^1_{0T,div}} E(\mathbf{H}+\mathbf{H}^p)$$

where $E(\mathbf{H}) = \int_\Omega \mathbf{H}^2 + |\textbf{curl } \mathbf{H}|^2$ and $Y^1_{0T,div} = \{\mathbf{H} \in \mathcal{H}^1(\Omega),\ \text{div } \mathbf{H} = 0,\ \mathbf{H}_T = 0\}$. Now E is continuous, strictly convex and infinite at infinity on $Y^1_{0T,div}$, then there exists a unique minimizer $\mathbf{H}^* \in Y^1_{0T,div}$. Moreover $\mathbf{H}^* + \mathbf{H}^p \not\equiv 0$ (because $(\mathbf{H}^* + \mathbf{H}^p)_T \not\equiv 0$ and $\mathbf{H}^* + \mathbf{H}^p \in \mathcal{H}^1(\Omega)$), then $E(\mathbf{H}^* + \mathbf{H}^p) > 0$. Now for $\mathbf{H}_T = \mu\mathbf{g}$ for some $\mu > 0$, if there exists a solution $\mathbf{H}$ to (1.1)-(1.2), then

$$0 < \mu^2 E(\mathbf{H}^* + \mathbf{H}^p) \le \int_\Omega \mathbf{H}^2 + \mathbf{J}^2 G'(\mathbf{J}^2) = \int_{\partial\Omega} F(\mathbf{J}_T^2)(\mathbf{J}_T \wedge \mathbf{H}_T)\cdot n \le \mu C|\mathbf{g}|_{L^1(\partial\Omega)}$$

Then for μ large enough there is no solution $\mathbf{H}$ to (1.1)-(1.2).

4 The case of a torus: an heuristical approach.

In this section we consider the particular case of a torus supercondutor Ω. Let us note the cylindrical basis $e_r = (\cos\phi, \sin\phi, 0)$, $e_\phi = (-\sin\phi, \cos\phi, 0)$ and $e_z = (0,0,1)$, and $\omega \subset \mathbf{R}^2$ a bounded smooth open set such that $\sup_{x\in\overline{\omega}} |x| < R$, which will be the section of the torus. Then we define

$$\Omega = \{M = (R+\rho\cos\theta)e_r + \rho\sin\theta e_z, \phi \in [0,2\pi[, (\rho\cos\theta, \rho\sin\theta) \in \omega\}$$

4.1 A particular submodel

In particular we define

$$\begin{cases} e_\rho = \cos\theta e_r + \sin\theta e_z \\ e_\theta = -\sin\theta e_r + \cos\theta e_z \end{cases}$$

If $\mathbf{H} = \mathbf{H}_\rho e_\rho + \mathbf{H}_\phi e_\phi + \mathbf{H}_\theta e_\theta$, then

$$\textbf{curl } \mathbf{H} = \begin{pmatrix} \frac{1}{r}\partial_\phi \mathbf{H}_\theta - \frac{1}{\rho}\partial_\theta \mathbf{H}_\phi + \frac{\sin\theta}{r}\mathbf{H}_\phi, \\ \frac{1}{\rho}\partial_\theta \mathbf{H}_\rho - \partial_\rho \mathbf{H}_\theta - \frac{1}{\rho}\mathbf{H}_\theta \\ \partial_\rho \mathbf{H}_\phi - \frac{1}{r}\partial_\phi \mathbf{H}_\rho + \frac{\cos\theta}{r}\mathbf{H}_\phi \end{pmatrix}$$

in the direct basis $(e_\rho, e_\phi, e_\theta)$. In particular the metric is $(dM)^2 = (d\rho)^2 + (\rho d\theta)^2 + (r d\phi)^2$. We are firstly interested in the solutions which take the form $\mathbf{H} = f(\rho,\theta)e_\phi$. If we note $(x,z) = (\rho\cos\theta, \rho\sin\theta)$, then we find $\textbf{curl } \mathbf{H} = (\frac{f}{R+x} + \partial_x f)e_z - \partial_z f e_\rho$. Then for $\mathcal{E}(\mathbf{H}) = \int_\Omega \frac{G(|\textbf{curl } \mathbf{H}|^2)}{2} + \frac{\mathbf{H}^2}{2}$, we get:

$$\mathcal{E}(\mathbf{H}) = 2\pi \int_\omega dxdz(R+x)\{\frac{f^2}{2} + \frac{1}{2}G((\partial_z f)^2 + (\frac{f}{R+x} + \partial_x f)^2)\}$$

It is easy to verify that the condition $\textbf{curl } \mathbf{H} \cdot n = 0$ is satisfied if we take the boundary condition $f = \frac{Const}{r} = \frac{Const}{R+x}$. In particular if we now introduce the new function $g = (R+x)f$, and minimizing the energy on g we find:

$$\begin{cases} \text{div } (G'((\frac{\nabla g}{R+x})^2)\frac{\nabla g}{R+x}) - \frac{g}{R+x} = 0 \text{ in } \omega \\ 0 \le g \le g_{|\partial\omega} = Const \end{cases}$$

In particular, as $R \to +\infty$, we find formally the model studied in [3].

4.2 A model without vortices, but with a phase parameter.

We recall that if $\mathbf{H} = \mathbf{H}_r e_r + \mathbf{H}_\phi e_\phi + \mathbf{H}_z e_z$, then

$$\textbf{curl } \mathbf{H} = \begin{pmatrix} \frac{1}{r}\partial_\phi \mathbf{H}_z - \partial_z \mathbf{H}_\phi, \\ \partial_z \mathbf{H}_r - \partial_r \mathbf{H}_z, \\ \frac{1}{r}\partial_r(r\mathbf{H}_\phi) - \frac{1}{r}\partial_\phi \mathbf{H}_r \end{pmatrix}$$

in the direct basis (e_r, e_ϕ, e_z). Let us recall that the Ginzburg-Landau energy is:

$$\mathcal{E}_\kappa(\mathbf{A},\psi) = \int_\Omega |(\frac{1}{\kappa}\nabla - i\mathbf{A})\psi|^2 + \frac{1}{2}(|\psi|^2-1)^2 + |\textbf{curl } \mathbf{A}|^2 \tag{4.1}$$

After a gauge transformation, we can assume that for some $N \in \mathbf{Z}$:

$$\begin{cases} \psi = e^{iN\phi} f, \;\; f > 0 \\ \mathbf{A} = \mathbf{Q} \text{ is real} \end{cases}$$

Then

$$\mathcal{E}_\kappa(\mathbf{A}, \psi) = \mathcal{E}_\kappa(N, f, \mathbf{Q}) = \int_\Omega \frac{|\nabla f|}{\kappa^2} + (\frac{N}{\kappa} e_\phi - \mathbf{Q})^2 f^2 + \frac{1}{2}(f^2 - 1)^2 + |\mathbf{curl}\ \mathbf{Q}|^2$$

In particular for $\kappa = +\infty$, we obtain formally for some $C \in \mathbf{R}$:

$$\mathcal{E}_\infty(C, f, \mathbf{Q}) = \int_\Omega (Ce_\phi - \mathbf{Q})^2 f^2 + \frac{1}{2}(f^2 - 1)^2 + |\mathbf{curl}\ \mathbf{Q}|^2$$

and we obtain the Euler-Lagrange equations:

$$\begin{cases} (Ce_\phi - \mathbf{Q})^2 + f^2 = 1 \\ (\mathbf{curl}\ )^2 \mathbf{Q} + (\mathbf{Q} - Ce_\phi) f^2 = 0 \end{cases}$$

For $\mathbf{H} = \mathbf{curl}\ \mathbf{Q}$, $\mathbf{B} = \mathbf{Q} - Ce_\phi$, then as long as $\mathbf{B}^2 < \frac{1}{3}$, or equivalently $|\mathbf{curl}\ \mathbf{H}|^2 < \frac{4}{27}$, then we get

$$\begin{cases} -\mathbf{curl}\ (F(|\mathbf{curl}\ \mathbf{H}|^2)\mathbf{curl}\ \mathbf{H}) - \mathbf{H} = -\frac{C}{r} e_z \text{ in } \Omega \\ \mathbf{H}_T = \mathbf{H}_T^{ext} \text{ on } \partial\Omega \end{cases}$$

Let us recall that the Ginzburg-Landau energy (4.1) is valuable for scale in space such that $\lambda = 1$. Now if we reintroduce the wave lenght of London λ, we get for Ω noted Ω_λ: $\Omega_1 = \lambda\Omega_\lambda$, $\tilde{x} \in \Omega_1$, $x \in \Omega_\lambda$, $\tilde{x} = \lambda x$, $\tilde{H}(\tilde{x}) = H(x)$, $C = C_\lambda$:

$$\begin{cases} -\lambda^2 \mathbf{curl}\ (F(\lambda^2 |\mathbf{curl}\ \tilde{\mathbf{H}}|^2)\mathbf{curl}\ \tilde{\mathbf{H}}) - \tilde{\mathbf{H}} = -\frac{\lambda C_\lambda}{\tilde{r}} e_z \text{ in } \Omega_1 \\ \tilde{\mathbf{H}}_T = \tilde{\mathbf{H}}_T^{ext} \text{ on } \partial\Omega_1 \end{cases} \tag{4.2}$$

where $\tilde{r} = \sqrt{\tilde{x}_1^2 + \tilde{x}_2^2}$. If we take

$$C_\lambda = \frac{C_1}{\lambda}$$

at the limit $\lambda \to 0$, we get formally

$$\tilde{\mathbf{H}} = \frac{C_1}{\tilde{r}} e_z \text{ on } \Omega_1$$

and near $\tilde{x}^* \in \partial\Omega_1$, the problem is close to (in other good coordinates)

$$\begin{cases} -\mathbf{curl}\ (F(|\mathbf{curl}\ \mathbf{H}|^2)\mathbf{curl}\ \mathbf{H}) - \mathbf{H} = \mathbf{V} = const \text{ in } \Pi \\ \mathbf{H}_T = H_0 e_3 \text{ on } \partial\Pi \end{cases}$$

where $\Pi = \{x_1 > 0\}$ with $H_0 = |\mathbf{H}_T(\tilde{x}^*)|$ and $\mathbf{V}$ is related to $\frac{C_1}{\tilde{r}(\tilde{x}^*)} e_z$. Now we can assume by symetry that $\mathbf{H} = \mathbf{H}(x_1)$, then $\mathbf{H}_1(x_1) = -V_1 = const$, and for $\overline{\mathbf{H}} = \mathbf{H} - \mathbf{V}$, we get:

$$\begin{cases} -\mathbf{curl}\ (F(|\mathbf{curl}\ \overline{\mathbf{H}}|^2)\mathbf{curl}\ \overline{\mathbf{H}}) - \overline{\mathbf{H}} = 0 \text{ in } \Pi \\ \overline{\mathbf{H}}_T = H_0 e_3 - \mathbf{V}_T \text{ on } \partial\Pi \end{cases}$$

and then $\overline{\mathbf{H}} = \phi(x_1)\overline{\mathbf{H}}_T$ on Π where $\phi(x_1)$ verifies an ordinary differential equation. It is possible to show that (see [6]), that $\phi(x_1)|\overline{\mathbf{H}}_T| = \sqrt{2}\frac{\sinh(x+a)}{\cosh^2(x+a)}$ for some $a \in \mathbf{R}$. In particular we can see that $|\mathbf{curl}\ \overline{\mathbf{H}}|^2 < \frac{4}{27}$ if and only if $|\overline{\mathbf{H}}_T| < \sqrt{\frac{5}{18}}$. Consequently we deduce that as $\lambda \to 0$, the limit $\tilde{\mathbf{H}}$ verifies:

$$\begin{cases} \tilde{\mathbf{H}} = \frac{C_1}{\tilde{r}} e_z \text{ on } \Omega_1 \\ |\tilde{\mathbf{H}}_T - \frac{C_1}{\tilde{r}}(e_z)_T| \leq \sqrt{\frac{5}{18}} \end{cases} \tag{4.3}$$

And outside Ω_1 we have the Maxwell equations

$$\begin{cases} \text{div } \tilde{\mathbf{H}} = 0 \text{ on } \Omega_1^c \\ \mathbf{curl}\ \tilde{\mathbf{H}} = 0 \text{ on } \Omega_1^c \end{cases} \tag{4.4}$$

and on $\partial\Omega_1$, $\tilde{\mathbf{H}}_T$ is discontinuous (because $\lambda = 0$), and

$$[\mathbf{H} \cdot n]_{int}^{ext} = 0 \text{ on } \partial\Omega_1 \tag{4.5}$$

Conclusion

This system (4.3)-(4.5) should permit to calculate the largest admissible constant C_1 which can be seen as the capacity of storage of magnetic flux in the superconducting annulus Ω.

REMARK 4.1 *The current $\mathbf{J}_T$ are so not arbitrar, that at the limit $\lambda \to 0$, we get a current $\mathbf{J} = \mathbf{J}_T$ on the surface $\partial\Omega$ of the superconductor. Moreover these currents are formally with free divergence on the surface $\partial\Omega$: there is no sources nor wells.*
In fact for $\lambda = 1$, we have div $\mathbf{J}_{|\partial\Omega} = a_\beta(s)\partial_{s_\beta} J_\beta + b_\beta(s) J_\beta + a_n(s)\partial_n J_n + b_n(s) J_n$, where $s = (s_1, s_2)$ parametrizes $\partial\Omega$ and for $\beta = 1, 2$, $a_\beta, b_\beta, a_n, b_n$ are some coefficients and J_β, J_n are the components of $\mathbf{J}$ such that $\mathbf{J} = J_\beta \partial_{s_\beta} + J_n n$ where n is the exterior normal to Ω. For $\lambda > 0$ we have $J_n = 0$ on $\partial\Omega$, and at the limit formally the term $\partial_n J_n$ disappears because we only have currents on the surface $\partial\Omega$. It means that formally for $\lambda = 0$, the currents are with free divergence on the surface $\partial\Omega$.

5 Appendix: Direct study by a variational formulation

We introduce a variational formulation for the model (3.2)-(3.3). This approach has the advantage to give us directly a weak solution, but the disadvantage that we have a very poor information on the solution.

It is natural to introduce the following energy

$$\mathcal{E}(\mathbf{H}) = \int_\Omega \frac{G(|\mathbf{curl}\ \mathbf{H}|^2)}{2} + \frac{\mathbf{H}^2}{2}$$

which is defined on the space

$$Y = \{\mathbf{H} \in \mathcal{H}^1(\Omega), \mathrm{div}\ \mathbf{H} = 0, \mathbf{H}_T = \mathbf{H}_T^{ext}\}$$

where $\mathbf{H}_T^{ext} \in \mathcal{H}^{\frac{1}{2}}(\partial\Omega)$ is the given boundary condition, such that

$$\mathbf{curl}\ \mathbf{H}_T^{ext} \cdot n = 0 \tag{5.1}$$

Let us remark that from lemma 2.2 in [8], there exists a (not unique) particular vector field $\mathbf{H}^p \in \mathcal{H}^1(\Omega)$, such that div $\mathbf{H}^p = 0$ and $\mathbf{H}_T^p = \mathbf{H}_T^{ext}$ on $\partial\Omega$. Then $Y \neq \emptyset$, and $Y = \mathbf{H}^p + Y^1_{0T,div}$, where $Y^1_{0T,div} = \{\mathbf{H} \in \mathcal{H}^1(\Omega), \mathbf{H}_T = 0, \mathrm{div}\ \mathbf{H} = 0\}$. In particular (see the proof of proposition 1.1) $\forall \mathbf{H} \in Y, \mathbf{curl}\ \mathbf{H} \cdot n = 0$ on $\partial\Omega$. In particular we expect that the minimizer verifies:

$$\begin{cases} -\mathbf{curl}\ (G'(|\mathbf{curl}\ \mathbf{H}|^2)\mathbf{curl}\ \mathbf{H}) - \mathbf{H} = 0 \text{ on } \Omega \\ \mathbf{H}_T = \mathbf{H}_T^{ext} \text{ on } \partial\Omega \end{cases} \tag{5.2}$$

Then we have the

THEOREM 5.1 *There exists a unique minimmizer* $\mathbf{H}$ *of* $\mathcal{E}$ *on* Y*. Moreover* $\mathbf{H}$ *is solution of (5.2).*

Proof of theorem 5.1

The existence and uniqueness of the solution comes from the fact that $\mathcal{E}$ is continuous, striclty convex and infinite at infinity on the space Y, because of the inequality (3.8). On the contrary the Euler-Lagrange equation (5.2) is not immediate. But with the same argument as in the proof of proposition 3.2, we can justify this Euler-Lagrange equation. This ends the proof of the theorem 5.1.

Let us remark that if **curl H** is continuous on Ω, then the results of Morrey (see [9]) permit us to deduce that the magnetic field **H** is C^∞ in Ω, and even analytic where $|\mathbf{curl}\ \mathbf{H}|^2 < \frac{4}{27} - \delta$. Nevertheless this variational formulation seems not so easy to use to deduce some qualitative properties of the solutions, because we have not found in the literature (except possibly some adapted Nirenberg translations (see [5])) some results which permit us directly to deduce that **curl H** is continuous, and then that **H** is a classical solution if $\mathbf{H}_T^{ext}$ is smooth enough. That is why we have used the inverse function theorem in section 3.2.

References

[1] A. Aftalion, *On the Minimizers of the Ginzburg-Landau Energy for High Kappa: the One-dimensional Case*, preprint.

[2] A. Aftalion, *On the Minimizers of the Ginzburg-Landau Energy for High Kappa: the Axially Symetric Case*, preprint.

[3] H. Berestycki, A. Bonnet and J. Chapman, *semilinear elliptic equation arising in a type-II superconductor for infinite κ.*

[4] A. Bonnet, J. Chapman and R. Monneau, *Convergence of the Meissner minimizers of the Ginzburg-Landau energy as $\kappa \to +\infty$.*

[5] H. Brezis, *Analyse fonctionnelle.*

[6] S.J. Chapman, *Superheating Field of type-II superconductors*, SIAM J. Appl. Math., to appear (1994).

[7] R. Dautray and J.-L. Lions, *Analyse mathématique et calcul numérique pour les sciences et les techniques*, Tome 2.

[8] V. Girault and P.-A. Raviard, *Finite Element Methods for Navier-Stokes Equations*, (1986), Springer-Verlag.

[9] Morrey, *Multiple Integrals in the Calculus of Variations*, Spronger-Verlag, 1966.