On the mushy region arising between two fluids in a porous medium

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Abstract

We study the mushy region arising between two fluids in a porous medium. We prove that the interior of the mushy region is an epigraph in the horizontal direction. Moreover when the interior of the mushy region is empty, we give a necessary and sufficient condition to claim that the Lebesgue measure of the mushy region is zero.

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1 Introduction

1.1 The physical problem

We study here the interface between two fluids for a steady flow in a porous medium.

One fluid

Let us recall that the steady flow of one fluid in a porous medium, is characterized by its own pressure $p \geq 0$ ($p = 0$ outside the fluid), and its velocity which is brought about by the Darcy law:

$$v = -k\nabla\phi$$

Here k is a permeability coefficient which depends on the porous medium and is assumed constant. The potential ϕ is given by $\phi = p + \rho g y$ (ρ is the volumic mass of the fluid, g the gravity, y the vertical axis upward oriented); if the soil is given by $\{y < 0\}$, then ϕ measures the difference of p to the hydrostatic pressure $p_0 = -\rho g y$. The coordinate x will denote the horizontal axis, and we will work in two dimensions $X = (x, y) \in \mathbf{R}^2$. Moreover we assume that the fluid is incompressible:

$$\text{div } v = 0$$

The research of the free surface of this monofluid can be then reduced (see [2]) to the equation

$$\Delta p = -\lambda \partial_y(\chi(p > 0)) \tag{1.1}$$

where $\chi(p) = \begin{cases} 1 \text{ if } p > 0 \\ 0 \text{ if } p = 0 \end{cases}$, $\lambda = \rho g$. We should add some boundary conditions.

Two fluids

If now we study in a porous medium two unmiscible fluids of density ρ_1 and ρ_2, we can give a formulation of the problem using the stream function $\psi \in \mathbf{R}$ in place of the potential ϕ, and defined by curl $\psi = \nabla \phi$, where curl $\psi = \begin{pmatrix} -\partial_y \psi \\ \partial_x \psi \end{pmatrix}$. Now a stream line is given by $\{\psi = const\}$. In particular the interface Γ between these two fluids is a stream line and (up to an additive constant) we can normalize ψ such that $\Gamma = \{\psi = 0\}$. Then the problem can be reduced (see [5]) to the equation:

$$\Delta \psi = -\lambda \partial_x(\chi(\psi > 0)) \tag{1.2}$$

where $\chi(\psi) = \begin{cases} 1 \text{ if } \psi > 0 \\ 0 \text{ if } \psi < 0 \end{cases}$, and $\lambda = (\rho_2 - \rho_1) g > 0$.

REMARK 1.1 *In particular for a uniform flow $v = v_0 e_x$ with $v_0 \in \mathbf{R}$ and for a horizontal interface $\Gamma = \{y = 0\}$, we get $\psi(x, y) = \frac{v_0}{k} y$. We see that the identification of the regions $\{\psi > 0\}$ and $\{\psi < 0\}$ to each one of these fluids depends on the sign of v_0.*

REMARK 1.2 *If $\psi_i; i = 1, 2$ is the restriction of ψ on the domain of density ρ_i, we have on the free boundary (and whatever are the relative position of the fluids 1 and 2):*

$$\frac{\partial \psi_1}{\partial n} - \frac{\partial \psi_2}{\partial n} + g(\rho_2 - \rho_1) < n, e_x >= 0 \text{ for } n = n_{2 \to 1} \tag{1.3}$$

It is equivalent to take $n = -n_{2\to 1}$. *When* $\rho_2 > \rho_1$, *then the fluid 1 is above the fluid 2 in a physical situation. But equations (1.2) and (1.3) continue to have an interpretation when the fluid 2 is above the fluid 1, although it does not correspond to a stable physical situation. In particular if* ψ *is a solution of (1.2) with* $\lambda = (\rho_2 - \rho_1)g > 0$, *and if* $\psi > 0$ *on one side of the free boundary* Γ, *and* $\psi < 0$ *on the other side, it can be asked where are the fluid 1 and the fluid 2? The answer is that we do not know. If one region, say* $\{\psi > 0\}$ *is always above the other region* $\{\psi < 0\}$, *then it would seem natural to say that the lighter fluid (the fluid 1) is in the region* $\{\psi > 0\}$, *and in this case the solution of the model (1.2) would describe a physical situation.*

But what can be said from a mathematical point of view? Mathematically in one case if we take ψ_1 *as the restriction of* ψ *on the region* $\{\psi > 0\}$ *and* ψ_2 *the restriction of* ψ *on the region* $\{\psi < 0\}$, *from (1.2) we can deduce the equality (1.3) on the free boundary* $\Gamma = \{\psi = 0\}$. *In another case it is easy to check that the other function* $\phi = -\psi$ *is also solution of (1.2), then if we take* ϕ_1 *as the restriction of* ϕ *on the region* $\{\phi > 0\}$ *and* ϕ_2 *the restriction of* ϕ *on the region* $\{\phi < 0\}$, *we deduce the equality (1.3) on the free boundary* $\Gamma = \{\phi = 0\}$ *with* ϕ_i *in place of* ψ_i. *Therefore it can be mathematically seen that we can chose the fluid 1 in* $\{\psi > 0\}$ *or in* $\{\psi < 0\}$, *i.e. the mathematical model (1.2) does not show in which region the lighter fluid is.*

REMARK 1.3 *In particular if* (ψ, γ) *is a solution of (1.2), then* $(\tilde{\psi}, \tilde{\gamma})$ *is also a solution with* $\tilde{\psi}(x, y) = \psi(x, -y), \tilde{\gamma}(x, y) = \gamma(x, -y)$. *It exchanges the relative position of the two fluids relatively to the gravity.*

REMARK 1.4 *One condition to derive the model (1.2) was that* $\{\psi > 0\}$ *and* $\{\psi < 0\}$ *are two connected components. In particular every solution of (1.2) with more than two connected components should be interpretated carefully.*

REMARK 1.5 *Let us note that we expect that* Γ *is a curve and then* $\mathcal{H}^2(\Gamma) = 0$. *In these case it is not necessary to precise the value of* $\chi(\psi)$ *on* Γ. *But up to our knowledge there is no general existence result of solutions with* $\mathcal{H}^2(\Gamma) = 0$. *The only known way to get a solution is to take the limit of solutions* u_ϵ *of the equation (1.2) with a smooth function* χ_ϵ *in place of* χ. *As* $\epsilon \to 0$, $\chi_\epsilon \to \chi$ *and* $\psi_\epsilon \to \psi$ *where* ψ *is a weak solution to (1.2). In* *particular* $\Gamma = \{\psi = 0\}$ *could be degenerated with* $\mathcal{H}^2(\Gamma) > 0$*, and* $\chi(\psi)_{|\Gamma}$ *could take every value between* 0 *and* 1*. In this case the values of* $\chi(\psi)_{|\Gamma}$ *would be important to claim that* ψ *is a solution of equation (1.2).*

1.2 The mathematical formulation

From now on, let us use the notations

$$\begin{cases} u = \psi \\ \gamma = \chi(\psi) \end{cases}$$

Then the weak solutions to (1.2) over an open set $\Omega \subset \mathbf{R}^2$ are given by the following variational formulation: search $(u, \gamma) \in H^1_{loc}(\Omega) \times L^\infty(\Omega)$ such that

$$\forall v \in C_0^\infty(\Omega), \ \int_\Omega \nabla u \nabla v + \lambda \gamma \partial_x v = 0, \text{ and } \gamma \in H(u) \tag{1.4}$$

where

$$H(u) = \begin{cases} \{1\} \text{ if } u > 0 \\ [0,1] \text{ if } u = 0 \\ \{0\} \text{ if } u < 0 \end{cases} \tag{1.5}$$

and

$$u = u_0 \text{ on } \partial\Omega \tag{1.6}$$

The existence of a solution (u, γ) to (1.4)-(1.6) is known under certain assumptions on $\partial\Omega$ and on u_0 (see [7], [14]).
Moreover let us recall:

PROPOSITION 1.6 *Every solution* (u, γ) *of (1.4) belongs to* $C^{0,1}_{loc}(\Omega) \times L^\infty(\Omega)$*.*

Proof of proposition 1.6
See [5] p 631, [12] p 52-53.

In this paper we are interested in getting information on the free boundary $\Gamma = \{u = 0\}$ and to know whether and when there exists a mushy region with $\mathcal{H}^2(\Gamma) > 0$. The nonexistence of a mushy region is intimely related to the question of the uniqueness of the solutions (u, γ) to (1.4)-(1.6) , as it is shown in [7] in the particular case of a strip $\Omega = \mathbf{R} \times (0, 1)$. In particular we study here the properties of Γ without assuming that the function u is monoton as in [5], or satisfies a property at $+\infty$ as in [7]. See also [13], [14]. Here we study the free boundary of the solution in the general case.

1.3 Main results

Let us recall that in the region $\{u=0\}$, γ can have any value between 0 and 1, which permits us to interprete γ as a coefficient of mixing of the two fluids. It justifies the terminology of mushy region (when $\mathcal{H}^2(\Gamma)>0$), that is sometimes given to the region $\{u=0\}$ for the Stephan problem (see [17], [18], [19], [20]).

DEFINITION 1.7 *Let $\omega\subset\mathbf{R}^2$ an open set convex in the e_x direction, i.e. $[(x,y),(x',y')]\subset\omega$ while $(x,y),(x',y)\in\omega$. Then we say that a set $A\subset\omega$ is a epigraph on ω in the e_x direction if $[(x,y),(x',y')]\subset A$ while $(x,y)\in A$ and $[(x,y),(x',y')]\subset\omega$, $x<x'$.*

We prove the

THEOREM 1.8 *If (u,γ) is a solution of (1.4) on an open set $\Omega\subset\mathbf{R}^2$, then for all open set ω convex in the e_x direction, $\omega\cap\{u=0\}^0$ is an epigraph on ω in the e_x direction.*

REMARK 1.9 *Shoshana Kamin has noticed that a similar result is true for the Stephan problem: the mushy region of a one-dimensionnal Stephan problem for $(x,t)\in\mathbf{R}\times\mathbf{R}$ can diseappear in finite time. We find the analogy with the change $(x,t)\to(y,-x)$.*

REMARK 1.10 *The function γ can be nonmonoton in y on a connected component of $\{u=0\}^0$ (see the counter-example of section 5).*

Moreover we prove

THEOREM 1.11 *If (u,γ) is a solution of (1.4) and if $\{u=0\}^0=\emptyset$, $\partial\{u>0\}\backslash\partial\{u<0\}=\emptyset$, $\partial\{u<0\}\backslash\partial\{u>0\}=\emptyset$, then $\mathcal{H}^2(\{u=0\})=0$.*

REMARK 1.12 *If $\mathcal{H}^2(\Gamma)=0$ and $\{u=0\}^0=\emptyset$ then $\Delta u=-\lambda\partial_x\gamma=0$ on $\{u\geq 0\}^0\cup\{u\leq 0\}^0$ and from maximum principle we deduce that $\partial\{u>0\}\backslash\partial\{u<0\}=\emptyset$, $\partial\{u<0\}\backslash\partial\{u>0\}=\emptyset$.*

REMARK 1.13 *We do not know if under general conditions there is uniqueness and/or even existence of a solution without a mushy region for problem (1.2)-(1.6).*

2 Preliminaries

The following proposition is obvious but usefull:

PROPOSITION 2.1 *If (u, γ) is a solution to (1.4), then $(-u, 1-\gamma)$ is a solution too.*

LEMMA 2.2 *(linear behaviour lemma, [9], [6]) Let $\Omega_1 \subset \mathbf{R}^n$ (respectively $\Omega_2 \subset \mathbf{R}^n$) such that there exists a ball B with*

$$B = B_r(re_n) \text{ and } B \subset \Omega_1$$

$$(\text{ respectively } B = B_r(-re_n) \text{ and } B \subset (\Omega_2)^c$$

Assume that u is a Lipschitz positive hamonic function in Ω_1 (respectively Ω_2) vanishing in $\partial\Omega_1$ (respectively $\partial\Omega_2$) and assume that $\partial\Omega_i \cap B = \{0\}$. Then near zero, u has the asymtotic development

$$u(X) = \alpha x_n + o(|X|) \text{ on } \Omega_i \text{ with } \alpha \geq 0$$

Furthermore $\alpha > 0$ in case Ω_1, because of Hopf lemma.

PROPOSITION 2.3 *If (u, γ) is a solution of (1.4) and if $u(X) = \alpha < X - X_0, \nu >^+$ $-\beta < X - X_0, \nu >^- + o(|X - X_0|)$ with $\nu \in \mathbf{S}^1$ and $\alpha, \beta \in \mathbf{R}$, then there exists two functions $0 \leq \gamma^0_\alpha(y), \gamma^0_\beta(y) \leq 1$ with $\gamma^0_\alpha \equiv 1$ if $\alpha > 0$, $\gamma^0_\alpha \equiv 0$ if $\alpha < 0$, $\gamma^0_\beta \equiv 0$ if $\beta > 0$, and $\gamma^0_\beta \equiv 1$ if $\beta < 0$, such that*

$$\alpha - \beta + \lambda < \nu, e_x > (\gamma^0_\alpha(y) - \gamma^0_\beta(y)) = 0 \tag{2.1}$$

PROPOSITION 2.4 *If locally $\{u = 0\}^0 = \emptyset$, then u is locally a solution for the free boundary problem (P_G) in the appendix with $G(\beta, \nu, X) = \beta - \lambda < e_x, \nu >$.*

Proof of proposition 2.4

From lemma 2.2 and proposition 2.3 we see (even for the particular cases $\alpha = 0$ or $\beta = 0$, because $\{u = 0\}^0 = \emptyset$) that u is a solution to problem (P_G) in the appendix with $G(\beta, \nu, X) = \beta - \lambda < e_x, \nu >$.

The main tool that is used by Caffarelli to prove regularity theorems in [8]-[9], is the monotonicity formula:

THEOREM 2.5 (lemma 5.1 [4]; lemma 18 [8]) *Let two continuous functions* $u_1, u_2 \geq 0$ *such that*

i) $\Delta u_i \geq 0$ *(*u_i *subharmonic)*

ii) $u_i(0) = 0$

iii) $u_1 u_2 \equiv 0$

Let

$$\phi(r) = \frac{\int_{B_r} |\nabla u_1|^2 \rho d\rho d\sigma \int_{B_r} |\nabla u_2|^2 \rho d\rho d\sigma}{r^4} \tag{2.2}$$

where (ρ, σ) *are the radial and spheric coordinates in* $\mathbf{R}^n$.

Then ϕ *is a nondecreasing function of* r. *Besides* ϕ *is bounded near* $r = 0$. *In particular if the functions* u_i *are defined on* $\mathbf{R}^2$, *and if* $\phi(r) = const > 0$, *then there exists* $\nu \in \mathbf{S}^1, \alpha_i > 0, i = 1, 2$ *such that* $u_1(X) = \alpha_1 < X, \nu >^+$, $u_2(X) = \alpha_2 < X, \nu >^-$.

Proof of proposition 2.3

From the assumption of proposition 2.3, let us consider the blow-up:

$$\left\{ \begin{array}{l} u^\epsilon(X) = \frac{u(X_0+\epsilon X)}{\epsilon} \\ \gamma^\epsilon = \gamma(X_0 + \epsilon X) \end{array} \right.$$

Let us recall that $u \in C^{0,1}_{loc}(\Omega)$ and $\gamma \in L^\infty(\Omega)$, then by Ascoli theorem up to extraction of some subsequence $(u^\epsilon, \gamma^\epsilon) \to (u^0, \gamma^0)$ on $C^{0,\alpha}(K) \times L^\infty_{weak*}(K)$ for every compact set $K \subset \mathbf{R}^2$ and every $\alpha \in (0,1)$. Then (u^0, γ^0) satisfies also (1.4) and is a solution on $\mathbf{R}^2$. We have $u^0(X) = \alpha < X, \nu >^+ -\beta < X, \nu >^-$ then

$$\int_{\mathbf{R}^2} (\alpha 1_{\{<X,\nu>>0\}} + \beta 1_{\{<X,\nu><0\}}) \nu \cdot \nabla v + \lambda \gamma^0 \partial_x v = 0 \tag{2.3}$$

In particular $\partial_x \gamma^0 = 0$ in $\{< X, \nu > \neq 0\}$, i.e.

$$\gamma^0(x,y) = \left\{ \begin{array}{l} \gamma^0_\alpha(y) \text{ in } \{< X, \nu >> 0\} \\ \gamma^0_\beta(y) \text{ in } \{< X, \nu >< 0\} \end{array} \right.$$

case 1: $< \nu, e_x >= 0$

From (2.3) we have $\lambda \partial_x \gamma^0 = -\Delta v = 0$ on $\{y > 0\}$ and $\{y < 0\}$. Then $\gamma^0 = \gamma^0(y)$ on $\mathbf{R}^2$. Consequently $\int_{\mathbf{R}^2} \gamma \partial_x v = 0$ and from (2.3) $\alpha = \beta$ and equation (2.1) is verified.

case 2: $< \nu, e_x > \neq 0$

We have $X = x e_x + y e_y = x_1 e_x + x_{\nu^\perp} \nu^\perp$, where $\nu^\perp = \begin{pmatrix} -\nu_y \\ \nu_x \end{pmatrix}$. Then

$$\left\{ \begin{array}{l} x = x_1 + x_{\nu^\perp} < \nu^\perp, e_x > \\ y = x_{\nu^\perp} < \nu^\perp, e_y > \end{array} \right. \qquad dxdy = dx_1 dx_{\nu^\perp} < \nu^\perp, e_y >$$

$$\left\{\begin{array}{l} x_{\nu^\perp} = \frac{y}{<\nu^\perp, e_y>} \\ x_1 = x - y\frac{<\nu^\perp, e_x>}{<\nu^\perp, e_y>} \end{array}\right. \qquad \left\{\begin{array}{l} \partial_x = \partial_{x_1} \\ \partial_y = \frac{1}{<\nu^\perp, e_y>}\partial_{x_{\nu^\perp}} - \frac{<\nu^\perp, e_x>}{<\nu^\perp, e_y>}\partial_{x_1} \end{array}\right.$$

Moreover $<\nu^\perp, e_x> = - <\nu, e_y>$, $<\nu^\perp, e_y> = <\nu, e_x>$, and

$$\nu \cdot \nabla v = \frac{1}{<\nu, e_x>}\partial_{x_1}\tilde{v} - \frac{<\nu, e_y>}{<\nu, e_x>}\partial_{x_{\nu^\perp}}\tilde{v} \text{ for } \tilde{v}(x_1, x_{\nu^\perp}) = v(x,y)$$

Similarly let $\tilde{\gamma}^0(x_1, x_{\nu^\perp}) = \gamma^0(x,y)$. Then from (2.3) we get

$$\int_{\mathbf{R}^2}(\alpha 1_{\{x_1<\nu,e_x>>0\}} + \beta 1_{\{x_1<\nu,e_x><0\}})(\frac{1}{<\nu, e_x>}\partial_{x_1}\tilde{v} - \frac{<\nu, e_y>}{<\nu, e_x>}\partial_{x_{\nu^\perp}}\tilde{v}) + \lambda\tilde{\gamma}\partial_{x_1}\tilde{v} = 0$$

Then $(\alpha-\beta)\delta_{\{x_1=0\}} + \lambda < \nu, e_x > \partial_{x_1}\tilde{\gamma}^0 = 0$, therefore $\alpha - \beta + \lambda < \nu, e_x > (\tilde{\gamma}^0_\alpha(x_{\nu^\perp}) - \tilde{\gamma}^0_\beta(x_{\nu^\perp})) = 0$ where $\tilde{\gamma}^0(x_{\nu^\perp}) = \left\{\begin{array}{l} \tilde{\gamma}^0_\alpha(x_{\nu^\perp}) \text{ on } \{x_1 > 0\} \\ \tilde{\gamma}^0_\beta(x_{\nu^\perp}) \text{ on } \{x_1 < 0\} \end{array}\right.$, i.e. $\alpha - \beta + \lambda < \nu, e_x > (\gamma^0_\alpha(y) - \gamma^0_\beta(y)) = 0$, which proves proposition 2.3.

3 $\{u=0\}^0$ is an epigraph in the e_x direction

Here we prove the theorem 1.8.

If the result is false, then we work in ω (we can forget Ω).

Step 1

LEMMA 3.1 *Let $P = (x_1, y_1) \in \{u = 0\}^0 \cap \omega$ such that $\exists x''_1 > x_1$, $P'' = (x''_1, y_1) \in (\{u=0\}^0)^c \cap \omega$. Let I_0 the connected component of $\{u=0\}^0$ which contains P. Then $\exists x'_1 > x_1, y'_1, \exists r' > 0$ such that for $I = [y'_1 - r', y'_1 + r']$, we have $\omega_0 = [x_1, x'_1] \times I \subset \omega$ and $\exists \delta_0 > 0$, $[x_1, x_1 + \delta_0] \times I \subset I_0$, $[x'_1 - \delta_0, x'_1] \times \subset \{u > 0\}$ (up to a change of sign for u).*

Proof of lemma 3.1

We know that $\exists B_r(P) \subset \{u=0\}^0$ and by definition of P'', $\exists P', d(P'', P') < \frac{r}{2}$ with say (up to a change of sign on u, see proposition 2.1) $u(P') > 0$. Then $\exists B_{r'}(P') \subset\subset \{u > 0\}$ with $r' < \frac{r}{2}$. We note $P' = (x'_1, y'_1)$ and $\omega_0 = [x_1, x'_1] \times I \subset \omega$ with $I = [y'_1 - r', y'_1 + r']$. We deduce the existence of a $\delta_0 > 0$ as in the lemma, decreasing r' if necessary. This ends the proof of lemma 3.1.

Step 2

LEMMA 3.2 *Let* $\Gamma_0 = \{\Gamma_0(y) = (f_0(y), y),\ f_0(y) = \sup\{x,\ (x,y) \in I_0 \cap \omega_0\}\}$. *Then up to a change of sign on* u *there exists a connected component* C_0 *of* $\{u > 0\} \cap \omega_0$, $\exists y_+, y_- \in I, y_+ > y_-$ *such that* $\Gamma_0(y_+), \Gamma_0(y_-) \in \partial I_0 \cap \partial C_0$.

Proof of lemma 3.2

Let $\mathcal{C}_{\omega_0}$ the set of all connected components of $(\{u > 0\} \cap \omega_0) \cup (\{u < 0\} \cap \omega_0)$. For each $C \in \mathcal{C}_{\omega_0}$, two cases appear:

i) either C is adherent to at most one point of Γ_0.

ii) or C is adherent to at least two points of Γ_0.

But $\mathcal{C}_{\omega_0}$ is a set of connected components at most denombrable, and Γ_0 is a set of non denombrable points. Then $\exists C_0 \in \mathcal{C}_{\omega_0}$ which verifies the case ii), and up to a change of sign on u we can assume that $u_{|C_0} > 0$, and there exist two points $\Gamma_0(y^-), \Gamma_0(y^+) \in \Gamma_0$, with $y^- < y^+$. This ends the proof of lemma 3.2.

Step 3

LEMMA 3.3 *Decreasing* ω *if necessary, we can assume (up to a change of sign on* u*) that* $u \geq 0$ *on* ω_0.

Proof of lemma 3.3

Let g_0 a continuous path which links together $\Gamma_0(y_-)$ to $\Gamma_0(y_+)$ in I_0. Precisely it means that there exists a injective and continuous map $\tilde{g}_0 : [-1, 1] \to \overline{I}_0$ with $Im(\tilde{g}_0) = g_0$, $\tilde{g}_0((-1, 1)) \subset I_0$, and $\tilde{g}_0(-1) = \Gamma_0(y_-)$, $\tilde{g}_0(+1) = \Gamma_0(y_+)$.

Let g_+ a continuous path which links together $\Gamma_0(y_-)$ to $\Gamma_0(y_+)$ in C_0. Then from the maximum principle $u \geq 0$ on the bounded component of boundary $g_0 \cup g_-$. This ends the proof of lemma 3.3.

Step 4: contradiction

Let us take a ball in $\{u > 0\}$ and slide it in direction $-e_x$. Then it touches ∂I_0 at a point X_0. Then from the linear behaviour lemma 2.2, we get $u(X) = \alpha < X - X_0, \nu >^+ -\beta < X - X_0, \nu >^- +o(|X - X_0|)$ for some $\alpha > 0$ (because of the Hopf lemma) and $\beta \leq 0$ (because $u \geq 0$) and with $< \nu, e_x >\geq 0$. Then from proposition 2.3 we get $\beta = \alpha + \lambda < \nu, e_x > (\gamma_\alpha^0(y) - \gamma_\beta^0(y)) > 0$ because $\gamma_\alpha^0(y) \equiv 1$ and $0 \leq \gamma_\beta^0(y) \leq 1$. Contradiction.

This ends the proof of the theorem 1.8.

4 Proof of theorem 1.11

4.1 Proof of theorem 1.11

DEFINITION 4.1 *We say that* $\Gamma(u)$ *is* ϵ*-flat in* 0 *for* $r \leq r_\epsilon$ *if and only if*

$$\forall r \in (0, r_\epsilon), \begin{cases} u > 0 \ \ on \ \{y \geq \epsilon r\} \cap B_r(0) \\ u < 0 \ \ on \ \{y \leq -\epsilon r\} \cap B_r(0) \end{cases}$$

We say that $X_0 \in \Gamma(u)$ *is a flat point if* $\forall \epsilon > 0, \exists r_\epsilon > 0$*, such that* $\Gamma(u)$ *is* ϵ*-flat in* X_0 *for* $r \leq r_\epsilon$*.*

DEFINITION 4.2 *Let* Γ_{flat} *the set of flat points of* Γ*, and let* Γ_{reg} *the set of points* $X_0 \in \Gamma$ *such that* Γ *is analytic in a neighbourhood of* X_0*.*

Then we have

PROPOSITION 4.3 *If* (u, γ) *is a solution of (1.4) on* ω *simply connected, if* $\{u = 0\}^0 \cap \omega = \emptyset$ *and if* $\{u > 0\} \cap \omega$ *and* $\{u < 0\} \cap \omega$ *are connected components such that* $(\partial\{u > 0\} \backslash \partial\{u < 0\}) \cap \omega = \emptyset$*,*$(\partial\{u < 0\} \backslash \partial\{u > 0\}) \cap \omega = \emptyset$*, then if* $X_0 \in \Gamma(u) \cap \omega$*, such that locally* $u(X) = o(|X - X_0|)$*, then* $X_0 \in \Gamma_{flat} \cap \omega$*.*

REMARK 4.4 *Here the Caffarelli theory [8]-[9] doesn't apply to improve the regularity of* Γ *because the solution is degenerate near* X_0*.*

PROPOSITION 4.5 *Under the same assumptions of proposition 4.3, if* $u(X) \neq o(|X - X_0|)$ *then* Γ *is analytic locally near* X_0*, i.e.* $X_0 \in \Gamma_{reg} \cap \omega$*.*

Proof of theorem 1.11

DEFINITION 4.6 *Let* C^+ *(resp.* C^-*) a connected component of* $\{u > 0\} \cap \Omega$ *(resp.* $\{u < 0\} \cap \Omega$*). Let* $P_1, P_2 \in \partial C^+ \cap \partial C^-$*. Then there exists a continuous path* $g^+ = g^+_{P_1,P_2} \subset C^+$ *which links together* P_1 *to* P_2*. Precisely it means that there exists a injective continuous map* $\tilde{g}^+ : [-1, 1] \to \overline{C}^+$ *such that* $Im(\tilde{g}^+) = g^+$*,* $g^+((-1, 1)) \subset C^+$*,* $g^+(-1) = P_1$*,* $g^+(+1) = P_2$*. Similarly there exists a continuous path* $g^- = g^-_{P_1,P_2} \subset C^-$ *which links together* P_1 *to* P_2*. We note* $\overline{\omega}(P_1, P_2)$ *every bounded closed component with boundary* $g^+ \cup g^-$*, with* g^+, g^- *as previously.*

We use the following lemma (Ω could be not simply connected, that is why we work on some ball $B \subset \Omega$):

LEMMA 4.7 *Let a ball $B \subset \Omega$. Let C^+ (resp. C^-) a connected component of $\{u > 0\} \cap B$ (resp. $\{u < 0\} \cap B$). Then $\exists P_1, P_2 \in \partial C^+ \cap \partial C^-$ and $\varpi(P_1, P_2)$ as in definition 4.6 such that $\partial C^+ \cap \partial C^- \subset \varpi(P_1, P_2)$.*

Then from proposition 4.5 and proposition 4.3 we have with $\omega = Int(\varpi(P_1, P_2))$:

$$\Gamma_0 := \partial C^+ \cap \partial C^- = \{P_0, P_0'\} \cup (\Gamma_{reg} \cap \omega) \cup (\Gamma_{flat} \cap \omega)$$

Let us consider a compact $K \subset\subset B$. If $\mathcal{H}^2(K \cap \Gamma_0) > 0$, then $\underline{\theta}(X) = \liminf_{r\to 0} \frac{|B_r(X) \cap (K \cap \Gamma_0)|}{|B_r(X)|} = 1$ $\mathcal{H}^2$-a.e. $X \in K \cap \Gamma_0$. In particular $\exists X_0 \in \Gamma_0, \underline{\theta}(X_0) = 1$. Then $X_0 \notin \Gamma_{reg}$, and then $X_0 \in \Gamma_{flat}$ which implies (from proposition 4.3) $\underline{\theta}(X_0) \leq \epsilon$ for all $\epsilon > 0$. Contradiction. Then $\mathcal{H}^2(K \cap \Gamma_0) = 0$ for all K, therefore $\mathcal{H}^2(\partial C^+ \cap \partial C^-) = 0$. The number of pair (C^+, C^-) is at most denombrable, therefore by denombrable summability, $\mathcal{H}^2(\{u = 0\} \cap B) = 0$ for every ball $B \subset \Omega$. Consequently $\mathcal{H}^2(\{u = 0\}) = 0$. This ends the proof of theorem 1.11.

Proof of lemma 4.7

It is easy to prove the lemma 4.7, using the connexity of C^+ and C^-, the fact that we work with topology in two dimensions, and the fact that B is simply connected. We proceed as follows. Let $P_0, P_1^0, P_2^0 \in \Gamma_0 = \partial C^+ \cap C^-$. We consider the sets $\mathcal{E}_{P_i^0}$ for $i = 1, 2$ of points P such that $P_i^0 \in \varpi(P_0, P)$. Each set $\mathcal{E}_{P_i^0}$ is ordened by the relation $P \leq P'$ if and only if $P \in \varpi(P_i^0, P')$ for some set $\varpi(P_i^0, P')$ as in definition 4.6. Let $P_i = \max \mathcal{E}_{P_i^0}; i = 1, 2$. To finish we prove that $\partial C^+ \cap \partial C^- \backslash \varpi(P_1, P_2) = \emptyset$.

4.2 Proof of proposition 4.5

Let us consider a point $X_0 \in \Gamma \cap \omega$ such that $u(X) \neq o(|X - X_0|)$. Then let $X_0 = 0$, $u^\epsilon(X) = \frac{u(\epsilon X)}{\epsilon}$. Then there exists a subsequence such that $u^\epsilon \to u^0$ and $u^0 \not\equiv 0$. In particular from theorem 2.5 if we set $u_1 = u^+$, $u_2 = u^-$, we get that $\phi_u(r) := \phi(r)$ is nondecreasing. Now $\phi_{u^\epsilon}(r) = \phi_u(\epsilon r)$, then $\phi_{u^0}(r) = \phi_u(0)$.

case 1: $\phi_u(0) > 0$

If $\phi_u(0) > 0$ we conclude that $u^0(X) = \alpha < X, \nu >^+ - \beta < X, \nu >^-$, with $\alpha, \beta > 0, \nu \in \mathbf{S}^1$. Then from proposition 2.4 and from theorem 6.3 i) in the appendix we conclude that Γ is locally $C^{1,\alpha}$ near X_0, and then from the result of Kinderlehrer-Mirenberg [16], Γ is locally analytic.

case 2: $\phi_u(0) = 0$

In this case $u^0 \geq 0$ or $u^0 \leq 0$. Up to a change of sign on u (see proposition 2.1) we can always assume that $u^0 \geq 0$ and $u^0 \not\equiv 0$. In particular from theorem 1.8, $\{u^0 = 0\}^0$ is an epigraph in the e_x direction. Here we prove:

LEMMA 4.8 *If $u^0 \not\equiv 0$, $u^0 \geq 0$ and (u^0, γ^0) is a solution of (1.4), then $\{u^0 = 0\}$ is an epigraph in the direction e_x.*

Proof of lemma 4.8

Let us assume that $u > 0$ on $B = B_r(x_0, y_0)$ and $\exists x_1 < x_0,\ u(x_1, y_0) = 0$. We have $u^0 \geq 0$ and $\Delta u^0 = 0$ in $\{u^0 > 0\}$ and u^0 is (Lipschitz-) continuous. Then u^0 is a subsolution and $\partial_x \gamma^0 = -\frac{1}{\lambda}\Delta u^0 \leq 0$. We know that $\gamma^0 \equiv 1$ in B and then $\Delta u^0 = 0$ on the left of B. From the hard maximum principle we deduce that $u^0 > 0$ on the left of B, because $u^0 > 0$ on B. Consequently $u^0(x_0, y_0) > 0$. Contradiction. This proves the lemma 4.8.

LEMMA 4.9 *If $u^0 \not\equiv 0$, $u^0 \geq 0$ and (u^0, γ^0) is a solution of (1.4), then $\forall y \in \mathbf{R}, \exists x \in \mathbf{R}, u^0(x, y) > 0$*

Proof of lemma 4.9

Let us assume that $\exists y_0 \in \mathbf{R}, \forall x \in \mathbf{R}, u^0(x, y_0) = 0$. Then up to a translation we can assume that $y_0 = 0$. Because $u^0 \not\equiv 0$, $\exists P \in \{y > 0\} \cup \{y < 0\}$, $u^0(P) > 0$.

Case 1: $\exists P^+ \in \{y > 0\}, \exists P^- \in \{y < 0\}, u^0(P^+) > 0, u^0(P^-) > 0$

Then let us consider the blow-in

$$\begin{cases} u^{0,\mu}(X) = \frac{u^0(\mu X)}{\mu} \\ \gamma^{0,\mu}(X) = \gamma^0(\mu X) \end{cases}$$

If we set $u_1 = u^0 1_{\{y>0\}}, u_2 = u^0 1_{\{y<0\}}$ we know from theorem 2.5 that $\phi_{u^0}(r) := \phi(r)$ is nondecreasing in r. In particular $\phi_{u^{0,\mu}}(r) = \phi_{u^0}(\mu r) > 0$ for $\mu > 0$ large enough. Then up to extraction of some subsequence $(u^{0,\mu}, \gamma^{0,\mu}) \to (u^{0,\infty}, \gamma^{0,\infty})$ which is a solution of (1.4) on $\mathbf{R}^2$, and $\phi_{u^{0,\infty}}(r) = \phi_{u^0}(+\infty) > 0$. Then from theorem 2.5 $u^{0,\infty}(X) = \alpha y^+ - \beta y^- \geq 0$ with $\alpha > 0, \beta < 0$ This is imposible from proposition 2.3.

Case 2: $u^0 = 0$ **on** $\{y > 0\}$ **or** $\{y < 0\}$

Let us assume that $u^0 = 0$ on $\{y > 0\}$. Let for some $\epsilon_0 > 0$:

$$\begin{cases} \tilde{u}^0(x,y) = \begin{cases} u^0(x, y+\epsilon_0) \text{ if } y \leq 0 \\ u^0(x, -y+\epsilon_0) \text{ if } y \geq 0 \end{cases} \\ \tilde{\gamma}^0(x,y) = \begin{cases} \gamma^0(x, y+\epsilon_0) \text{ if } y \leq 0 \\ \gamma^0(x, -y+\epsilon_0) \text{ if } y \geq 0 \end{cases} \end{cases}$$

Then $(\tilde{u}^0, \tilde{\gamma}^0)$ is a solution of (1.4) on $\mathbf{R}^2$ and we get a contradiction as in case 1.

This ends the proof of lemma 4.9.

Now we will use the following result:

PROPOSITION 4.10 $\forall \eta_0 > 0, \exists \epsilon > 0$, *such that if* (u, γ) *is a solution of (1.4) on* Ω *such that* $\partial\{u > 0\} \backslash \partial\{u < 0\} = \emptyset, \partial\{u < 0\} \backslash \partial\{u > 0\} = \emptyset$, *and* $|u| < \epsilon$ *on* $R_1 \subset \Omega$, *then for* $y^+ = \sup\{y, (x,y) \in \partial\{u < 0\} \cap R_1$, $y^- = \inf\{y, (x,y) \in \partial\{u < 0\} \cap R_1$. *we have* $\partial\{u < 0\} \cap R_1 \subset (-1,1) \times ([y^-, y^- + 5\eta_0] \cup [y^+ - 5\eta_0, y^+])$.

We have $0 \in \partial\{u^0 = 0\}$, then from lemma 4.8, $u^0(x,0) = 0$ for $x \geq 0$. Let for $\epsilon \geq 0$, $\lambda \geq 1$, $t \geq 0$:

$$\begin{cases} u^\epsilon_{t,\lambda}(x,y) = \frac{u^\epsilon(\lambda x + t, \lambda y)}{\lambda} \\ \gamma^e_{t,\lambda}(x,y) = \gamma^\epsilon(\lambda x + t, \lambda y) \end{cases}$$

In particular $u^0_{t,1} \to u^0_{\infty,1}$ uniformly on every compact sets (up to extraction of some subsequence), and $\gamma^0_{t,1} \to \gamma^0_{\infty,1}$ in L^∞_{weak*}. Then $(u^0_{\infty,1}, \gamma^0_{\infty,1})$ is a solution of (1.4), $u^0_{\infty,1} \geq 0$, and $\forall x \in \mathbf{R}, u^0_{\infty,1}(x,0) = 0$, therefore from lemma 4.9 we have $u^0_{\infty,1} \equiv 0$. Now we deduce that $|u^{\epsilon_1}_{t,\lambda} - u^0_{\infty,\lambda}| \leq \epsilon$ on R_2 if ϵ_1 is small enough and t large enough. Then from proposition 4.10 we deduce that there exist $y^\pm = (y^{\epsilon_1}_{t,\lambda})^\pm$ such that $|u^{\epsilon_1}_{t,\lambda}| > 0$ on $R_1 \backslash [-1,1] \times ([y^-, y^- + 5\eta_0] \cup [y^+ - 5\eta_0, y^+])$, and $\gamma^{\epsilon_1}_{t,\lambda} = 0$ where $u^{\epsilon_1}_{t,\lambda} < 0$, $\gamma^{\epsilon_1}_{t,\lambda} = 1$ where $u^{\epsilon_1}_{t,\lambda} > 0$. If we pass to the limit firstly on $\epsilon_1 \to 0$, we deduce that $\gamma^0_{t,\lambda} = u^0_{t,\lambda} = 0$ on certain regions of the form $[-1,1] \times [a,b]$. But $\gamma^0_{t,\lambda}$ is nonincreasing in x, therefore $\forall t' > t, \gamma^0_{t',\lambda} = u^0_{t',\lambda} = 0$ on $[-1,1] \times [a,b]$.

Now if we take λ large enough we deduce that there exist $y^+, y^- \in \mathbf{R} \cup \{-\infty\} \cup \{+\infty\}$ such that at $x = +\infty$, γ^0 is equal to 0 or 1 on each interval $(-\infty, y^-), (y^-, y^+), (y^+, +\infty)$.

Case 1:$\gamma^0_{\infty,1} = 1$ on $(a,0)$ or $(0,a)$

It implies by construction, and because of lemma 4.9 that $u^0 > 0$ on $\mathbf{R} \times (0,a)$. Then we get a contradiction as in the proof of lemma 4.9 because $u^0(x,0)=0$ for $x\geq 0$.

Case 2:$\gamma^0_{\infty,1}=0$ on $(a,b)\ni 0$

Then there exists a ball $B_r(\Lambda e_x)\subset\{u^0=0\}^0$ for some $r,\Lambda>0$, where $\gamma^0=0$. Then we define $v^0(x,y)=\int_x^\Lambda u^0(s,y)ds$ which verifies

$$\begin{cases} v^0\geq 0 \\ \Delta v^0=\lambda 1_{\{v>0\}} \\ |D^2v^0|_{L^\infty}\leq \max(Lip(u),\lambda)<+\infty \end{cases} \tag{4.1}$$

Moreover $(v^0)'_x=-u^0\leq 0$ and $(v^0)'_x<0$ in $\{v^0>0\}$. Then from a result of Alt [1] (see also lemma 5.2 with $\lambda(x')\equiv\lambda=const$), $\partial\{v^0>0\}=\partial\{u^0>0\}$ is locally Lipschitz. We made a new blow-up $u^{0,\epsilon}(X)=\frac{u^0(\epsilon X)}{\epsilon}\to u^{00}(X)$, $v^{0,\epsilon}(X)=\frac{v^0(\epsilon X)}{\epsilon^2}\to v^{00}(X)$. From Caffarelli theory [11] we get $v^{00}(x,y)=\frac{(<X,\nu>^+)^2}{2}$ or $v^{00}(x,y)=\frac{<X,\nu>^2}{2}$ where $<e_x,\nu>\neq 0$ because Γ is Lipschitz. But $\partial_x v^{00}=-u^{00}\leq 0$, then $u^{00}(X)=<X,\nu>^+$ for $<e_x,\nu><0$. Once more, from Caffarelli theory [11] we get that $\partial\{u^0>0\}$ is analytic near 0. We conclude (see lemma 6.4 and the proof of theorem 6.3 iii) in the appendix) that $\Gamma(u)$ is analytic near X_0. Let us remark that if $\Gamma(u)$ is analytic then from the Hopf lemma locally $u(X)=\alpha<X-X_0,\nu>^+-\beta<X-X_0,\nu>^-+o(|X|)$ with $\alpha,\beta>0$ and then $\phi_u(0)>0$, which proves that case 2 is impossible (for the two phases problem). This ends the proof of proposition 4.5.

4.3 Flat points: proof of proposition 4.3

REMARK 4.11 *Let us remark that* $\forall\lambda\in\mathbf{R}$, $u(x,y)=e^{-\lambda x}\sin\lambda y,\gamma\in H(u)$ *is locally a solution of problem (1.4).*

We assume that $u(X)=o(|X|)$ near 0.

For $\mu>0$, let $R_\mu=(-\mu,\mu)^2$ of center 0, and $\overline{R}_\mu=[-\mu,\mu]^2$. We assume that $|u|<\epsilon$ on R_2.

LEMMA 4.12 $\forall\eta_0>0,\exists\epsilon>0$, *such that if* $\exists y_0\in\mathbf{R}$, $|u|<\epsilon$ *on* $[-1,1]\times[y_0-\eta_0,y_0+\eta_0]$, $\exists x_0\in(-1,1)$, *and if there exists a continuous path* $g_0\subset\{u>0\}\cap[x_0,1]\times[y_0-\eta_0,y_0+\eta_0]$ *with* $\{P_1=(x_1,y_0-\eta_0)\}=g_0\cap[x_0,1]\times\{y_0-\eta_0\}$, $\{P_2=(x_2,y_0+\eta_0)\}=g_0\cap[x_0,1]\times\{y_0+\eta_0\}$, *then* $\{y=y_0\}\cap R^{g_0}\subset\{u>0\}$, *where* $P_1'=(-1,y_0-\eta_0),P_2'=(-1,y_0+\eta_0)$ *and* R^{g_0} *is the bounded connected component of boundary* $[P_1P_1']\cup[P_1'P_2']\cup[P_2'P_2]\cup g_0$, *i.e. the component of* $[-1,1]\times[y_0-\eta_0,y_0+\eta_0]$ *at the left of* g_0.

Proof of lemma 4.12

Step 1: Construction of a subsolution

Let $\alpha_0 > 0$ very small. Let $\mathcal{C}(\alpha_0) = \{\rho(\cos\phi, \sin\phi), \rho > 0,\ \phi \in [-\alpha_0, \alpha_0]\}$, and for $\delta > 0$, $\mathcal{C}_\delta(\alpha_0) = \{X,\ d(X, \mathcal{C}(\alpha_0)) < \delta\}$. For $L > 0$, let $\mathcal{C}_\delta^L(\alpha_0) = \{X \in \mathcal{C}_\delta(\alpha_0),\ x < L\}$. We want to construct a subsolution of the problem 1.4 on $\mathcal{C}_{2\delta}^L(\alpha_0)$. For this, we introduce the function v_1 defined on $\overline{\mathcal{C}_{2\delta}^L(\alpha_0)\backslash\mathcal{C}_\delta^L(\alpha_0)}$ by

$$\left\{\begin{array}{l} \Delta v_1 = 0 \text{ on } \mathcal{C}_{2\delta}^L(\alpha_0)\backslash\overline{\mathcal{C}_\delta^L(\alpha_0)} \\ v_1 = 0 \text{ on } (\partial\mathcal{C}_\delta(\alpha_0)) \cap \{x < L\} \\ v_1 = 1 \text{ on } (\partial\mathcal{C}_{2\delta}(\alpha_0)) \cap \{x < L\} \\ v_1 = \frac{d(x, \mathcal{C}_\delta(\alpha_0))}{\delta} \text{ on } \{x = L\} \cap \overline{\mathcal{C}_{2\delta}(\alpha_0)\backslash\mathcal{C}_\delta(\alpha_0)} \end{array}\right. \tag{4.2}$$

And on $\mathcal{C}_\delta^L(\alpha_0)$ we define v^1 by

$$\left\{\begin{array}{l} \Delta v^1 = 0 \text{ on } \mathcal{C}_\delta^L(\alpha_0) \\ v^1 = 0 \text{ on } (\partial\mathcal{C}_\delta(\alpha_0)) \cap \{x < L\} \\ v^1 = \cos(\frac{y}{y_L}) \text{ on } \mathcal{C}_\delta(\alpha_0) \cap \{x = L\} \\ \text{where } 2y_L = \text{ lenght of } \mathcal{C}_\delta(\alpha_0) \cap \{x = L\} \end{array}\right. \tag{4.3}$$

Then let

$$v_\epsilon^\eta = \left\{\begin{array}{l} -\epsilon v_1 \text{ on } \overline{\mathcal{C}_{2\delta}^L(\alpha_0)\backslash\mathcal{C}_\delta^L(\alpha_0)} \\ \eta v^1 \text{ on } \overline{\mathcal{C}_\delta^L(\alpha_0)} \end{array}\right. \tag{4.4}$$

Then on the free boundary, $\forall X_0 \in \Gamma(v_\epsilon^\eta) = (\partial\mathcal{C}_\delta(\alpha_0)) \cap \{x < L\}$, we have $u(X) = \alpha < X - X_0, \nu_0 >^+ -\beta < X - X_0, \nu_0 >^- + o(|X - X_0|)$, where ν_0 is the normal to $\Gamma(v_\epsilon^\eta)$ in X_0. But here $\alpha = \eta\alpha_1$, $\beta = \epsilon\beta_1$, $< e_x, \nu > \geq \sin\alpha_0 > 0$, and we search to verify the condition of subsolution on the boundary

$$\alpha > \beta - \lambda < e_x, \nu > \tag{4.5}$$

Then we see that $\exists\epsilon = \epsilon(\alpha_0, \delta, L) > 0$ (and $\epsilon \to 0$ as $\alpha_0, \delta_0 \to 0$) such that we obtain for all $\eta \geq 0$ a strict subsolution v_ϵ^η of problem (P_G) in the appendix with $G(\beta, x, \nu) = \beta - \lambda < e_x, \nu >$. Let us chose δ and α_0 such that $\eta_0 \geq 2\delta + L\sin\alpha_0$ and $L = 4$.

Step 2

Let $v^t(x, y) = v_\epsilon^\eta(x + t, y - y_0)$. For $t < -2$ we have $R^{g_0} \cap supp(v^t) = \emptyset$. Then we apply a sliding method, increasing t continuously. By hypothesis v^t is a subsolution on $supp(v^t) \cap R^{g_0}$ (see figure 1), and v^t can not touch u on $\partial(supp(v^t) \cap R^{g_0}) \subset supp(v^t) \cup g_0$, because

i) $v^t = -\epsilon$ and $u > -\epsilon$ on $\partial(supp(v^t))$.

ii) $u > 0$ on g_0 and then we can chose $\eta > 0$ such that $u > \eta \geq v^t$ on g_0.

Then v^t can only touch u on $\Gamma(v^t) = \partial\{v^t > 0\}$. But v^t is a strict subsolution on $\Gamma(v^t)$, consequently it is impossible (see lemma 7 in [8]). Then if we have chosen δ and α_0 small enough we can increase t until $\Gamma(v^t)$ touches $\{x = -2 + \delta\}$, which proves lemma 4.12.

REMARK 4.13 *The lemma 4.12 is true too if we change* $\{u > 0\}$ *by* $\{u < 0\}$ *(see proposition 2.1).*

figure 1

LEMMA 4.14 *Let us assume that* $|u| < \epsilon$ *on* R_2. *Then it does not exist three points* $P_i = (x_i, y_i) \in \partial\{u < 0\} \cap R_1, i = 1, 2, 3$ *such that* $-1 \leq y_1 < y_1 + 5\eta_0 \leq y_2 < y_2 + 5\eta_0 \leq y_3 \leq 1$.

Then proposition 4.10 is a corollary of lemma 4.14.

Proof of lemma 4.14

Let us assume that there exists three such points. Then there exists $P_{12} = (x_{12}, y_{12}) \in (-1, 1) \times (y_1 + 2\eta_0, y_2 - 2\eta_2)$ such that $u(P_{12}) \neq 0$. By symmetry let us assume that $u(P_{12}) > 0$. Then there exists a continuous path $g_{12} \subset \{u < 0\}$ which links together P_1 to P_2. It is possible that g_{12} goes outside R_2, but in every cases $u < \epsilon$ on the bounded component of boundary $[P_1P_2] \cup g_{12}$, because of the maximum principle.

case 1):g_{12} goes on the right of P_{12}

Because g_{12} goes on the right of P_{12}, we can apply the proof of lemma 4.12: let $P_1' = (-1, y_1), P_2' = (-1, y_2)$ and $R^{g_{12}}$ the bounded component of boundary $[P_1P_1'] \cup [P_1'P_2'] \cup [P_2'P_2] \cup g_{12}$. Then $P_{12} \in R^{g_{12}}_{\eta_0} = R^{g_{12}} \cap \{y_1 + \eta_0 \leq y \leq y_2 - \eta_0\} \subset \{u < 0\}$. Contradiction.

case 2):g_{12} goes on the left of P_{12}

Let $g_{23} \subset \{u > 0\}$ a continuous path which links together P_{12} to P_3.

subcase 2)a):g_{23} goes on the left of g_{12} (see figure 2)

Let $R^{g_{23}}$ the bounded component of boundary $g_{23} \cup [P_{12}P_3]$. Then as previously we get $P_1 \in R^{g_{23}}_{\eta_0} = R^{g_{23}} \cap \{y \leq y_{12} - \eta_0\} \subset \{u < 0\}$. Contradiction.

subcase 2)a):g_{23} goes on the right of g_{12} (see figure 2)

Then $P_2 \in R^{g_{23}}_{\eta_0} = R^{g_{23}} \cap \{y_{12} + \eta_0 \leq y \leq y_3 - \eta_0\} \cap R_1 \subset \{u < 0\}$. Contradiction.

In every case we get a contradiction. Then it proves the lemma 4.14.

figure 2

Proof of proposition 4.3

Up to consider $u^{\epsilon_1}(X) = \frac{u(\epsilon_1 X)}{\epsilon_1}, \gamma^{\epsilon_1}(X) = \gamma(\epsilon_1 X)$ with $0 < \epsilon_1 < 1$ in place of (u, γ), we deduce from proposition 4.10 that we are in one the following cases:

Case C3

We have three parts: $|u| > 0$ on $(-1,1) \times (-1, y^-), (-1,1) \times (y^- + 5\eta_0, y^+ - 5\eta_0), (-1,1) \times (y^+, 1)$, where $-1 < y^- < y^- + 10\eta_0 < y^+ < 1$ and $0 \in [y^-, y^- + 5\eta_0] \cup [y^+ - 5\eta_0, y^+]$.

Case C2

We have two parts: $|u| > 0$ on $(-1,1) \times (-1,-10\eta_0), (-1,1) \times (10\eta_0, 1)$.

And each case has subcases: we see the sign of u on each part from the top to the bottom. For example we note $C3++-$ a situation in case C3 where $u > 0$ on the two parts above and $u < 0$ on the last part below. We will note more generally $C3aab$ the case $C3++-$ or the case $C3--+$.

subcase C3aaa

Let us consider for example the case $C3---$. Then the method of proof of lemma 4.14 applies and gives a contradiction (see figure 3).

figure 3

subcase C3aab or C3abb

Let us consider for example the case $C3--+$. Then the method of proof of lemma 4.14 applies and gives a contradiction (see figure 4).

figure 4

subcase C3aba

Let us consider for example the case $C3+-+$. Then with a zoom with some $0<\epsilon_1<1$ we get the case C2ab for $(u^{\epsilon_1},\gamma^{\epsilon_1})$ (see figure 5).

figure 5

subcase C2ab

Let us consider for example the case $C2+-$. Then with a continuous zoom with $0<\epsilon_1<1$ we get the case C2ab for $(u^{\lambda},\gamma^{\lambda})$ (see figure 6), because the only other cases C3aab or C3abb are impossible. Then the configuartion C2ab is "stable" by zoom.

figure 6

subcase C2aa (see figure 7)

Let us consider for example the case $C2--$. Then with a zoom with $0 < \epsilon_1 < 1$ we can only get cases:

i) C3aaa: impossible

ii) C3aba, and then C2ab

iii) C2aa

Let us assume that we keep the case C2aa for every $0 < \epsilon_1 < 1$. Then for the cones $C^{\pm} = \{X, \pm\frac{<X,e_y>}{|X|} > \frac{1}{2}\}$ we have

$$(C^+ \cup C^-) \cap B_r(0) \subset \{u < 0\} \tag{4.6}$$

if η_0 is small enough. But $0 \in \partial\{u > 0\}$, and if $C_l = \{u > 0\} \cap \{x < 0\} \cap B_r(0) \neq \emptyset$, $C_d = \{u > 0\} \cap \{x > 0\} \cap B_r(0) \neq \emptyset$ for every $r > 0$ small enough, then there exists a continuous path $g \subset \{u > 0\}$ which connects C_l to C_d. This path can not go near 0 because of (4.6), then g goes round one of the components $(-1, 1) \times (-1, -10\eta_0), (-1, 1) \times (10\eta_0, 1)$ where $u < 0$. Contradiction. Then we have $C_l = \emptyset$ or $C_d = \emptyset$. Then $u < 0$ locally on $\{x < 0\}$ or $\{u > 0\}$. Thus the Hopf lemma gives a contradiction to the fact that $u(X) = o(|X|)$.

figure 7

Consequenlty in every case for $0 < \epsilon_1 < 1$ small enough $(u^{\epsilon_1}, \gamma^{\epsilon_1})$ is in the case C2ab, and it proves the proposition 4.3.

5 Examples

Let us recall an example of a mushy region which is given in [7]. Let $\Omega = \mathbf{R} \times (0,1)$, $u = f_0$ on $\mathbf{R} \times \{0\}$, and $u = f_1$ on $\mathbf{R} \times \{1\}$, where $f_1(x) = -f_0(x) = a \inf(1, exp(-x))$, for a constant $a > 0$ to be fixed. Let $g(y) = \frac{1}{y(1-y)}$. Let v on $\{x < g(y)\}$ equal to the harmonic function which vanishes on $x = g(y)$ and takes the values f_0 and f_1 on $\partial\Omega$; and $v = 0$ on $\{x > g(y)\}$. Let ν the exterior unit normal to $\{x > g(y)\}$, and $\gamma_v = \chi_{[\frac{1}{2},1]}(y) + \frac{v_\nu^+(g(y),y)}{\nu \cdot e_x}$. On the free boundary one has $v_\nu^+ = O(e^{-|x|})$ and $e_x \cdot \nu = O(\frac{1}{x})$ as $x \to +\infty$. Therefore for a small enough $\gamma_v \in [0,1]$ and (v, γ_v) is a solution.

Remark that when $u \geq 0$, the problem (1.4) reduced to the problem (1.1) with a generalised function $\tilde{\chi}(p) = \gamma(u)$, and $u(x,y) = p(y,-x)$. Recall that if we assume that $\tilde{\chi}(0) = 0$, then it is known (see [1]) that the boundary $\partial(\{p = 0\}^0)$ (i.e. $\partial(\{u = 0\}^0)$) is an analytical graph.
In the general case we have the

PROPOSITION 5.1 *If (u, γ) is a slution of (1.4), and locally*

$$\begin{cases} u \geq 0, \ \ \gamma \in C^{0,1} \\ \gamma \leq 1 - \delta < 1 \ \ on \ \{u = 0\}^0 \end{cases} \tag{5.1}$$

Then locally $\Gamma_0(u) = \partial(\{u = 0\}^0)$ is a $C^{1,\alpha}$ graph in direction e_x and $\gamma = \gamma(y)$ on $\{u = 0\}^0$.

This proposition can be proved using for the function $v(x,y) = \int_x^a v(x', y)dx'$ the following lemma which is an adapted version of a result of Alt [1].

LEMMA 5.2 *For $B_1 \subset \mathbf{R}^n$, if $v \in C^1(B_1)$,$\lambda \in C^{0,1}$, $0 \in \Gamma = \partial\{v > 0\}$, and for $x = (x', x_n) \in \mathbf{R}^n$:*

$$\begin{cases} \Delta v = \lambda(x') > 0 \ \ in \ B_1 \cap \{v > 0\} \\ \partial_{x_n} v \geq 0 \ \ in \ B_1 \\ \partial_{x_n} v > 0 \ \ in \ B_1 \cap \{v > 0\} \end{cases}$$

Then Γ is Lipschitz in $B_{\frac{1}{2}}$.

In proposition 5.1, the condition $\delta > 0$ is necessary, because if not, we can construct a counter-example u solution of (1.4) such that $\Gamma_0(u)$ has a cusp.

Counter-example

We use the holomorphic function $F(z) = -\exp(-\sqrt{-\ln(z)})$, with $z = re^{i\theta}, \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}], r \geq 0$. Then F is a diffeomorphism from $\{x \geq 0\}$ on its range $\mathcal{C}$ which is a cusp because $F(re^{i\theta}) = Re^{i\Theta}$, and a calculus gives $R = e^{-\sqrt{-\ln r}(1+o(\frac{1}{-\ln r}))}$, $\Theta - \pi = \frac{\theta}{2\sqrt{-\ln r}}(1+o(1))$. Now let $u = u_1 \circ F^{-1}$ where $u_1(x+iy) = x^+$. Then $\Delta u = 0$ on $\mathcal{C}$ by composition of holomorphic function. We must verify that u is locally Lipschitz, it means u_ν^+ is locally bounded, and construct a function $\gamma^0(y) = \gamma$ in $\{u = 0\}^0$ such that $\Delta u + \lambda \partial_x \gamma = 0$ in a neighbourhood of 0. In particular from proposition 2.3, $\gamma_0(y)$ must verify:

$$0 \leq 1 - \gamma_0(y) = -\frac{u_\nu^+}{\lambda < e_x, \nu >} \leq 1$$

But a calculus gives $0 \leq -\frac{u_\nu^+}{\lambda < e_x, \nu>} = \frac{1}{\lambda Re(F_z'(z))}$ where $z = re^{i\theta}$ with $\theta = \pm\frac{\pi}{2}$. But $F_z'(z) = \frac{X}{2}e^{\frac{1}{X^2}-\frac{1}{X}} \to +\infty$ where $X = \frac{1}{\sqrt{-\ln z}} = \frac{1}{\sqrt{-(\ln r \pm i\frac{\pi}{2})}} \to 0$. Moreover $< e_x, \nu > \to 0$ as $r \to 0$, then $u_\nu' \to 0$ as $r \to 0$ and locally u is Lipschitz (and positive harmonic). Then locally it is a solution to (1.4). Remark that here $\gamma_0 = 1 + \frac{u_\nu^+}{\lambda e_x \cdot \nu} \in C^\infty$. Moreover $\Gamma_0(u) = \partial(\{u = 0\}^0) \in C^{0,\beta}$ for every $\beta \in (0, 1)$.

REMARK 5.3 *We do not know if there exists a solution (u, γ) to problem (1.4) on an open set Ω, such that $u \geq 0$ on Ω and $(\partial\{u > 0\}) \cap \Omega \neq \emptyset$. In what follows we give a possible candidate but we do not know if it is a posteriori a solution.*
Let $\Omega = (0, +\infty) \times (0, 1)$. We give us a sequence $(\rho_n)_n$ of positive real numbers, such that $0 < \sum_{n=1}^{+\infty} 2^{n-1}\rho_n < 1$. Then we will build a sequence of functions $(u_n)_n$ which converge to a function u_∞ such that $(\partial\{u_\infty > 0\}) \cap \Omega \neq \emptyset$. But we do not know if there exists a function $\gamma_\infty \in L^\infty(\Omega)$ such that $(u_\infty, \gamma_\infty)$ is a solution of problem (1.4) on Ω.
Step 0
Let $y_0 = z_0 = 0$, $y_1 = z_1 = 1$ and $y_{01} = \frac{z_0+z_1}{2}$. For some $x_{01} > 0$ let $P_{01} = (x_{01}, y_{01})$, and $\Gamma^-(P_{01}, z_0, \mu_1) = \{x \geq x_{01}, y = y_{01} + (z_0 - y_{01})(1 - e^{-\mu_1(x-x_{01})})\}$, $\Gamma^-(P_{01}, z_1, \mu_1) = \{x \geq x_{01}, y = y_{01} + (z_1 - y_{01})(1 - e^{-\mu_1(x-x_{01})})\}$, and because $y_{01} > z_0$, $y_{01} < z_1$, $G^-(P_{01}, z_0, \mu_1) = \{x \geq x_{01}, y_{01} \geq y \geq y_{01} + (z_0 - y_{01})(1 - e^{-\mu_1(x-x_{01})})\}$, $G^+(P_{01}, z_1, \mu_1) = \{x \geq x_{01}, y_{01} \leq y \leq y_{01} + (z_1 - y_{01})(1 - e^{-\mu_1(x-x_{01})})\}$, for $\mu_1 = \frac{\pi}{y_1 - y_0}$.
Now we search a function u_0 such that $u_0 = 0$ on $(0, +\infty) \times \{0, 1\}$ and $u_0 = \lambda_0 \cos(\pi(y - y_{01}))$ on $\{0\} \times (0, 1)$ for some $\lambda_0 > 0$. We assume that $u_0 = 0$ on ∂G_0 and u_0 is harmonic on $\Omega \backslash G_0$ for $G_0 = G^-(P_{01}, z_0, \mu_1) \cup G^+(P_{01}, z_1, \mu_1)$.

Step 1

Let $z_{001} = y_{01} - \frac{\rho_1}{2}$, $z_{011} = y_{01} + \frac{\rho_1}{2}$, *and* $y_{001} = \frac{z_0+z_{001}}{2}$, $y_{011} = \frac{z_1+z_{011}}{2}$. *Let* $P_{001} = \{y = y_{001}\} \cap \Gamma^-(P_{01}, z_0, \mu_1)$, $P_{001} = \{y = y_{011}\} \cap \Gamma^+(P_{01}, z_1, \mu_1)$, *and* $\mu_2 = \frac{\pi}{y_{011}-y_{001}}$. *Now we define*

$$u_1 = \begin{cases} u_0 \ \ on\ \partial\Omega \\ 0 \ \ on\ \partial G_1 \end{cases}$$

and u_1 *is harmonic on* $\Omega \backslash G_1$ *where* $G_1 = G^-(P_{001}, z_0, \mu_1) \cup G^+(P_{001}, z_{001}, \mu_2) \cup G^-(P_{011}, z_{011}, \mu_2) \cup G^+(P_{011}, z_1, \mu_1)$.

Step 2

Let $x_{0001} = y_{001} - \frac{\rho_2}{2}$, $z_{0011} = y_{001} + \frac{\rho_2}{2}$, $z_{0101} = y_{011} - \frac{\rho_2}{2}$, $z_{0111} = y_{011} + \frac{\rho_2}{2}$, *and* $y_{0001} = \frac{z_0+z_{0001}}{2}$, $y_{0011} = \frac{z_{0011}+z_{001}}{2}$, $y_{0101} = \frac{z_{011}+z_{0101}}{2}$, $y_{0111} = \frac{z_{0111}+z_1}{2}$. *Let* $P_{0001} = \{y = y_{0001}\} \cap \Gamma^-(P_{001}, z_0, \mu_1)$, $P_{0011} = \{y = y_{0011}\} \cap \Gamma^+(P_{001}, z_{001}, \mu_2)$, $P_{0101} = \{y = y_{0101}\} \cap \Gamma^-(P_{011}, z_{011}, \mu_2)$, $P_{0111} = \{y = y_{0111}\} \cap \Gamma^+(P_{011}, z_1, \mu_1)$, *and* $\mu_3 = \frac{\pi}{y_{0011}-y_{0001}} = \frac{\pi}{y_{0111}-y_{0101}}$. *Now we define*

$$u_2 = \begin{cases} u_0 \ \ on\ \partial\Omega \\ 0 \ \ on\ \partial G_2 \end{cases}$$

and u_2 *is harmonic on* $\Omega \backslash G_2$ *where* $G_2 = G^-(P_{0001}, z_0, \mu_1) \cup G^+(P_{0001}, z_{0001}, \mu_3) \cup G^-(P_{0011}, z_{0011}, \mu_3) \cup G^+(P_{0011}, z_{001}, \mu_2) \cup G^-(P_{0101}, z_{011}, \mu_2) \cup G^+(P_{0101}, z_{0101}, \mu_3) \cup G^-(P_{0111}, z_{0111}, \mu_3) \cup G^+(P_{0111}, z_1, \mu_1)$.

Step $n \geq 3$

As previoulsy we build all the functions $u_n, n \geq 3$, *and this sequence converges to a function* u_∞ *which is positive except on horizontal half lines where* $u_\infty = 0$. *If the sequence* $(\rho_n)_n$ *converges rapidly to* 0, *we can see that near a tip* P^* *of a half line the "free boundary" is locally essentially vertical because the sequence* $\mu_n \to +\infty$. *In particular if a blow-up is possible we find* $\frac{u_\infty(P^*+\epsilon X)}{\epsilon} \to \alpha x^-$ *for some* $\alpha \geq 0$ *with* $x^- = max(0, -x)$, *which is coherent with proposition 2.3.*

6 Appendix: extension of Caffarelli results for free boundaries with general function $G(u_\nu^+, \nu, X)$

DEFINITION 6.1 *A function* u *is a solution of the problem* (P_G) *on the open set* $\Omega \subset \mathbf{R}^n$ *if and only if:*

i) $u \in C^{0,1}_{loc}(\Omega)$

ii) $\Delta u = 0$ *in* $\Omega^+(u) := \{u > 0\}, \Omega^-(u) := \{u \leq 0\}^0$

iii) On $\Gamma(u) := (\partial\Omega^+(u)) \cap \Omega$, *we have* $u_\nu^+ = G(u_\nu^-, \nu, X_0)$ *in the following weak sense.*

For every ball $B = B_r(Y_0)$ *with* $X_0 \in \partial B \cap \Gamma(u)$ *and* $r = |X_0 - Y_0|$*:*

a) If $B \subset \Omega^+(u)$, *let* $\nu = \frac{Y_0 - X_0}{r} \in \mathbf{S}^{n-1}$. *Then*

$$\exists \alpha > 0, \beta \geq 0, u(X) \leq \alpha < X - X_0, \nu >^+ - \beta < X - X_0, \nu >^- + o(|X - X_0|)$$

b)If $B \subset \Omega^-(u)$, *let* $\nu = -\frac{Y_0 - X_0}{r} \in \mathbf{S}^{n-1}$. *Then*

$$\exists \alpha \in \mathbf{R}, \beta \geq 0, u(X) \geq \alpha < X - X_0, \nu >^+ - \beta < X - X_0, \nu >^- + o(|X - X_0|)$$

where in each case $\alpha = G(\beta, \nu, X_0)$.

We assume that G verifies the hypothesis:

HYPOTHESIS 6.2 *i)* $G(\beta, x, \nu) \in \mathbf{R}$

ii) G *is strictly increasing in* β.

iii) $\forall M > 0$, G *is a lipschitz continuous function in* $(\beta, \nu, X) \in [-M, M] \times \mathbf{S}^{n-1} \times \overline{\Omega}$

Then we have the following two local results:

THEOREM 6.3 *i)If locally* $u(X) = \alpha_0 < X - X_0, \nu_0 >^+ - \beta_0 < X - X_0, \nu_0 >^- + o(|X - X_0|)$ *and* $\alpha_0, \beta_0 > 0$, *then locally* $\Gamma(u) \in C^{1,\alpha}$, *and for every* $X_1 \in \Gamma(u)$ *near* X_0, *there exist* $\nu_1 \in \mathbf{S}^{n-1}$ *and* $\alpha, \beta \geq const(\alpha_0, \beta_0) > 0$ *such that we have locally* $u(X) = \alpha < X - X_1, \nu_1 >^+ - \beta < X - X_1, \nu_1 >^- + o(|X - X_1|)$.

ii)Let us assume that $\forall \epsilon > 0, \exists r_\epsilon > 0, \forall r < r_\epsilon, \Omega^-(u) \supset B_r(X_0) \cap \{< X - X_0, \nu_0 >\leq -\epsilon r\}$. *If* $u \geq 0$ *locally near* X_0 *with* $u(X) = \alpha_0 < X - X_0, \nu_0 >^+ + o(|X - X_0|)$, $\alpha_0 > 0$, *then locally* $\Gamma \in C^{1,\alpha}$, *and for* $X_1 \in \Gamma(u)$ *near* X_0, *there exists* $\nu_1 \in \mathbf{S}^{n-1}$ *and* $\alpha \geq const(\alpha_0) > 0, \beta \geq 0$ *such that we have locally* $u(X) = \alpha < X - X_1, \nu_1 >^+ - \beta < X - X_1, \nu_1 >^- + o(|X - X_1|)$.

iii)The conclusion of ii) is also true if we change the condition $u \geq 0$ *in a neighbourhood of* X_0 *by the condition* $G(0, \nu_0, X_0) > 0$.

LEMMA 6.4 *If* u *is a solution of the problem* (P_G), *with the condition i) of the theorem 6.3 (resp. ii) or iii)), then locally,* $\forall \theta_0 \in (0, \frac{\pi}{2})$, $\exists C_{\theta_0} > 0$, $\exists \epsilon > 0$, $\forall \tau \in C^+(\theta_0, \nu_0) \cap \mathbf{S}^{n-1}$ *where* $C^+(\theta_0, \nu_0) = \{\tau \in \mathbf{R}^n \backslash \{0\},\ angle\ (\tau, \nu_0) \leq \theta_0\}$, *we have locally* $u(X + \epsilon\tau) - u(X) \geq C_{\theta_0}\epsilon$ *(resp.* $u(X + \epsilon\tau) - u(X) \geq C_{\theta_0}\epsilon$ *locally in* $\{u > 0\}$*).*

Proof of lemma 6.4

It is an easy consequence of lemma 1, lemma 5, lemma 4 of [8], and of an adaptation of the proof of the theorem 2' of [9].

Proof of the theorem 6.3

Cases i) and ii)

Under the conditions i) or ii) , the difficulty, is to avoid the values of $G(u_\nu^-, \nu, X) \leq 0$, with the help of a control on the normal ν of Γ. So we adapt the proof of Caffarelli [9], using the fact that initially for $\mathcal{C}_M = B_1^{n-1} \times [-M, M] \subset \mathbf{R}^n$ where $e_n = \nu_0$, $\theta_0 \sim \frac{\pi}{2}$, $\alpha_1 > 0$, there exists $\epsilon_0 > 0, \epsilon_0 << 1$, such that

$$\forall \epsilon > \epsilon_0, \ \begin{cases} v = \sup_{|Y|<\sin\theta_0} u(X - \epsilon(e_n + Y)) \leq u(X) - \epsilon\alpha_0 \cos\theta_0 \text{ in } \{v > 0\} \cap \mathcal{C}_M \\ v = \sup_{|Y|<\sin\theta_0} u(X - \epsilon(e_n + Y)) \leq u(X) \text{ in } \mathcal{C}_M \end{cases} \tag{6.1}$$

What is important in the proof, is the condition on the boundary $\partial\{\overline{v_t} > 0\}$. We recall that $\overline{v_t} = v_t + \eta w$ is a subsolution for the free boundary problem, where $v_t(X) = \sup_{Y \in B_{\sigma\phi_t(X)}} u(Y)$, $\eta = C\epsilon^{\frac{1}{4}}$, and $w \geq 0$ is a corrector function to permit to satisfy the boundary condition of subsolution on the free boundary of $\overline{v_t}$:

$$\overline{v_t}(X) \geq \alpha < X - \tilde{X}_1, \tilde{\nu} >^+ -\beta < X - \tilde{X}_1, \tilde{\nu} >^- +o(| < X - \tilde{X}_1, \tilde{\nu} > |) \tag{6.2}$$

with $\alpha = G(\beta, \tilde{\nu}, \tilde{X}_1)$.

1)Firstly, from the lemma 2 [9], v_t is monoton in a cone $\mathcal{C}(\overline{\theta}_0)$, where $\overline{\theta}_0$ is very close to $\frac{\pi}{2}$, if θ_0 is close enough to $\frac{\pi}{2}$ initially. Then the normal $\tilde{\nu}$ to $\partial\{v_t > 0\}$ is very close to e_n. This fact permits us to control the free boundary in a neighbourhood of X_0. In particular the normal ν in the proof of lemma 4 in [9] is close to e_n because $\tilde{\nu} = \frac{\nu + \sigma\nabla\phi_t}{|\nu + \sigma\nabla\phi_t|}$ and $|\sigma\nabla\phi_t| \leq C\epsilon^{\frac{1}{2}}$.

2) Secondly, we must satisfy the boundary condition 6.2. But here we have changed the condition

$$\exists C > 0, \ \beta^{-C} G(\beta, x, \nu) \text{ is a decreasing function in } \beta \tag{6.3}$$

which was required in the proof of Caffarelli, by the condition iii) of hypothesis 6.2. The boundary condition is satisfied, because the function w constructed in [9] is now nondegenerate because of (6.1). More precisely we can find a ball $B \subset \Omega^+(v_t)$ with B tangent to $\partial\Omega^+(v_t)$ and the diameter of B is of order ϵ. Now $w \geq v$ and at a distance $\frac{CM\epsilon}{2}$ where

$v \sim u \geq \epsilon \frac{CM}{2} \alpha_0 \cos \theta_0$ and by Harmack inequality on B, we can construct a barrier subsolution which proves that $\exists C = C(\alpha_1 \cos \theta_0) > 0, \partial_{\tilde{\nu}} w \geq C > 0$.

REMARK 6.5 *Remark that this modification of the proof of Caffarelli is suffisant to apply his proof without other knowledge on the sign of G in a neighboorhood of X_0 (a priori G could be negative in some points).*

REMARK 6.6 *In particular it proves that the results of Caffarelli in [8]-[9] with* $\inf_{\nu,X} G(0,\nu,X) > 0$ *are true without the conditions (6.3), but only assuming that u is Lipschitz and hypothesis iii).*

Then independantly on ϵ, $\exists \lambda \in (0,1)$ such that we obtain (6.1) with $\epsilon, \theta_0, \alpha_1, \mathcal{C}_M$ respectively changed by $\lambda\epsilon, \theta_0 - \epsilon^{\frac{1}{4}}, \alpha_1 - \epsilon^{\frac{1}{16}}, \mathcal{C}_{M(1-C\epsilon^{\frac{1}{8}})}$. Then the proof of Caffarelli applies and proves that Γ is Lipschitz. Moreover the proof of [8] applies with the same modification.

REMARK 6.7 *We haven't used the fact that $G(\beta_0, \nu_0, X_0) > 0$. In fact it is a consequence of (6.1) which implies at the limit $\epsilon \to 0^+$,*

$$\exists \theta_0^*, \alpha_1^*, \ \forall \tau \in C^+(\theta_0^*), \ (u^+)'_\tau \geq C\alpha_1^* \cos \theta_0^* > 0$$

This proves that $\alpha \geq const(\alpha_0) > 0$. Particularly, under condition i) we have $\beta_0 > 0$, then the same raisonning applies in $\{u < 0\}$ which proves that $\beta \geq const(\beta_0) > 0$.

Case iii)

In this case we would like to apply the proof of theorem 2 in [9]. But here is a new difficulty: we do not know a priori if u^+ in nondegenerate, i.e. in a neighbourhood of X_0, $u^+_\nu \geq C > 0$. To prove the iii) we must modify the proof of lemma 6 in [9] as follows. Let us take the new criteria (for some $1 >> \delta_1 > 0$ fixed): $u^-(-\frac{1}{2}e_n) \geq C\epsilon^{1-\delta_1}$ for the alternative a), and $u^-(-\frac{1}{2}e_n) < C\epsilon^{1-\delta_1}$ for the alternative b).

alternative a)

Then in $C_{1-C\epsilon^{\tau_1}}$ for some $\tau_1 > 0$ small enough and for $C_1\epsilon^{\tau_1} < |X_1 - X_2| < C_2\epsilon^{\tau_1}$ we get (see p 72 in [9] for the function v defined page 70) $v(X_2) - v(X_1) \geq \frac{C}{\delta_0} u^-(-\frac{1}{2}e_n)\epsilon^{(\alpha+1)\tau_1} \geq C\epsilon^{1-(\delta_1+(\alpha+1)\tau_1)}$, and because u is Lipschitz and $d(X_3, \Gamma) \leq \epsilon$ for every $X_3 \in A$, we get

$u^- \leq C\epsilon$ on A and then $v \leq u^- \leq v + C\epsilon$ (as in p 70 in [9]). We conclude that $u(X_2) \geq u(X_1)$ for $X_1 - X_2 \in \Gamma(\theta_1, e_n)$ and u is $C\epsilon^{\tau_1}$-monoton.

alternative b)

We have $u_\nu^+ \geq G(u_\nu^-, \nu, X) \sim G(0, \nu_0, X_0) > 0$ in the points of interest of $\Gamma(u)$ (point of comparison for $\overline{v}_t$, see 1)). In particular for these points we get (from the monotonicity formula) $0 \leq u_\nu^- \leq C\epsilon^{\frac{1-\delta_1}{2}-\mu}$ and then we conclude similarly as in [9] with $\eta \geq C\epsilon^{\frac{1-\delta_1}{2}-\mu}$.

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