Superprocesses and nonlinear partial differential equations

We would like to describe some links between superprocesses and some semi-linear partial differential equations. Let L be a second order differential operator defined on $\mathbb{R}^d$ and ψ a general function from $\mathbb{R}_+$ to $\mathbb{R}_+$. We will consider the nonnegative solutions of the parabolic equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = Lu - \psi(u) \quad \text{in} \quad [0,\infty)\times\mathbb{R}^d \\ u(0,\cdot) = f(\cdot), \end{cases} \tag{1}$$

where f is a bounded nonnegative function defined on $\mathbb{R}^d$. We will mostly be interested with $L = \frac{1}{2}\Delta$ and $\psi(u) = \gamma u^2$, $\gamma > 0$. We will also consider the Dirichlet problem associated with the elliptic equation. Let D be an open set of $\mathbb{R}^d$, we want to study the nonnegative solutions of:

$$\begin{cases} Lu = \psi(u) \qquad \text{in} \quad D \\ u_{|\partial D} = \varphi, \end{cases} \tag{2}$$

where φ is a bounded nonnegative function defined on the boundary ∂D. Let me stress that we are only interested in the NONNEGATIVE solutions of (1) or (2). Those solutions can be related to the Laplace transform of measure valued Markov processes called superprocesses.

In a first part we will recall some basic results on superprocesses which have already been introduced by Alison Etheridge and Robert Adler in September.

In a second part, we will present one of the most important tool associated with superprocesses: the so-called exit measures. They are closely related to (2). From there we will focus on the particular case $L = \frac{1}{2}\Delta$ and $\psi(u) = \gamma u^2$. We will also describe the solutions of the Dirichlet problem with blow-up condition at the boundary. Then we will look at the maximal and minimal solutions of (2) with infinite boundary condition.

In a third part we will describe the polar sets for the superprocesses and give a characterization in terms of removable singularities for solutions of (2).

In the last part we will describe the trace of the solutions of $\Delta u = u^2$ in a planar domain and present a representation formula. We will also give some extensions in higher dimension.

1 Superprocesses

1.1 Construction via the semi-group

We consider an homogeneous càdlàg Markov process $(\xi_t, t \geq 0)$ taking values in a polish space (E, d). (Typically ξ is a diffusion in $\mathbb{R}^d$.) Let $(P_t, t \geq 0)$ denote its transition semi-group and P_x the law of ξ starting at $x \in E$. We consider a branching mechanism $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ of the form

$$\psi(u) = au + bu^2 + \int_{(0,\infty)} n(dr)[\mathrm{e}^{-ru} - 1 + ru],$$

where $a \geq 0$, $b \geq 0$ and n is a Radon measure on $(0, \infty)$ satisfying $\int_{(0,\infty)}(r \wedge r^2)\, n(dr) < \infty$. Notice this includes the following cases:

- $\psi(u) = bu^2$ (take $a = 0, n = 0$);
- $\psi(u) = cu^{1+\beta}$ (take $a = 0, b = 0$ and $n(dr) = c' r^{-2-\beta} dr$) for $\beta \in (0, 1)$.

The function ψ is nonnegative, convex and locally Lipschitz.

Let $\mathcal{B}_+(E)$ be the set of bounded nonnegative measurable functions defined on E. We then consider the following integral equation

$$u(t, x) + \mathrm{E}_x \left[\int_0^t \psi(u(t - s, \xi_s)) ds \right] = \mathrm{E}_x f(\xi_t) = P_t f(x),$$

where $f \in \mathcal{B}_+(E)$. This equation is the mild form of (1), with L replaced by the infinitesimal generator of ξ. Using the method of Picard iteration, we can prove there exists a unique measurable (jointly in $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$) nonnegative function solution to the above integral equation. We denote by $V_t f(x)$ this solution. Using the Markov property of ξ, it is easy to show that $(f \to V_t f, t \geq 0)$ forms a nonlinear semi-group of operator on $\mathcal{B}_+(E)$.

Let $M_f(E)$ be the set of all finite measures on E endowed with the topology of the weak convergence. Fitzsimmons [11, 12] proved there exists a Markov process $((X_t, t \geq 0), (\mathbb{P}_\mu, \mu \in M_f(E)))$ taking values in $M_f(E)$, such that for every $f \in \mathcal{B}_+(E)$, $t \geq 0$,

$$\mathbb{E}_\mu \left[\mathrm{e}^{-(X_t, f)} \right] = \mathrm{e}^{-(\mu, V_t f)},$$

where $(\mu, f) = \int f(x)\mu(dx)$. We called this process the (ξ, ψ)-superprocess.

Computing $\mathbb{P}_\mu[X_t = 0]$ from the Laplace transform, we can see the process X dies out in finite time if and only if $\int_1^\infty \psi(u)^{-1} du < \infty$. We can easily deduce from the Laplace transform the first moment for X:

$$\mathbb{E}_\mu [(X_t, f)] = (\mu, P_t f)\, \mathrm{e}^{-at}.$$

In the case $\psi(u) = \gamma u^2$ we get for the second moment formula:

$$\mathbb{E}_\mu [(X_t, f)^2] = (\mu, P_t f)^2 + 2\gamma \int \mu(dx)\; E_x \left[\int_0^t P_{t-s} f(\xi_s)^2 ds \right]. \tag{3}$$

It can also be proved that in this case, the total mass process $((X_t, \mathbf{1}), t \geq 0)$ is a Feller diffusion, whose Laplace transform is given by

$$\mathbb{E}_\mu [\mathrm{e}^{-\lambda(X_t, \mathbf{1})}] = \mathrm{e}^{-\lambda(\mu, \mathbf{1})/[1 + \lambda\gamma(\mu, \mathbf{1})t]}.$$

1.2 Construction via the branching particle system

Although we have a direct definition for superprocesses, they can be viewed as limit of branching particle systems (cf [5]). Let ξ be a continuous Markov process. Let us assume for simplicity that $\psi(u) = \frac{1}{2}u^2$. Let n be an integer which will tend to infinity. We consider $N_0^{(n)}$ particles starting at time 0 at points $\{x_0^i, 1 \leq i \leq N_0^{(n)}\}$. Each particle evolves, independently of the others, according to the law of ξ. At independent exponential times of parameter n, each particle dies and give birth to two children with probability 1/2 or to none with probability 1/2. This is a critical binary branching mechanism. Conditionally on the fact that each newborn particle starts from the death point of its parent, the particles evolve independently from the past and from the other particles. The law of their trajectory is the law of ξ. They will also die at random independent exponential times (of parameter n), and the branching occurs again. We repeat again and again this procedure. Since we have a critical branching mechanism, the particle system dies out in finite time. At time t we have $N_t^{(n)}$ particles located at points $\{x_t^i, 1 \leq i \leq N_t^{(n)}\}$. To study this system we look at the measure valued process

$$X_t^{(n)} = \frac{1}{n} \sum_{i=1}^{N_t^{(n)}} \delta_{x_t^i},$$

where δ_x stands for the Dirac measure at point x. (Notice we consider particles of weight $1/n$.) The process $(X_t^{(n)}, t \geq 0)$ is a Markov process taking values in $M_f(E)$ and starting at $X_0^{(n)} = \frac{1}{n}\sum_{i=1}^{N_0^{(n)}} \delta_{x_0^i}$. Let us assume that $X_0^{(n)}$ converges, as n goes to infinity, to $\mu \in M_f(E)$ (for the weak topology). Then the finite dimensional distribution of $X^{(n)}$ converges in law to the finite dimensional distribution of the $(\xi, \frac{1}{2}u^2)$-superprocess X starting at μ (see also [2] and [3] for a stronger convergence and general results on superprocesses).

2 Exit measure

The exit measures have been introduced by Dynkin [6]. Let D be a given domain in E. Intuitively the exit measure of D describes, in the particle setting, the repartition of the particles frozen when they leave the domain D for the first time. There are three ways to construct the exit measures.

- We can define a semi-group of operator on $\mathcal{B}_+(E)$ indexed not by the time but by the open sets of E.
- We can consider the historical superprocess, that is the superprocess with underlying (inhomogeneous) Markov process $\tilde{\xi}_t = (\xi_s, s \in [0,t])$ and then consider the paths which just end when they go out of D for the first time.
- In the particular case where $\psi(u) = \gamma u^2$, we can use the Brownian snake approach introduced by Le Gall. It will enable us to have some nice proofs and representations for further results.

2.1 The Brownian snake

We assume from now on that ξ is a "nice" continuous Markov process in E (for example the d-dimensional Brownian motion). Let $\tilde{E}$ denotes the set of all continuous stopped paths in E. An element $w \in \tilde{E}$ is a continuous path $w : [0, \zeta] \to E$, $\zeta \geq 0$ is called the lifetime of the path w. We denote by $\underline{x}$ the trivial path of lifetime 0 such that $\underline{x}(0) = x$. The space $(\tilde{E}, \tilde{d})$ is a polish space for the distance

$$\tilde{d}(w, w') = \left|\zeta - \zeta'\right| + \sup_{t \geq 0} d(w(t \wedge \zeta), w'(t \wedge \zeta')).$$

The Brownian snake $W = (W_s, s \geq 0)$ is a strong continuous Markov process with values in $\tilde{E}$. We denote by ζ_s the lifetime of the path W_s. The law of W can be characterized as follows:

- The lifetime process $\zeta = (\zeta_s, s \geq 0)$ is a Brownian motion reflecting at 0.
- Conditionally on $(\zeta_s, s \geq 0)$, W is an inhomogeneous Markov process. Its transition kernel is characterized by: for $0 \leq s < s'$,
  - $W_{s'}(t) = W_s(t)$ for $t \leq m(s, s')$, where $m(s, s') = \inf_{r \in [s, s']} \zeta_r$.
  - Conditionally on $W_{s'}(m(s, s'))$, the path $(W_{s'}(t), m(s, s') \leq t \leq \zeta_{s'})$ is independent of W_s and has the same distribution as ξ started at time $m(s, s')$ at point $W_{s'}(m(s, s'))$.

We denote by $\mathbb{P}_x$ the law of W started at time 0 at the trivial path $\underline{x}$. Let $\hat{W}_s = W_s(\zeta_s)$ denotes the end point of the path W_s. It can be proved that the function $s \mapsto \hat{W}_s$ is a.s. continuous.

Notice that for $s \in [s', s'']$ the paths $(W_s(t), 0 \leq t \leq \zeta_s)$ coincide for $t \in [0, m(s', s'')]$. We will refer later to this property as the snake property.

We also introduce $(L^r_s, s \geq 0)$ the local time of ζ at level r. Let $\sigma = \inf\left\{s \geq 0; L^0_s > 1\right\}$. Then the process $(L^t_\sigma, t \geq 0)$ is a Feller diffusion and its Laplace transform is given by:

$$\mathrm{E}\left[\mathrm{e}^{-\lambda L^t_\sigma} \mid \zeta_0 = 0\right] = \mathrm{e}^{-\lambda/[1+2\lambda t]}.$$

We can now give a construction of superprocesses via the Brownian snake.

Proposition 1. *The process defined under $\mathbb{P}_x$ by: for $t \geq 0$, for $f \in \mathcal{B}_+(E)$,*

$$(X_t, f) = \int_0^\sigma f(\hat{W}_s) dL^t_s,$$

is the $(\xi, 2u^2)$-superprocess started at δ_x.

Notice that for all $t \geq 0$, a.s. $\operatorname{supp} X_t = \left\{\hat{W}_s; s \in [0, \sigma], \zeta_s = t\right\}$.

Remark. It is also easy to built the so-called historical process $(\tilde{X}_t, t \geq 0)$ a $M_f(\tilde{E})$-valued process: for $F \in \mathcal{B}_+(\tilde{E})$,

$$(\tilde{X}_t, F) = \int_0^\sigma F(W_s) dL^t_s.$$

2.2 Exit measures for the Brownian snake

For simplicity, let us assume from now on that ξ is the d-dimensional Brownian motion. We are now ready to build the exit measure of a connected open set $D \subset \mathbb{R}^d$ for the Brownian snake.

For $w \in \tilde{\mathbb{R}}^d$, let $\tau_D(w) = \inf\{t \geq 0; w(t) \notin D\}$ with the convention that $\inf \emptyset = +\infty$. Let us assume that for $x \in D$, $\mathrm{P}_x[\tau_D(\xi) < \infty] > 0$. For $\varepsilon > 0$, we consider the following measures on $\mathbb{R}^d$: for $f \in \mathcal{B}_+(\mathbb{R}^d)$,

$$(X_D^\varepsilon, f) = \frac{1}{\varepsilon}\int_0^\sigma \mathbf{1}_{\{\tau_D(W_s)<\zeta_s<\tau_D(W_s)+\varepsilon\}} f(\hat{W}_s)ds = \int_0^\sigma f(\hat{W}_s)dL_s^{D,\varepsilon},$$

where

$$L_s^{D,\varepsilon} = \frac{1}{\varepsilon}\int_0^s \mathbf{1}_{\{\tau_D(W_r)<\zeta_r<\tau_D(W_r)+\varepsilon\}}dr = \frac{1}{\varepsilon}\int_0^s \mathbf{1}_{\{0<\gamma_r<\varepsilon\}}dr,$$

and $\gamma_s = (\zeta_s - \tau_D(W_s))_+$. We introduce the functions $K_s = \int_0^s \mathbf{1}_{\{\gamma_r>0\}}dr$ and $A_t = \inf\{s \geq 0; K_s > t\}$. And we consider the time change process $\Gamma_s = \gamma_{A_s}$. We have $\Gamma_{K_s} = \gamma_s$. Using the snake property, we can prove that the process $(\Gamma_s, s \geq 0)$ is a reflecting Brownian motion in $\mathbb{R}^+$. Furthermore we have

$$L_s^{D,\varepsilon} = \frac{1}{\varepsilon}\int_0^s \mathbf{1}_{\{0<\gamma_r<\varepsilon\}}dr = \frac{1}{\varepsilon}\int_0^{K_s} \mathbf{1}_{\{0<\Gamma_r<\varepsilon\}}dr.$$

Clearly this quantity converges a.s. to the local time of Γ at level 0 up to time K_s. This implies that a.s. the measure $dL_s^{D,\varepsilon}$ converges to a measure, which we denote by dL_s^D. Since the local time of Γ at level 0 increases only when $\Gamma_r = 0$, we deduce that the measure dL_s^D increases only when $\zeta_s = \tau_D(W_s)$.

The measure defined by: for $f \in \mathcal{B}_+(\mathbb{R}^d)$,

$$(X_D, f) = \int_0^\sigma f(\hat{W}_s)dL_s^D,$$

is the exit measure of D. Notice that supp $X_D \subset \partial D$ since $\hat{W}_s$ is continuous. We can compute the Laplace transform of the exit measure.

Theorem 2. *Let φ be a bounded nonnegative measurable function defined on ∂D. Then we have*

$$\mathbb{E}_x\left[\mathrm{e}^{-(X_D,\varphi)}\right] = \mathrm{e}^{-u(x)}, \quad x \in D,$$

where the function u, defined on D, solves the equation

$$u(x) + 2\mathrm{E}_x\left[\int_0^{\tau_D} u(\xi_s)^2 ds\right] = \mathrm{E}_x[\varphi(\xi_{\tau_D})].$$

Remarks. i) The above integral equation is the mild form of (2) with $L = \frac{1}{2}\Delta$ and $\psi(u) = 2u^2$. The exit measure has been built in the case $\psi(u) = 2u^2$, this restriction is due to the Brownian snake approach. The approach of Dynkin [8] yields the general ψ.

ii) We have been looking at elliptic equation in domain $D \subset \mathbb{R}^d$, but using the process (t, ξ_t) instead of (ξ_t) leads to parabolic equation in domains of $\mathbb{R}_+ \times \mathbb{R}^d$.

2.3 Dirichlet problem in D

Let D be an open subset of $\mathbb{R}^d$. A point $a \in \partial D$ is said to be regular if $\mathrm{P}_a\left[\tau_D(\xi) = 0\right] = 1$. We say that D is regular if all points $a \in \partial D$ are regular. Let us recall some well-known results.

a) If φ is a bounded measurable function defined on ∂D then the function $h(x) = \mathrm{E}_x[\varphi(\xi_{\tau_D})]$ is in $C^2(D)$ and $\Delta h = 0$ in D. Furthermore if φ is continuous at $a \in \partial D$ and if a is regular, then $\lim_{x \in D; x \to a} h(x) = \varphi(a)$.

b) If ρ is a bounded measurable function defined in a bounded domain D, then the function $F(x) = \mathrm{E}_x\left[\int_0^{\tau_D} \rho(\xi_s)\right]$ is in $C^1(D)$. Let $a \in \partial D$ be regular, then we have $\lim_{x \in D; x \to a} F(x) = 0$. Furthermore if ρ is Hölder continuous in D, then $F \in C^2(D)$ and $\frac{1}{2}\Delta F = -\rho$ in D.

We then deduce from a) and b) that if D is a bounded regular domain and if φ is nonnegative continuous defined on ∂D, then the function u defined in theorem 2 is in $C^2(D) \cap C(\overline{D})$ and solves

$$\begin{cases} \Delta u = 4u^2 \quad \text{in} \quad D \\ u_{|\partial D} = \varphi. \end{cases} \tag{4}$$

We also will need the following comparison principle for elliptic differential equations.

Comparison principle. Let D be a bounded open set of $\mathbb{R}^d$. Let L be an elliptic differential operator. Let ψ be a nondecreasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$. Let u, v in $C^2(D)$ such that

$$\begin{aligned} &Lu - \psi(u) \geq Lv - \psi(v) \quad \text{in} \quad D, \\ &\limsup_{x \in D; x \to a} \left[u(x) - v(x)\right] \leq 0 \quad \text{for all } a \in \partial D, \end{aligned}$$

then $u(x) \leq v(x)$ in D.

We deduce from this comparison principle that the solution of (4) is unique.

2.4 Dirichlet problem in D with blows-up boundary condition

We keep the same hypothesis as in (2.2). We define the range of the superprocess as the closed set $\mathcal{R} = \overline{\bigcup_{t \geq 0} \operatorname{supp} X_t}$. Recall that a.s. $\operatorname{supp} X_t = \left\{\hat{W}_s; s \in [0, \sigma], \zeta_s = t\right\}$. It is then clear, using the continuity of the path $s \mapsto \hat{W}_s$, that a.s. $\mathcal{R} = \left\{\hat{W}_s; s \in [0, \sigma]\right\}$. Notice the range is a compact set. Let us now consider the two functions defined on D:

$$\begin{aligned} u_D(x) &= -\log \mathbb{P}_x\left[X_D = 0\right] \\ v_D(x) &= -\log \mathbb{P}_x\left[\mathcal{R} \subset D\right]. \end{aligned}$$

For any nonnegative function u defined in theorem 2, with φ bounded nonnegative, we have $u \leq u_D$ in D. Since $\operatorname{supp} X_D \subset \mathcal{R} \cap \partial D$, we get that $u_D \leq v_D$ in D. Since $s \mapsto \hat{W}_s$ is continuous and σ is finite, for every $\eta > 0$ the quantity $\mathbb{P}_x\left[\sup_{s \in [0,\sigma]} \left|\hat{W}_s - x\right| \leq \eta\right]$ is positive. It is even independent of x by space translation invariance. Let K be a compact set. Hence

$$\inf_{x\in K} \mathbb{P}_x\left[\mathcal{R}\subset B(x,\eta)\right] > 0,$$

where $B(x,\eta)$ is the ball with center x and radius η. Thus we deduce that the function v_D is uniformly bounded on every compact subset of D. We can now give a result on the maximal and minimal solutions of $\Delta u = 4u^2$ with blow-up boundary condition.

Proposition 3.

1. *Let $D\subset\mathbb{R}^d$ be a bounded regular open set, then the function u_D is the minimal solution of*

$$\begin{cases} \Delta u = 4u^2 \quad \textit{in} \quad D \\ \lim\limits_{x\in D; x\to\partial D} u(x) = \infty. \end{cases} \tag{5}$$

2. *Let $D\subset\mathbb{R}^d$ be an open set, then the function v_D is the maximal nonnegative solution of $\Delta u = 4u^2$ in D. This means that if $u\in C^2(D)$ is nonnegative and solves $\Delta u = 4u^2$ then $u\leq v_D$.*

We deduce from this proposition that if D is regular and bounded, then v_D blows up at the boundary.

Before going into the proof, let us prove a nonlinear "mean-value" property. Let $D\subset\mathbb{R}^d$ be an open set. Let $v\in C^2(D)$ be a solution of $\Delta u = 4u^2$ in D. Consider O a regular bounded open set such that $\overline{O}\subset D$. The function $u(x) = -\log \mathbb{E}_x\left[\mathrm{e}^{-(X_O,v)}\right]$ defined in O is a nonnegative solution of (4) in O with $\varphi = v$. But so is v. We deduce from the comparison principle that $u=v$ in O, that is

$$v(x) = -\log \mathbb{E}_x\left[\mathrm{e}^{-(X_O,v)}\right] \quad \text{for} \quad x\in O.$$

Proof 1. Let $D\subset\mathbb{R}^d$ be a regular bounded open set. Consider the following increasing sequence of functions $u_n(x) = -\log\mathbb{E}_x\left[\mathrm{e}^{-n(X_D,\mathbf{1})}\right]$. This sequence converges to u_D in D. Recall that the function v_D is uniformly bounded on every compact subset $K\subset D$. Since $u_D\leq v_D$, we deduce that u_D is bounded on K. Let $O\subset D$ be a regular bounded open set such that $\overline{O}\subset D$. By the non-linear "mean-value" property, we have $u_n(x) = -\log\mathbb{E}_x\left[\mathrm{e}^{-(X_O,u_n)}\right]$ for $x\in O$. By monotone convergence, we get that the nonlinear "mean-value" property also holds for u_D. This implies that $u_D\in C^2(O)$ and solves $\Delta u = 4u^2$ in O and thus in D. Since $u_n(x)$ converges to n as x goes to ∂D, we deduce that $u_D(x)$ blows up as x goes to ∂D. Let v a solution of (5). By the comparison principle, we have $v\geq u_n$ in D. Thus we get $v\geq u_D$ in D. $\square$

Proof 2. A.s. we have

$$\{\mathcal{R}\subset D\}\subset\{X_D=0\}\subset\left\{\mathcal{R}\subset\bar{D}\right\}. \tag{6}$$

The last inclusion is non trivial and will be admitted here. We now consider an increasing sequence of bounded regular domains D_n such that $\bar{D}_n \subset D_{n+1} \subset D$ and $D = \bigcup_{n\geq 1} D_n$. According to the inclusions (6), we have for $x \in D_n$

$$\mathbb{P}_x [\mathcal{R} \subset D_n] \leq \mathbb{P}[X_{D_n} = 0] \leq \mathbb{P}_x \left[\mathcal{R} \subset \bar{D}_n\right] \leq \mathbb{P}_x [\mathcal{R} \subset D_{n+1}].$$

Thus we have $v_{D_n}(x) \geq u_{D_n}(x) \geq v_{D_{n+1}}(x)$ in D_n. The sequences $(v_{D_n}, n \geq 1)$ and $(u_{D_n}, n \geq 1)$ are nonincreasing. They converge to the same limit $v_D(x) = -\log \mathbb{P}_x [\mathcal{R} \subset D]$ which is defined on D (we used the fact that $D_{n+1} \uparrow D$ and that $\mathcal{R}$ is compact). By the nonlinear "mean value" property, we have for every open bounded regular set O such that $\bar{O} \subset D_{n_0}$, $\forall x \in O$, $\forall n \geq n_0$, $u_{D_n}(x) = -\log \mathbb{E}_x \left[\mathrm{e}^{-(X_O, u_{D_n})}\right]$. By dominated convergence, we get $\forall x \in O$, $v_D(x) = -\log \mathbb{E}_x \left[\mathrm{e}^{-(X_O, v_D)}\right]$. Since v_D is bounded on $\bar{O}$, we get that it solves $\Delta u = 4u^2$ in O, and thus in D. It remains to prove that v_D is the maximal solution in D. Let g a nonnegative solution of $\Delta u = 4u^2$ in D. The function g is at least bounded on every D_n. By the comparison principle, we get that on D_n, $g \leq u_{D_n}$. This implies that $g \leq v_D$ in D. □

We are now led to two natural questions:

1. On what condition on the open set D do we have the existence of a solution with blow-up at the boundary?

2. On what condition on D, do we have $u_D = v_D$?

The first question has been answered in the case $\Delta u = 4u^2$ by Dhersin and Le Gall [4] (see also [18] for a more general setting). A point $x \in \partial D$ is said to be super-regular if $\mathbb{P}_x[\tau = 0] = 1$, where $\tau = \inf\{t > 0, \operatorname{supp} X_t \cap D^c \neq \emptyset\}$. We now define the following capacities. Let $\beta \geq 0$. Let A be a Borel set of $\mathbb{R}^d$.

$$\operatorname{cap}_\beta(A) = \left[\inf_{\nu(A)=1} \iint \nu(dx)\nu(dy) h_\beta(x-y)\right]^{-1},$$

$$\text{where} \quad h_\beta(x-y) = \begin{cases} 1 + \log_+[1/|x-y|] & \text{if } \beta = 0, \\ |x-y|^{-\beta} & \text{if } \beta > 0, \end{cases}$$

with $\log_+ x = (\log x) \vee 0$. Then we have the next result which includes the Wiener's test for the $(\xi, 2u^2)$-superprocess.

Theorem 4. *Let $D \subset \mathbb{R}^d$ be a domain. Let $x \in \partial D$. For $n \geq 1$ let*

$$F_n(x) = \left\{y \in D^c; 2^{-n} \leq |x-y| < 2^{-n+1}\right\}.$$

Then the next three properties are equivalent.

1. *The point x is super-regular.*

2. *Either $d \leq 3$, or $d \geq 4$ and*

$$\sum_{n\geq 1} 2^{n(d-2)} \operatorname{cap}_{d-4}(F_n(x)) = \infty.$$

3. *There exists a solution of $\Delta u = 4u^2$ in D such that $\lim_{y\in D; y\to x} u(y) = \infty$.*

Furthermore, if every point $x \in \partial D$ is super regular, then there exists a solution to (5).

The answer to the second question does not seem optimal yet (see [15] and [18]).

3 Polar sets

We still assume $L = \frac{1}{2}\Delta$ and $\psi(u) = 2u^2$. Let K be a compact set of $\mathbb{R}^d$. We say that the set K is polar if

$$\forall x \in \mathbb{R}^d \backslash K, \quad \mathbb{P}_x[\mathcal{R} \cap K \neq \emptyset] = 0.$$

Since the function $v_{K^c} = -\log \mathbb{P}_x[\mathcal{R} \cap K = \emptyset]$ is the maximal solution of $\Delta u = 4u^2$ in K^c. We deduce the next three statements are equivalent:

- K is polar.
- $v_{K^c} = 0$.
- There is no non trivial solution $u \geq 0$ of $\Delta u = 4u^2$ in K^c (i.e. K is a removable singularity).

We now give a characterization for polar sets.

Theorem 5. *The set K is a polar set (for the super Brownian motion) if and only if*

- $K = \emptyset$ *if* $d \leq 3$,
- $\text{cap}_{d-4}(K) = 0$ *if* $d \geq 4$.

This theorem is due to Perkins [19] ($\Rightarrow$) and Dynkin [7] ($\Leftarrow$) (see also [1]). Those results have been extended by Dynkin to the general equation $Lu = u^{1+\beta}$, $\beta \in (0, 1]$. We will only give the proof of ($\Rightarrow$).

Proof. Let us assume $d \geq 4$ and $\text{cap}_{d-4} K > 0$. Then there exists a probability ν on K such that $\iint \nu(dx)\nu(dy)h_{d-4}(x-y)$ is finite. We consider a continuous nonnegative function f on $\mathbb{R}^d$ with support in $B(0,1)$, which is radial (i.e. $f(y) = f(x)$ if $|y| = |x|$) and such that $\int f(y)dy = 1$. And we set $f_\varepsilon(y) = \varepsilon^{-d} f(y/\varepsilon)$ (thus $f_\varepsilon(y)dy \Rightarrow \delta_0$). Let $g_\varepsilon(x) = \int f_\varepsilon(x-y)\nu(dy)$. We consider the occupation measure $\Gamma(dx) = \int_0^\infty dt\ X_t(dx)$, and we compute the first two moments of (Γ, g_ε). We have for $x \in K^c$:

$$\mathbb{E}_x\left[(\Gamma, g_\varepsilon)\right] = \int G(x-y)g_\varepsilon(y)dy,$$

where G is the Green function $G(x) = \alpha_d |x|^{2-d}$. Since the right hand side converges to $\int G(x-y)\nu(dy)$ which is finite positive, we deduce there exists a constant c_1, depending on $x, K, \varepsilon_0 > 0$, such that for every $\varepsilon \in (0, \varepsilon_0]$,

$$\mathbb{E}_x\left[(\Gamma, g_\varepsilon)\right] \geq c_1 > 0.$$

The formula (3) for the second moment gives

$$\mathbb{E}_x\left[(\Gamma, g_\varepsilon)^2\right] = \left(\int G(x-y)g_\varepsilon(y)dy\right)^2 + 4\int dz\ G(x,z) \iint dydy'\ G(z-y)G(z-y')g_\varepsilon(y)g_\varepsilon(y').$$

The first right hand side term is bounded (use the above remark on $\mathbb{E}_x[(\Gamma, g_\varepsilon)]$). Now using the properties of f_ε and the fact that the function $G(y)$ is super-harmonic on $\mathbb{R}^d$, we get

$$\int dz\, G(x,z)\left(\int dy\, G(z-y)f_\varepsilon(y-a)\right)\left(\int dy'\, G(z-y')f_\varepsilon(y'-a')\right)$$
$$\leq \int dz\, G(x,z)G(z-a)G(z-a')$$
$$\leq c_2 h_{d-4}(a-a').$$

Thanks to the assumption on ν, we deduce there exists a constant c_3, depending on $x, K, \varepsilon_0 > 0$, such that for every $\varepsilon \in (0, \varepsilon_0]$,

$$\mathbb{E}_x\left[(\Gamma, g_\varepsilon)^2\right] \leq c_3.$$

Now, using Cauchy-Schwarz inequality, we get for $\varepsilon \in (0, \varepsilon_0]$

$$\mathbb{P}_x\left[(\Gamma, g_\varepsilon)\ > 0\right] \geq \frac{\mathbb{E}_x\left[(\Gamma, g_\varepsilon)\right]^2}{\mathbb{E}_x\left[(\Gamma, g_\varepsilon)^2\right]} \geq \frac{c_1^2}{c_3} > 0.$$

But since the support of f is in $B(0,1)$, we get that

$$\{(\Gamma, g_\varepsilon)\ > 0\} \subset \{\mathcal{R} \cap K_\varepsilon \neq \emptyset\},$$

where $K_\varepsilon = \left\{y \in \mathbb{R}^d; d(y,K) \leq \varepsilon\right\}$. Letting ε goes to 0, we get that $\mathbb{P}_x[\mathcal{R} \cap K \neq \emptyset] > 0$. Thus K is not polar.

Let $d \leq 3$. In fact it is sufficient to consider the case $d = 3$. Now using the definition of the capacity, it is easy to check that a segment is not polar in dimension $d = 4$. Thus by projection, we deduce that points are not polar in dimension $d = 3$. □

Remark. The points are polar if and only if $d \geq 4$. In the case $\Delta u = u^{1+\beta}$ for $\beta \in (0,1]$ the points are polar if and only if $d \geq 2(1+\beta)/\beta$.

4 Representation theorems

We use the Brownian snake approach. We want to describe all the solutions of $\Delta u = 4u^2$ in $D \subset \mathbb{R}^d$, where D is a smooth open set. Let us consider the set of exit points of D:

$$\mathcal{E}_D = \left\{\hat{W}_s; s \leq \sigma, \tau_D(W_s) = \zeta_s\right\}.$$

Notice that supp $X_D \subset \mathcal{E}_D$ a.s.

4.1 The critical case $d = 2$

The next theorem is due to Le Gall [17].

Theorem 6. *Let D be a domain of class C^2 (not necessarily bounded). There is a one-to-one correspondence between nonnegative solution of $\Delta u = 4u^2$ in D and pairs (K, ν), where K is a closed subset of ∂D and ν is a Radon measure on $\partial D \backslash K$. The correspondence is characterized as follows:*

- *On one hand, for* $x \in D$,

$$u(x) = -\log \mathbb{E}_x \left[\mathbf{1}_{\{\mathcal{E}_D \cap K = \emptyset\}} \, \mathrm{e}^{-\int Y_D(y)\nu(dy)} \right],$$

where $(Y_D(y), y \in \partial D)$ *is the continuous density of* X_D *with respect to the Lebesgue measure on* ∂D $\sigma(dy)$.

- *On the other hand*

$$K = \{z \in \partial D; \limsup_{x \in D; x \to z} d(x, \partial D)^2 u(x) > 0\},$$
$$(\nu, \varphi) = \lim_{r \downarrow 0} \int_{\partial D \setminus K} u(z + rN_z)\varphi(z)\sigma(dz),$$

where φ *is continuous with compact support on* $\partial D \setminus K$ *and* N_z *denotes the inward pointing vector normal to* ∂D *at* z.

The pair (K, ν) will be called the trace of the solution u.

Remarks. i) If $K = \emptyset$, D bounded, and $\nu(dy) = \varphi(y)\sigma(dy)$, where the function φ is continuous, then we recover the fact that the function

$$u(x) = -\log \mathbb{E}_x \left[\mathrm{e}^{-(X_D, \varphi)} \right],$$

is the only solution of the Dirichlet problem (4).

ii) If $K = \partial D$, D bounded, then we recover the fact that the function

$$u(x) = -\log \mathbb{P}_x [\mathcal{E}_D \cap K = \emptyset] = -\log \mathbb{P}_x [\mathcal{R} \cap D^c = \emptyset]$$

is the maximal solution of $\Delta u = 4u^2$ in D.

4.2 The ∂-polar sets

The above formula cannot be extended in higher dimension the same way. The density of the exit measure does not exist if $d \geq 3$. Furthermore, there exist compact sets $K \subset \partial D$ such that

$$\mathbb{P}_x[\mathcal{E}_D \cap K \neq \emptyset] = 0.$$

Such sets are called ∂-polar sets ("boundary polar sets"). This means that a.s. no path W_s will exit D through K. We can prove the following result (see [14]).

Proposition 7. *Let* D *be a bounded Lipschitz domain. Let* K *be a compact subset of* ∂D. *The function*

$$u(x) = -\log \mathbb{P}_x [\mathcal{E}_D \cap K = \emptyset]$$

is the maximal nonnegative solution of

$$\begin{cases} \Delta u = 4u^2 \quad \text{in} \quad D \\ \lim_{x \in D; x \to y} u(x) = 0 \quad \forall y \in \partial D \setminus K. \end{cases} \tag{7}$$

We will now give a characterization of the ∂-polar sets. The next theorem is due to Le Gall [16] and has been extended by Dynkin and Kuznetsov [9] in the general case. Recall the capacity defined in the previous section.

Theorem 8. *Let $D \subset \mathbb{R}^d$ be an open bounded set, sufficiently smooth (C^5). A compact set K of ∂D is ∂-polar if and only if*

- *$K \neq \emptyset$ if $d \leq 2$,*

- *$\mathrm{cap}_{d-3}(K) = 0$ if $d \geq 3$.*

Thus we deduce that (7) has a non trivial nonnegative solution if and only if $d \leq 2$ and $K \neq \emptyset$ or $d \geq 3$ and $\mathrm{cap}_{d-3}(K) > 0$.

4.3 Moderate solutions

We would like to characterize all the nonnegative solutions of $\Delta u = 4u^2$ in D, bounded by harmonic function. Such a solution will be called a moderate solution. For example assume D is a bounded regular domain, and consider a continuous nonnegative function φ defined on ∂D. We know that the function $u(x) = -\log \mathbb{E}_x\left[\mathrm{e}^{-(X_D,\varphi)}\right]$ is the only solution of (4). And we have seen in section 2.2 that

$$u(x) + 2\mathrm{E}_x\left[\int_0^{\tau_D} u(\xi_s)^2 ds\right] = \mathrm{E}_x[\varphi(\xi_{\tau_D})],$$

where $\tau_D = \inf\{t > 0, \xi_t \in D^c\}$. Now the function $h(x) = \mathrm{E}_x[\varphi(\xi_{\tau_D})]$ is harmonic in D and $u \leq h$. Furthermore u is the maximal solution of $\Delta u = 4u^2$ in D, bounded by h. Indeed, let v be an other nonnegative solution bounded by h. And assume that $u \leq v \leq h$ in D. By continuity we get $v_{|\partial D} = \varphi$. Since the nonnegative solution of (4) is unique, we get that $v = u$.

We now give a complete description of all the moderate solutions. Let D be a bounded domain in $\mathbb{R}^d$, $d \geq 2$.

Proposition 9. *Let u be a moderate solution of $\Delta u = 4u^2$ in D. Then there exists a unique harmonic function h such that for all $x \in D$,*

$$u(x) + 2\mathrm{E}_x\left[\int_0^{\tau_D} u(\xi_s)^2 ds\right] = h(x). \tag{8}$$

Furthermore the function h is the smallest harmonic function dominating u.

Proof. In order to prove this proposition, we consider an increasing sequence of regular open set D_n such that $\bar{D}_n \subset D_{n+1} \subset D$ and $D = \bigcup_{n \geq 1} D_n$. Let u be a moderate solution. Since u solves (4) in D_n with boundary condition $\varphi = u$, we deduce that for all $x \in D_n$,

$$u(x) + 2\mathrm{E}_x\left[\int_0^{\tau_{D_n}} u(\xi_s)^2 ds\right] = \mathrm{E}_x[u(\xi_{\tau_{D_n}})]. \tag{9}$$

Let h denotes the limit of the nondecreasing sequence of functions $\mathrm{E}_x[u(\xi_{\tau_{D_n}})]$. ($h$ is defined in D.) Since u is bounded by an harmonic function let say g, we get for $x \in D$:

$$h(x) = \lim_{n\to\infty} \mathrm{E}_x[u(\xi_{\tau_{D_n}})] \leq \lim_{n\to\infty} \mathrm{E}_x[g(\xi_{\tau_{D_n}})] = g(x).$$

Now by dominated convergence, h is clearly harmonic in D. Letting n goes to infinity in (9), we get (8). And by construction h is clearly the smallest harmonic function dominating u. $\square$

The next step is to characterize all the harmonic function h such that there exists a function $u \geq 0$ satisfying (8). Let us assume that D is bounded and sufficiently smooth (at least C^5).

Theorem 10. *There exists a one-to-one correspondence between the moderate solutions of $\Delta u = 4u^2$ in D and finite measures on ∂D that does not charge ∂-polar sets. The correspondence is given by*

$$u(x) + 2\int G_D(x,y)u(y)^2 dy = \int_{\partial D} P_D(x,y)\nu(dy),$$

where G_D and P_D are respectively the Green function on D and the associated Poisson kernel.

Let $\mathcal{N}$ be the set of finite measures on ∂D which doesn't charge ∂-polar set. We will write u_ν for the moderate solution corresponding to the measure $\nu \in \mathcal{N}$.

In fact we have a probabilistic representation of u_ν. Let D_n be an increasing sequence of bounded open sets such that $D = \cup_{n\geq 1} D_n$. Consider the harmonic function $h(x) = \int_{\partial D} P_D(x,y)\nu(dy)$. We can built a continuous additive functional $(A^h_s, s \geq 0)$ of the Brownian snake in the following way: for all $s \geq 0$,

$$A^h_s = \lim_{n\to\infty} \int_0^s h(\hat{W}_s) dL^{D_n}_s.$$

We have $A^h_\infty = \lim_{n\to\infty} < X_{D_n}, h >$. This convergence can be proved by martingale techniques using the so called special Markov property. The function u_ν is then defined by:

$$u_\nu(x) = -\log \mathbb{E}_x \left[\mathrm{e}^{-A^h_\infty} \right] \quad x \in D.$$

4.4 Trace of solutions in the super critical case

The following results are recent work from Dynkin and Kuznetsov [10] and Kuznetsov [13]. They have been presented by their author in September. I will only present them for the nonnegative solutions of $\Delta u = 4u^2$ in $D \subset \mathbb{R}^d$, a smooth bounded domain. We first define the Borel set of singular points of a nonnegative solution u:

$$\mathrm{SG}(u) = \left\{ y \in \partial D; \mathrm{P}^D_{x\to y} a.s. \int_0^\tau u(\xi_s) ds = \infty \right\},$$

where $\mathrm{P}^D_{x\to y}$ is the law of the h-transform of the Brownian motion started at point $x \in D$ with respect to the Poisson kernel $P_D(\cdot, y)$. Intuitively $\mathrm{P}^D_{x\to y}$ is the law of the Brownian motion started at x "conditionally" on going out of D at point y. In dimension $d = 2$, the set of singular points of the solution with trace (K, ν) is $\mathrm{SG}(u) = K$. In higher dimension, if u is a moderate solution then $\mathrm{SG}(u)$ is a ∂-polar set.

Let $\mathcal{N}'$ be the set of all measures obtained as increasing limits of measures of $\mathcal{N}$. (Notice the measures of $\mathcal{N}'$ are not necessarily σ-finite.) A solution u is called σ-moderate, if it is an increasing limit of moderate solutions u_{ν_n}. Let $\mu \in \mathcal{N}'$ be the increasing limit of ν_n. Since the limit u is independent of the sequence (ν_n) which converge to μ we write u_μ for the limit.

For every Borel set $\Gamma \subset \partial D$, we put $u_\Gamma(x) = \sup\{u_\nu(x); \nu(\Gamma^c) = 0, \nu \in \mathcal{N}\}$. The function u_Γ is σ-moderate. We say that Γ is finely closed if $\mathrm{SG}(u_\Gamma) \subset \Gamma$. This define a finer topology on ∂D than the induced topology. We write $u_{\Gamma,\mu}$ for the maximal solution dominated by $u_\Gamma + u_\mu$.

The trace of a nonnegative solution u is the pair (Γ, μ) defined by $\Gamma = \mathrm{SG}(u)$ and μ is a function defined on $\mathcal{B}(\partial D)$ by: for every Borel set $B \subset \partial D$,

$$\mu(B) = \sup\{\nu(B); \nu \in \mathcal{N}, \nu(\Gamma) = 0, u_\nu \le u\}.$$

Theorem 11. *The trace (Γ, μ) of a nonnegative solution of $\Delta u = 4u^2$ in D has the following properties*

i) *Γ is finely closed.*

ii) *μ is a σ-finite measure, $\mu \in \mathcal{N}'$, $\mu(\Gamma) = 0$ and $\mathrm{SG}(u_\mu) \subset \Gamma$.*

Moreover, $u_{\Gamma,\mu}$ is the maximal σ-moderate solution dominated by u.

We say that two pairs (Γ, μ) and (Γ', μ') are equivalent if $\mu = \mu'$ and $\Gamma \triangle \Gamma'$ is ∂-polar. Clearly $u_{\Gamma,\mu} = u_{\Gamma',\mu'}$ if the two pairs (Γ, μ) and (Γ', μ') are equivalent.

Theorem 12. *Let (Γ, μ) satisfy i) and ii) of the above theorem, then the trace of $u_{\Gamma,\mu}$ is equivalent to (Γ, μ). Furthermore, $u_{\Gamma,\mu}$ is the minimal solution with this property and the only σ-moderate one.*

There is still an open question: Are the σ-moderate solutions the only nonnegative solutions of $\Delta u = 4u^2$ in D? (The answer is yes if the dimension is less or equal to 2 (cf section 4.1).)

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