On a simple model of the interaction between bus and traffic flow[1].

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ABSTRACT:

The aim of this paper is to provide a simple model of the interaction between buses and the surrounding traffic flow. The traffic flow is assumed to be described by a first order macroscopic model of the LWR type. As a consequence of their kinematics, which in a large measure can be considered to be independent of the flow of other vehicles, buses should be considered as a moving capacity restriction from other drivers point of view. This simple interaction model is analysed, mainly by considering the moving frame associated to the bus in order to derive analytical computation rules for the derivation of the effects of the buses present in the traffic flow. After deriving traffic equations in the moving frame associated to a bus, the usual basic concepts of first order models are generalized to the moving frame, including those of relative traffic supply and demand. The solution to the classical Riemann problem in the moving frame yields then a simple model for bus-traffic interaction, assuming that the dimension of the bus can be neglected. Finally, some tentative results concerning the inclusion of buses into discretized first order traffic flow models are given, under the asumption that the discretization results from Godunov's scheme.

KEYWORDS:

Bus, traffic flow, macroscopic theory, first order model, discretization.

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1   Introduction.

The new interest for dynamic traffic network simulation models due to the deployment of intelligent transport systems has resulted in recent years in the apparition of various types of models. New generations of microscopic models have thus been developed, such as INTEGRATION [VA 94] or PARAMICS [CD 96], which are explicitly devoted to repesenting wide area networks. However, constraints due to the microscopic nature of these models make them difficult to calibrate and use for wide areas. This is the reason for which a renewed interest in macroscopic models is also observed, resulting in the development of models such as DYNASMART [JMH 94] or the Cell Transmission model proposed by Daganzo [DA 94], METACOR [EHP 94], STRADA [LB 97], or the extensions of CONTRAM [HRRS 92].

To deal with urban area networks, and particularly to make possible studies concerning various transport modes, these models need to take into account public transport. This has been carried out for years in microscopic models [CGHT 77], but does not constitute a classical approach in dynamic macroscopic models.

The purpose of this paper is to describe a possible way to integrate buses into a first order macroscopic flow model, in order to describe the interactions between the bus and the flow around it.

The contents of the paper are the following. After a short bibliographical review, we introduce the model of the bus as a moving capacity restriction and procede by analysing the propagation of a single moving singularity whithin the framework of the LWR (Lighthill-Whitham-Richards) model [LW 55], [RI 56]. We concentrate on the generalization of the concepts of local traffic supply and demand as introduced in [LE 96], and correlatively the solution to the generalized Riemann problem, which constitutes the key to the calculation of analytical solutions. As an application, we then develop a theory of interactions between a bus and its surrounding traffic flow, using the simplifying hypothesis that the bus can be modelled as a point, i.e. that its dimensions can be neglected. Finally, we address the problem of analyzing numerically the interaction of a moving bus with the surrounding traffic flow, whose dynamics are assumed to be discretized according to the Godunov scheme ([LE 96]), as in the model [DA 94] or in the STRADA model [BLL 95-96], [BLLM 96], [LB 97]. The difficulties involved are described, some ideas aiming to solve them are then exposed, and some tentative numerical results for simple cases are presented.

2   Modelling buses: general considerations.

There are two series of reasons why buses must be specifically taken into account in a traffic model:

1. their behaviour is different from other vehicles,

2. they are subject to specific actions (priority at traffic signals...).

The progression of buses through a road network presents several specific features:

- their routes are specific (bus lines),

- their kinematic characteristics (speed, acceleration...) are different,

- they stop at particular locations inside the links of the network (bus stops),

- in most cases, buses are isolated vehicles; it is only at particular places that buses can be considered as a continuous flow.

Since buses share the same road links as other vehicles and have different characteristics, the interactions between both types of vehicles may be complex.

In a microscopic model, the buses particular features may be taken into account with few difficulties, as has been done long ago [CGHT 77], NETSIM [WL 74]: the car-following law has to be adapted to the bus characteristics (minimum headway, acceleration, free speed...), the overtaking model, if any, has also to be adapted, bus stops have to be introduced. However, this is mainly an engineering problem with few theoretical implications.

In macroscopic models, the question is much more complex as the bus is an heterogeneous object in the flow. Some models used in programs computing traffic signals settings such as BUS-TRANSYT [PW 77], which can be considered macroscopic, do take buses into account, but in a very simple way: in the TRANSYT model, buses are considered as a separate continuous flow, with a specific computation of travel times along the network link taking account of their specific speeds, their stops and particular dispersion features. A 'shared stopline' facility is introduced to maintain some kind of FIFO discipline in the queue at the end of each link. On the other hand, there is no consideration of the effect of buses over traffic flow conditions. The CONTRAM model [LTB 78] represents buses in a very similar way.

In the present work, we attempt to modelize the dynamics of buses, especially their interaction with the surrounding traffic flow, whithin the framework provided by a first order macroscopic traffic flow model of the LWR type. The rationale behind this approach is the following. As noted above, the bus, owing to its size and to the special properties of its kinematics, does not partake of the general flow of traffic. On the contrary, it affects the general traffic flow adversely, and should be considered, from the point of view of regular drivers, as a *moving capacity restriction*. This is the very simple model of the interaction between bus and general traffic which we shall develop in the sequel. Therefore, as far as the inclusion of buses into macroscopic models goes, it makes sense to consider a first order macroscopic traffic model for the general traffic, since such models, in contrast to second order models, accomodate capacity restrictions in a straightforward and even intrinsic manner. The only difficulties here result from the fact that the capacity restriction associated to the bus is *moving*, and that its movement may or may not be restricted by the movement of the general traffic, depending on the nature of the interaction between the bus and the surrounding traffic.

3   Moving capacity restrictions in the LWR model.

In this section we set notations, recall the main elements of the LWR (Lighthill-Whitham-Richards) [LW 55], [RI 56] model and the concepts of local traffic supply and demand, introduce more precisely the concept of the bus as a moving capacity restriction, and develop the technical tools for the analysis of the effects on traffic flow of moving capacity restriction.

3.1   The LWR model and some of its extensions.

The basic variables of the LWR model [LW 55], [RI 56] are $K(x,t)$, the density at point x and at time t, $Q(x,t)$, the flow at point x and at time t and $V(x,t)$, the space-time average speed at point x and at time t, defined by the relationship $Q = KV$. The basic

equations of the model are:

$$\frac{\partial K}{\partial t} + \frac{\partial Q}{\partial x} = 0 \quad \text{(conservation of vehicles)},$$
(1)

$$Q = KV \quad \text{(definition equation of } V\text{)},$$
(2)

$$V = V_e(K, x) \quad \text{(equilibrium speed-density relationship)}.$$
(3)

Of course these equations can be rewriten as

$$\frac{\partial K}{\partial t} + \frac{\partial}{\partial x} Q_e(K, x) = 0$$
(4)

with $Q_e(K, x) \stackrel{def}{=} KV_e(K, x)$ the equilibrium flow-density relationship. Typically, the equilibrium flow-density has the aspect illustrated by Figure 1.

The dependency of Q_e on x reflects the fact that the fundamental physical parameters of the model, i.e. the maximum flow and density Q_{max} and K_{max}, the critical density K_{crit} and the maximum speed V_{max}, depend generally on the position x, as the characteristics of the infrastructure change with the position. Let us recall that, following [LE 96] we define the *equilibrium traffic demand and supply functions* as the greatest possible outflow respectively inflow at a given point x, as a function of the local density κ; these functions are defined by:

(5)

— the equilibrium traffic demand function
$$\Delta_e(\kappa, x) \stackrel{def}{=} \begin{cases} Q_e(\kappa, x-) & \text{if} \quad \kappa \le K_{crit}(x-) \quad \text{(undercritical flow)} \\ Q_{max}(x-) & \text{if} \quad \kappa \ge K_{crit}(x-) \quad \text{(overcritical flow)} \end{cases}$$

— the equilibrium traffic supply function
$$\Sigma_e(\kappa, x) \stackrel{def}{=} \begin{cases} Q_{max}(x+) & \text{if} \quad \kappa \le K_{crit}(x+) \quad \text{(undercritical flow)} \\ Q_e(\kappa, x+) & \text{if} \quad \kappa \ge K_{crit}(x+) \quad \text{(overcritical flow)} \end{cases} .$$

The symbols $x-$ and $x+$ imply as usual that the left and right limits are taken at point x for the quantities that contain these symbols among their arguments. The functions can be prolonged in a consistent fashion to the case $\kappa > K_{max}(x*)$ with $* = +$ or $-$: see [LE 97]. The corresponding equilibrium demand and supply functions are illustrated by Figures 2 and 3. For a piecewise continuous solution of the LWR model, it is then possible to define the *local traffic demand and supply* as:

(6)
$$\Delta(x, t) \stackrel{def}{=} \Delta_e(K(x-, t), x)$$
$$\Sigma(x, t) \stackrel{def}{=} \Sigma_e(K(x+, t), x) .$$

Further, for any piecewise continuous solution of the LWR model,

(7)
$$Q(x, t) = Min\,[\,\text{local upstream demand}\,,\,\text{local downstream supply}\,]$$
$$= Min\,[\,\Delta(x, t)\,,\,\Sigma(x, t)\,] .$$

This is of course the calculation rule of the *entropy solution* of the LWR model (see [GR 91]); its application ensures existence, unicity and continuous dependence on initial conditions of the solutions of this model. The above definitions and relations (5), (6), and (7) accomodate easily spatial discontinuities of Q_e, can be naturally generalized to yield

the Godunov discretization scheme for the LWR model, and allow generalization of the LWR model to networks.

If the equilibrium relationship Q_e admits a spatial discontinuity, the calculation rules that apply at, and in the vicinity of, the discontinuity are the following ([LE 96]):

- conservation of the flow at the discontinuity,

- the characteristics carry a constant flow value, and the state of the traffic (over- or undersaturated) varies smoothly along a characteristic,

- in case of multiple solutions to (4), the flow should be maximised, i.e. (7) should apply at the discontinuity, yielding the proper boundary condition for both upstream and downstream traffic.

These rules guarantee existence and unicity for the solutions of (4).

3.2 The bus model: basic ideas.

Let us now address the problem of modelling a bus in interaction with the surrounding traffic flow. The trajectory will be defined by the position of one point of the bus in time, say $y(t)$. The extremities of the bus (front and back) are $f(t)$ and $b(t)$, with of course:

$$\dot{y}(t) = \dot{f}(t) = \dot{b}(t) \quad \forall t \quad .$$

We assume that at point x, the maximum density $K_{max}(x)$ (one of the parameters of Q_e) is reduced by a fixed amount K_{bus} whenever a bus is present at that point; of course, K_{bus} represents the width of track occupied by the bus (≈ 0.2 vh/m). Hence, for all x in $[b(t), f(t)]$, the residual maximum density (i.e. storage capacity) for the regular traffic is

$$(8) \qquad K_{max}^{\beta}(x) \stackrel{def}{=} K_{max}(x) - K_{bus} \quad \forall x \in [b(t), f(t)] \quad .$$

If say the track has 4 lanes and the bus occupies one lane, it follows that $K_{max}^{\beta}(x)/K_{max}(x) = 3/4$. The Figure 4 illustrates this idea.

We assume then, following the analysis of incidents whithin the framework of the LWR model carried out in [BLLM 96] and [MO 97], that the maximum density constraint (8) affects the equilibrium flow-density relationship by affecting its fundamental parameters but *respecting its functional form*. We assume further that the desired speed of traffic remains unchanged. It follows that the equilibrium flow-density relationship Q_e^{β} in presence of a bus is given by:

$$(9) \qquad Q_e^{\beta}(\kappa, x) \stackrel{def}{=} \alpha Q_e(\kappa/\alpha, x) \quad ,$$

with

$$\alpha(x) \stackrel{def}{=} K_{max}^{\beta}(x)/K_{max}(x) \quad .$$

Actually, it would be straightforward to modelize the effect of a desired speed constraint, say $V_{max}^{\beta}(x)$, in case empirical evidence pointed in that direction. Indeed the resulting equilibrium flow-density relationship would be:

$$(10) \qquad Q_e^{\beta}(\kappa, x) \stackrel{def}{=} \beta Q_e(\kappa/\alpha, x) \quad ,$$

with

$$\gamma(x) \stackrel{def}{=} V_{max}^{\beta}(x)/V_{max}(x)$$

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and

$$\beta(x) \overset{def}{=} \alpha(x)\gamma(x) \quad .$$

The parameters α, β, $\gamma = \beta/\alpha$ are called incidents severity level coefficients in [BLLM 96] and [MO 97]. The following relationships result for the physical parameters:

$$
\begin{aligned}
K^\beta_{max} &= \alpha\, K_{max} \\
Q^\beta_{max} &= \beta\, Q_{max} \\
V^\beta_{max} &= \gamma\, V_{max} \\
K^\beta_{crit} &= \alpha\, K_{crit} \\
V^\beta_{crit} &= \gamma\, V_{crit} \quad .
\end{aligned}
$$

As far as the trajectory of the bus is concerned, it will be approximated by assuming that $\dot{y} = 0$ (bus-stops) or that $\dot{y}$ is equal to the desired bus-speed V_b (if the bus enjoys special lanes) or is the minimum of the desired bus-speed V_b and the local traffic speed, i.e.:

(11) $$\dot{y} = Min\,[\,V_b\,,\,V(x,t)\,] \quad .$$

4 Analysis of a moving singularity.

4.1 Introduction

The basic problem we address in this section is simple. As we have just seen, the equilibrium flow-density relationship changes at the extremities of the bus, i.e. at points $b(t)$ and $f(t)$. Hence in order to calculate the interaction of the bus with the surrounding traffic, we must first determine the calculation rules for the local traffic dynamics in the vicinity of each of the extremities of the bus. In order to achieve this, the necessary step is to consider a single moving singularity and the associated inhomogeneous Riemann problem. The notations for this problem are the following:

- $z(t)$ the singularity,

- the superscripts u and d, refering to quantities upstream and downstream of the singularity,

- $Q^u_e(\kappa, x)$ and $Q^d_e(\kappa, x)$ the equilibrium flow-density relationships upstream and downstream of the singularity,

- K^u and K^d the density upstream and downstream of the singularity, these densities are assumed to be uniform. See Figure 5 for an illustration of the data of the inhomogeneous Riemann problem.

Typically,

- $\dot{z}(t) = \dot{y}(t) \quad \forall t$,

- if $z(t) = b(t)$, $Q^u_e = Q_e$ and $Q^d_e = Q^\beta_e$,

- if $z(t) = f(t)$, $Q^u_e = Q^\beta_e$ and $Q^d_e = Q_e$.

Considering that the rules for the solution of the Riemann problem are well-known if the singularity is static, the natural way to solve the moving singularity problem is to solve it in a frame moving at the speed of the singularity, i.e. $\dot{y}(t)$.

4.2   The moving frame.

The moving frame is defined by the change of variables:

(12)
$$\begin{cases} \xi &=& x - y(t) \\ \tau &=& t \end{cases}$$

which implies, for the partial derivation operators:

(13)
$$\begin{cases} \frac{\partial}{\partial x} &=& \frac{\partial}{\partial \xi} \\[2mm] \frac{\partial}{\partial t} &=& \frac{\partial}{\partial \tau} - \dot{y}(\tau)\frac{\partial}{\partial \xi} \end{cases} .$$

Let us denote:

(14)
$$q(\xi, \tau) \stackrel{def}{=} Q(\xi, \tau) - \dot{y}(\tau) K(\xi, \tau)$$

the *relative flow* (it is the flow that passes the bus), and let us define:

(15)
$$q_e(\kappa, \xi, \tau) \stackrel{def}{=} Q_e(K, \xi + y(\tau)) - \dot{y}(\tau)\kappa$$

the *relative equilibrium flow-density relationship*. The relative flow is related to the density by:

$$q(\xi, \tau) = q_e(K(\xi, \tau), \xi, \tau) .$$

It follows from (13) that the conservation of vehicles (1) applies in the moving frame under the form:

(16)
$$\frac{\partial K}{\partial \tau} + \frac{\partial q}{\partial \xi} = 0$$

and the fundamental equation (4) is equivalent to:

(17)
$$\frac{\partial K}{\partial \tau} + \frac{\partial}{\partial \xi} q_e(K, \xi, \tau) = 0 .$$

This equation (17) means that the traffic flow in the moving frame is described by a conservation equation associated with the flux function q_e given by (15), and illustrated by Figure 6. It must nevertheless be noted that the similarity between the equations of traffic in the moving and the fixed frame cannot be pushed to far. Indeed, in the moving frame, negative (relative) flows are acceptable, and the equilibrium flow-density relationship q_e is not concave.

Important values are:

- $q_{max}(\xi, \tau) \stackrel{def}{=} Max_\kappa\, q_e(\kappa, \xi, \tau)$ the *maximum relative flow*, a value depending both on position ξ (by virtue of the dependence of Q_e on x) and on time τ (by virtue of the dependence of the bus speed on time),

- $k_{crit}(\xi, \tau) \stackrel{def}{=} Arg_\kappa Max\, q_e(\kappa, \xi, \tau)$ the value of density for which the maximum relative flow is obtained, depending on both position ξ and time τ, for the same reasons as q_{max}.

One of the main reasons it is actually necessary to consider the moving frame is that in general:

(18)
$$\begin{array}{rcl} q_{max} &\neq& Q_{max} - \dot{y}K_{crit} , \\ k_{crit} &\neq& K_{crit} . \end{array}$$

Density values such that $K \leq k_{crit}$ will be called *relatively under-critical*, density values such that $K \geq k_{crit}$ will be called *relatively over-critical*.

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4.3 Shock-waves and Characteristics.

Shock-waves and characteristics are the same in the moving and the fixed frame.

Let us consider shock-waves first, and let us consider a discontinuity in flow $[Q]$ and density $[K]$ moving in the fixed frame (x, t) at speed

$$v = [Q]/[K] \quad .$$

In the moving frame (ξ, τ), the corresponding discontinuity will carry jump values $[q]$ in relative flow and $[K]$ in densities, at (relative) speed

$$\nu \overset{def}{=} v - \dot{y} \quad .$$

By definition (14), it follows that

$$[q] = [Q] - \dot{y}[K]$$

and consequently also that

$$\nu = [q]/[K] \quad .$$

As a consequence, since $z(t)$ is fixed in the moving frame, any discontinuity in density and relative flow resulting from the difference between the equilibrium relationships upstream and downstream of this point satisfies to

$$[q]_{z(t)} = 0 \quad ,$$

since in that case $\nu = 0$. Hence, *the relative flow is conserved* at $z(t)$.

In the same spirit, characteristics in the fixed frame carry an invariant which is the flow Q when the equilibrium flow-density relationship Q_e depends on both the density K and the position x. For instance, a characteristic carrying flow value $\underline{Q}$ and originating at point $(\underline{x}, \underline{t})$ would be given by the following equation:

$$\left|
\begin{array}{l}
\frac{dx}{dt} = \frac{\partial Q_e}{\partial K}(K(x,t), x) \quad \forall t \geq 0 \\[2mm]
Q_e(K(x(t), t), x(t)) = \underline{Q} \quad \forall t \geq 0 \\[2mm]
x(\underline{t}) = \underline{x}
\end{array}
\right.$$

with the additional condition that the state of traffic (under- or over-critical) should vary continuously, in order for the density on the characteristic to be uniquely determined by the flow value $\underline{Q}$. It follows from equations (12), (14), (15), that the same characteristic lines in the moving frame carry the same invariant $\underline{Q} = q + \dot{y}K$. Hence, in the moving frame, the characteristic originating at point $(\underline{\xi}, \underline{\tau})$, carrying the invariant $\underline{Q}$ would be given by the following equation:

$$\left|
\begin{array}{l}
\frac{d\xi}{d\tau} = \frac{\partial q_e}{\partial K}(K(\xi, \tau), \xi, \tau) \quad \forall \tau \geq 0 \\[2mm]
q_e(K(\xi(\tau), \tau), \xi, \tau) = \underline{Q} - \dot{y}(\tau)K(\xi(\tau), \tau) \quad \forall \tau \geq 0 \\[2mm]
\xi(\underline{\tau}) = \underline{\xi}
\end{array}
\right.$$

with the additional condition that the relative state of traffic (relative under- or over-critical) should vary continuously, in order for the density on the characteristic to be

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uniquely determined by the invariant value $\underline{Q}$. It should be noted, an this is another peculiarity of the traffic equations in the moving frame, that the invariant transmitted by characteristics, i.e. $Q = q + \dot{y}K$, is not the same as the quantity conserved at the singularity, i.e. q.

If the dependence of Q_e on x is piecewise constant, and the bus speed $\dot{y}$ constant, characteristic lines in both the fixed and the moving frame are piecewise straight lines, a fact which we shall use for analytical calculations. Another salient fact is that, since q_e is not a concave function of density, one might have to consider in some special cases *acceleration shock-waves* between states of traffic situated in the non-concave region, as shown in [LE 97] and following rules explained in section II.6 of [GR 91].

4.4   Supply and demand in the moving frame.

These quantities are defined as usual as the the greatest possible inflow and outflow at any given point. In the present case, it is necessary to consider *relative* inflows and outflows. The relative supply and demand at a given point say z are easiest calculated by solving a Riemann problem at that point. For relatively undercritical density K^u upstream of z, the greatest possible relative outflow at z is necessarily $q_e(K^u)$, since otherwise there would be an initial shockwave originating from z with positive speed if the outflow were greater, implying a contradiction. If we consider the relative supply at point z associated to relatively overcritical density K^d downstream of z, we can show that the greatest possible inflow, for similar reasons, is necessarily $q_e(K^d)$ (if the inflow were greater, there would be an initial shockwave originating from z with negative speed, implying a contradiction). Since relative supply and demand cannot by definition exceed q_{max}, we define, by analogy with (5), the equilibrium demand and supply functions in the moving frame, δ_e and σ_e, as functions of the density κ, the position ξ and the time τ:

(19)
$$
\begin{cases}
\text{—} & \text{the equilibrium traffic demand function (in the moving frame)} \\
& \delta_e(\kappa, \xi, \tau) \stackrel{def}{=} \begin{cases} q_e(\kappa, \xi-, \tau) & \text{if } \kappa \le k_{crit}(\xi-, \tau) & \text{(undercritical flow)} \\ q_{max}(\xi-, \tau) & \text{if } \kappa \ge k_{crit}(\xi-, \tau) & \text{(overcritical flow)} \end{cases} \\
\text{—} & \text{the equilibrium traffic supply function (in the moving frame)} \\
& \sigma_e(\kappa, \xi, \tau) \stackrel{def}{=} \begin{cases} q_{max}(\xi+, \tau) & \text{if } \kappa \le k_{crit}(\xi+, \tau) & \text{(undercritical flow)} \\ q_e(\kappa, \xi+, \tau) & \text{if } \kappa \ge k_{crit}(\xi+, \tau) & \text{(overcritical flow)} \end{cases} .
\end{cases}
$$

Thes functions are illustrated by Figure 7. In the above formulas, the symbols - and + imply the usual limits, and under- and over-critical should be understood as relative. It must be emphasized that
$$
\begin{bmatrix} \delta_e \ne \Delta_e - \dot{y}\kappa \\ \sigma_e \ne \Sigma_e - \dot{y}\kappa \end{bmatrix}
$$

To check the formulas (19), it suffices to check that the above values of relative demand and supply, which constitute upper bounds of relative outflow and inflow can be attained. Hence it suffices to consider two special Riemann problems, one with initial conditions $K^u = \kappa$, $K^d = 0$, yielding a flow at the origin $q_0 = \delta_e(\kappa)$, and one with initial conditions $K^u = k_{crit}$, $K^d = \kappa$, yielding a flow at the origin $q_0 = \sigma_e(\kappa)$. These results are checked on Figures 8,9 and 10,11, representing characteristics charts in the moving frame. The bus speed $\dot{y}$ is supposed constant and q_e homogeneous, which enables analytical calculation but does not impair the generality of the result. The rules applied for the calculation are the same as those applying in a fixed frame to a fixed singularity (recalled at the end of subsection 3.1), but for the modifications required by the moving frame setting:

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- conservation of the relative flow at the discontinuity,

- the characteristics carry a constant flow value in the fixed frame (or the above-mentionned invariant in the moving frame), and the state of the traffic (over- or undersaturated) varies smoothly along a characteristic,

- in case of multiple solutions to (17), the relative flow should be maximised, yielding the proper boundary condition for both upstream and downstream traffic.

As a consequence, it is then natural to define the relative local traffic demand $\delta(\xi, \tau)$ and supply $\sigma(\xi, \tau)$ as:

(20)
$$
\left|
\begin{array}{ll}
\delta(\xi, \tau) & \overset{def}{=} \quad \delta_e(K(\xi-, \tau), \xi, \tau) \\
\sigma(\xi, \tau) & \overset{def}{=} \quad \sigma_e(K(\xi+, \tau), \xi, \tau) \quad .
\end{array}
\right.
$$

4.5 The generalized Riemann Problem.

By analogy with (7), we shall now show that for any piecewise continuous solution of the LWR model, the following result applies in the moving frame:

(21)
$$
\begin{aligned}
q(\xi, \tau) \;=\;& Min\,[\,\text{local upstream demand , local downstream supply}\,] \\
\;=\;& Min\,[\,\delta(\xi, \tau)\,,\,\sigma(\xi, \tau)\,] \quad .
\end{aligned}
$$

This result generalizes to the moving frame the classical method of construction of the entropy solutions of the LWR model. It is self-evident anywhere except at discontinuities of q_e. To check (21) at such a discontinuity, it suffices to consider a Riemann problem with piecewise homogeneous data $q_e^u = Q_e^u - \dot{y}\kappa$, $q_e^d = Q_e^d - \dot{y}\kappa$ (q_e constant upstream and downstream of the singularity z, fixed in the moving frame and taken as the origin) and initial data K^u, K^d. Calling q_0 the flow at the origin, it suffices to check that it is possible to construct a solution to the Riemann problem satisfying

$$
q_0 = Min[\delta_e(K^u, u), \sigma_e(K^d, d)]
$$

with $\delta_e(., u)$ and $\sigma_e(., d)$ the upstream demand and downstream supply functions in the moving frame associated to q_e^u and q_e^d by (19). The construction method is straightforward: the above flow value of q_0 is imposed as a boundary condition at the origin z, and the Riemann problem is solved separately upstream and downstream of the singularity. Let us denote:

$$
\delta^u \overset{def}{=} \delta_e(K^u, u)
$$

the upstream demand, and

$$
\sigma^d \overset{def}{=} \sigma_e(K^d, d)
$$

the dowstream supply. Two sets of boundary conditions result, set 1 when $q_0 = \delta^u \leq \sigma^d$, including (BC_u^1) for the upstream solution and (BC_d^1) for the downstream solution, and set 2 when $q_0 = \sigma^d \leq \delta^u$, including (BC_u^2) for the upstream solution and (BC_d^2) for the downstream solution. The description of the boundary conditions and their associated solution is the following:

(BC_u^1) : $q_0 = \delta^u$, $K(z-) = Arg_{\kappa \leq k_{crit}}[\delta_e(\kappa, u) = q_0]$. The corresponding solutions, depending on the values of K^u, are described on Figure 12.

(BC_d^1) : $q_0 = \delta^u$, $K(z+) = Arg_{\kappa \leq k_{crit}}[\delta_e(\kappa, d) = q_0]$. The corresponding solutions, depending on the values of K^u, are described on Figure 13.

(BC_u^2) : $q_0 = \sigma^d$, $K(z-) = Arg_{\kappa \geq k_{crit}}[\sigma_e(\kappa, u) = q_0]$. The corresponding solutions, depending on the values of K^u, are described on Figure 14.

(BC_d^2) : $q_0 = \sigma^d$, $K(z+) = Arg_{\kappa \geq k_{crit}}[\sigma_e(\kappa, d) = q_0]$. The corresponding solutions, depending on the values of K^u, are described on Figure 15.

To obtain the relevant solutions of the generalized Riemann problem, any solution associated to (BC_u^1) must be combined with any solution associated to (BC_d^1), and any solution associated to (BC_u^2) must be combined with any solution associated to (BC_d^2).

5   Dynamics of a bus: the approximation of the bus as a point.

5.1   Principle of the analysis.

Macroscopic models are only adequate at some relevant physical scale, if only by virtue of the fact that the basic macroscopic variables, flow, density and speed, only make sense when defined as (local) averages. Hence, making allowance to the fact that the size of a bus is actually of the order of magnitude of the lower admissible values for the length scale of macroscopic models, it is only natural to try to modelize the bus-traffic flow interaction by neglecting the size of the bus, i.e. considering the bus as a point. This is what we shall do now.

We must give a rigorous content to the idea of neglecting the size of the bus. It is simple, for homogeneous conditions upstream and dowstream of the bus: solutions between the extremities $b(t)$ and $f(t)$ tend to a stationnary state, and if the dimensions of the bus can be neglected, then transitory solutions between the extremities $b(t)$ and $f(t)$ can be ignored, and only homogeneous solutions are relevant. Henceforth, we shall denote by:

- $K^\beta(t)$ the density along the bus, i.e. between points $b(t)$ and $f(t)$,

- $Q^\beta(t)$ the flow along the bus,

- $q^\beta(t) = Q^\beta(t) - \dot{y}(t)K^\beta(t)$ the relative flow along the bus.

In the general case, the same approximation applies, since locally, at an adequate space and time scale, conditions upstream and dowstream of the bus can be considered homogeneous. Indeed, neglecting the size of the bus allows us to look for solutions that are locally scale invariant.

5.2   Analytical solutions.

Let us consider then the case of a single bus, with initial conditions K^u upstream and K^d downstream. Since we look only for scale invariant solutions, we must also assume the bus speed constant, and we shall determine by the same token the initial condition K^β for the traffic along the bus (which should really be the limit of the transitory solution),

and the relative flow along the bus, q^β (see Figure 16). Both K^β and q^β are constant. The relative demand upstream of the bus is called δ^u and given by:

$$\delta^u \stackrel{def}{=} \delta_e(K^u, u)$$

whereas the relative downstream supply σ^d

$$\sigma^d \stackrel{def}{=} \sigma_e(K^d, d) \quad .$$

Finally, the maximum relative flow along the bus will be denoted q^β_{max} as usual. It follows from the calculation rules given in the preceding section that:

$$q^\beta = Min\left[q^\beta_{max}, \delta^u, \sigma^d \right] \quad .$$

To determine K^β as well as the rest of the solution, it is necessary to consider 3 cases and to apply again the calculation rules developed in the preceding section. The three cases will simply be described, and the corresponding solution described by its characteristics chart in the moving frame.

Case 1 : $q^\beta_{max} \leq Min\left[\delta^u, \sigma^d \right]$. Then: $K^\beta = k^\beta_{crit}$ (let us recall that k^β_{crit} is the critical density in the moving frame (Figure 17)).

Case 2 : $\delta^u \leq Min\left[q^\beta_{max}, \sigma^d \right]$. Then:

$$K^\beta = Arg_{\kappa \leq k^\beta_{crit}}[\delta^\beta_e(\kappa) = q^\beta] \quad .$$

An example of solution is depicted on Figure 18.

Case 3 : $\sigma^d \leq Min\left[q^\beta_{max}, \delta^u \right]$. Then:

$$K^\beta = Arg_{\kappa \geq k^\beta_{crit}}[\sigma^\beta_e(\kappa) = q^\beta] \quad .$$

An example of solution is depicted on Figure 18.

Not all possible cases have been depicted for the upstream and downstream traffic, since shock-waves or rarefaction waves might both be possible in several instances. Nevertheless, the density immediately upstream and downstream of the bus results in all cases easily from the flow value (q^β) and the traffic state (over- or under-critical) in the moving frame.

Remarks.

1. It should be noted that the bus, when considered as a point, has an effect on the surrounding traffic only in case 1. Otherwise, the flow being the same whether the bus is present or not, and the traffic state upstream and downstream being independent of the presence of the bus, the traffic flow is unaffected by the bus. The limit of this remark is of course set by what has been neglected in the above analysis: the transitories and the bus size.

2. If we consider the general case (non homogeneous and non constant data, $f(t) = b(t) = y(t)$) it follows that:

(22) $$ q^\beta(t) = Min\big[\, q^\beta_{max}(t), \delta(y(t), t), \sigma(y(t), t)\,\big] $$

with, following (20):

$$ \left| \begin{array}{lll} \delta(y(t), t) & \stackrel{def}{=} & \delta_e(K(y(t)-, t), 0, t) \\ \sigma(y(t), t) & \stackrel{def}{=} & \sigma_e(K(y(t)+, t), 0, t) \end{array} \right. . $$

Formula (22), considering that the relative flow is continuous in the vicinity of the bus, results in a boundary condition for the traffic flow upstream and downstream of the bus.

3. If the speed of the bus varies in time, it may be convenient to consider the fixed frame. The construction of analytical solutions is easy, since horizontal lines of the fundamental diagram in the moving frame (q_e, q^β_e as functions of S) become lines of slope $\dot{y}$ of the fundamental diagram in the fixed frame (Q_e, Q^β_e as functions of S).The reader is refered to Figure 17bis for an illustration.

6 Discretization of the model.

Except for a few straightforward configurations, of which the preceding section gives some examples, and for which analytical solutions may be determined, simulation is necessary to implement the model. This makes necessary a time and space discretization of the model. The basis of the discretization schemes studied here is the STRADA model [BLL 95-96], [BLLM 96], [LB 97].

6.1 The STRADA model

STRADA is a discretized first-order macroscopic model based on the Godunov scheme, as explained in [LE 96] (see also [GR 91], [DA 95], [LE 84]). It belongs to the same family of models as DYNASMART [JMH 94] or the model proposed by Daganzo [DA 94]. Other features of the model include a complete modelling of intersections and the possibility to implement dynamic assignments procedures. STRADA also uses a particular flow/density diagram including two segments of parabola (Figure 1).

In order to study the integration of buses in the model, a simplified version has been used :

- A single link with constant geometric characteristics and no intersection is considered

- A unique bus is introduced

However, the discretization scheme has to be designed in a way such as to allow for geometric discontinuities such as changes in the number of lanes.

Basically, the STRADA link model is based on the discretization equation (4).

Let us consider two successive discretization cells i and $i + 1$, with a length δx and a time step δt:

Q_{i-1} Q_i Q_{i+1}

K_i K_{i+1}

Cell i Cell $i+1$
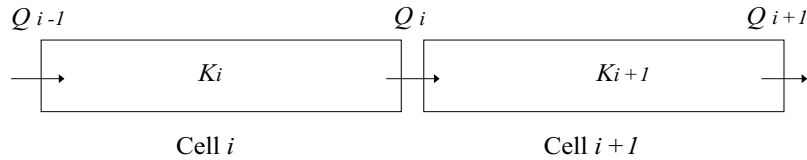
The discretization of the conservation equation is straightforward (the Godunov scheme is conservative):

(23)
$$K_i(t + \delta t) = K_i(t) + \frac{\delta t}{\delta x}\left(Q_{i-1}(t) - Q_i(t)\right)$$

The discretization of the flow/density relationship is based on the Godunov scheme. It results in the definition of *demand* at the exit of each cell and *supply* at the entrance :

- The demand $\Delta_i(K_i)$ willing to exit cell i is equal to the equilibrium flow $Q_e(K_i)$ if K_i is subcritical, and equal to the maximum flow otherwise.

- The supply $\Sigma_i(K_i)$ associated to cell i is equal to the maximum flow if K_i is subcritical and to the equilibrium flow $Q_e(K_i)$ otherwise.

These definitions are straightforward generalizations of formulas (5) and (6) and can be figured out as depicted on Figure 19). The flow exiting cell i, $Q_i(t)$, is computed as

$$Q_i(t) = Min\left[\Delta_i(K_i(t)), \Sigma_{i+1}(K_{i+1}(t))\right]$$

which generalizes (7). It can be shown ([LE 96]) that this last expression yields exactly the Godunov flux.

It can be noticed, as it will be used further on, that this discretization scheme consists of computing at each time step the exact solution of an approximated problem (this is the principle of the Godunov scheme). The approximation lies in considering the concentration as piece-wise constant in space (see Figure 20). Starting from this situation and examining the evolution of flows at the cells limits, it can be seen that a shock wave will start from the limit between cells i and $i+1$ when $K_i < K_{i+1}$, moving forward or backward, and that a fan-like characteristic scheme will be observed when $K_i > K_{i+1}$ (in the homogeneous case). In all cases, the flow at the $i \rightarrow i+1$ limit will be constant during time δt, provided the following constraint is respected :

$$\delta x \geq V_{max}\delta t$$

where V_{max} is the maximum speed (CFL (Courant-Friedrichs-Lewy) type condition).

The value of the flow is the one given by the supply/demand computation. One of the interesting features of this discretization scheme is that it is well suited to the description of space or time discontinuities such as road characteristics variations or traffic incidents [MO 97]. This is due to the fact that the computations of demand and supply can be made using different flow/density relationships. On the other hand, discontinuities considered till now were fixed ones. The problem raised by the bus model is the presence of a discontinuity that is moving relatively to otherwise fixed geometry and discretization.

6.2 Discretization of the bus model

6.2.1 Moving discretization

It has been demonstrated previously that the supply/demand scheme was still valid in the moving coordinates attached to the bus. The most natural idea the is to realise a moving discretization, with cells moving at the same speed as the bus. The bus itself is then a particular limit between two cells, which restrains the flow between these two cells, as depicted on Figure 21

The flow and concentration computations can the be made, in the moving frame coordinates, exactly as they are made for the basic model in the fixed frame. There is a difference in the definition of the demand and supply, which must be defined in the moving referential, in a fashion similar to formulas (19), as depicted on Figure 22 and also Figures 6, 7.

The constraints on the discretization parameters which guarantee that the flow between two cells is constant during δt (CFL constraints) are also slightly different:

$$\delta x \geq (V_{max} - V_b)\delta t$$
$$\text{and}$$
$$\delta x \geq (Q'(K_{max}) - V_b)\,\delta t$$

with $V_b \overset{def}{=} \dot{y}$.

For the cell limit corresponding to the bus location, the computation is similar except that an additional constraint applies on the flow in the moving coordinates.

Unfortunately, this discretization scheme can only be used to describe the flow around a single bus on a link with no singularity, i.e. a homogeneous link. If there is for instance a change in the number of lanes, this discontinuity will be considered as moving in the moving coordinates and the discretization scheme does not hold any longer. The real problem is to combine a discretization scheme with a discontinuity moving in its coordinates . From this viewpoint, it seems preferable to come back to a discretization into fixed cells, and to try to describe the effects of a moving singularity.

6.2.2 A moving singularity in a fixed discretization

The basic fixed discretization scheme is composed of cells having a length δx consistent with the time step δt :

- It must respect the constraint $\frac{\delta x}{\delta t} \geq V_{max}$

- if δx is too high, the model will present a high degree of numerical viscosity, which leads to undesirable effects such as unrealistically high speeds or infinite travel-times.

Actually, it is reasonable to consider that the only correct value is

$$\frac{\delta x}{\delta t} = V_{max}$$

The difficulty to represent the progression of a bus in such a scheme is that the bus speed is not necessarily consistent with the scheme. Since the bus is a singularity in traffic, the concentrations in front of it and behind it have to be considered separately, i.e it is necessary to consider two separate sub-cells in the cell where the bus is located, as depicted on Figure 23.

This leads to creating two cells whose length do not respect the discretization constraints : the assumption that the flow at the cell frontiers is constant during δt does not hold anymore, as there may be interactions between the entrance and the exit of the same cell. The solution to that problem could be to consider explicitly these interactions and to make a comprehensive computation of flows, but this leads to consider a high number of cases and would result in too high computational effort. On the other hand, since the model itself is not devoted to reproduce in detail the traffic flow around the bus since its resolution is simply $\delta x \times \delta t$, it must be possible to represent these interaction in a simplified way, without obtaining too important undesirable effects. Research is going on in that direction.

In order to produce some numerical examples, an oversimplified discretization procedure has been used : It consists of considering the concentration in the cell where the bus is located as a whole, without making any distinction between the back and front of the bus. This concentration is used to determine wether the bus causes a perturbation in the traffic flow, i.e if the concentration is included in the interval $]K_1, K_2[$, as depicted on Figure 24(see remark 2 of subsection 5.2).

If there is no perturbation, the flows are computed as usual. If there is one, constraints corresponding to the maximum flow around the bus are applied to the exit demand and to the entrance supply.

This scheme presents little interest by itself, as it is correct only in very specific situations in which the bus spends an integer number of time slices in each cell and does stop only at the cells limits, but is mentionned for the sake of giving some numerical results on a simple case. The case presented above represents a bus riding at a constant speed along a uniform uncongested link, then stopping inside the link, then starting again at its original speed. In a simple case like this, following the calculation rules derived in subsection 5.2, it is possible to draw the complete evolution of the characteristics scheme and to compute analytically the trajectory of shockwaves, as depicted on Figure 25.

The flows computed by simulation of the discretized model lead to a very similar shape, as depicted on Figure 26. It can be seen both on the characteristics plot and on the simulation results that the resulting perturbations are rather complex but correctly represented by the simulation.

7   Conclusion.

The model presented here is only the beginning of the research. The main result is the feasability of introducing isolated vehicles in a macroscopic model in a way consistent with the model itself. Research is still going on to improve the discretization scheme, particularly to take account of discontinuities inside the links (changes in the number of lanes...). It is also clear that such a model needs to be validated by field measurements, to guarantee that the predicted effects of the presence of bus over traffic flow conditions are actually observed in practice.

A possible direction for future research could also be to adapt such a type of model to the description of lorries in motorway traffic. This raises the additional problem of dealing with the interactions of the vehicles themselves.

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Equilibrium flow, demand and supply functions

Q_e

Q_{max}

Figure 1

K_{crit} K_{max} K

Δ_e

Q_{max}

Figure 2

K_{crit} K_{max} K

Σ_e

Q_{max}

Figure 3

K_{crit} K_{max} K
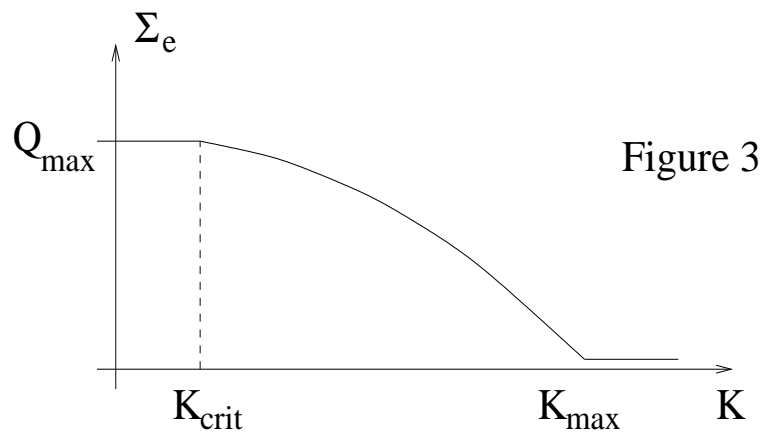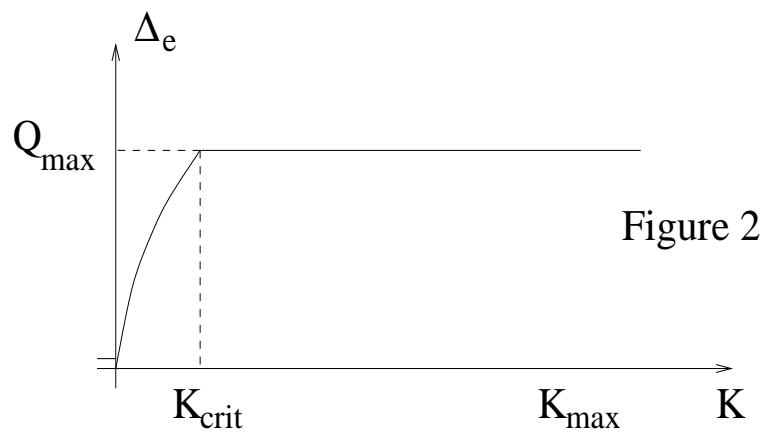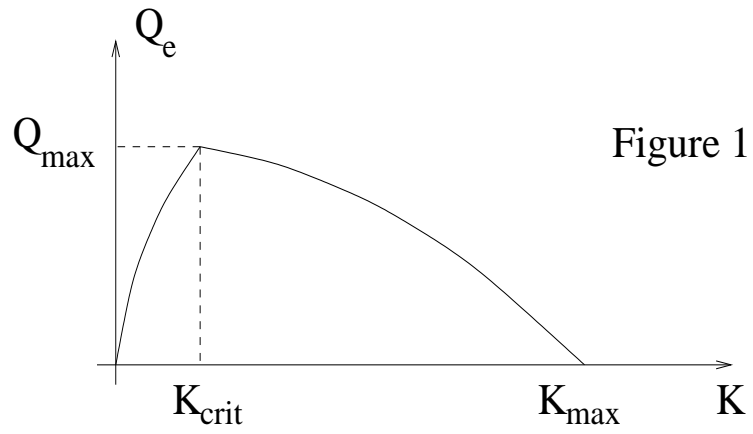
Figure 4: equilibrium flow-density relationship in the vicinity of a bus
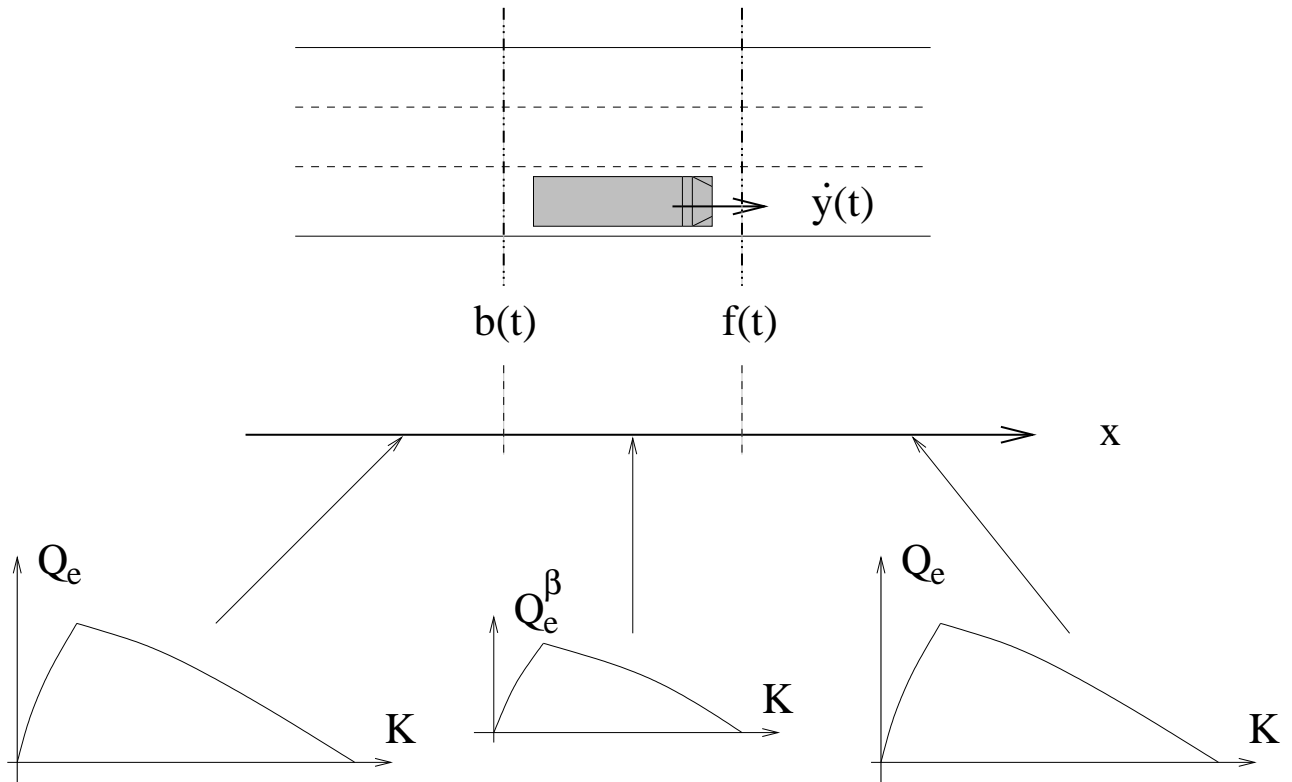
Figure 5: data for the inhomogeneous Riemann problem
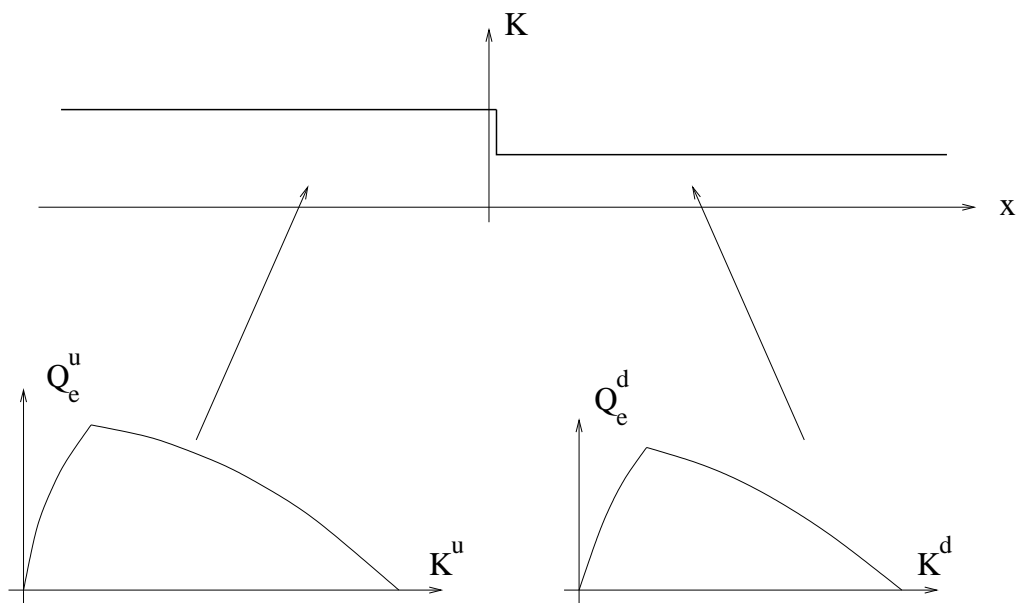
Figure 6: equilibrium flow-density relationship in the moving frame
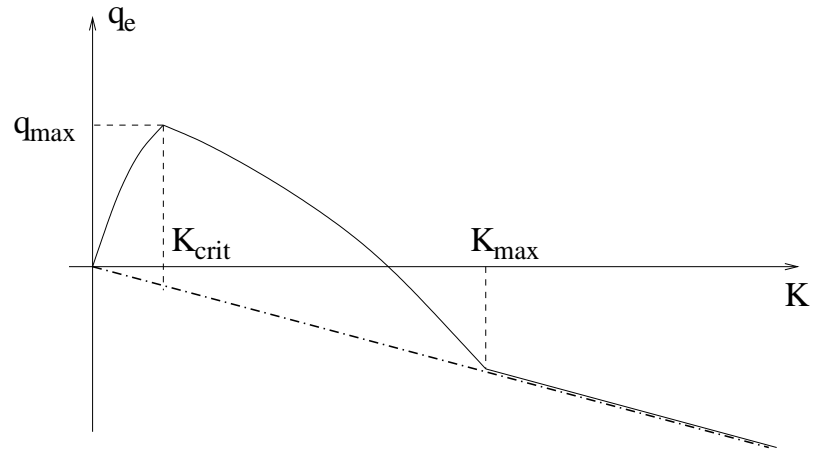
Figure 7: equilibrium demand and supply relationships in the moving frame
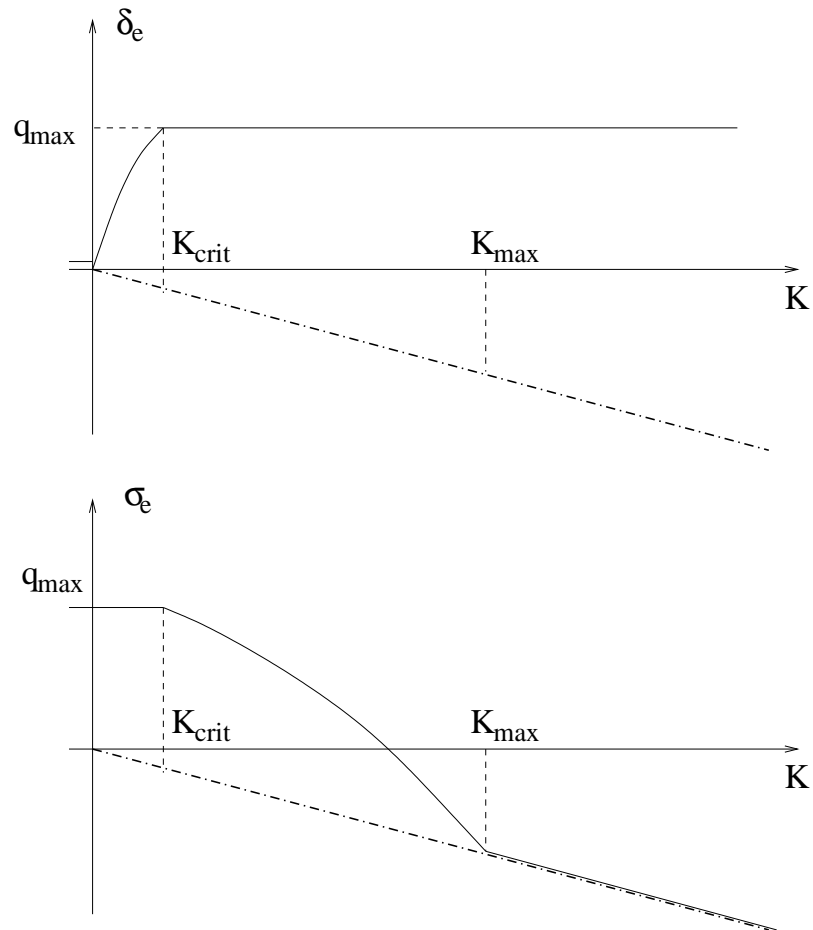
Figure 8: Relative demand 1
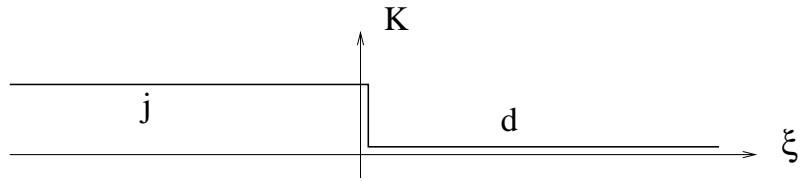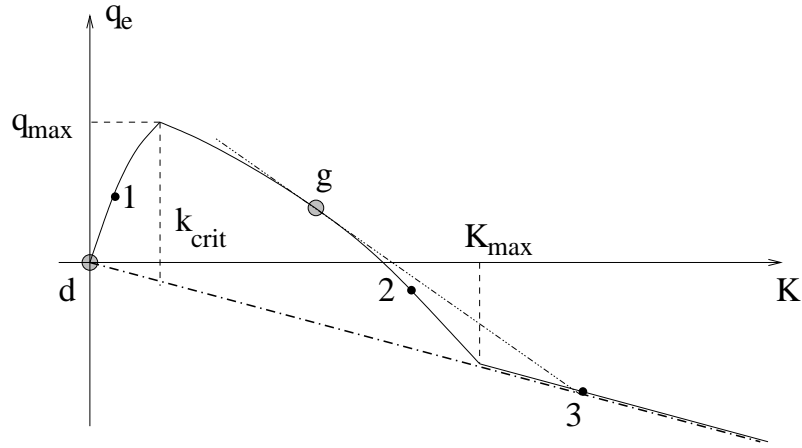
Initial Conditions. j=1,2,3

Figure 9: Relative demand 2
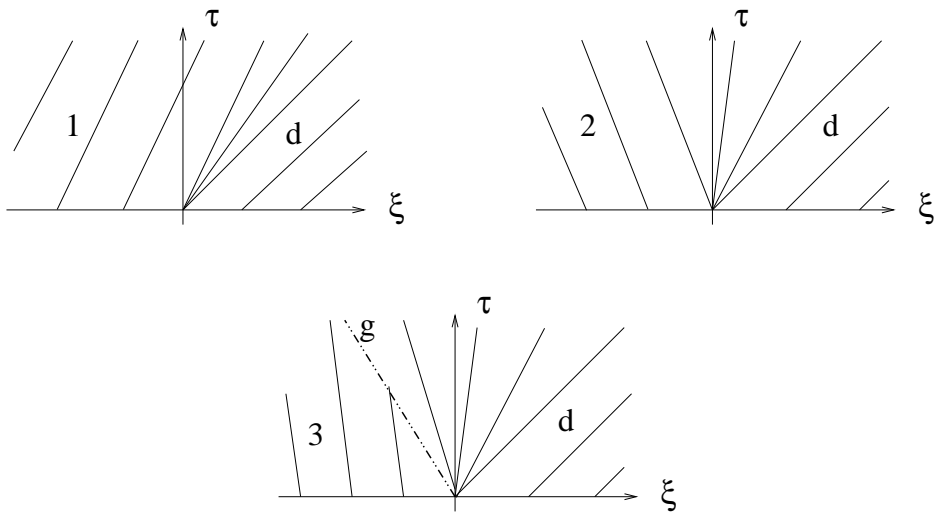
Figure 10: Relative supply 1
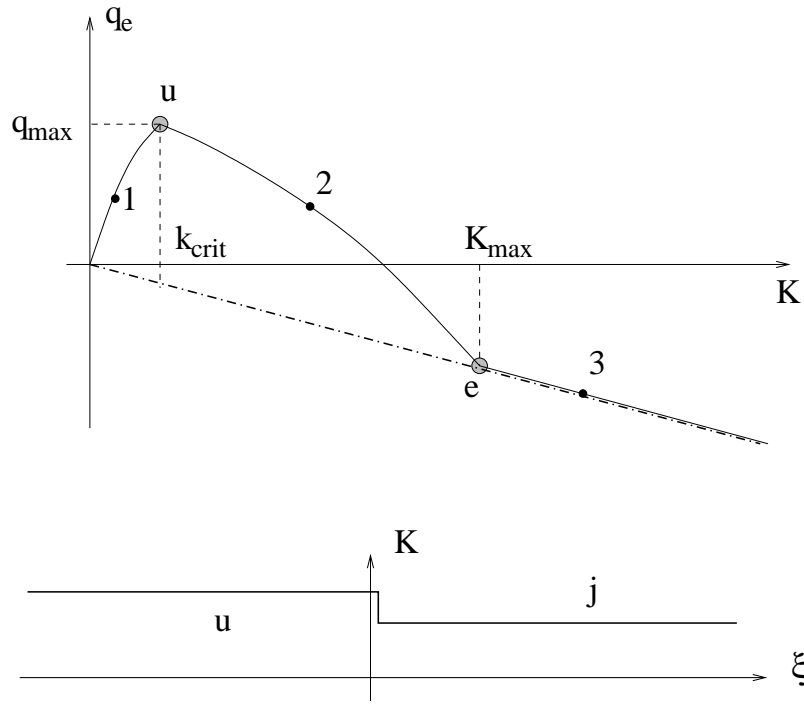
Initial Conditions. j=1,2,3

Figure 11: Relative supply 2
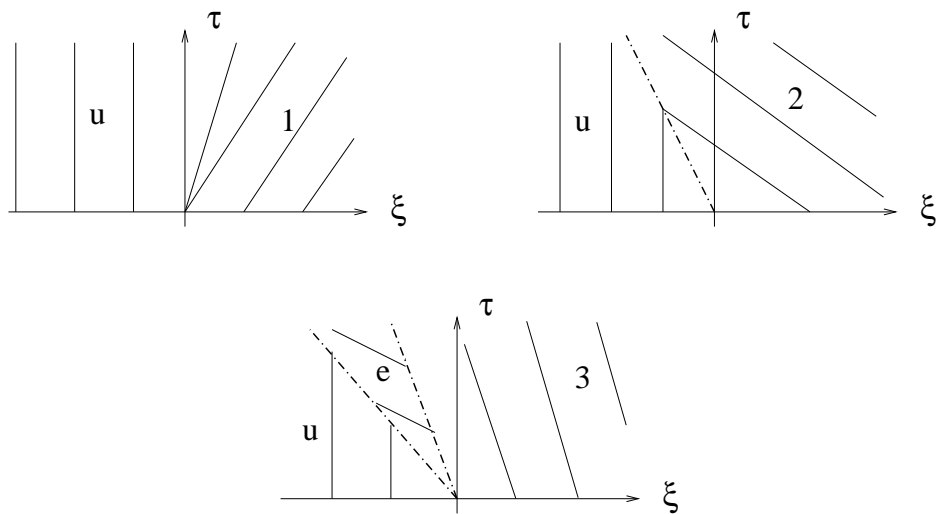
Figure 12: Riemann problem, BC_1^u
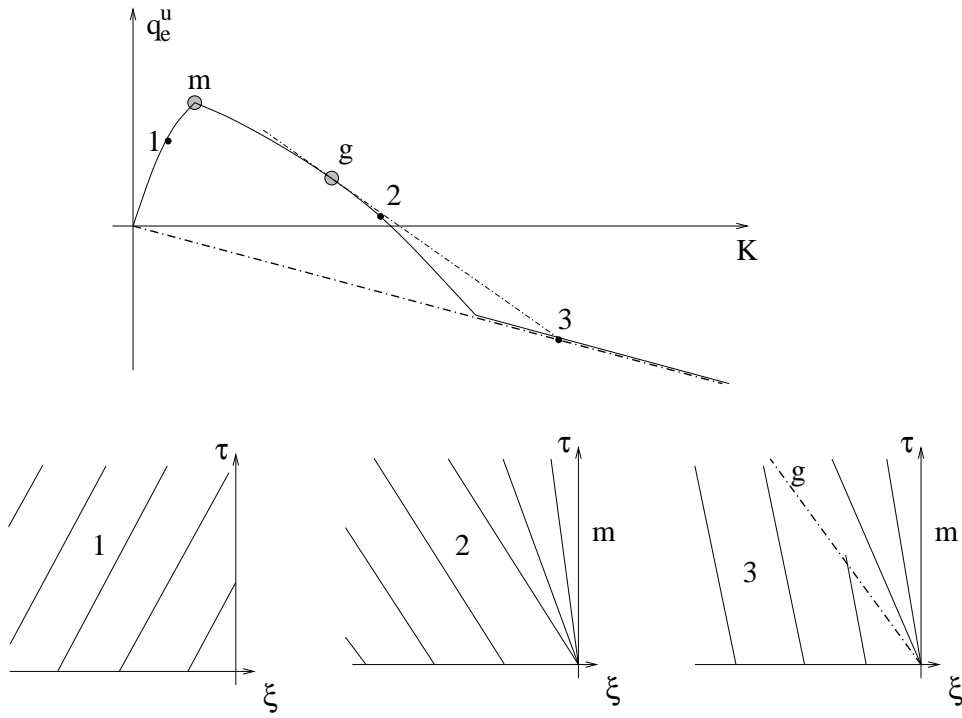
Figure 13: Riemann problem, BC_1^d
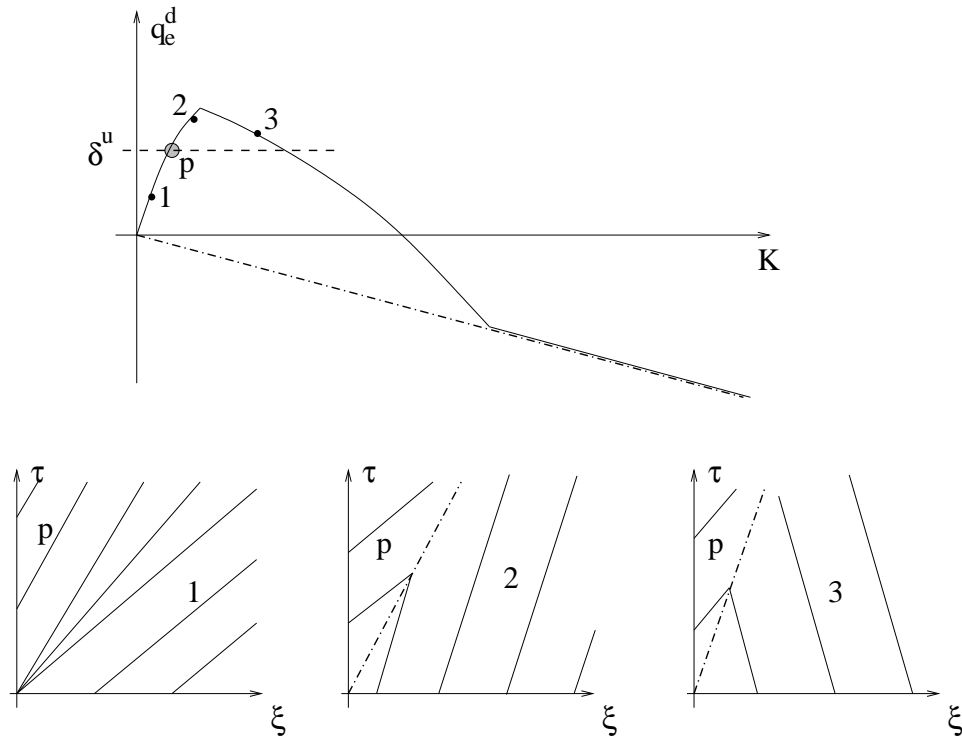
Figure 14: Riemann problem, BC_2^u
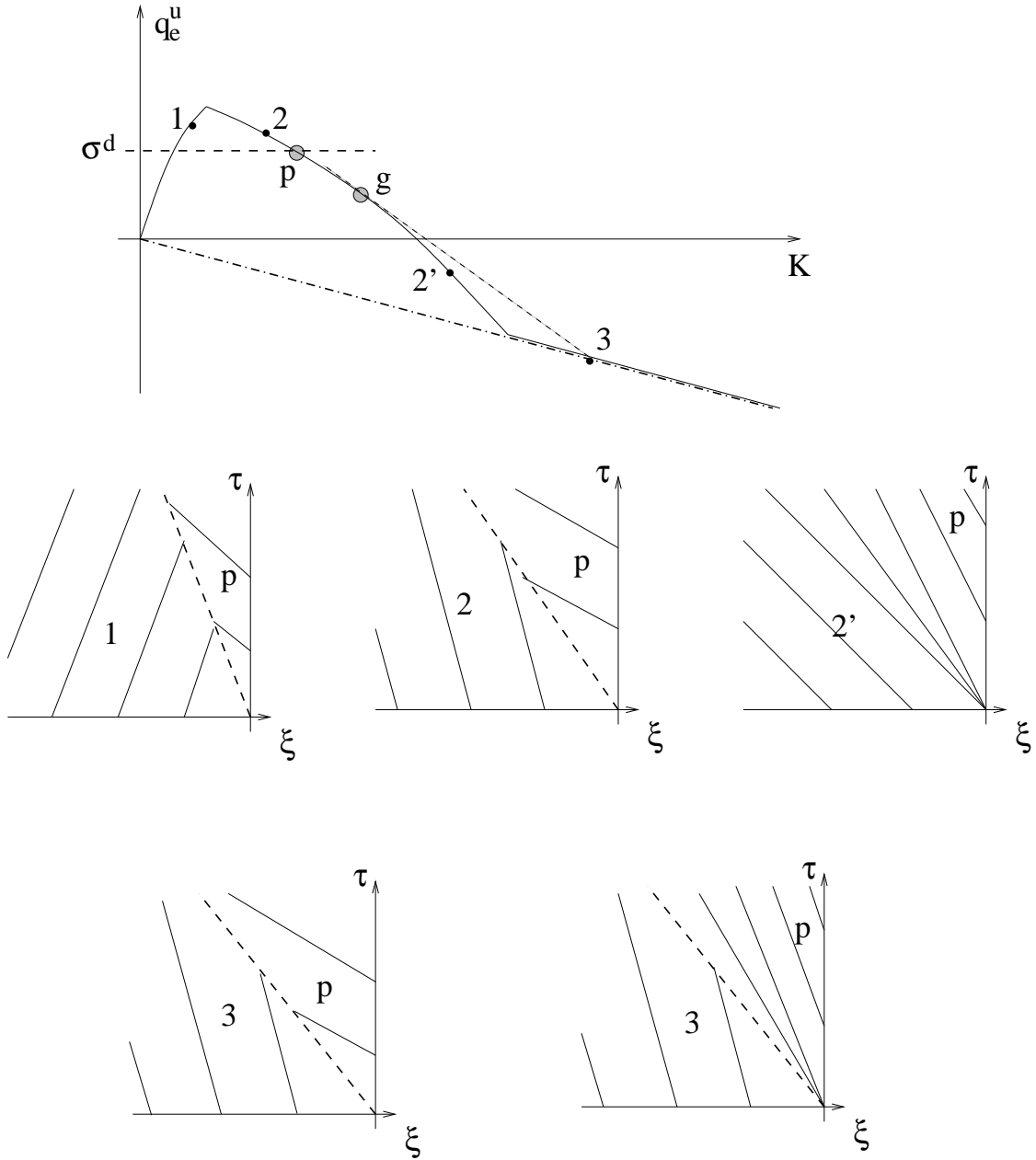
Figure 15: Riemann problem, BC_2^d
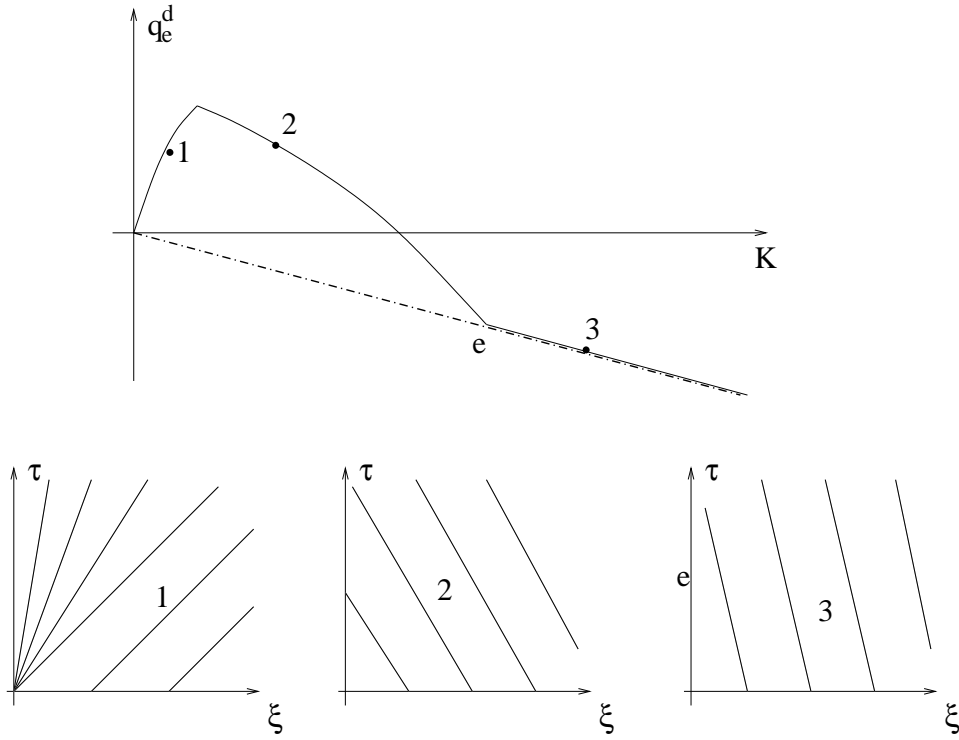
Figure 16: Riemann problem for the bus
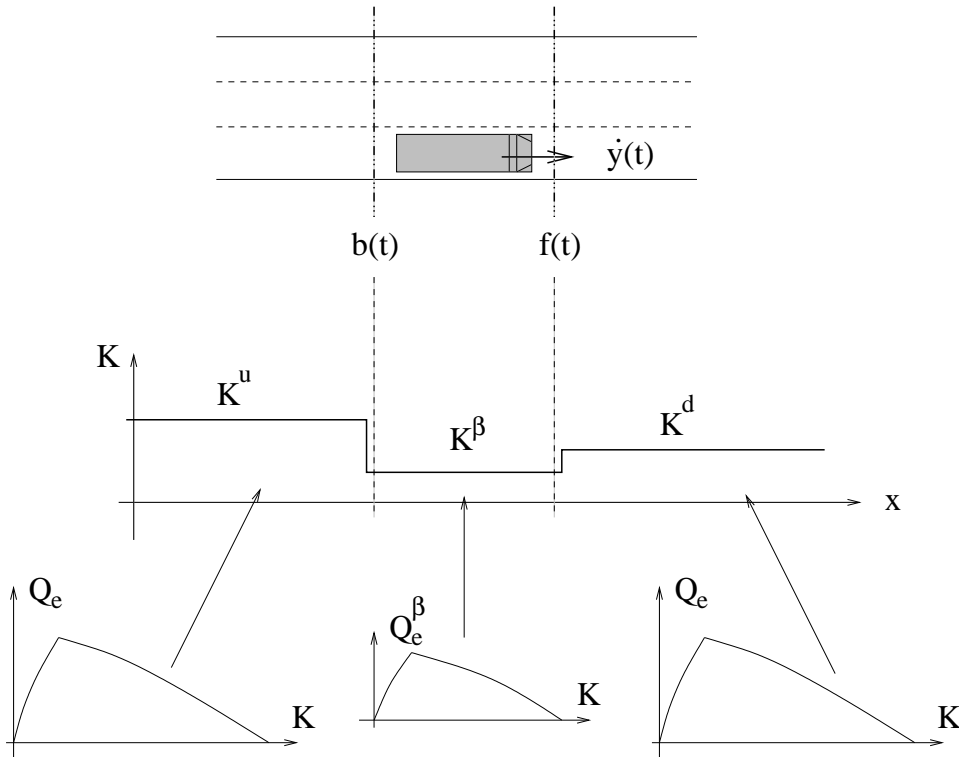
Figure 17: Perturbation resulting from the presence of a bus
Moving frame

Case 1

Supply greater than Demand

Supply less than Demand
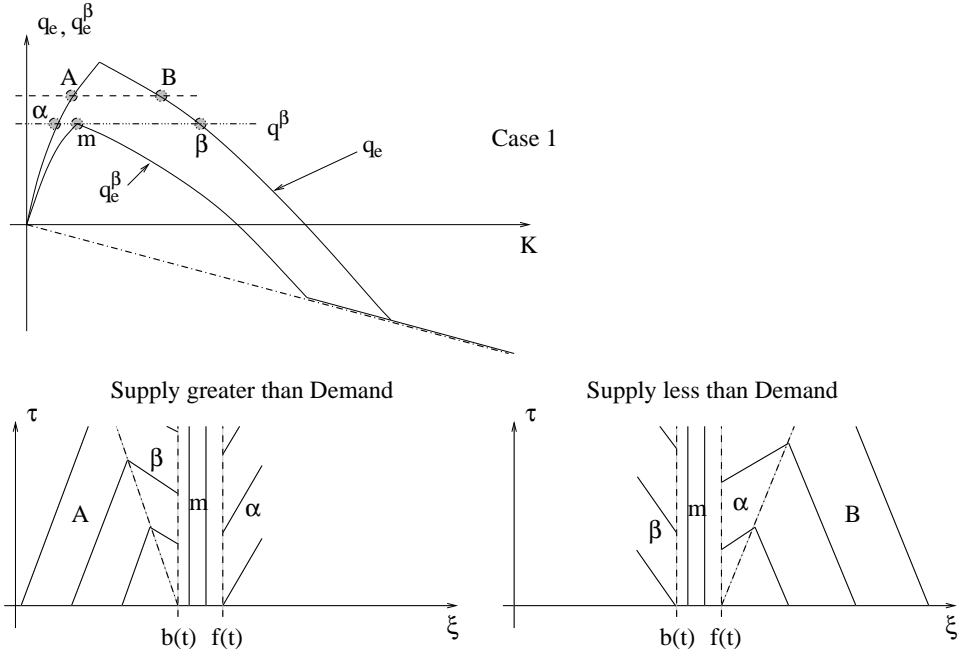
Figure 17bis: Perturbation resulting from the presence of a bus
Fixed frame

Case 1

Supply greater than Demand

Supply less than Demand
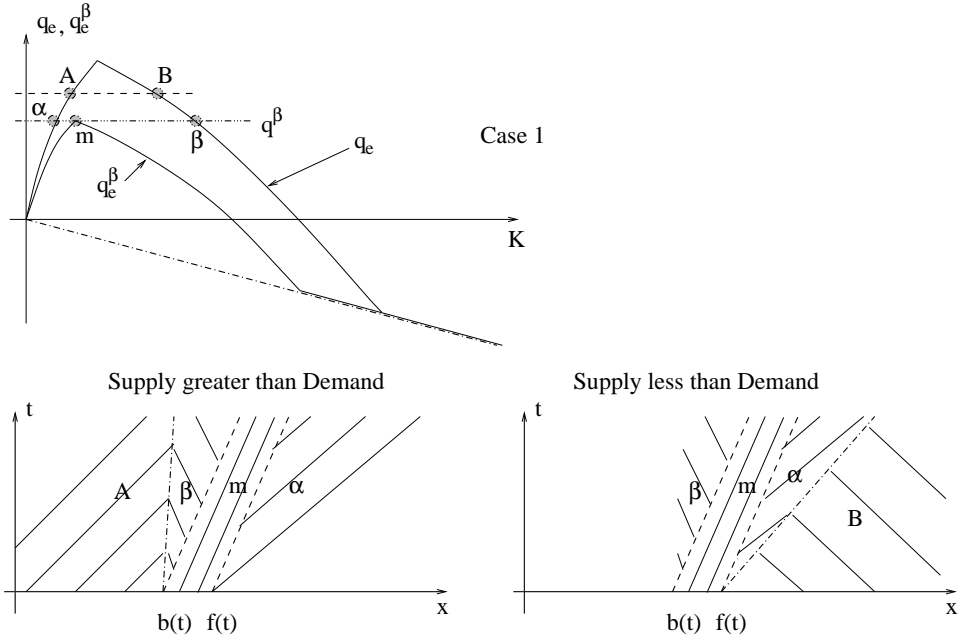
Figure 18: Bus exerting no effect on traffic flow
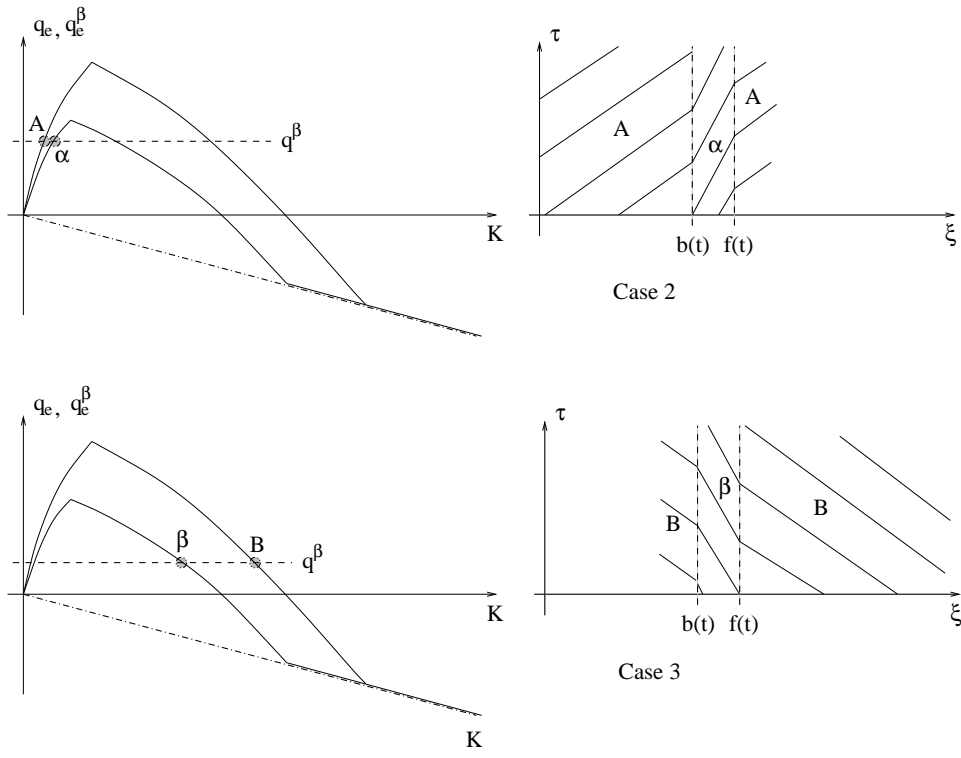
Figures: The Godunov scheme and its application to buses

Figure 19:

Flow/density relationship   Supply   Demand
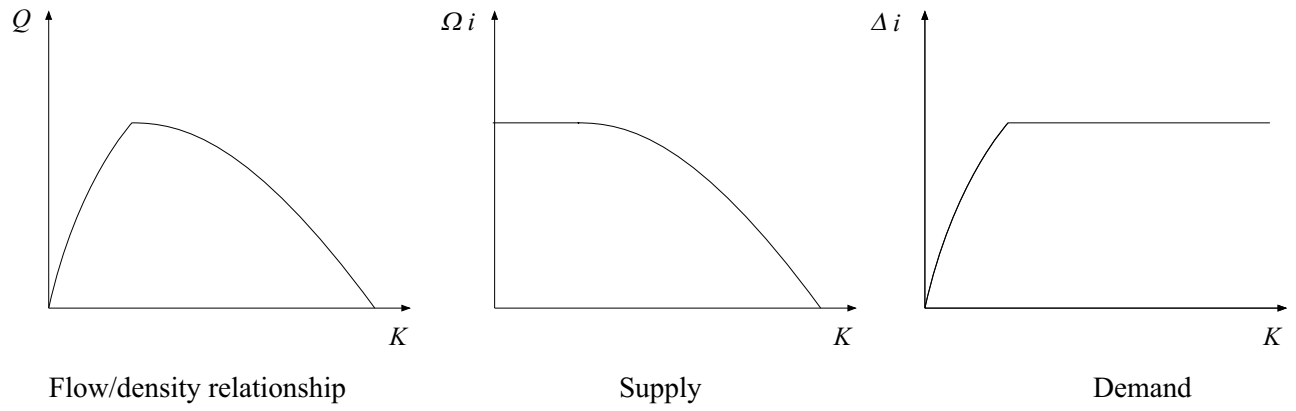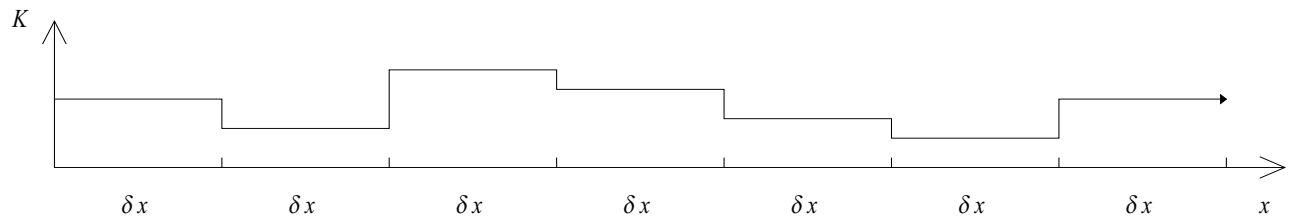
Figure 20:

The moving discretization

Figure 21:

δx　　　　δx　　Bus　　δx

Figure 22:

Flow/density relationship
in the moving coordinates

Supply

Demand

Figure 23:

K

Bus position

δx δx δx δx
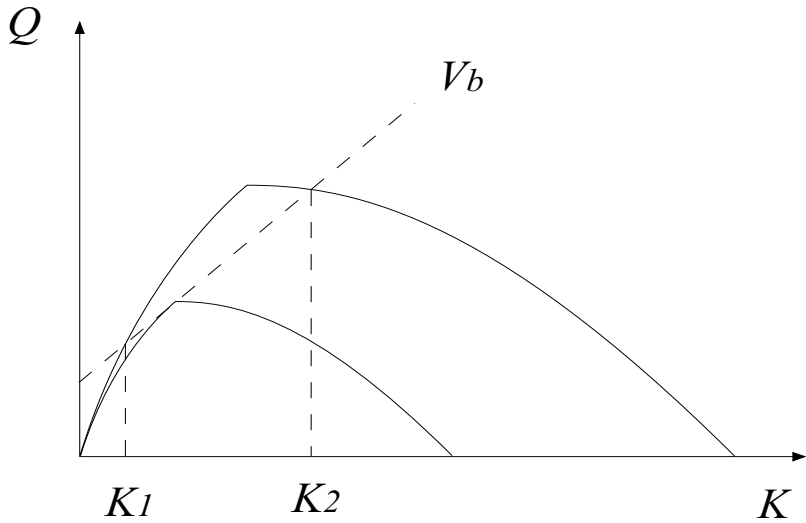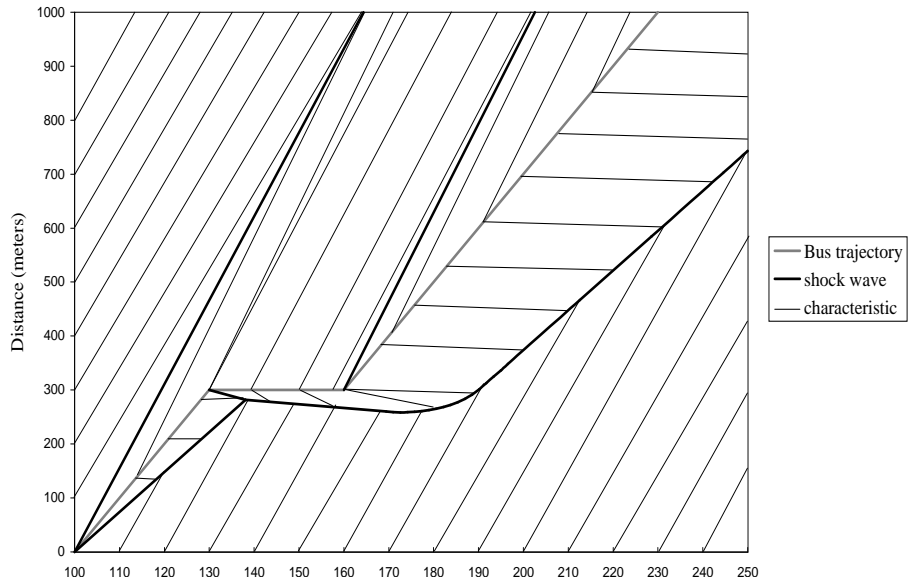
Figure 24:

Q

V_b

K_1 K_2 K
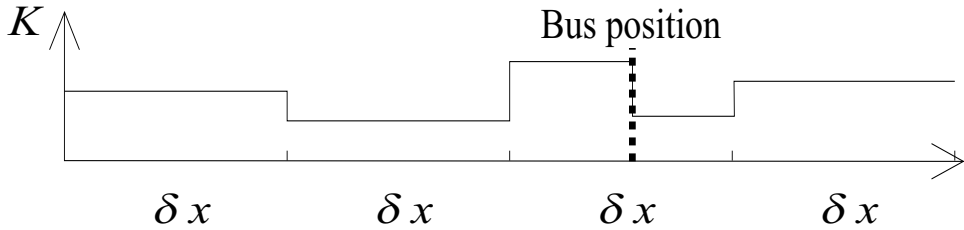
Flow/density relationship

Figures 25,26: Analytical versus numerical calculations

Graph3