A Formal Approach to Optimal Control

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Abstract: We use recent improvements in the parametrization of controllable multidimensional control systems to show how to transform the study of a linear quadratic optimal problem into that of a variational problem without constraints. We give formal conditions on the differential module determined by the control system, to pass from the Pontryagin method to a purely Euler-Lagrange variational problem. This formal approach uses the cost function in order to link the formally exact sequence formed by the controllable system and its parametrizations with the sequence formed by their adjoint operators. In the case of partial differential equations, this scheme is typical for any problem of elasticity.

Keywords: Optimal control, linear quadratic control, elasticity theory, controllability, parametrization, differential modules, formal duality, formal integrability, homological algebra.

1 Introduction

In this paper, we show how new results on parametrization of linear multidimensional control systems can be used to find formal results on optimal control and variational calculus. In particular, we are interested in knowing how the structural properties of a multidimensional control system, described in the framework of module theory, can be useful in order to reduce a constrained variational problem to a free one.

We first recall the fact that a controllable system, in the sense that the system determines a torsion-free differential module, is parametrizable and we show how to find effectively its parametrization [13, 15, 17]. The problem investigated in this paper is to extremize a functional under the constraint given by a linear multidimensional control system. We prove that if the control system generates a torsion-free module and if the differential sequence formed by the system and one of its parametrizations is locally exact, then by substitution, we are led to a simple variational problem without constraints. In particular, if the control system determines a projective module, one can always reduce our problem to this case. Moreover, if the system is defined by a surjective operator and determines a projective module, the Lagrange multipliers can always be found explicitly, without any integrations. Many examples illustrate this approach and, in particular, we show how this formal method can be used to linear quadratic problem, elasticity theory, electromagnetism... We hope to convince the reader that these algebraic and geometric methods, developped for control system theory in [13, 14, 15, 17], and using as main ingredients, formal adjoint of an operator, differential sequences and module theory, are in fact closely related to some physics principles, for example, duality existing between geometry and physics, in the sense of Poincaré.

2 Formal Tools

Let us expose and recall some results about the formal theory of differential operators [9, 10, 13, 24] and its dual approach in terms of differential modules [1, 8, 11, 12, 15].

Let E and F be trivial vector bundles over a differential manifold X of dimension n with local coordinates $x = (x^1, ..., x^n)$. In the course of this paper, we shall take $\mathbb{R}^n$ for X or open subsets. Let

$$\begin{array}{lcl} E & \xrightarrow{\mathcal{D}} & F \\ (x, \xi^k(x)) & \longmapsto & (x, \eta^\tau(x) = \sum_{0\leq|\mu|\leq q, 1\leq k\leq m} a_k^{\tau\mu}(x)\partial_\mu \xi^k, 1 \leq \tau \leq l), \end{array} \tag{1}$$

be a *differential operator* from E to F, where the fibered dimension of E (resp. F) is equal to m (resp. equal to l), $\mu = (\mu_1, ..., \mu_n)$ is a multi-index of length $|\mu| = \mu_1 + ... + \mu_n$ and we adopt the notation $\partial_\mu = \partial_1^{\mu_1} ... \partial_n^{\mu_n}$. If we denote by Θ the kernel of the operator $\mathcal{D}$, then we have the following exact sequence:

$$0 \longrightarrow \Theta \longrightarrow E \xrightarrow{\mathcal{D}} F. \tag{2}$$

Now, we associate with any differential operator $\mathcal{D}$ an algebraic object, namely a differential module M, in the following way (see [15] for more details). For that, when K is a differential field [7, 22], let us introduce the ring $D = K[d_1, ..., d_n]$ of differential operators, i.e. the ring of elements of the form $P = \sum_{|\mu|<\infty} a^\mu(x) d_\mu$, where the coefficients $a^\mu(x)$ belong to K and where the derivations d_i satisfy:

$$d_i(a(x)d_j) = a(x)d_i\, d_j + \partial_i a(x) d_j.$$

We associate with (1) the D-homomorphism $.\mathcal{D}$ defined as follows

$$\begin{array}{lcl} D^l & \xrightarrow{.\mathcal{D}} & D^m \\ (P_\tau) & \longmapsto & (\sum_{0\leq|\mu|\leq q, 1\leq\tau\leq l} P_\tau\, a_k^{\tau\mu}(x) d_\mu, 1 \leq k \leq m), \end{array} \tag{3}$$

i.e. we let operate a row vector of D^l on the left of $\mathcal{D}$ to obtain a row vector of D^m. Now, we associate with (2), the *finitely presented* left D-module M defined by the following exact sequence:

$$D \otimes_K F^\star \xrightarrow{.\mathcal{D}} D \otimes_K E^\star \longrightarrow M \longrightarrow 0,$$

or simply, because the vector bundles are trivial:

$$D^l \xrightarrow{.\mathcal{D}} D^m \longrightarrow M \longrightarrow 0, \tag{4}$$

i.e. $M = D^m/D^l\,\mathcal{D}$ (see [8, 9, 10, 11, 12, 15] for more details).

When $\mathcal{D} : \xi \to \eta$ is a sufficiently regular differential operator, the *compatibility conditions* of the inhomogeneous system

$$\mathcal{D}\,\xi = \eta, \tag{5}$$

are defined by an operator $\mathcal{D}_1 : F_0 \to F_1$, with $F = F_0$ and $l = l_0$. In other words, all the necessary conditions on η, in order to have the local existence of ξ such that (5) is satisfied, are generated by $\mathcal{D}_1\eta = 0$. The operator $\mathcal{D}_1$ can be constructed by bringing the operator $\mathcal{D}$ to *involutiveness* [13]. A historical problem was to construct effectively the operator $\mathcal{D}_1$ and it was investigated by Riquier and Cartan at the beginning of the century [2, 20, 21] but received a nice improvement with Janet's work in the twenties [3, 4] and a final achievement with the works of Spencer in the seventies [13, 24]. Then, we can construct the *formally exact sequence* [13, 15, 24]:

$$0 \longrightarrow \Theta \longrightarrow E \xrightarrow{\mathcal{D}} F_0 \xrightarrow{\mathcal{D}_1} F_1. \tag{6}$$

In the differential module langage, this means that we have computed the beginning of the *free resolution* (see e.g [23]) of the D-module M corresponding to $\mathcal{D}$, i.e. we have the following exact sequence:

$$D^{l_1} \xrightarrow{.\mathcal{D}_1} D^{l_0} \xrightarrow{.\mathcal{D}} D^m \longrightarrow M \longrightarrow 0. \tag{7}$$

We can repeat the same thing with $\mathcal{D}_1$ instead of $\mathcal{D}$ and we obtain a long formally exact sequence of compatibility conditions in the operator language, or a free resolution of M in the algebraic one. Moreover, we know, from the works of Spencer [24], that we can find a resolution of M of length equal to n, where n is the number of derivatives ∂_i, or equivalently, the number of derivations d_i in D. However, the Spencer resolution is in general very difficult to compute [13] and thus it is much easier to compute the *Janet sequence* [13, 15]

$$0 \longrightarrow \Theta \longrightarrow E \xrightarrow{\mathcal{D}_0} F_0 \xrightarrow{\mathcal{D}_1} F_1 \xrightarrow{\mathcal{D}_2} \ldots \xrightarrow{\mathcal{D}_{n-1}} F_{n-1} \xrightarrow{\mathcal{D}_n} F_n \longrightarrow 0,$$

giving rise to a free resolution of M of length equal to $n+1$:

$$0 \longrightarrow D^{l_n} \xrightarrow{.\mathcal{D}_n} D^{l_{n-1}} \longrightarrow ... \longrightarrow D^{l_1} \xrightarrow{.\mathcal{D}_1} D^{l_0} \xrightarrow{.\mathcal{D}_0} D^m \longrightarrow M \longrightarrow 0, \tag{8}$$

obtained by replacing $\mathcal{D}$ by an *involutive* operator $\mathcal{D}_0 : E \to F_0$ with the same kernel Θ [13] and where $\mathcal{D}_i$ are involutive first order operators. In this case, we know that the last operator $\mathcal{D}_n : F_{n-1} \to F_n$ defines a projective D-module (see e.g. [23] and definition 1).

Applying the functor $\hom_D(\cdot, D)$ to (8), we obtain the *dual sequence*

$$0 \longleftarrow D^{l_n} \xleftarrow{\mathcal{D}_n .} D^{l_{n-1}} \longleftarrow ... \longleftarrow D^{l_1} \xleftarrow{\mathcal{D}_1 .} D^{l_0} \xleftarrow{\mathcal{D}_0 .} D^m \longleftarrow \hom_D(M, D) \longleftarrow 0, \tag{9}$$

where $\mathcal{D}_i\,.$ means that we make $\mathcal{D}_i$ operate on the left of a column vector of $D^{l_{i-1}}$ in order to obtain a column vector of D^{l_i}. The defect of cohomology at D^{l_i} is denoted by $H(D^{l_i}) = \mathrm{ext}^i_D(M, D) = \ker \mathcal{D}_{i+1}/\mathrm{im}\, \mathcal{D}_i$. The defects of cohomology $\mathrm{ext}^i_D(M, D)$ do only depend in fact on M and not on its resolution (8), that is, if we have two different resolutions of the same D-module M, then we obtain the same defect of cohomology from the two different dual sequences [23]. Now, we have to notice that, using the fact that D is both a left and right D-module, we can endow $\hom_D(M, D)$ with the structure of a right D-module:

$$\forall\, a \in D, \forall\, \phi \in \hom_D(M, D) : \forall\, m \in M, (\phi\, a)(m) = \phi(m)\, a.$$

The cokernel of $\mathcal{D}_0$ is the right D-module N_r defined by:

$$0 \longleftarrow N_r \longleftarrow D^{l_0} \xleftarrow{\mathcal{D}_0 .} D^m \longleftarrow \hom_D(M, D) \longleftarrow 0. \tag{10}$$

It can be shown that N_r only depends on M up to a *projective equivalence* [18]. If we want to give an interpretation of the *extension functor* coming from the functor $\hom_D(\cdot, D)$ in terms of differential operators, we have to use the notion of *formal adjoint* [13, 15, 17]: if $T^\star$ denotes the cotangent bundle of X and $\mathcal{D} : E \to F$ is a differential operator, then its formal adjoint is the operator $\tilde{\mathcal{D}} : \bigwedge^n \tilde{F} = \bigwedge^n T^\star \otimes F^\star \to \tilde{E} = \bigwedge^n T^\star \otimes E^\star$, defined by using the three following formal rules equivalent to integration by parts:

- the adjoint of a matrix (zero order operator) is the transposed matrix,
- the adjoint of ∂_i is $-\partial_i$,
- for two linear PD operators P, Q that can be composed, then: $\widetilde{P \circ Q} = \tilde{Q} \circ \tilde{P}$.

Moreover, we have the relation

$$< \lambda,\, \mathcal{D}_1\, \eta > - < \tilde{\mathcal{D}}_1\, \lambda,\, \eta >= d(\cdot),$$

expressing a difference of n-forms and where d is the standard exterior derivative. In homological language, the functor $\bigwedge^n T \otimes_K \cdot$ is the *side changing functor* [1] and it allows to pass from a right D-module N_r to a left D-module. Thus the left D-module $N = \bigwedge^n T \otimes_K N_r$ is the module determined by the adjoint $\tilde{\mathcal{D}}$ of $\mathcal{D}$ and we have:

$$0 \longleftarrow N \longleftarrow \bigwedge^n T \otimes_K D^l \xleftarrow{.\tilde{\mathcal{D}}} \bigwedge^n T \otimes_K D^m. \tag{11}$$

Now, let us start with an involutive operator $\mathcal{D}_0 : E \to F_0$ and let us denote by M the D-module determined by $\mathcal{D}$. We give a formal test to check whether or not $\mathrm{ext}^i_D(M, D)$ is equal to zero or not.

Computation of $\mathrm{ext}^i_D(M, D)$:

1. Start with $\mathcal{D}_0$.
2. Find the sequence of the compatibility condition operators $\mathcal{D}_r$ up to $\mathcal{D}_i$.
3. Construct the adjoint sequence formed by the operators $\tilde{\mathcal{D}}_i$ and $\tilde{\mathcal{D}}_{i-1}$.
4. Find the compatibility conditions $\tilde{\mathcal{D}}'_{i-1}$ of $\tilde{\mathcal{D}}_i$.
5. Check whether or not $\tilde{\mathcal{D}}_{i-1}$ generates all the compatibility conditions $\tilde{\mathcal{D}}'_{i-1}$ of $\tilde{\mathcal{D}}_i$. If yes, then $\mathrm{ext}^i_D(M, D) = 0$ else $\mathrm{ext}^i_D(M, D)$ is defined by all the compatibility conditions which are in $\tilde{\mathcal{D}}'_{i-1}$ and not in $\tilde{\mathcal{D}}_{i-1}$.

We can represent the above algorithm by the following diagram

$$\begin{array}{llllllll}
1 \;\; E \xrightarrow{\mathcal{D}_0} F_0 & \xrightarrow{\mathcal{D}_1} \dots & \dots \xrightarrow{\mathcal{D}_{i-2}} & F_{i-2} & \xrightarrow{\mathcal{D}_{i-1}} & F_{i-1} & \xrightarrow{\mathcal{D}_i} & F_i \quad 2 \\
 & & 3 & \tilde{F}_{i-2} & \xleftarrow{\tilde{\mathcal{D}}_{i-1}} & \tilde{F}_{i-1} & \xleftarrow{\tilde{\mathcal{D}}_i} & \tilde{F}_i, \\
 & & 4 & \tilde{F}'_{i-2} & \xleftarrow{\tilde{\mathcal{D}}'_{i-1}} & & &
\end{array}$$

where the number indicates the step of the algorithm.

More generally, from algebraic analysis [6, 11, 17], we have the following theorem:

Theorem 1. *We can embed the D-module M into an exact sequence*

$$0 \longrightarrow M \longrightarrow D^m \xrightarrow{.\mathcal{D}_{-1}} D^{l_{-1}} \xrightarrow{.\mathcal{D}_{-2}} \dots \xrightarrow{.\mathcal{D}_{-r+1}} D^{l_{-r+1}} \xrightarrow{.\mathcal{D}_{-r}} D^{l_{-r}}, \tag{12}$$

if and only if $\mathrm{ext}^i_D(N, D) = 0,\ \forall\, i = 1, ..., r$, where N is the left D-module corresponding to the right D-module defined by (10). Equivalently, we have, in the framework of differential operator, the following formally exact sequence:

$$E_{-r} \xrightarrow{\mathcal{D}_{-r}} E_{-r+1} \xrightarrow{\mathcal{D}_{-r+1}} \dots \xrightarrow{\mathcal{D}_{-2}} E_{-1} \xrightarrow{\mathcal{D}_{-1}} E_0 \xrightarrow{\mathcal{D}} F,$$

where $E_0 = E$ and each operator generates all the compatibility conditions of the preceding one if and only if $\mathrm{ext}^i_D(N, D) = 0,\ \forall\, i = 1, ..., r$, where N is the left D-module determined by $\tilde{\mathcal{D}}$.

Definition 1.

- A finitely generated D-module M is *free* if it is isomorphic to copies of D.
- A finitely generated D-module M is *projective* if there exist a free D-module F and a D-module N such as $F = M \oplus N$.
- A finitely generated D-module M is *reflexive* if $M \cong \hom_D(\hom_D(M, D), D)$.
- A finitely generated D-module M is *torsion-free* if $t(M) = \{m \in M \mid \exists\, p \neq 0,\, p\, m = 0\} = 0$. We call $t(M)$ the *torsion submodule* of M.

We have the following inclusions of D-modules:

$$\text{free} \subseteq \text{projective} \subseteq \ldots \subseteq \text{reflexive} \subseteq \text{torsion-free}. \tag{13}$$

Proposition 1. *Let M be a finitely generated left D-module determined by the operator $\mathcal{D}$ and N the left D-module defined by $\tilde{\mathcal{D}}$. We have the following propositions [17]:*

- *M is a torsion-free D-module $\Leftrightarrow \operatorname{ext}^1_D(N, D) = 0$.*
- *M is a reflexive D-module $\Leftrightarrow \operatorname{ext}^i_D(N, D) = 0,\ i = 1, 2$.*
- *M is a projective D-module $\Leftrightarrow \operatorname{ext}^i_D(N, D) = 0,\ i = 1, ..., n$.*

Let us notice that, if $n = 1$, then any torsion-free D-module is projective. It has been shown in [19, 17] that the notions of torsion-freeness and projectiveness were the intrinsic formulation of the notion of *minor left coprimeness* and *zero minor coprimeness*, used in multidimensional systems theory [26, 27, 25, 28], for matrices with maximal generic rank.

Theorem 2.

- *If D is a principal ideal ring (for example $D = K[\frac{d}{dt}]$), then any torsion-free D-module is free.*
- *If k is the field of constants, i.e. $\forall\, a \in k : d_i\, a = 0$, then any projective $D = k[d_1, ..., d_n]$-module is free.*

The first point is a well-known result and one can find the proof in any textbook on module theory. The second point is the famous and difficult Quillen-Suslin theorem (see e.g. [23] for a proof).

Example 1. Let $\mathcal{D} : \xi \to \eta$ be defined by

$$\begin{cases} \partial_{12}\, \xi = \eta^1, \\ \partial_{22}\, \xi = \eta^2, \end{cases}$$

$D = \mathbb{R}[d_1, d_2]$ and let $M = D/D^2 \begin{pmatrix} d_1 d_2 \\ d_2^2 \end{pmatrix}$ be the D-module determined by $\mathcal{D}$. Let us check whether or not $\operatorname{ext}^1_D(M, D)$ is equal to zero. We first find that the compatibility condition of $\mathcal{D}\,\xi = \eta$ is given by $\mathcal{D}_1 : \eta \to \zeta$ defined by $\partial_1 \eta^2 - \partial_2 \eta^1 = \zeta$. Its adjoint $\tilde{\mathcal{D}}_1 : \lambda \to \mu$, obtained by multiplying $\mathcal{D}_1$ on the left by a row vector λ and integrating by parts, is defined by:

$$\begin{cases} \partial_2 \lambda = \mu_1, \\ -\partial_1 \lambda = \mu_2. \end{cases}$$

The compatibility conditions $\tilde{\mathcal{D}}' : \mu \to \nu$ of $\tilde{\mathcal{D}}_1$ are generated by

$$\partial_1\, \mu_1 + \partial_2\, \mu_2 = \nu',$$

whereas the adjoint $\tilde{\mathcal{D}}$ of $\mathcal{D}$ is defined by:

$$\partial_{12}\,\mu_1 + \partial_{22}\,\mu_2 = \nu.$$

Thus, $\tilde{\mathcal{D}}$ does not generate all the compatibility conditions of $\tilde{\mathcal{D}}_1$ and we have the relation

$$\partial_2\,\nu' = \nu.$$

We let the reader check by himself that the left D-module determined by the operator $\tilde{\mathcal{D}}$ is not a torsion-free D-module and $z = \partial_1\,\mu_1 + \partial_2\,\mu_2$ is a torsion element because it satisfies the equation $\partial_2\,z = 0$.

Example 2. We let the reader check by himself that the sequence of compatibility conditions of the operator $\mathcal{D} : \xi \to \eta$, defined by the gradient in $\mathbb{R}^3$, i.e. $\nabla \xi = \eta$, is formed respectively by the curl and the divergence operator. Moreover, we can easily verify that, up to a sign, the differential sequence is self-adjoint, i.e. the formal adjoint of the gradient is minus the divergence... Now, if we start with the divergence operator and call $M = D^3/D(d_1\ d_2\ d_3)$, the corresponding left D-module, then we easily verify that $\mathrm{ext}^1_D(N, D) = 0$ because the divergence is parametrized by the curl, $\mathrm{ext}^2_D(N, D) = 0$ because the curl is parametrized by the gradient and $\mathrm{ext}^3_D(N, D) = D/D^3(d_1\ d_2\ d_3)^t \neq 0$ because the gradient is not a formally injective operator. Hence, using proposition 1, we obtain that the D-module M is reflexive but not projective. Similarly, one can prove that the D-module determined by the curl is only torsion-free and the gradient determines a torsion D-module.

Example 3. Let us consider the operator $\mathcal{D} : \xi \to \eta$ defined by

$$\partial_1\,\xi^1 + \partial_2\,\xi^2 - x^2\xi^1 = \eta,$$

$D = \mathbb{R}(x^1, x^2)[d_1, d_2]$ and let $M = D^2/D\left(\ d_1 - x^2 \quad d_2\ \right)$ be the left D-module determined by $\mathcal{D}$. Let us determine the algebraic nature of M. First of all, we have to notice that $\mathcal{D}$ is *formally surjective*, i.e. $\mathcal{D}$ has no compatibility conditions. The operator $\tilde{\mathcal{D}} : \mu \to \nu$ is defined by:

$$\begin{cases} -\partial_1\mu - x^2\mu = \nu_1, \\ -\partial_2\mu = \nu_2. \end{cases} \tag{14}$$

We easily verify that we have $\mu = \partial_1\nu_2 - \partial_2\nu_1 + x^2\nu_2$, which implies that $\tilde{\mathcal{D}}$ is an injective operator. Let us define the operator $\tilde{\mathcal{P}} : \nu \to \mu$ by $\partial_1\nu_2 - \partial_2\nu_1 + x^2\nu_2 = \mu$, then $\tilde{\mathcal{P}} \circ \tilde{\mathcal{D}} = \mathrm{id}_{\tilde{F}}$, i.e. $\tilde{\mathcal{P}}$ is a *left-inverse* of $\tilde{\mathcal{D}}$. The $\bigwedge^n T \otimes_K D$-morphism $.\tilde{\mathcal{D}} : \bigwedge^n T \otimes_K D^m \to \bigwedge^n T \otimes_K D^{l_0}$ is then surjective because for all $a \in \bigwedge^n T \otimes_K D^{l_0}$, we can define $b = a\,\tilde{\mathcal{P}}$ and we easily verify that $a = b\,\tilde{\mathcal{D}}$. Hence, the left D-module N, defined by (11), verifies $N = \mathrm{coker}\,.\tilde{\mathcal{D}} = 0 \Rightarrow \mathrm{ext}^1_D(N, D) = 0$, $i = 1, 2$ and M is projective by proposition 1. Dualizing the operator $\tilde{\mathcal{P}}$, we obtain a *right-inverse* $\mathcal{P}$ of $\mathcal{D}$, i.e. $\mathcal{D} \circ \mathcal{P} = \mathrm{id}_F$. We refer the reader to [15] for the applications of left and right-inverses to the *generalized Bezout identity.* Substituting the expression of μ in functions of ν_1 and ν_2 in (14), we obtain the operator $\tilde{\mathcal{D}}_{-1} : \nu \to \pi$ defined by:

$$\begin{cases} \partial_{11}\nu_2 - \partial_{12}\nu_1 + 2x^2\partial_1\nu_2 - x^2\partial_2\nu_1 + (x^2)^2\nu_2 + \nu_1 = \gamma_1, \\ \partial_{12}\nu_2 - \partial_{22}\nu_1 + x^2\partial_2\nu_2 + 2\nu_2 = \gamma_2. \end{cases}$$

Dualizing $\tilde{\mathcal{D}}_{-1}$, we obtain $\mathcal{D}_{-1} : \theta \to \xi$ given by:

$$\begin{cases} -\partial_{22}\,\theta^2 - \partial_{12}\,\theta^1 + x^2\partial_2\,\theta^1 + 2\,\theta^1 = \xi^1, \\ \partial_{12}\,\theta^2 + \partial_{11}\,\theta^1 - x^2\partial_2\theta^2 - 2x^2\partial_1\,\theta^1 + (x^2)^2\,\theta^1 + \theta^2 = \xi^2. \end{cases}$$

We let the reader check by himself that the compatibility conditions of $\mathcal{D}_{-1}\,\theta = \xi$ are exactly generated by the operator $\mathcal{D}\,\xi = 0$. Hence, $\mathcal{D}$ is parametrized by $\mathcal{D}_{-1}$ in agreement with the fact that any projective module is torsion-free.

3 Optimal Control

We first recall how the preceding section can be used for the analysis of control systems. We refer the reader to [13, 15, 17] for more details and examples.

3.1 Controllablity

In agreement with the notion of controllability used in multidimensional control theory, we have the following definition [19, 13, 25, 28]:

Definition 2. A control system, described by the operator $\mathcal{D}_1 : F_0 \to F_1$, is *controllable* if the module M determined by $\mathcal{D}_1$ is a torsion-free D-module.

By Proposition 1, a control system, defined by the operator $\mathcal{D}_1$, is controllable if and only if $\mathrm{ext}^1_D(N, D) = 0$, where N is the left D-module determined by $\tilde{\mathcal{D}}_1$. In the case where the system is controllable, using theorem 1, we know that $\mathcal{D}_1$ can be parametrized by an operator $\mathcal{D}_0$, i.e. $\mathcal{D}_1$ represents exactly all the compatibility conditions of $\mathcal{D}_0$. If we want to check whether or not a system is controllable and to compute effectively the operator $\mathcal{D}_0$ or the torsion elements, we have to proceed in the following way:

Controllability test:

1. Start with $\mathcal{D}_1$.
2. Construct its adjoint $\tilde{\mathcal{D}}_1$.
3. Find the compatibility conditions of $\tilde{\mathcal{D}}_1 \lambda = \mu$ and denote this operator by $\tilde{\mathcal{D}}_0$.
4. Construct its adjoint $\mathcal{D}_0$ $(= \tilde{\tilde{\mathcal{D}}}_0)$.
5. Find the compatibility conditions of $\mathcal{D}_0\, \xi = \eta$ and call this operator $\mathcal{D}'_1$.

This leads to two different cases:

- If $\mathcal{D}_1$ is exactly the compatibility conditions $\mathcal{D}'_1$ of $\mathcal{D}_0$, then the system $\mathcal{D}_1$ determines a torsion-free D-module M and $\mathcal{D}_0$ is a parametrization of $\mathcal{D}_1$.
- Otherwise, the operator $\mathcal{D}_1$ is among, but not exactly, the compatibility conditions of $\mathcal{D}_0$ and we shall write $\mathcal{D}_1 < \mathcal{D}'_1$. The torsion elements of M are all the new compatibility conditions modulo the equations $\mathcal{D}_1 \eta = 0$.

Remark 1. For a matrix with polynomial entries and maximal generic rank, it is well-known that this matrix determines a torsion-free module if there is no common factor on all the maximal minors [19, 17, 25]. The above test can be used for more general systems (variable coefficients case, non surjective operator). Moreover, if the D-module is torsion-free, it gives effectively an explicit parametrization and, if the module is not torsion-free, it gives a basis of torsion elements.

Example 4. We have seen in example 3 that, up to a change of notations, the system defined by $\mathcal{D}_1 : \eta \to \zeta$ by

$$\partial_1 \eta^1 + \partial_2 \eta^2 - x^2 \eta^1 = \zeta,$$

determines a projective D-module and thus is controllable. Moreover, we have found a parametrization $\mathcal{D}_0 : \xi \to \zeta$, defined by

$$\begin{cases} -\partial_{22}\, \xi^2 - \partial_{12}\, \xi^1 + x^2 \partial_2\, \xi^1 + 2\, \xi^1 = \eta^1, \\ \partial_{12}\, \xi^2 + \partial_{11}\, \xi^1 - x^2 \partial_2 \xi^2 - 2x^2 \partial_1\, \xi^1 + (x^2)^2\, \xi^1 + \xi^2 = \eta^2. \end{cases}$$

of $\mathcal{D}_1$. This concept of parametrization generalizes the notion of *controller form* and *partial state* [5] to non surjective operator and to multidimensional systems. We refer the interested reader to [15, 18] for more details.

We use similarly the differential operator language to compute explicitly $\mathrm{ext}^i_D(N, D)$ and, therefore, to know whether a multidimensional system determines a reflexive, ..., projective D-module. Applications of projective modules to generalized Bezout identity are shown in [15].

Theorem 3. An OD control system defined by a surjective operator $\mathcal{D}_1$, i.e. the operator $\mathcal{D}_1$ has no compatibility conditions, is controllable iff its adjoint $\tilde{\mathcal{D}}_1$ is an injective operator, i.e. $\tilde{\mathcal{D}}_1\lambda = 0 \Rightarrow \lambda = 0$.

Proof. Let M be the $D = k[\frac{d}{dt}]$-module determined by the surjective operator $\mathcal{D}_1$. The left D-module N is then defined by:

$$0 \longleftarrow N \longleftarrow T \otimes_K D^l \xleftarrow{\tilde{\mathcal{D}}_1} T \otimes_K D^m.$$

If $\tilde{\mathcal{D}}_1$ is an injective operator, then there exists an operator $\tilde{\mathcal{P}}_1 : \tilde{F}_1 \to \tilde{F}_0$ such that $\tilde{\mathcal{P}}_1 \circ \tilde{\mathcal{D}}_1 = \mathrm{id}_{\tilde{F}_1}$. This implies that the operator $.\tilde{\mathcal{D}}_1 : T \otimes_K D^m \to T \otimes_K D^{l_0}$ is surjective. Indeed, for all $a \in T \otimes_K D^{l_0}$, we define $b = a\,\tilde{\mathcal{P}}_1$ and we easily verify that $a = b\,\tilde{\mathcal{D}}_1$. Thus $N = \mathrm{coker}\,.\tilde{\mathcal{D}}_1 = 0 \Rightarrow \mathrm{ext}^1_D(N, D) = 0 \Rightarrow M$ is a torsion-free D-module. Reciprocally, suppose that M is a torsion-free D-module then, since D is a principal ideal ring, by theorem 2, M is a projective D-module. Thus, the sequence

$$0 \longrightarrow D^l \xrightarrow{.\mathcal{D}_1} D^m \longrightarrow M \longrightarrow 0,$$

splits [23], i.e. there exists an operator $.\mathcal{P}_1 : D^l \to D^m$ such that $.\mathcal{D}_1 \circ \mathcal{P}_1 = .\mathrm{id}_{D^l}$, that is to say, $\mathcal{D}_1 \circ \mathcal{P}_1 = \mathrm{id}_{F_1}$. Hence, $\tilde{\mathcal{D}}_1$ is injective with left-inverse $\tilde{\mathcal{P}}_1$. Notice that, in this case, we have $N = 0$. □

Example 5. Let us consider the system in the Kalman form $-\dot{y} + A(t)\,y + B(t)\,u = 0$, where A is a square $n \times n$ matrix and B is $n \times m$. The OD surjective operator $\mathcal{D}_1 : \eta \to \zeta$, defined by $-\dot{\eta}^1 + A(t)\,\eta^1 + B(t)\,\eta^2 = \zeta$, determines a torsion-free D-module M iff M is projective. The adjoint operator $\tilde{\mathcal{D}}_1 : \lambda \to \mu$ is given by:

$$\begin{cases} \dot{\lambda} + \lambda\,A(t) = \mu_1, \\ \lambda\,B(t) = \mu_2. \end{cases}$$

Differentiating the zero order equation and using the first one, we obtain that $\lambda\,(AB - \dot{B}) = 0 \Rightarrow \lambda\,(A^2B - \dot{A}B - 2A\dot{B} + \ddot{B}) = 0\ldots$ Therefore, the operator $\tilde{\mathcal{D}}_1$ is injective, i.e. $N = 0 \Leftrightarrow M$ is projective, iff the rank over K of the controlability matrix $\mathrm{rk}\,(B\ AB - \dot{B}\ \ldots\ A^{n-1}B + \ldots\ \ldots)$ is equal to n. Of course, we can proceed similarly if A and B do not depend on the time t, and we recover the classical Kalman test. See [17] for more details.

The controllability of a linear multidimensional control system with variable or unknown coefficients may depend on some differential relations on the coefficients. We refer the reader to [14] where examples of trees of conditions are exhibited.

3.2 Linear Quadratic Case

In the course of the text, we shall use the following jet notation $\eta_q = (\eta_\mu, 0 \le |\mu| \le q)$. For example, if we take $X = \mathbb{R}$, i.e. in the OD case, we have $\eta_q = (\eta, ..., \dot{\eta}, ..., \eta^{(q)})$. Let us consider the differential operator $\mathcal{D}_1 : \eta \to \zeta$ of order q and the Lagrangian function

$$L(\eta_q) = \frac{1}{2}\,\eta_q^t\,R\,\eta_q,$$

where R is a symmetric matrix ($R^{\alpha,\beta}_{k,l} = R^{\beta,\alpha}_{l,k}$) with entries in K and $\eta = (\eta^k, 1 \le k \le m)$. Let us consider the problem of minimizing

$$\int L(\eta_q)\, \mathrm{d}x,$$

with $dx = dx^1 \wedge ... \wedge dx^n$, under the constraint

$$\mathcal{D}_1\, \eta = 0.$$

The variation of the Lagrangian function is given by $\delta L(\eta_q) = \sum_{|\alpha|\le q, 1\le k\le m} \pi^\alpha_k \delta\, \eta^k_\alpha$, where

$$\pi^\alpha_k = \frac{\partial L(\eta_q)}{\partial\, \eta^k_\alpha} = \sum_{1\le l\le m,\, |\beta|\le q} R^{\alpha,\beta}_{k,l}\, \eta^l_\beta.$$

We define the operator $\mathcal{B} : \eta \to \mu$ by

$$\mathcal{B}\,\eta = (\sum_{|\alpha|\le q} (-1)^{|\alpha|} d_\alpha \pi^\alpha_k) = \mu,$$

and for any section η of F_0, $\mathcal{B}\,\eta$ belongs to $\tilde{F}_0 = \bigwedge^n T^\star \otimes F^\star_0$ and we have the following diagram:

$$\begin{array}{ccc} F_0 & \xrightarrow{\mathcal{D}_1} & F_1 \\ \downarrow \mathcal{B} & & \\ \tilde{F}_0. & & \end{array}$$

Proposition 2. *The operator $\mathcal{B} : F_0 \to \tilde{F}_0$ is a self-adjoint operator, i.e. $\tilde{\mathcal{B}} = \mathcal{B}$.*

Proof. We multiply $\mathcal{B}\,\eta$ on the left by a vector $\theta \in F_0$ and we integrate by parts, we obtain with implicit summation on the dumb indices :

$$\begin{aligned} < \mathcal{B}\,\eta, \theta > &= (-1)^{|\alpha|}(d_\alpha \pi^\alpha_k)\, \theta^k = \pi^\alpha_k\, d_\alpha\, \theta^k + d(\cdot) \\ &= R^{\alpha,\beta}_{k,l}\, \eta^l_\beta\, d_\alpha \theta^k + d(\cdot) \\ &= R^{\alpha,\beta}_{k,l} \theta^k_\alpha \eta^l_\beta + d(\cdot) \\ &= ((-1)^{|\beta|} d_\beta \pi^\beta_l) \eta^l + d(\cdot) \\ &=< \eta, \mathcal{B}\,\theta > + d(\cdot). \end{aligned}$$

Finally, we notice that $\widetilde{(\tilde{F}_0)} = F_0$. $\square$

Theorem 4. *The optimal system is given by*

$$\left\{ \begin{array}{l} \mathcal{D}_1\, \eta = 0, \\ \mathcal{B}\,\eta - \tilde{\mathcal{D}}_1 \lambda = 0. \end{array} \right. \tag{15}$$

where λ is a Lagrange multiplier. Moreover, if the compatibility condition of $\tilde{\mathcal{D}}_1$ are written by means of an operator $\tilde{\mathcal{D}}_0$ and if the sequence $\tilde{E} \xleftarrow{\tilde{\mathcal{D}}_0} \tilde{F}_0 \xleftarrow{\tilde{\mathcal{D}}_1} \tilde{F}_1$ is locally exact at $\tilde{F}_0$, then the optimal system is

$$\left\{ \begin{array}{l} \mathcal{D}_1\, \eta = 0, \\ (\tilde{\mathcal{D}}_0 \circ \mathcal{B})\, \eta = 0, \end{array} \right. \tag{16}$$

and we have the following diagram:

$$\begin{array}{ccccc} & & F_0 & \xrightarrow{\mathcal{D}_1} & F_1 \\ & \swarrow & \downarrow \mathcal{B} & & \\ \tilde{E} & \xleftarrow{\tilde{\mathcal{D}}_0} & \tilde{F}_0 & \xleftarrow{\tilde{\mathcal{D}}_1} & \tilde{F}_1. \end{array}$$

Proof. If we denote by λ a Lagrange multiplier, then we have:

$$\delta \int (L(\eta_q) - \lambda \mathcal{D}_1 \eta)\, dx = \int (\mathcal{B}\,\eta - \tilde{\mathcal{D}}_1 \lambda)\delta\,\eta\; dx + \ldots$$

Accordingly, a necessary condition of optimality is:

$$\mathcal{B}\,\eta - \tilde{\mathcal{D}}_1 \lambda = 0, \tag{17}$$

and we obtain (15). Eliminating the Lagrange multiplier by composing (17) on the left by $\tilde{\mathcal{D}}_0$, we obtain $(\tilde{\mathcal{D}}_0 \circ \mathcal{B})\,\eta = 0$ and we have (16). Now, if we have (16) and the sequence $\tilde{E} \xleftarrow{\tilde{\mathcal{D}}_0} \tilde{F}_0 \xleftarrow{\tilde{\mathcal{D}}_1} \tilde{F}_1$ is locally exact at $\tilde{F}_0$, then we have $\tilde{\mathcal{D}}_0 \circ \mathcal{B}\,\eta = 0 \Rightarrow \tilde{\mathcal{D}}_0\,(\mathcal{B}\,\eta) = 0$ and thus $\exists\,\lambda \in \tilde{F}_0$ such that $\mathcal{B}\eta - \tilde{\mathcal{D}}_1\lambda = 0$. It leads back to (15). □

Example 6. Let us minimize

$$\int \frac{1}{2}((\eta^1)^2 + (\eta^2)^2)\; dx,$$

under the constraint:

$$\partial_1\eta^1 + \partial_2\eta^2 - x^2\eta^1 = 0.$$

The operator $\mathcal{B} : F_0 \to \tilde{F}_0$ is defined by $\mathcal{B}\,\eta = \eta$ and the adjoint $\tilde{\mathcal{D}}_1$ of $\mathcal{D}_1$ is (see example 3):

$$\left\{\begin{array}{l} -\partial_1\lambda - x^2\lambda = \mu_1, \\ -\partial_2\lambda = \mu_2. \end{array}\right.$$

Thus, the optimal system has to satisfy:

$$\left\{\begin{array}{l} \partial_1\eta^1 + \partial_2\eta^2 - x^2\eta^1 = 0, \\ \eta^1 + \partial_1\lambda + x^2\lambda = 0, \\ \eta^2 + \partial_2\lambda = 0. \end{array}\right.$$

We have seen in example 3 that the operator $\mathcal{D}_1 : \eta \to \partial_1\eta^1 + \partial_2\eta^2 - x^2\eta^1 = \zeta$ determines a projective D-module and thus that the sequence $\tilde{E} \xleftarrow{\tilde{\mathcal{D}}_0} \tilde{F}_0 \xleftarrow{\tilde{\mathcal{D}}_1} \tilde{F}_1$ is locally exact at $\tilde{F}_0$. The optimal system is equivalently given by:

$$\left\{\begin{array}{l} \partial_{11}\eta^2 - \partial_{12}\eta^1 + 2x^2\partial_1\eta^2 - x^2\partial_2\eta^1 + (x^2)^2\eta^2 + \eta^1 = 0, \\ \partial_{12}\eta^2 - \partial_{22}\eta^1 + x^2\partial_2\eta^2 + 2\eta^2 = 0, \\ \partial_1\eta^1 + \partial_2\eta^2 - x^2\eta^1 = 0. \end{array}\right.$$

We let the reader check by himself that the above system is formally integrable (see [13] for more details) and that its solution space depends on two arbitrary functions of one variable.

Theorem 5. *If the control system $\mathcal{D}_1$ is controllable, that is parametrizable by an operator $\mathcal{D}_0 : E \to F_0$ in such a way that the sequences $\tilde{E} \xleftarrow{\tilde{\mathcal{D}}_0} \tilde{F}_0 \xleftarrow{\tilde{\mathcal{D}}_1} \tilde{F}_1$ and $E \xrightarrow{\mathcal{D}_0} F_0 \xrightarrow{\mathcal{D}_1} F_1$ are locally exact at $\tilde{F}_0$ and F_0, then the optimal system is given by:*

$$\left\{\begin{array}{l} \mathcal{A}\,\xi = 0, \\ \mathcal{D}_0\,\xi = \eta, \end{array}\right. \tag{18}$$

where $\mathcal{A}$ is defined by

$$\mathcal{A} = \tilde{\mathcal{D}}_0 \circ \mathcal{B} \circ \mathcal{D}_0, \tag{19}$$

and we have the following diagram:

$$\begin{array}{ccccc} E & \xrightarrow{\mathcal{D}_0} & F_0 & \xrightarrow{\mathcal{D}_1} & F_1 \\ \mathcal{A}\downarrow & & \downarrow\mathcal{B} & & \\ \tilde{E} & \xleftarrow{\tilde{\mathcal{D}}_0} & \tilde{F}_0 & \xleftarrow{\tilde{\mathcal{D}}_1} & \tilde{F}_1. \end{array}$$

Proof. Let us show the equivalence between (16) and (18). Suppose that (16) is satisfied, then we have $\mathcal{D}_1\,\eta = 0 \Leftrightarrow \mathcal{D}_0\,\xi = \eta$ because the sequence formed by $\mathcal{D}_0$ and $\mathcal{D}_1$ is a locally exact at F_0. Moreover, $(\tilde{\mathcal{D}} \circ \mathcal{B})\,\eta = (\tilde{\mathcal{D}}_0 \circ \mathcal{B} \circ \mathcal{D}_0)\,\xi = \mathcal{A}\,\xi = 0$ and (16) and (18) are equivalent. □

Remark 2. Such an assumption gives the possibility to transfer a variational problem for η with constraint into a variational problem for ξ without any constraint.

Corollary 1. *If the control system $\mathcal{D}_1$ determines a projective D-module M, then the optimal system is determined by (18) where $\mathcal{A}$ is defined by (19). This is in particular the case if $\mathcal{D}_1$ is a controllable OD system.*

Proof. If $\mathcal{D}_1$ determines a projective D-module M, then the operator $\mathcal{D}_1$ can be parametrized by an operator $\mathcal{D}_0$ such that the two following sequences $\tilde{E} \xleftarrow{\tilde{\mathcal{D}}_0} \tilde{F}_0 \xleftarrow{\tilde{\mathcal{D}}_1} \tilde{F}_1$ and $E \xrightarrow{\mathcal{D}_0} F_0 \xrightarrow{\mathcal{D}_1} F_1$ are locally exact at $\tilde{F}_0$ and F_0 respectively and thus we have the result. If $\mathcal{D}_1$ is an OD control system, then M is a torsion-free D-module and thus a projective one because $D = k\,[\frac{d}{dt}]$ is a principal ideal ring. □

Remark 3. This result explains the importance of the controllability condition of an OD control system in the study of the variational formulation of an optimal control problem. In the PD case, the controllability is a necessary condition but, in general, not a sufficient one, whereas the fact that the D-module M, determined by the control system, is projective is a sufficient condition but not a necessary one because, for example, the Poincaré sequence, induced by the exterior derivative, is a locally exact sequence [13] but none of its operators determines a projective module.

Example 7. Let us take back example 6. The operator $\mathcal{D}_1$ determines a projective D-module and thus the sequence $E \xrightarrow{\mathcal{D}_0} F_0 \xrightarrow{\mathcal{D}_1} F_1$ is locally exact at F_0. The optimal system is then defined by $\mathcal{A}\,\xi = 0$ but in this case, as $\mathcal{B} = \mathrm{id}$, it follows that, with a slight abuse of language, we can write $\mathcal{A} = \tilde{\mathcal{D}}_0 \circ \mathcal{D}_0$ and the fourth order square operator $\mathcal{A}$ is trivially self-adjoint.

Example 8. Let the control system be defined by $D(\frac{d}{dt})y + N(\frac{d}{dt})u = 0$, with $\det\ D(\frac{d}{dt}) \neq 0$, and $(D\ \ N)$ left-coprime, i.e. controllable. Thus, the operator $\mathcal{D}_1 : (y\ u) \to \zeta$, defined by

$$D(\frac{d}{dt})y + N(\frac{d}{dt})u = \zeta$$

has a right-inverse and the $D = \mathbb{R}[\frac{d}{dt}]$-module M determined by $\mathcal{D}_1$ is a projective D-module (see example 14 and see [15] for more details). But, $D = \mathbb{R}[\frac{d}{dt}]$ is a principal ideal domain and thus M is a free D-module, i.e. $\mathcal{D}_1$ admits an injective parametrization $\mathcal{D}_0$, which is in fact the *controller form*

$$\left\{ \begin{array}{l} \overline{N}(s)\,\xi = y, \\ \overline{D}(s)\,\xi = u, \end{array} \right.$$

where ξ is the basis of M, called the *partial state* [5]. The adjoint $\tilde{\mathcal{D}}_0 : (\mu_1, \mu_2) \to \nu$ of $\mathcal{D}_0$ is defined by:

$$\overline{N}(-s)^{\,t}\,\mu_1 + \overline{D}(-s)^{\,t}\,\mu_2 = \nu$$

Now, let us find the optimal system minimizing

$$\int \frac{1}{2}\,\eta^t\,R\,\eta\,dt,$$

where $\eta = (y, u)^t$ and R is a symmetric matrix. We easily check that $\mathcal{B} = R$. Finally, we obtain:

$$\tilde{\mathcal{D}}_0 \circ \mathcal{B} = (\overline{N}(-s)^{\,t}\ \ \overline{D}(-s)^{\,t}) \circ R,$$

and the operator $\mathcal{A} : \xi \to \nu$ is defined by:

$$(\overline{N}(-s)^{\,t} \;\; \overline{D}(-s)^{\,t}) \circ R \circ \begin{pmatrix} \overline{N}(s) \\ \overline{D}(s) \end{pmatrix} \xi = \nu. \tag{20}$$

In particular, if we take $R = \begin{pmatrix} I & 0 \\ 0 & S \end{pmatrix}$, where S is a symmetric and definite positive matrix acting on the inputs, we find that the dynamics of the optimal system is given by: (compare with [5] where one needs many technical results on determinants of matrices):

$$(\overline{N}(-s)^{\,t} \circ \overline{N}(s) + \overline{D}(-s)^{\,t} \circ S \circ \overline{D}(s))\, \xi = 0.$$

Corollary 2. *The operator $\mathcal{A} : E \to \tilde{E}$ defined by (18) is self-adjoint, i.e. $\tilde{\mathcal{A}} = \mathcal{A}$.*

Proof. We have:

$$\tilde{\mathcal{A}} = \widetilde{\tilde{\mathcal{D}}_0 \circ \mathcal{B} \circ \mathcal{D}_0} = \tilde{\mathcal{D}}_0 \circ \tilde{\mathcal{B}} \circ \mathcal{D}_0 = \mathcal{A},$$

because we know from Proposition 2 that $\mathcal{B}$ is a self-adjoint operator, i.e. $\tilde{\mathcal{B}} = \mathcal{B}$. □

Example 9. We take again example 8. We have seen that the dynamics of the optimal system is given by $\mathcal{A}\xi = 0$, where $\mathcal{A}$ is defined by (20). We easily verified that $\mathcal{A}$ was a self-adjoint operator. If we denote by $\Delta(s) = (\overline{N}(-s)^{\,t} \;\; \overline{D}(-s)^{\,t}) \circ R \circ \begin{pmatrix} \overline{N}(s) \\ \overline{D}(s) \end{pmatrix}$ and $\delta(s) = \det \Delta(s)$, thus we have:

$$\begin{aligned} \delta(s) &= \det\,(\Delta(s)^t) \\ &= \det\,(\Delta(-s)) \\ &= \delta(-s). \end{aligned}$$

Hence, if there exists $s_0 \in \mathbb{C}$ such that $\delta(s_0) = 0$, then $\delta(-s_0) = 0$, showing that the eigenvalues of the dynamics $\mathcal{A}\xi = 0$ are symmetric with respect to the real axis.

Proposition 3. *If the operator $\mathcal{D}_1 : \eta \to \zeta$ is surjective and determines a projective D-module M, then we can express λ as differential combination of η, i.e. $\lambda = \tilde{\mathcal{P}}_1 \circ \mathcal{B}\,\eta$, where $\tilde{\mathcal{P}}_1$ is a left-inverse of the injective operator $\tilde{\mathcal{D}}_1$. The operator $\tilde{\mathcal{P}}_1 \circ \mathcal{B} : F_0 \to \tilde{F}_1$ allows to observe λ and we have the diagram:*

$$\begin{array}{ccccc} F_0 & \xrightarrow{\mathcal{D}_1} & F_1 & \longrightarrow & 0 \\ \mathcal{B} \downarrow & & \searrow & & \\ \tilde{F}_0 & \xleftarrow{\tilde{\mathcal{D}}_1} & \tilde{F}_1 & \longleftarrow & 0. \\ & \xrightarrow{\tilde{\mathcal{P}}_1} & & & \end{array}$$

Proof. The fact that the operator $\mathcal{D}_1$ is surjective and that it determines a projective D-module implies that $\tilde{\mathcal{D}}_1$ is an injective operator and thus $\tilde{\mathcal{D}}_1$ admits a left-inverse $\tilde{\mathcal{P}}_1$, i.e. $\tilde{\mathcal{P}}_1 \circ \tilde{\mathcal{D}}_1 = \mathrm{id}_{\tilde{F}_1}$. Accordingly, we have $\mathcal{B}\,\eta = \tilde{\mathcal{D}}_1 \lambda \Rightarrow \lambda = (\tilde{\mathcal{P}}_1 \circ \mathcal{B})\,\eta$. □

Remark 4. In the OD case, a control system defined by a surjective operator $\mathcal{D}_1$ is controllable if and only if $\tilde{\mathcal{D}}_1$ is injective.

Example 10. We take back example 6. We have seen in example 3 that $\tilde{\mathcal{D}}_1$ was an injective operator and that $\tilde{\mathcal{D}}_1$ had a left-inverse $\tilde{\mathcal{P}}_1 : \mu \to \lambda$ defined by:

$$\partial_2 \mu_1 + \partial_1 \mu_2 - x^2 \mu_2 = \lambda.$$

Thus, the equation $\tilde{\mathcal{P}}_1 \circ \mathcal{B} : \eta \to \lambda$ is defined by:

$$\partial_2 \eta^1 + \partial_1 \eta^2 - x^2 \eta^2 = \lambda.$$

The following particular case is motivated by elasticity theory as we shall see later on and by the fact that we closed the diagram of theorem 5 on the left and not on the right.

Proposition 4. *If the operator $\mathcal{B} : F_0 \to \tilde{F}_0$ is invertible, then the optimal system is given by:*

$$\left\{ \begin{array}{l} \mathcal{C}\lambda = 0, \\ (\mathcal{B}^{-1} \circ \tilde{\mathcal{D}}_1)\,\lambda = \eta. \end{array} \right. \tag{21}$$

where $\mathcal{C}$ is defined by:

$$\mathcal{C} = \mathcal{D}_1 \circ \mathcal{B}^{-1} \circ \tilde{\mathcal{D}}_1, \tag{22}$$

and we have the following diagram:

$$\begin{array}{ccc} F_0 & \stackrel{\mathcal{D}_1}{\longrightarrow} & F_1 \\ \mathcal{B} \downarrow\uparrow \mathcal{B}^{-1} & & \uparrow \mathcal{C} \\ \tilde{F}_0 & \stackrel{\tilde{\mathcal{D}}_1}{\longleftarrow} & \tilde{F}_1. \end{array}$$

In particular, it is the case if $\mathcal{D}_1$ is a first order operator (e.g. Kalman system) and $\mathcal{B} = R$ a non negative square matrix with constant entries.

Proof. We have seen that the optimal system is given by:

$$\left\{ \begin{array}{l} \mathcal{D}_1\,\eta = 0, \\ \mathcal{B}\,\eta - \tilde{\mathcal{D}}_1\lambda = 0, \end{array} \right.$$

From the second equation, we obtain $\eta = (\mathcal{B}^{-1} \circ \tilde{\mathcal{D}}_1)\lambda$, and thus $\mathcal{D}_1\,\eta = (\mathcal{D}_1 \circ \mathcal{B}^{-1} \circ \tilde{\mathcal{D}}_1)\lambda = 0$ and we obtain (21). Reciproquely, if we have (21), then $0 = \mathcal{C}\lambda = (\mathcal{D}_1 \circ \mathcal{B}^{-1} \circ \tilde{\mathcal{D}}_1)\lambda = \mathcal{D}_1\,\eta$, which concludes the proof. □

We let the reader do the computations for Kalman systems and invertible cost.

Example 11. We take back example 6. The operator $\mathcal{B} : F_0 \to \tilde{F}_0$ is invertible and $\mathcal{B}^{-1}\mu = \eta$. Thus, the operator $\mathcal{C} : \tilde{F}_1 \to F_1$ is defined by

$$\Delta\,\lambda - (x^2)^2\lambda = 0.$$

The optimal system is governed by the following system:

$$\left\{ \begin{array}{l} \Delta\,\lambda - (x^2)^2\lambda = 0, \\ \eta^1 = -\partial_1\lambda - x^2\lambda, \\ \eta^2 = -\partial_2\lambda. \end{array} \right.$$

We find back that the optimal problem only depends on two arbitrary functions of one variable needed for integrating the first equation above.

Let us finish with the *Riccati equation* and *integrability conditions*. We have seen that if we start with a control system $\mathcal{D}_1\,\eta = 0$, then resolving the variational problem associated with the optimal control problem adds the new equations $\tilde{\mathcal{B}}\,\eta - \tilde{\mathcal{D}}_1\lambda = 0$ (see theorem 4) and the new system becomes (16). The solution of (15) does not depend, in general, on arbitrary functions of n variables, i.e. the module determined by (15) is a torsion D-module, i.e. the system (15) is determined. If we add new equations in η and λ to the system (15), then there exists a new solution if and only if some integrability conditions are satisfied (see [14] for trees of integrability conditions). This is the way which leads to the Riccati equation: let us find the solution of

$$\min \int \frac{1}{2} \begin{pmatrix} x & u \end{pmatrix}^t \begin{pmatrix} Q & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} dt,$$

where R is a definite positive matrix, Q is a positive one while x and u satisfy the Kalman system:

$$\dot{x} - A\,x - B\,u = 0.$$

We obtain the following system:

$$\begin{cases} \dot{x} - A\,x - B\,u = 0, \\ \dot{\lambda} + A^t\lambda + Qx = 0, \\ R\,u + B^t\,\lambda = 0. \end{cases}$$

Using the fact that R is invertible, we have:

$$\begin{cases} \dot{x} - A\,x - B\,R^{-1}B^t\lambda = 0, \\ \dot{\lambda} + A^t\lambda + Q\,x = 0, \\ -R^{-1}\,B^t\lambda = u. \end{cases}$$

The system in x and λ is determined and thus, if we add to this system the new equation $\lambda - Px = 0$ as a kind of feedback for the total system, *it becomes non formally integrable*, i.e. we cannot find step by step the solution of the system as a formal power series (see for more informations [13, 15]). Indeed, if we differentiate the zero order equation and take into account the other equations, we find the following new zero order equation:

$$(\dot{P} + A^tP + PA - PBR^{-1}B^tP + Q)\,x = 0.$$

Hence, the system

$$\begin{cases} \dot{x} - A\,x - B\,R^{-1}B^t\lambda = 0, \\ \dot{\lambda} + A^t\lambda + Q\,x = 0, \\ -R^{-1}\,B^t\lambda = u, \\ \lambda - P\,x = 0, \end{cases}$$

has a solution different from zero iff the following integrability condition on P is satisfied

$$\dot{P} + A^tP + PA - PBR^{-1}B^tP + Q = 0,$$

that is the Riccati equation for P. In this case, we can rewrite the system of equations as:

$$\begin{cases} \dot{P} + A^tP + PA - PBR^{-1}B^tP + Q = 0, \\ \dot{x} - (A - B\,R^{-1}\,B^t\,P)\,x = 0, \\ -R^{-1}\,B^t\,P\,x = u, \\ P\,x = \lambda. \end{cases}$$

We have recently shown in [14, 16] that the controllablity of a system with indetermined coefficients depended on trees of integrability conditions on the coefficients. The same thing may happen in optimal control and we provide an illustrative example.

Example 12. Let us consider the following system

$$\min \int \frac{1}{2}(y^2 - u^2)\,dt,$$

where y and u satisfy the system

$$\dot{y} + a\,y - \dot{u} - u = 0,$$

in which a is a constant coefficient. The system (15) is given by:

$$\left\{ \begin{array}{l} \dot{\lambda} - a\,\lambda + y = 0, \\ -\dot{\lambda} + \lambda - u = 0, \\ \dot{y} + a\,y - \dot{u} - u = 0. \end{array} \right.$$

Let us eliminate λ in order to find (16): summing the first two equations, we obtain the new zero order equation $(1-a)\lambda = u - y$. Therefore, two cases may happen depending on the value of the parameter a:

1. if $a = 1$, then $y - u = 0$ and the optimal system is thus given by:

$$\left\{ \begin{array}{l} \dot{y} + y - \dot{u} - u = 0, \\ y - u = 0, \end{array} \right.$$

   i.e. $y - u = 0$.

2. if $a \neq 1$, then $\lambda = (y-u)/(a-1)$ and, after substituting, we get:

$$\left\{ \begin{array}{l} \dot{y} + a\,y - \dot{u} - u = 0, \\ \dot{y} - y - \dot{u} + a\,u = 0, \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \dot{y} + a\,y - \dot{u} - u = 0, \\ (a+1)\,(y-u) = 0. \end{array} \right.$$

   We are led to new integrability conditions:

   (a) if $a = -1$, then the optimal system is given by the only equation $\dot{y} + a\,y - \dot{u} - u = 0$. In fact, we can notice that in this case a parametrization $\mathcal{D}_0 : \xi \rightarrow \left(\begin{array}{cc} y, & u \end{array} \right)$ of the system is given by:

$$\left\{ \begin{array}{l} \dot{\xi} + \xi = y, \\ \dot{\xi} - \xi = u, \end{array} \right.$$

   and thus $\frac{1}{2}(y^2 - u^2) = 2\,\xi\,\dot{\xi} = \dfrac{d}{dt}(\xi^2)$.

   (b) if $a \neq -1$, then the only solution is $y = 0 = u$.

We notice that the condition $a \neq 1$ is, in fact, the condition on a for the system to be controllable (if $a = 1$, the element $z = y - u$ satisfies $(\frac{d}{dt} + 1)z = 0$).

4 Applications

We show in this section how all the preceding sections can be used for applications, specially in elasticity theory.

4.1 Elasticity Theory

Let us denote the displacement in $\mathbb{R}^n$ by $\xi = (\xi^i)_{1 \leq i \leq n}$ and contract the index of ξ^i by the euclidean metric $\omega_{ij} = \omega_{ji} = \delta_{ij}, 1 \leq i, j \leq n$, of $\mathbb{R}^n$ in order to lower the index with $\xi_i = \omega_{ij}\,\xi^j$. The so-called small strain tensor is then given by the operator

$$\begin{array}{rcll} \mathcal{L}(\cdot)\,\omega : & T & \longrightarrow & S_2 T^\star, \\ & \xi & \longrightarrow & (\epsilon_{ij} = \frac{1}{2}(\mathcal{L}(\xi)\omega)_{ij} = \frac{1}{2}(\partial_i \xi_j + \partial_j \xi_i))_{1 \leq i,j \leq n} \end{array}$$

where $\mathcal{L}$ is the Lie derivative of the euclidean metric. Let us only consider the case $n = 2$. Thus, the small strain tensor is given by

$$\left\{\begin{array}{l} \epsilon_{11} = \partial_1 \xi_1, \\ \epsilon_{12} = \epsilon_{21} = \frac{1}{2}(\partial_1\xi_2 + \partial_2\xi_1), \\ \epsilon_{22} = \partial_2 \xi_2. \end{array}\right.$$

This system has only one compatibility condition of order two, namely

$$\partial_{11}\, \epsilon_{22} + \partial_{22}\, \epsilon_{11} - 2\, \partial_{12}\, \epsilon_{12} = 0, \tag{23}$$

and we have the following sequence of differential operators

$$0 \longrightarrow \Theta \longrightarrow E \xrightarrow{\mathcal{D}} F_0 \xrightarrow{\mathcal{D}_1} F_1 \longrightarrow 0, \tag{24}$$

where $E = T, F_0 = S_2 T^\star, \Theta$ is the field of small rigid displacements and $\mathcal{D}\,\xi = \frac{1}{2}(\mathcal{L}(\xi)\omega)$. In the spirit of Poincaré, this sequence is only based on geometry whereas the adjoint sequence, i.e. the sequence formed by the adjoint operators, gives the physics. Indeed, the adjoint $\tilde{\mathcal{D}} : \sigma \rightarrow f$ of the operator $\mathcal{D}$ is defined by multiplying ϵ by σ and integrating by parts, i.e.

$$\begin{aligned} \sigma^{11}\epsilon_{11} + \sigma^{12}\epsilon_{12} + \sigma^{21}\epsilon_{21} + \sigma^{22}\epsilon_{22} &= \sigma^{11}\epsilon_{11} + 2\,\sigma^{12}\epsilon_{12} + \sigma^{22}\epsilon_{22} \\ &= -(\partial_1\, \sigma^{11} + \partial_2\, \sigma^{12})\, \xi_1 - (\partial_1\, \sigma^{12} + \partial_2\, \sigma^{22})\, \xi_2 + \dots \end{aligned}$$

where we have supposed that $\sigma^{12} = \sigma^{21}$. Thus, up to a sign, $-\tilde{\mathcal{D}} : \sigma \rightarrow f$ is given by

$$\left\{\begin{array}{l} \partial_1\, \sigma^{11} + \partial_2\, \sigma^{12} = f^1, \\ \partial_1\, \sigma^{12} + \partial_2\, \sigma^{22} = f^2, \end{array}\right. \tag{25}$$

where σ is the stress tensor and f is a density of forces. Similarly, the adjoint $\tilde{\mathcal{D}}_1$ of the operator $\mathcal{D}_1$ is obtained by multiplying (23) by λ and integrating by parts

$$\lambda\, (\partial_{11}\, \epsilon_{22} + \partial_{22}\, \epsilon_{11} - 2\, \partial_{12}\, \epsilon_{12}) = \partial_{11}\lambda\, \epsilon_{22} + \partial_{22}\lambda\, \epsilon_{11} - 2\, \partial_{12}\lambda\, \epsilon_{12} + \dots$$

and thus $\tilde{\mathcal{D}}_1 : \lambda \rightarrow \sigma$ is given by :

$$\left\{\begin{array}{l} \partial_{22}\, \lambda = \sigma^{11}, \\ -\partial_{12}\, \lambda = \sigma^{12}, \\ \partial_{11}\, \lambda = \sigma^{22}, \end{array}\right. \tag{26}$$

and we check easily that all the compatibility conditions of $\tilde{\mathcal{D}}_1$ are generated by $\tilde{\mathcal{D}}$ or $-\tilde{\mathcal{D}}$. We find back the well-known parametrization of the stress tensor by the Airy function λ. Finally, we have the formally exact sequence:

$$0 \longleftarrow \tilde{E} \xleftarrow{-\tilde{\mathcal{D}}} \tilde{F}_0 \xleftarrow{\tilde{\mathcal{D}}_1} \tilde{F}_1. \tag{27}$$

In fact, it can be shown that the sequence (24) is locally equivalent to the Poincaré sequence $\bigwedge^0 T^\star \xrightarrow{d} \bigwedge^1 T^\star \xrightarrow{d} \bigwedge^2 T^\star \longrightarrow 0$ (see [13]), which is a locally exact and self-adjoint sequence. Hence, the sequences (24) et (27) are locally exact. Moreover, the Poincaré sequence being a self-adjoint sequence, this is the reason why the kernel Θ of the operator $\mathcal{D}$ and the kernel Ω of $\tilde{\mathcal{D}}_1$ both depend on three arbitrary constants. Finally, we can link these two differential sequences with the constitutive law, namely the Hooke law $\mathcal{B} : \epsilon \rightarrow \sigma$, defined by

$$\left\{\begin{array}{l} \sigma^{11} = (\alpha + 2\,\beta)\, \epsilon_{11} + \alpha\, \epsilon_{22}, \\ \sigma^{12} = \sigma^{21} = 2\,\beta\, \epsilon_{12}, \\ \sigma^{22} = \alpha\, \epsilon_{11} + (\alpha + 2\,\beta)\epsilon_{22}, \end{array}\right.$$

where (α, β) are the Lamé constants and we obtain the following locally exact diagram:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \Theta & \longrightarrow & E & \xrightarrow{\mathcal{D}} & F_0 & \xrightarrow{\mathcal{D}_1} & F_1 & \longrightarrow & 0 \\ & & & & & & \downarrow{\scriptstyle \mathcal{B}} & & & & \\ & & 0 & \longleftarrow & \tilde{E} & \xleftarrow{-\tilde{\mathcal{D}}} & \tilde{F}_0 & \xleftarrow{\tilde{\mathcal{D}}_1} & \tilde{F}_1 & \longleftarrow & \Omega \longleftarrow 0. \end{array} \tag{28}$$

With such a diagram, we naturally want to associate the operator $\mathcal{A} = -\tilde{\mathcal{D}} \circ \mathcal{B} \circ \mathcal{D} : E \to \tilde{E}$ to it. Now, let us notice that $\mathcal{B}$ is a symmetric matrix, i.e. $\tilde{\mathcal{B}} = \mathcal{B}$, and thus $\mathcal{A}$ is a self-adjoint operator. This fact can easily be verified on the direct expression of the operator $\mathcal{A}$

$$\begin{cases} (\alpha + 2\,\beta)\,\partial_{11}\,\xi_1 + \beta\,\partial_{22}\,\xi_1 + (\alpha + \beta)\,\partial_{12}\,\xi_2 = f^1, \\ (\alpha + \beta)\,\partial_{12}\,\xi_1 + \beta\,\partial_{11}\,\xi_2 + (\alpha + 2\,\beta)\,\partial_{22}\,\xi_2 = f^2, \end{cases}$$

or equivalently on the so-called Navier equations:

$$\begin{cases} (\alpha + \beta)\,\partial_1\,(\partial_1\,\xi_1 + \partial_2\,\xi_2) + \beta\,\Delta\,\xi_1 = f^1, \\ (\alpha + \beta)\,\partial_2\,(\partial_1\,\xi_1 + \partial_2\,\xi_2) + \beta\,\Delta\,\xi_2 = f^2. \end{cases}$$

The Hook law is in fact invertible and the operator $\mathcal{B}^{-1}$ is given by

$$\begin{cases} \epsilon_{11} = \dfrac{(\alpha + 2\,\beta)}{4\,\beta\,(\alpha + \beta)}\,\sigma^{11} - \dfrac{\alpha}{4\,\beta\,(\alpha + \beta)}\,\sigma^{22}, \\ \epsilon_{12} = \dfrac{1}{2\,\beta}\,\sigma^{12}, \\ \epsilon_{22} = -\dfrac{\alpha}{4\,\beta\,(\alpha + \beta)}\,\sigma^{11} + \dfrac{(\alpha + 2\,\beta)}{4\,\beta\,(\alpha + \beta)}\,\sigma^{22}. \end{cases}$$

and thus, we have an operator $\mathcal{C} = \mathcal{D}_1 \circ \mathcal{B}^{-1} \circ \tilde{\mathcal{D}}_1 : \tilde{F}_1 \to F_1$, defined by:

$$\frac{(\alpha + 2\,\beta)}{4\,\beta\,(\alpha + \beta)}\,\Delta\Delta\,\lambda = \zeta.$$

Finally, we can sum up the different operators by the following locally exact diagram :

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \Theta & \longrightarrow & E & \xrightarrow{\mathcal{D}} & F_0 & \xrightarrow{\mathcal{D}_1} & F_1 & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \mathcal{A}} & & \downarrow{\scriptstyle \mathcal{B}} & & \uparrow{\scriptstyle \mathcal{C}} & & \\ & & 0 & \longleftarrow & \tilde{E} & \xleftarrow{-\tilde{\mathcal{D}}} & \tilde{F}_0 & \xleftarrow{\tilde{\mathcal{D}}_1} & \tilde{F}_1 & \longleftarrow & \Omega \longleftarrow 0 \end{array}$$

To finish this section, let us connect the above results to controllability of multidimensional systems. We can conclude, from the controllability test and what precedes, that the operator $\mathcal{D}_1$ determines a torsion-free D-module M, with $\mathcal{D}_0$ as a parametrization (it is not surprising because, by definition, $\mathcal{D}_1$ is the compatibility condition of $\mathcal{D}_0$). More surprisingly, we have proved that the D-module determined by the operator $\tilde{\mathcal{D}}_0$ is also a torsion-free module with parametrization given by the operator (26) and a "potential" λ called Airy function in this case.

4.1.1 Case without forces

In the case where there is no force, let us mimimize the energy of deformation given by

$$\int \frac{1}{2}\,\epsilon^t\,\mathcal{B}\,\epsilon\;\mathrm{d}x^1\mathrm{d}x^2,$$

under the constraint $\mathcal{D}_1\epsilon = 0$. Introducing a new unknown λ as a Lagrange multiplier, it is equivalent to extremize the new integral

$$\int (\frac{1}{2}\,\epsilon^t\,\mathcal{B}\,\epsilon - \lambda\,\mathcal{D}_1\,\epsilon)\,\mathrm{d}x^1\mathrm{d}x^2,$$

where the ϵ are now considered as independent variables. Thus, by theorem 4, we have to solve the system

$$\left\{\begin{array}{l} \mathcal{B}\,\epsilon - \tilde{\mathcal{D}}_1\lambda = 0, \\ \mathcal{D}_1\,\epsilon = 0, \end{array}\right. \tag{29}$$

or, in another words, the system:

$$\left\{\begin{array}{l} \sigma^{12} + \partial_{11}\,\sigma^{22} = 0, \\ \sigma^{11} - \partial_{22}\,\lambda = 0, \\ \sigma^{12} + \partial_{12}\,\lambda = 0, \\ \sigma^{22} - \partial_{11}\,\partial_{11}\,\lambda = 0, \\ \partial_{22}\,\sigma^{11} - 2\,\partial_{12}. \end{array}\right.$$

We can solve $\epsilon = (\mathcal{B}^{-1} \circ \tilde{\mathcal{D}}_1)\,\lambda$ in the first equation of (29), and, substituting it in the second, we obtain $\mathcal{C}\,\lambda = (\mathcal{D}_1 \circ \mathcal{B}^{-1} \circ \tilde{\mathcal{D}}_1)\lambda = 0$. Finally, we have to solve the following system:

$$\left\{\begin{array}{l} \Delta\,\Delta\,\lambda = 0, \\ \epsilon_{11} = \dfrac{(\alpha + 2\,\beta)}{4\,\beta\,(\alpha+\beta)}\,\partial_{22}\,\lambda - \dfrac{\alpha}{4\,\beta\,(\alpha+\beta)}\,\partial_{11}\,\lambda, \\ \epsilon_{12} = -\dfrac{1}{2\,\beta}\,\partial_{12}\,\lambda, \\ \epsilon_{22} = -\dfrac{\alpha}{4\,\beta\,(\alpha+\beta)}\,\partial_{22}\,\lambda + \dfrac{(\alpha + 2\,\beta)}{4\,\beta\,(\alpha+\beta)}\,\partial_{11}\,\lambda. \end{array}\right.$$

and, from the first equation of the above system, λ is biharmonic.

Moreover, the sequence (24) is locally exact, therefore, as we saw in theorem 5, the solution of the equivalent unconstrained problem

$$\min \int \frac{1}{2}\,\xi^t\,(\tilde{\mathcal{D}} \circ \mathcal{B} \circ \mathcal{D})\,\xi\,\mathrm{d}x^1\,\mathrm{d}x^2$$

is

$$\left\{\begin{array}{l} \mathcal{A}\,\xi = 0, \\ \mathcal{D}\,\xi = \epsilon, \end{array}\right.$$

or equivalently, we have to solve a system of PDE only in the displacements ξ:

$$\left\{\begin{array}{l} (\alpha+\beta)\,\partial_1\,(\partial_1\,\xi_1 + \partial_2\,\xi_2) + \beta\,\Delta\,\xi_1 = 0, \\ (\alpha+\beta)\,\partial_2\,(\partial_1\,\xi_1 + \partial_2\,\xi_2) + \beta\,\Delta\,\xi_2 = 0, \\ \partial_1\,\xi_1 = \epsilon_{11} \\ \frac{1}{2}(\partial_1\,\xi_2 + \partial_2\,\xi_1) = \epsilon_{12}, \\ \partial_2\,\xi_2 = \epsilon_{22}. \end{array}\right.$$

4.1.2 Forces coming from a potential

If the force f comes from a potential ψ, i.e.

$$\left\{\begin{array}{l} f^1 = \partial_1\,\psi, \\ f^2 = \partial_2\,\psi, \end{array}\right.$$

then, from (25), we have the following system

$$\left\{ \begin{array}{l} \partial_1 \sigma^{11} + \partial_2 \sigma^{12} - \partial_1 \psi = 0, \\ \partial_1 \sigma^{12} + \partial_2 \sigma^{22} - \partial_2 \psi = 0, \end{array} \right. \tag{30}$$

and, if we introduce by $\overline{\sigma}^{11} = \sigma^{11} - \psi$, $\overline{\sigma}^{12} = \sigma^{12}$ and $\overline{\sigma}^{22} = \sigma^{22} - \psi$, we find the new system without forces :

$$\left\{ \begin{array}{l} \partial_1 \overline{\sigma}^{11} + \partial_2 \overline{\sigma}^{12} = 0, \\ \partial_1 \overline{\sigma}^{12} + \partial_2 \overline{\sigma}^{22} = 0, \end{array} \right. \tag{31}$$

Moreover, we have $\tilde{\mathcal{D}}\, \overline{\sigma} = 0 \Longleftrightarrow \overline{\sigma} = \tilde{\mathcal{D}}_1 \lambda$ because the differential sequence $\tilde{F}_1 \xrightarrow{\tilde{\mathcal{D}}_1} \tilde{F}_0 \xrightarrow{\tilde{\mathcal{D}}} \tilde{E}$ is locally exact at $\tilde{F}_0$. Finally, we find the system

$$\left\{ \begin{array}{l} \sigma^{11} = \partial_{22}\, \lambda + \psi, \\ \sigma^{12} = -\partial_{12}\, \lambda, \\ \sigma^{22} = \partial_{11}\, \lambda + \psi. \end{array} \right. \tag{32}$$

Therefore, solving the system (30), where σ satisfies $\epsilon = \mathcal{A}\, \sigma$ and $\mathcal{D}_1\, \epsilon = 0$, is the same as solving the system (32) with $\epsilon = \mathcal{A}\, \sigma$ and $\mathcal{D}_1\, \epsilon = 0$. Hence, we have to solve the following system of PDE

$$\Delta \Delta\, \lambda + \frac{\beta}{(\alpha + 2\, \beta)}\, \Delta\, \psi = 0,$$

and to substitute the result in (32) to obtain the corresponding stress tensor σ. As a matter of fact, when the only forces involved are of gravitational type, then $\Delta\, \psi = 0$, and we are brought back to the preceding situation.

5 Conclusion

We hope to have convinced the reader about the possibility to extend optimal control theory from the study of variational problems with linear ordinary differential constraints to variational problems with linear partial differential constraints. At the same time, we have explicitely stressed out the role of the controllability condition imposed on the control system as a differential constraint.

References

[1] Bjork, J. E. (1993). *Analytic $\mathcal{D}$-modules and Applications*, Kluwer.

[2] Cartan, E. (1945). *Les systèmes différentiels extérieurs et leurs applications géométriques*, Hermann.

[3] Janet, M. (1920). "Sur les systèmes aux dérivées partielles", *Journal de Math.*, 8$^{\text{ème}}$ série, III, pp. 65-151.

[4] Janet, M. (1929). *Leçons sur les systèmes d'équations aux dérivées partielles*, Cahiers Scientifiques IV, Gauthier-villars.

[5] Kailath, T. (1980). *Linear Systems*, Prentice-Hall.

[6] Kashiwara, M. (1970). *Algebraic Study of Systems of Partial Differential Equations*, Mémoires de la Société Mathématiques de France, no. 63 (1995).

[7] Kolchin, E. R. (1973). *Differential Algebra and Algebraic Groups*, Pure Appl. Math., No. 54, Academic Press.

[8] Maisonobe, P. and Sabbah, C. (1993). *$\mathcal{D}$-Modules Cohérents et Holonomes*, Travaux en cours No. 45, Hermann.

[9] Malgrange, B. (1962/63). "Systèmes à coefficients constants", Séminaire Bourbaki, 246, pp. 1-11.

[10] Malgrange, B. (1966). "Cohomologie de Spencer (d'après Quillen)", *Sém. Math. Orsay*, France.

[11] Palamodov, V. P. (1970). *Linear Differential Operators with Constant Coefficients*, 168, Springer-Verlag.

[12] Pham, F. (1980). *Singularités des systèmes différentiels de Gauss-Manin*, Progress in Math. Vol. 1 et 2, Birkhaüser.

[13] Pommaret, J. F. (1994). *Partial Differential Equations and Group Theory: New Perspectives for Applications*, Kluwer.

[14] Pommaret, J. F. and Quadrat, A. (1997). "Formal obstructions to the controllability of partial differential control systems", *IMACS World Congress*, édité par A. Sydow, Berlin, août, Vol. 5, pp. 209-214.

[15] Pommaret, J. F. and Quadrat, A. (1998). "Generalized Bezout identity", *Applicable Algebra in Engineering, Communication and Computing*, Vol. 9 (2), pp. 91-116.

[16] Pommaret, J. F. and Quadrat, A. (1998). "Formal elimination theory. Applications for control theory", *preprint CERMICS*, No. 97-108.

[17] Pommaret, J. F. and Quadrat, A. (1999). "Localization and parametrization of linear multidimensional control systems", *Systems and Control Letters*, Vol. 37 (1999), pp. 247-260.

[18] Pommaret, J. F. and Quadrat, A. (1999). "Algebraic analysis of linear multidimensional control systems", *IMA J. Control and Information*, Vol. 16. pp. 275-297.

[19] Oberst, U. (1990). "Multidimensional Constant Linear Systems", *Acta Appl. Math.*, Vol. 20, pp. 1-175.

[20] Riquier, Ch. (1910). *Les systèmes d'équations aux dérivées partielles*, Gauthier-Villars.

[21] Riquier, Ch. (1928). *La méthode des fonctions majorantes et les systèmes d'équations aux dérivées partielles*, Mémorial Sci. Math. XXXII, Gauthier-Villars.

[22] Ritt, J. F. (1950). *Differential Algebra*, AMS Colloq. Publ., Vol. 33.

[23] Rotman, J. J. (1979). *An Introduction to Homological Algebra*, Academic Press.

[24] Spencer, D. C. (1965). "Overdetermined Systems of Partial Differential Equations", *Bull. Amer. Math. Soc.*, Vol. 75, pp. 1-114.

[25] Wood, J., Rogers, E. and Owens, D. (1998). "Formal theory of matrix primeness", *Mathematics of Control Signals, and Systems*, Vol. 11, pp. 40-78.

[26] Youla, D. C. and Gnavi, G. (1979). "Notes on n-dimensional system theory", *IEEE Trans. Circuits Syst.*, Vol CAS-26, No. 2, pp. 105-111.

[27] Youla, D. C. and Pickel, P. F. (1984). "The Quillen-Suslin theorem and the structure of n-dimensional elementary polynomial matrices", *IEEE Trans. Circuits Syst.*, Vol CAS-31, No. 6, pp. 513-518.

[28] Zerz, E. (1996). "Primeness of multivariate polynomial matrices", *Systems & Control Letters*, Vol. 29, No. 3, pp. 139-145.