American prices embedded in European prices

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Abstract

In this paper, we are interested in American option prices in the Black-Scholes model. For a large class of payoffs, we show that in the region where the European price increases with the time to maturity, this price is equal to the American price of another claim. We give examples in which we explicit the corresponding claims. The characterization of the American claims obtained in this way remains an open question.

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Introduction

Consider the classical Black-Scholes model:

$$\begin{aligned} dX_t^x &= \rho X_t^x dt + \sigma X_t^x dB_t \\ X_0^x &= x > 0 \\ \rho \in \mathbb{R} \quad &, \quad \sigma > 0 \end{aligned} \tag{0.1}$$

where B is a standard Brownian motion, ρ the instantaneous interest rate and σ the volatility of X and denote by

$$\mathcal{A}f(x) = \frac{\sigma^2 x^2}{2} f''(x) + \rho x f'(x) - \rho f(x)$$

the corresponding infinitesimal generator. Given a continuous function $\psi : \mathbb{R}_+^* \to \mathbb{R}_+$ satisfying some growth assumptions, the price of the so-called American option with payoff ψ, maturity $t > 0$ and spot x is given by the expression

$$v_\psi^{am}(t,x) = \sup_{\tau \in \mathcal{T}(0,t)} \mathbb{E}\left[e^{-\rho\tau} \psi\left(X_\tau^x\right)\right] \tag{0.2}$$

where τ runs across the set of stopping times of the Brownian filtration such that $\tau \leq t$ almost surely. Except for some very particular class of payoffs ψ (e.g. payoffs satisfying $\forall x >$

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$0, \mathcal{A}\psi(x) \geq 0$ or $\forall x > 0, \mathcal{A}\psi(x) \leq 0$), in general, there is no closed-form expressions for $v_\psi^{am}(t,x)$. The computation of $v_\psi^{am}(t,x)$ usually relies either on finite-difference type methods or Markov-chain approximation methods to solve the corresponding optimal stopping problem in a discrete time-space framework. There is also a huge literature on special approximation methods designed for some particular payoffs, among which the case of the Put option, given by $\psi(x) = (K-x)^+$ where K is some positive constant (the strike of the option) has received much attention.

The purpose of this paper is to exhibit a new class of payoffs ψ for which a closed-form expression for $v_\psi^{am}(t,x)$ is available. The idea originates from the analytic properties of the function v_ψ^{am} : this function is greater than ψ by (0.2) and typically the space $]0,\infty[\times\mathbb{R}_+^*$ splits into two regions, the so-called Exercise region where by definition $v_\psi^{am} = \psi$ and its complement the Continuation region where $v_\psi^{am} > \psi$. It is known that v_ψ^{am} solves the evolution equation associated with (0.1)

$$\partial_t v_\psi^{am} = \mathcal{A} v_\psi^{am}$$

in the Continuation region (at least in the distribution sense). Moreover, as from (0.2) $t \mapsto v_\psi^{am}(t,x)$ is non-decreasing, $\partial_t v_\psi^{am} \geq 0$ holds. In fact, since v_ψ^{am} is a continuous function, it may be remarked that the knowledge of v_ψ^{am} in the Continuation region is enough to get v_ψ^{am} everywhere.

This leads to the natural idea to build American prices (i.e. functions v_ψ^{am} for some ψ) by picking up the classical solution $v_\varphi(t,x)$ of the evolution equation :

$$\begin{cases} \forall t, x > 0,\ \partial_t v_\varphi(t,x) = \mathcal{A} v_\varphi(t,x) \\ \forall x > 0,\ v_\varphi(0,x) = \varphi(x) \end{cases}$$

in the region where it increases with time. From a financial point of view, $v_\varphi(t,x)$ is the Black-Scholes price of the European option with payoff φ and maturity t i.e. $v_\varphi(t,x) = \mathbb{E}\left[e^{-\rho t}\varphi\left(X_t^x\right)\right]$. This embedding idea has been worked out in [2] in case $\rho = 0$. A similar approach has also been developped in the different context of the free boundary arising in a two-phases problem (see [1] [1]). Trying to generalize things to the case $\rho \neq 0$, we ran across a probabilistic proof which allows a very compact statement of the embedding result.

The first section of the paper is devoted to some basic properties of European and American prices within the Black-Scholes model. Next we state and prove our embedding theorem (section 2). Then we give some examples (section 3). Lastly, we discuss some properties of the map which takes a payoff φ to the payoff $\widehat{\varphi}$ the American price of which is embedded in its European price (section 4). The characterization of the payoffs $\widehat{\varphi}$ obtained in this way remains an open question.

1 European and American prices in the Black-Scholes model

In this section we recall the very few properties of European and American Black-Scholes prices we shall need in the next section.

Let $\alpha = \frac{2\rho}{\sigma^2}$. The invariant functions of the semigroup associated with (0.1) are easily seen to be the vector space generated by x and $x^{-\alpha}$. We shall consider payoffs φ such that

$$x \in \mathbb{R}_+^* \mapsto \varphi(x) \in \mathbb{R}_+ \text{ is continuous and } \sup_{x>0} \frac{\varphi(x)}{x + x^{-\alpha}} < \infty \qquad (H0)$$

[1] We thank Régis Monneau (CERMICS) for bringing out this work to our attention.

The growth assumption is only there to grant the existence of the various expectations involved. It seems that the continuity assumption could be removed, but the connection with American options would be more intricate, so we keep this hypothesis.

Proposition 1 *[3] Under* $(H0)$ *the price of the European option with maturity* $t \geq 0$ *and payoff* φ *is given by*

$$v_{\varphi}(t,x) = \mathbb{E}\left[e^{-\rho t}\varphi\left(X_t^x\right)\right]$$

In particular $v_{\varphi}(0,x) = \varphi(x)$.

The function v_{φ} *is continuous from* $[0,\infty[\times \mathbb{R}_+^*$ *into* $\mathbb{R}$, *and for any* $t > 0$, *the process* $\left(e^{-\rho u} v_{\varphi}\left(t-u, X_u^x\right)\right)_{0 \leq u \leq t}$ *is a continuous square-integrable martingale.*

Let us now turn to American options:

Proposition 2 *[3] If* ψ *satisfies* $(H0)$, *the price of the American option with maturity* $t \geq 0$ *and payoff* ψ *is given by*

$$v_{\psi}^{am}(t,x) = \sup_{\tau \in \mathcal{T}(0,t)} \mathbb{E}\left[e^{-\rho\tau}\psi\left(X_\tau^x\right)\right]$$

where τ *runs across the set of stopping times of the Brownian filtration such that* $\tau \leq t$ *almost surely. In particular* $v_{\psi}^{am}(0,x) = \psi(x)$.

The function v_{ψ}^{am} *is continuous from* $[0,\infty[\times \mathbb{R}_+^*$ *into* $\mathbb{R}$, *and for any* $x \in \mathbb{R}_+^*$ *the map* $t \mapsto v_{\psi}^{am}(t,x)$ *is non-decreasing.*

2 Embedding American prices in European prices

Our main results relates the price $v_{\varphi}(t,x)$ of the European option with payoff φ to the price $v_{\widehat{\varphi}}^{am}(t,x)$ of the American option with payoff $\widehat{\varphi}(x) = \inf_{t \geq 0} v_{\varphi}(t,x)$:

Theorem 3 *Under (H0) let*

$$\widehat{\varphi}(x) = \inf_{t \geq 0} v_{\varphi}(t,x)$$

Then

$$\forall (t,x) \in [0,+\infty) \times \mathbb{R}_+^*, \quad \sup_{\tau \in \mathcal{T}(0,t)} \mathbb{E}\left[e^{-\rho\tau}\widehat{\varphi}(X_\tau^x)\right] \leq v_{\varphi}(t,x) \tag{2.1}$$

where the supremum is taken over all the stopping times τ *of the filtration of the Brownian motion smaller than* t.
Moreover, if there exists a continuous function $\widehat{t}\colon \mathbb{R}_+^* \to [0,+\infty]$ *such that*

$$\forall x > 0, \ \inf_{t \geq 0} v_{\varphi}(t,x) = v_{\varphi}\left(\widehat{t}(x), x\right)$$

(where $v_{\varphi}(\infty,x)$ *is defined as* $\liminf_{t \to +\infty} v_{\varphi}(t,x)$*) and either:*

$$\forall x > 0, \ \widehat{t}(x) < +\infty$$

or:

$$\exists C > 0, \forall x, y > 0,\ |\varphi(x) - \varphi(y)| \leq C(|x - y| + |x^{-\alpha} - y^{-\alpha}|)$$

then the function $\widehat{\varphi}$ satisfies (H0) and

$$\forall (t, x) \in [0, +\infty) \times \mathbb{R}_+^*,\ v_{\widehat{\varphi}}^{am}(t, x) = v_\varphi\left(t \vee \widehat{t}(x), x\right) \tag{2.2}$$

Proof : Let $(t, x) \in \mathbb{R}_+ \times \mathbb{R}_+^*$. According to Proposition 1, the process $(e^{-\rho u} v_\varphi\left(t - u, X_u^x\right))_{u \in [0,t]}$ is a martingale. If $\tau \leq t$ is a stopping time, by Doob optional sampling theorem

$$v_\varphi(t, x) = \mathbb{E}\left[e^{-\rho\tau} v_\varphi\left(t - \tau, X_\tau^x\right)\right] \geq \mathbb{E}\left[e^{-\rho\tau}\widehat{\varphi}(X_\tau^x)\right].$$

Since τ is arbitrary, we deduce that (2.1) holds.

To prove the other statements, we suppose the existence of $\widehat{t} : \mathbb{R}_+^* \to [0, +\infty]$ continuous such that

$$\forall x > 0,\ \widehat{\varphi}(x) = v_\varphi\left(\widehat{t}(x), x\right).$$

If $\forall x > 0,\ \widehat{t}(x) < +\infty$ then the continuity of $\widehat{\varphi}$ is a consequence of the continuity of $(t, x) \in [0, +\infty) \times \mathbb{R}_+^* \to v_\varphi(t, x)$. If $\forall x, y > 0,\ |\varphi(x) - \varphi(y)| \leq C(|x - y| + |x^{-\alpha} - y^{-\alpha}|)$, then

$$\begin{aligned}
\forall t \geq 0, |v_\varphi(t, x) - v_\varphi(t, y)| &\leq \mathbb{E}\left[e^{-\rho t}|\varphi(X_t^x) - \varphi(X_t^y)|\right] \\
&\leq C\left(|x - y|\mathbb{E}\left[e^{-\rho t} X_t^1\right] + |x^{-\alpha} - y^{-\alpha}|\mathbb{E}\left[e^{-\rho t}(X_t^1)^{-\alpha}\right]\right) \\
&\leq C(|x - y| + |x^{-\alpha} - y^{-\alpha}|)
\end{aligned}$$

Hence the functions $x \to v_\varphi(t, x)$ indexed by $t \geq 0$ are equicontinuous, which ensures the continuity of $x \to \inf_{t \geq 0} v_\varphi(t, x) = \widehat{\varphi}(x)$.

To show (2.2), we make a distinction between the two following situations :

• case $\widehat{t}(x) = +\infty$: since $s \to v_{\widehat{\varphi}}^{am}(s, x)$ is increasing, for $u \geq t$

$$\widehat{\varphi}(x) = v_{\widehat{\varphi}}^{am}(0, x) \leq v_{\widehat{\varphi}}^{am}(t, x) \leq v_{\widehat{\varphi}}^{am}(u, x) \leq v_\varphi(u, x) \quad \text{by (2.1).}$$

Letting $u \to +\infty$, we deduce that

$$\forall t \geq 0,\ v_{\widehat{\varphi}}^{am}(t, x) = \widehat{\varphi}(x) = v_\varphi(\infty, x) = v_\varphi\left(t \vee \widehat{t}(x), x\right).$$

• case $\widehat{t}(x) < +\infty$: let $t \geq \widehat{t}(x)$ and $\tau_0 = \inf\{u : t - u - \widehat{t}(X_u^x) \leq 0\}$. Since $\widehat{t}(X_t^x) \geq 0$, the stopping time τ_0 is smaller than t. By continuity of $u \to t - u - \widehat{t}(X_u^x)$, $t - \tau_0 = \widehat{t}(X_{\tau_0}^x)$. Hence

$$\begin{aligned}
v_\varphi(t, x) &= \mathbb{E}\left[e^{-\rho\tau_0} v_\varphi\left(t - \tau_0, X_{\tau_0}^x\right)\right] = \mathbb{E}\left[e^{-\rho\tau_0} v_\varphi\left(\widehat{t}(X_{\tau_0}^x), X_{\tau_0}^x\right)\right] \\
&= \mathbb{E}\left[e^{-\rho\tau_0}\widehat{\varphi}(X_{\tau_0}^x)\right] \leq v_{\widehat{\varphi}}^{am}(t, x).
\end{aligned}$$

The converse inequality (2.1) is already proved. Hence $\forall t \geq \widehat{t}(x)$, $v_{\widehat{\varphi}}^{am}(t, x) = v_\varphi(t, x)$ and τ_0 is an optimal stopping time. Since $t \to v_{\widehat{\varphi}}^{am}(t, x)$ is increasing, for $t \leq \widehat{t}(x)$, $\widehat{\varphi}(x) \leq v_{\widehat{\varphi}}^{am}(t, x) \leq v_{\widehat{\varphi}}^{am}(\widehat{t}(x), x) = v_\varphi\left(\widehat{t}(x), x\right) = \widehat{\varphi}(x)$, which ends the proof. ∎

Remark 4 *The continuity of the argument of the infimum is granted in the following uniqueness situation: suppose that* $\forall x > 0,\ \exists! \widehat{t}(x) \leq T(x),\ \widehat{\varphi}(x) = v_\varphi\left(\widehat{t}(x), x\right)$ *where* $T : \mathbb{R}_+^* \to \mathbb{R}_+$ *is continuous. Then by the continuity of* T *and* v_φ*, it is easy to see that* $\widehat{\varphi}(x) = \inf_{t\in[0,T(x)]} v_\varphi(t,x)$ *is continuous. Moreover, since* $\widehat{t}(x) = \inf\{t \geq 0 : \widehat{\varphi}(x) = v_\varphi(t,x)\}$ *(resp.* $\widehat{t}(x) = \sup\{t \leq T(x) : \widehat{\varphi}(x) = v_\varphi(t,x)\}$*),* $\widehat{t}$ *is lower semi-continuous (resp. upper semi-continuous) i.e.* $\widehat{t}$ *is continuous and (2.2) holds.*

In the above theorem it may happen that the function $\widehat{\varphi}$ is nil: in case $\limsup_{x\to 0} x^\alpha \varphi(x) = 0$ and $\limsup_{x\to+\infty} \varphi(x)/x = 0$, we easily check that

$$\forall x > 0,\ \lim_{t\to+\infty} v_\varphi(t,x) = 0.$$

In such a situation, the following localized version of our main result is far more interesting than Theorem 3. It is proved by the same arguments, after noticing that the continuity of $(t,x) \in [0,+\infty) \times \mathbb{R}_+^* \to v_\varphi(t,x)$ implies the continuity of $x \to \widehat{\varphi}^T(x) = \inf_{0\leq t\leq T} v_\varphi(t,x)$ where $T > 0$.

Theorem 5 *Let* $T > 0$*. The function* $\widehat{\varphi}^T(x) = \inf_{0\leq t\leq T} v_\varphi(t,x)$ *satisfies (H0) and*

$$\forall (t,x) \in [0,T] \times \mathbb{R}_+^*,\ v^{am}_{\widehat{\varphi}^T}(t,x) \leq v_\varphi(t,x).$$

Moreover, if there exists a continuous function $\widehat{t} : \mathbb{R}_+^* \to [0,T]$ *such that*

$$\forall x > 0,\ \inf_{0\leq t\leq T} v_\varphi(t,x) = v_\varphi\left(\widehat{t}(x), x\right),$$

then

$$\forall (t,x) \in [0,T] \times \mathbb{R}_+^*,\ v^{am}_{\widehat{\varphi}^T}(t,x) = v_\varphi\left(t \vee \widehat{t}(x), x\right).$$

Remark 6 *The only feature of the Black-Scholes model which is required in the above results is time-homogeneity. In fact, Propositions 1 and 2 and Theorems 3 and 5 can be adapted to the so-called generalized Black-Scholes model :*

$$\begin{aligned} &X_t^x = x\exp\left(\sigma B_t + (\rho - \delta - \sigma^2/2)t\right) \\ &v_\varphi(t,x) = \mathbb{E}\left[e^{-\rho t}\varphi(X_t^x)\right] \quad \textit{and} \quad v^{am}_\psi(t,x) = \sup_{\tau\in\mathcal{T}(0,t)} \mathbb{E}\left[e^{-\rho\tau}\psi(X_\tau^x)\right], \end{aligned}$$

or to the more general time-homogeneous model :

$$\begin{aligned} &X_0^x = x,\ \ dX_t^x = X_t^x(\sigma(X_t^x)dB_t + (\rho(X_t^x) - \delta(X_t^x))dt) \\ &v_\varphi(t,x) = \mathbb{E}\left[e^{-\int_0^t \rho(X_s^x)ds}\varphi(X_t^x)\right] \quad \textit{and} \quad v^{am}_\psi(t,x) = \sup_{\tau\in\mathcal{T}(0,t)} \mathbb{E}\left[e^{-\int_0^\tau \rho(X_s^x)ds}\psi(X_\tau^x)\right] \end{aligned}$$

and also to the multidimensional versions of these models.

Of course it would be of great interest to give conditions on φ which ensure the existence of a continuous curve in the argument of the infimum. One way is to perform explicit computations, since the Black-Scholes semigroup is explicit. Nevertheless this is not very illuminating. We ran across the following statement, for the local embedding result, which is maybe the simplest in this direction:

Proposition 7 *Let φ be a C^4 function which satisfies $(H0)$ and $\exists x_c \in \mathbb{R}_+^*$ such that*

(i) $\mathcal{A}\varphi(x_c) = 0$ *and either* $\forall x > 0,\ (x - x_c)\mathcal{A}\varphi(x) \geq 0$ *or* $\forall x > 0,\ (x - x_c)\mathcal{A}\varphi(x) \leq 0$

(ii) $\mathcal{A}^2\varphi(x_c) > 0$ *and* $\mathcal{A}\partial_x\varphi(x_c) \neq 0$

Then there exists a constant $T > 0$ such that the assumptions of Theorem 5 are satisfied.

Proof : Since φ is C^4, the function $v_\varphi(t,x)$ belongs to $C^{2,4}(\mathbb{R}_+ \times \mathbb{R}_+^*)$ (C^2 in t, C^4 in x) and satisfies the Black-Scholes partial differential equation $\partial_t v_\varphi = \mathcal{A}v_\varphi$ for $t \geq 0$ and not only $t > 0$. Consider the equation $\partial_t v_\varphi(t,x) = 0$ in a neighbourhood of $(0, x_c)$ in $\{(t,x),\ t \geq 0\}$. By derivation of the Black-Scholes evolution equation, $\partial^2_{tx} v_\varphi(0, x_c) = \mathcal{A}\partial_x\varphi(x_c) \neq 0$. Hence, by the implicit functions theorem, there is for some $\varepsilon > 0$ a curve $\widehat{x}$:

$$\widehat{x} : [0, \varepsilon] \to \mathbb{R}_+^*$$

continuous on $[0,\varepsilon]$, with $\widehat{x}(0) = x_c$, such that $\partial_t v_\varphi(t, \widehat{x}(t)) = 0$, and C^1 on $]0,\varepsilon[$ with

$$\partial^2_{t^2} v_\varphi(t, \widehat{x}(t)) + \partial^2_{tx} v_\varphi(t, \widehat{x}(t))\, \widehat{x}'(t) = 0$$

Moreover by taking ε small enough we can assume that $\widehat{x}'(t)$ does not vanish and keeps the same sign as $\widehat{x}'(0^+) = -\frac{\mathcal{A}^2\varphi(x_c)}{\mathcal{A}\partial_x\varphi(x_c)}$. We deduce that there exists a continuous function $\widehat{t} : [x_c, \widehat{x}(\varepsilon)] \to [0,\varepsilon]$ such that $\widehat{x}\left(\widehat{t}(x)\right) = x$.

Assume $\widehat{x}'(0^+) > 0$. Then the function $\widehat{t}$ is increasing. Moreover, $\mathcal{A}\partial_x\varphi(x) < 0$ which ensures $\forall x < x_c,\ \mathcal{A}\varphi(x) \geq 0$ and $\forall x > x_c,\ \mathcal{A}\varphi(x) \leq 0$. We set $T = \varepsilon$ and extend $\widehat{t}$ to $\mathbb{R}_+^*$ by setting $\widehat{t}(x) = T$ for $x > x_c + \varepsilon$ and $\widehat{t}(x) = 0$ for $x < x_c$. The obtained function is continuous and the whole point is to show that for every x, the infimum of $t \longmapsto v_\varphi(t,x)$ on $[0,T]$ is reached at $\widehat{t}(x)$. This amounts to show that $\partial_t v_\varphi(t,x) = \mathcal{A}v_\varphi(t,x)$ is non-positive for (t,x) above $\widehat{x}$ (i.e. for $t \leq T$ and $x \geq \widehat{x}(t)$) and non-negative below. If $(P_t)_{t\geq 0}$ denotes the semigroup associated with (0.1),

$$\mathcal{A}v_\varphi(t,x) = \mathcal{A}P_t\varphi(x) = P_t\mathcal{A}\varphi(x).$$

Let (t,x) belong to the above (resp. below) region. By the optimal stopping theorem, $\mathcal{A}v_\varphi(t,x)$ is equal to the expectation of the value of the martingale $(e^{-\rho u}P_{t-u}\mathcal{A}\varphi(X_u^x))_{0\leq u\leq t}$ stopped at the border of the above (resp. below) region $\{(u,\widehat{x}(u)),\ u \in [0,\epsilon]\} \cup \{(0,x),\ x \geq x_c\}$ (resp. $\{(u,\widehat{x}(u)),\ u \in [0,\epsilon]\} \cup \{(0,x),\ x \leq x_c\}$) which is non-positive (resp. non-negative) since $\forall t \in]0,\epsilon],\ P_u\mathcal{A}\varphi(\widehat{x}(u)) = \partial_u v_\varphi(\widehat{x}(u)) = 0$ and $\forall x \geq x_c,\ \mathcal{A}\varphi(x) \leq 0$ (resp. $\forall x \geq x_c,\ \mathcal{A}\varphi(x) \geq 0$).

The case $\widehat{x}'(0^+) < 0$ is handled in the same way. ∎

Example 8 *As an application, consider the family of payoffs*

$$\varphi_{a,b}(x) = x^{-\alpha} + x^a - x^b$$

where $1 > a > b > -\alpha$. Then for $x \geq 1$, $x^a \geq x^b$, for $x < 1$, $x^{-\alpha} > x^b$ so that $\varphi_{a,b}$ is non-negative. Moreover

$$\lim_{x\to 0} \frac{\varphi_{a,b}(x)}{x + x^{-\alpha}} = 1,\ \lim_{x\to+\infty} \frac{\varphi_{a,b}(x)}{x + x^{-\alpha}} = 0$$

and $\varphi_{a,b}$ satisfies (H0). Let $\lambda(y) = \left(\frac{\sigma^2}{2}y + \rho\right)(y-1)$. Then

$$\mathcal{A}\varphi_{a,b}(x) = \lambda(a)\, x^a - \lambda(b)\, x^b$$

which gives, since $\lambda(a) < 0$ and $\lambda(b) < 0$, $\mathcal{A}\varphi_{a,b}(x) < 0$ for $x > x_c$ and $\mathcal{A}\varphi_{a,b}(x) > 0$ for $x < x_c$ with $\lambda(a)\, x_c^a = \lambda(b)\, x_c^b$. Moreover $\mathcal{A}^2\varphi_{a,b}(x_c) = \lambda(a)^2\, x_c^a - \lambda(b)^2\, x_c^b = (\lambda(a) - \lambda(b))\,\lambda(b)\, x_c^b$ and $\mathcal{A}^2\varphi_{a,b}(x_c) > 0$ as soon as $\lambda(b) > \lambda(a)$. Lastly $\mathcal{A}\partial_x\varphi_{a,b}(x_c) = \lambda(a)\, a x_c^{a-1} - \lambda(b)\, b x_c^{b-1} = \left(\frac{a}{x_c} - \frac{b}{x_c}\right)\lambda(a)\, x_c^a \neq 0$.

Of course, in this example, since $v_\varphi(t,x) = x^{-\alpha} + x^a e^{t\lambda(a)} - x^b e^{t\lambda(b)}$, everything can be computed explicitely and it is even possible to check the hypotheses of the global embedding result:

$$\left(\frac{x \vee x_c}{x_c}\right)^{a-b} = e^{\widehat{t}(x)(\lambda(b)-\lambda(a))} \text{ and } \widehat{\varphi}_{a,b}(x) = x^{-\alpha} + x^a \left(\frac{x \vee x_c}{x_c}\right)^{\frac{\lambda(a)(a-b)}{\lambda(b)-\lambda(a)}} - x^b \left(\frac{x \vee x_c}{x_c}\right)^{\frac{\lambda(b)(a-b)}{\lambda(b)-\lambda(a)}}.$$

Similarly the hypothesis of Proposition 7 are satisfied by the payoff $x + x^b - x^a$ where $1 > a > b > -\alpha$ in case $\lambda(a) > \lambda(b)$.

In the global case, we could not find any simple condition on φ ensuring the existence of a continuous curve in the argument of $\inf_{t\geq 0} v_\varphi(t,x)$. Nevertheless, it is worth mentioning the following interesting class of European payoffs : if φ is a non-negative function equal to an invariant function $ax + bx^{-\alpha}$ with $a, b \geq 0,\ a + b > 0$, less a non-negative function ϕ satisfying $\limsup_{x\to 0} x^\alpha \phi(x) = \limsup_{x\to+\infty} \phi(x)/x = 0$, then

$$\forall x > 0,\ \forall t \geq 0,\ v_\varphi(t,x) \leq ax + bx^{-\alpha} \quad \text{and} \quad \lim_{t\to+\infty} v_\varphi(t,x) = ax + bx^{-\alpha},$$

which implies that $\widehat{\varphi}(x) = \inf_{t\geq 0} v_\varphi(t,x)$ is not trivial and that $\forall x \in \mathbb{R},\ \exists \widehat{t}(x) < +\infty,\ \widehat{\varphi}(x) = v_\varphi(\widehat{t}(x), x)$. The only assumption missing to apply Theorem 3 is the continuity of $\widehat{t}$.
The next section is dedicated to a family of payoffs φ included in the above class. In these examples, we explicit some American prices with a non-trivial Exercise region thanks to Theorem 3. We also check that the above-mentioned continuity of $\widehat{t}$ is not always satisfied.

3 Case study: $\varphi(x) = x(1_{\{x<K_1\}} + 1_{\{x>K_2\}})$

This payoff is equal to the invariant function x less the bounded function $\phi(x) = x1_{\{K_1\leq x\leq K_2\}}$. Since

$$\forall x > 0,\ \forall t > 0,\ 0 < v_\varphi(t,x) < x \quad \text{and} \quad \lim_{t\to+\infty} v_\varphi(t,x) = x.$$

the function $t \to v_\varphi(t,x)$ is likely to be increasing for $K_1 < x < K_2$ and decreasing then increasing otherwise. This remark together with the easiness of computations motivate the choice of this example. The function φ satisfies the growth assumption in (H0) but is not continuous. Therefore, even if we make the computations for φ, we shall after all apply our results to a suitable regularization of φ.

3.1 The case $K_1 = 0$

To simplify notations, we replace K_2 by K and write $\varphi(x) = x1_{\{x>K\}}$. This payoff corresponds to the sum of one Call and K Digit options with common strike K. Its simplicity allows to compute explicitely $\widehat{\varphi}$ and $\widehat{t}$.

Proposition 9 *Let $\varphi(x) = x1_{\{x>K\}}$. Then*

$$v_\varphi(t,x) = xN(d_1(t,x)) \quad \textit{where} \quad d_1(t,x) = \frac{\ln(\frac{x}{K}) + (\rho + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}$$

and $N(d) = \int_{-\infty}^{d} e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}}$ is the cumulative distribution function of the normal law. Moreover,

$$\widehat{\varphi}(x) = x1_{\{x>K\}} N\left(\frac{2}{\sigma}\sqrt{\left(\rho + \frac{\sigma^2}{2}\right)\ln\left(\frac{x}{K}\right)}\right) = v_\varphi\left(\widehat{t}(x), x\right) \quad \textit{and} \quad \widehat{t}(x) = \frac{\ln(x/K)1_{\{x\geq K\}}}{\rho + \frac{\sigma^2}{2}}$$

and $\forall x > 0$, $t \to v_\varphi(t,x)$ is strictly decreasing on $[0, \widehat{t}(x)]$ and strictly increasing on $[\widehat{t}(x), +\infty)$.

Proof : Using Girsanov theorem, we get

$$v_\varphi(t,x) = \mathbb{E}\left[xe^{\sigma B_t - \frac{\sigma^2}{2}t} 1_{\left\{xe^{\sigma B_t + (\rho - \frac{\sigma^2}{2})t} \geq K\right\}}\right] = xP\left(xe^{\sigma B_t + (\rho + \frac{\sigma^2}{2})t} \geq K\right) = xN(d_1(t,x))$$

By the chain rule, $\partial_t v_\varphi(t,x) = xN'(d_1(t,x))\partial_t d_1(t,x)$. Since $\forall x, t > 0,\ xN'(d_1(t,x)) > 0$ and

$$\partial_t d_1(t,x) = \frac{(\rho + \frac{\sigma^2}{2})t - \ln(\frac{x}{K})}{2\sigma t^{\frac{3}{2}}},$$

we obtain that

$$\forall x > 0, \begin{cases} \forall t \in \left]0, \widehat{t}(x)\right[,\ \partial_t v_\varphi(t,x) < 0 \\ \forall t > \widehat{t}(x),\ \partial_t v_\varphi(t,x) > 0 \end{cases}$$

Hence $\inf_{t\geq 0} v_\varphi(t,x) = v_\varphi\left(\widehat{t}(x), x\right)$ and the explicit expression of this function is easily computed. ∎

Let us now regularize things in order to apply our Theorem. Let $u > 0$. The function $x \to v_\varphi(u,x)$ is continuous. Let $(P_t)_{t\geq 0}$ denote the semigroup associated with (0.1). By the semigroup property, the price of the European option with payoff $v_\varphi(u,x)$ is $P_t(P_u\varphi) = P_{t+u}\varphi$. If we set $\widehat{\varphi}_u = \inf_{t\geq 0} P_t(P_u\varphi)$, then by the previous Proposition, $\widehat{\varphi}_u(x) = v_\varphi\left(u \vee \widehat{t}(x), x\right) = P_{0\vee(\widehat{t}(x)-u)}(P_u\varphi)(x)$. Since $\widehat{t}$ is a continuous function with values in $[0, +\infty)$, so is $\widehat{t}_u(x) = 0 \vee (\widehat{t}(x) - u)$. Applying Theorem 3, we obtain the price of the American option with payoff $\widehat{\varphi}_u$:

Corollary 10 *Let $u > 0$. The price of the American option with payoff $\widehat{\varphi}_u(x) = v_\varphi\left(u \vee \widehat{t}(x), x\right)$ is*

$$\begin{aligned} v^{am}_{\widehat{\varphi}_u}(t,x) &= v_\varphi\left((t+u) \vee \widehat{t}(x), x\right) \\ &= x\Bigg(N\left(\frac{2}{\sigma}\sqrt{\left(\rho + \frac{\sigma^2}{2}\right)\ln\left(\frac{x}{K}\right)}\right) 1_{\{t+u\leq \ln(x/K)/(\rho+\sigma^2/2)\}} \\ &\quad + N\left(\frac{\ln(\frac{x}{K}) + (\rho + \frac{\sigma^2}{2})(t+u)}{\sigma\sqrt{t+u}}\right) 1_{\{t+u>\ln(x/K)/(\rho+\sigma^2/2)\}}\Bigg) \end{aligned}$$

and the Exercise region is given by $\{(t,x) : t + u \leq \ln(x/K)/(\rho + \sigma^2/2)\}$.

Remark 11 *Although the payoff $\widehat{\varphi}_u$ has no financial meaning, this example provides a very interesting benchmark for numerical procedures devoted to American options since the price and the Exercise boundary are explicit. Let us also notice that this is a two-parameter (K and u) family of closed-formula. The payoff is of course obtained by setting t to zero in $v^{am}_{\widehat{\varphi}_u}(t,x)$.*

3.2 The case $K_1 > 0$

The main purpose of this subsection is to design an example where there is no continuous curve in the argument of the infimum (Proposition 13). By a slight modification of the computations made in the proof of Proposition 9, we get

$$v_\varphi(t,x) = x(N(-d_1(t,x)) + N(d_2(t,x))), \quad \text{where for } i=1,2 \;\; d_i(t,x) = \frac{\ln(x/K_i) + \left(\rho + \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}.$$

It is not possible to compute $\widehat{\varphi}$ explicitely but using the implicit functions theorem, we can study the sign of $\partial_t v_\varphi(t,x)$ to obtain :

Lemma 12 *There exist two differentiable functions $t \in \mathbb{R}_+^* \to \xi_1(t) < \xi_2(t)$ satisfying*

1. $\lim_{t\to 0} \xi_i(t) = K_i$ *($i=1,2$)*
2. $\forall t > 0,\ \xi_2'(t) > 0$ *and* $\exists(\beta,T), 0 < \beta < T \leq \frac{1+\ln\sqrt{K_2/K_1}}{2\rho+\sigma^2}$, $\forall t < \beta,\ \xi_1'(t) > 0$ *and* $\forall t > T,\ \xi_1'(t) < 0$
3. $\forall t > 0,\ \xi_2(t) > K_2 e^{\left(\rho+\frac{\sigma^2}{2}\right)t}$ *and* $\xi_1(t) < K_1 e^{\left(\rho+\frac{\sigma^2}{2}\right)t} \wedge \sqrt{K_1K_2}e^{-\left(\rho+\frac{\sigma^2}{2}\right)t}$

and such that

$$\forall t > 0,\ \forall x \in]\xi_1(t), \xi_2(t)[,\ \partial_t v_\varphi(t,x) > 0 \ \text{ and } \ \forall x \notin [\xi_1(t), \xi_2(t)],\ \partial_t v_\varphi(t,x) < 0.$$

Proof : An easy computation yields that $\partial_t v_\varphi(t,x)$ is equal to the product of a strictly positive function with $f(t, \ln x)$ where

$$f(t,y) = (y-a_1)e^{b_1 y + c_1} + (a_2 - y)e^{b_2 y + c_2} \quad \text{where for } i=1,2$$
$$a_i(t) = \ln K_i + \left(\rho + \frac{\sigma^2}{2}\right)t, \quad b_i(t) = \frac{\ln K_i}{\sigma^2 t}, \quad c_i(t) = \left(\frac{\rho}{\sigma^2} + \frac{1}{2}\right)\ln K_i - \frac{\ln^2 K_i}{2\sigma^2 t}$$

Since $a_1 < a_2$, $f(t,a_2) = (a_2 - a_1)e^{b_1 a_2 + c_1} > 0$. Hence the function $y \to f(t,y)$ vanishes at the same points as

$$y \to g(t,y) = e^{(b_2-b_1)y + (c_2 - c_1)} - \frac{y - a_1}{y - a_2}$$

As $a_1 < a_2$ and $b_1 < b_2$, the function $y \to g(t,y)$ is strictly increasing from -1 to $+\infty$ on $]-\infty, a_2[$ and from $-\infty$ to $+\infty$ on $]a_2, +\infty[$, so it vanishes exactly twice. Let $y_1 < a_2 < y_2$ denote the corresponding points. Since $e^{(b_2-b_1)y_1 + (c_2-c_1)} > 0$ and $\frac{y_1 - a_1}{y_1 - a_2} < 1$, we obtain respectively $y_1 < a_1$ and $(b_2 - b_1)y_1 < c_1 - c_2$. We combine these upper-bounds to get

$$y_1(t) < a_1(t) \wedge \left(\ln\sqrt{K_1K_2} - \left(\rho + \frac{\sigma^2}{2}\right)t\right) \tag{3.1}$$

We deduce that $x \to \partial_t v_\varphi(t,x)$ vanishes exactly twice, at the points $\xi_1(t) = e^{y_1(t)}$ and $\xi_2(t) = e^{y_2(t)}$ which satisfy statement 3.. As $f(t,a_2) > 0$, $\partial_t v_\varphi(t,x)$ is strictly positive for $x \in (\xi_1(t), \xi_2(t))$. Moreover as $b_1 < b_2$, $f(t,y) < 0$ for $|y|$ large and $\partial_t v_\varphi(t,x)$ is strictly negative for $0 < x < \xi_1(t)$ and for $x > \xi_2(t)$.

Let us study more precisely the functions $y_1(t)$ and $y_2(t)$. Since $\forall t > 0,\ \forall y \neq a_2(t),\ \partial_y g(t,y) > 0$, by the implicit function theorem, for $i = 1, 2$, $y_i(t)$ is continuously differentiable and $y_i'(t)$ has the same sign as $-\partial_t g(t, y_i(t))$. Expliciting the dependence of g on the time variable, we get

$$g(t,y) = \exp\left(\frac{\ln(K_2/K_1)}{\sigma^2 t}\left(y + \left(\rho + \frac{\sigma^2}{2}\right)t - \ln\sqrt{K_1K_2}\right)\right) - 1 + \frac{\ln(K_1/K_2)}{y - \ln(K_2) - (\rho + \frac{\sigma^2}{2})t}$$

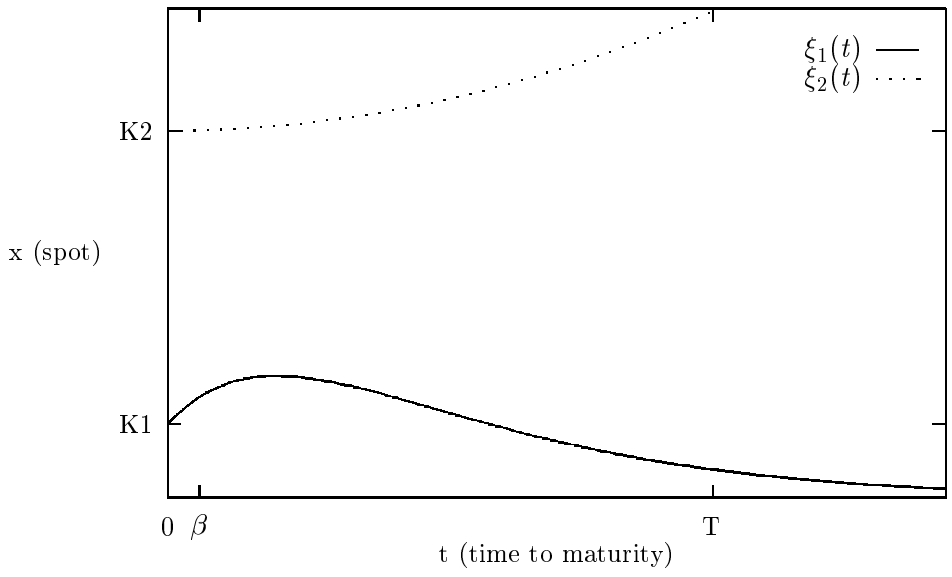
$$\partial_t g(t,y) = \frac{\ln(K_1/K_2)}{\sigma^2 t^2}(y - \ln\sqrt{K_1K_2})\exp\left(\frac{\ln(K_2/K_1)}{\sigma^2 t}\left(y + \left(\rho + \frac{\sigma^2}{2}\right)t - \ln\sqrt{K_1K_2}\right)\right)$$
$$+ \frac{(\rho + \frac{\sigma^2}{2})\ln(K_1/K_2)}{(y - \ln K_2 - (\rho + \frac{\sigma^2}{2})t)^2}$$

Since $y_2(t) > a_2(t) > \ln\sqrt{K_1K_2}$, $\partial_t g(t, y_2(t))$ is strictly negative and $\forall t > 0,\ y_2'(t) > 0$. Moreover, when $t \to 0$ the first term in $g(t, y_2(t))$ has a limit equal to $+\infty$ and the equation $g(t, y_2(t)) = 0$ implies that the second term goes also to ∞ which gives $\lim_{t\to 0} y_2(t) = \ln K_2$. By (3.1), $y_1(t) < a_1(t) = \ln K_1 + (\rho + \frac{\sigma^2}{2})t$. Hence when $t \to 0$ the first term in $g(t, y_1(t))$ has a limit equal to 0. By considering the other terms we deduce that $\lim_{t\to 0} y_1(t) = \ln K_1$. Hence the first term in $\partial_t g(t, y_1(t))$ goes to 0 and $\lim_{t\to 0} \partial_t g(t, y_1(t)) < 0$. Therefore $\exists \beta > 0$, $\forall t \in]0, \beta[\ ,\ y_1'(t) > 0$.
Using the equality $g(t, y_1(t)) = 0$ to replace the exponential in $\partial_t g(t, y_1(t))$ and multiplying by $(y_1(t) - \ln K_2 - (\rho + \frac{\sigma^2}{2})t)^2 / \ln(K_2/K_1)$, we obtain that $\partial_t g(t, y_1(t))$ has the same sign as

$$\frac{-1}{\sigma^2 t^2}(y_1(t) - \ln\sqrt{K_1K_2})\left(y_1(t) - \ln K_1 - \left(\rho + \frac{\sigma^2}{2}\right)t\right)\left(y_1(t) - \ln K_2 - \left(\rho + \frac{\sigma^2}{2}\right)t\right) - \left(\rho + \frac{\sigma^2}{2}\right).$$

As by (3.1), $y_1(t) < \ln\sqrt{K_1K_2} - \left(\rho + \frac{\sigma^2}{2}\right)t$, we conclude that for some $T \leq \frac{1 + \ln\sqrt{K_2/K_1}}{2\rho + \sigma^2}$,
$\forall t > T,\ \partial_t g(t, y_1(t)) > 0$ and $y_1'(t) < 0$. ∎

So the situation looks like:



Let $u > 0$. The payoff $v_\varphi(u,x)$ satisfies $(H0)$. Let $\widehat{\varphi}_u(x) = \inf_{t\geq 0} v_\varphi(t+u,x)$. Since $(t,x) \to v_\varphi(t+u,x)$ is continuous and $t \to v_\varphi(t,x)$ is increasing for $t \geq t(x)$ where $t(x)$ is locally bounded (see Lemma 12), the function $\widehat{\varphi}_u(x)$ is continuous. According to Lemma 12, there exist $0 < \beta < T < +\infty$ such that $t \to \xi_1(t)$ is strictly increasing on $[0,\beta]$ and strictly decreasing on $[T,+\infty)$. Concerning the existence of a continuous function $\widehat{t}_u$ such that $\widehat{\varphi}_u(x) = v_\varphi\left(\widehat{t}_u(x)+u,x\right)$ the situation depends on whether $u < \beta$ or $u \geq T$.

Proposition 13

- *If $u \geq T$, then $\widehat{\varphi}_u(x) = v_\varphi\left(\widehat{t}_u(x)+u,x\right)$ for the continuous function*

$$\widehat{t}_u(x) = 1_{\{x\leq \xi_1(u)\}}(\xi_1^{-1}(x)-u) + 1_{\{x\geq \xi_2(u)\}}(\xi_2^{-1}(x)-u)$$

where ξ_1^{-1} denotes the inverse of the restriction of ξ_1 to $[T,+\infty)$ and the price of the American option with payoff $\widehat{\varphi}_u$ is

$$\begin{aligned} v^{am}_{\widehat{\varphi}_u}(t,x) &= v_\varphi\left((t\vee \widehat{t}_u(x))+u,x\right) \\ &= v_\varphi\left((t+u)\vee \xi_1^{-1}(x),x\right)1_{\{x\leq\xi_1(u)\}} + v_\varphi(u,x)\,1_{\{\xi_1(u)<x<\xi_2(u)\}} \\ &\quad + v_\varphi\left((t+u)\vee \xi_2^{-1}(x),x\right)1_{\{x\geq\xi_2(u)\}} \end{aligned}$$

- *If $u < \beta$, there is no continuous function $\widehat{t}_u$ such that $\widehat{\varphi}_u(x) = v_\varphi\left(\widehat{t}_u(x)+u,x\right)$. Moreover,*

$$\forall t>0,\ \forall x \in \left]\xi_1(t+u),\xi_2(t+u)\right[,\ v_\varphi(t+u,x) > v^{am}_{\widehat{\varphi}_u}(t,x)$$

Proof : We first suppose that $u \geq T$. According to Lemma 12, $t \in [0,+\infty) \to \xi_1(t+u)$ (resp. $t \in [0,+\infty) \to \xi_2(t+u)$) is decreasing (resp. increasing), and $\forall x \in \left]0,\xi_1^{-1}(u)\right[$ (resp $\forall x \in \left]\xi_2^{-1}(u),+\infty\right[$) $t \to v_\varphi(t+u,x)$ is decreasing on $[0,\xi_1^{-1}(x)-u]$ (resp. $[0,\xi_2^{-1}(x)-u]$) and increasing on $[\xi_1^{-1}(x)-u,+\infty[$ (resp. $[\xi_2^{-1}(x)-u,+\infty[$). Moreover $\forall x \in [\xi_1(u),\xi_2(u)]$, $t \to v_\varphi(t+u,x)$ is increasing. Hence $\widehat{\varphi}_u(x) = v_\varphi\left(\widehat{t}_u(x)+u,x\right)$ for the continuous function

$$\widehat{t}_u(x) = 1_{\{x\leq \xi_1(u)\}}(\xi_1^{-1}(x)-u) + 1_{\{x\geq \xi_2(u)\}}(\xi_2^{-1}(x)-u),$$

and we deduce the price of the American option with payoff $\widehat{\varphi}_u$ by Theorem 3.

We turn to the case $u < \beta$. Let $F = \{(t,x) : v_\varphi(t+u,x) = \widehat{\varphi}_u(x)\}$. According to Lemma 12, $t \to \xi_1(t)$ is increasing on $[0,\beta]$. We deduce that $\forall t \in \left]u,\beta\right[,\ v_\varphi(t,\xi_1(t)) > v_\varphi(u,\xi_1(t))$ and $(t-u,\xi_1(t)) \notin F$. Hence

$$\begin{aligned} F \subset F_1 \cup F_2 \quad &\text{where} \quad F_1 = \{(t-u,\xi_1(t)),\ t \geq \beta\} \\ &\text{and} \quad F_2 = \{(t-u,\xi_2(t)),\ t\geq u\} \cup \{(0,x),\ x \in [\xi_1(u),\xi_2(u)]\} \end{aligned}$$

Let $\widehat{t}_u$ be such that $\forall x>0,\ \widehat{\varphi}_u(x) = v_\varphi\left(\widehat{t}_u(x)+u,x\right)$ i.e. $(\widehat{t}_u(x),x) \in F$. For x small enough $(\widehat{t}_u(x),x) \in F_1$ and for x big enough $(\widehat{t}_u(x),x) \in F_2$. Since F_1 and F_2 are not connected, the function $\widehat{t}_u$ is discontinuous.
Let $t>0$ and $x \in (\xi_1(t+u),\xi_2(t+u))$. The positive continuous function

$$w \in W \to \Phi(w) = \inf_{s\in[0,t]} e^{-\rho s}\left(v_\varphi\left(t+u-s, xe^{\sigma w_s+(\rho-\frac{\sigma^2}{2})s}\right) - \widehat{\varphi}_u\left(xe^{\sigma w_s+(\rho-\frac{\sigma^2}{2})s}\right)\right)$$

where $W = \{w \in C([0,t],\mathbb{R}),\ w(0)=0\}$, is not constantly equal to 0. Indeed, when

$$\begin{aligned} &\forall s \leq 0\vee(t+u-\beta),\ \xi_1(t+u-s) < xe^{\sigma w_s+(\rho-\frac{\sigma^2}{2})s}, \quad xe^{\sigma w_t+(\rho-\frac{\sigma^2}{2})t} < \xi_1(u) \\ &\text{and} \quad \forall s\leq t,\ xe^{\sigma w_s+(\rho-\frac{\sigma^2}{2})s} < \xi_2(t+u-s), \end{aligned}$$

then $\forall s \in [0,t],\ (t-s, xe^{\sigma w_s + (\rho - \frac{\sigma^2}{2})s}) \notin F$ and $\Phi(w) > 0$. As the support of the Wiener measure is W, $\mathbb{E}\left[\Phi\left((B_s)_{s\leq t}\right)\right] > 0$. Let τ be a stopping time smaller than t. Then

$$v_\varphi(t+u,x) = \mathbb{E}\left[e^{-\rho\tau} v_\varphi(t+u-\tau, X_\tau^x)\right] \geq \mathbb{E}\left[e^{-\rho\tau}\widehat{\varphi}_u(X_\tau^x)\right] + \mathbb{E}\left[\Phi\left((B_s)_{s\leq t}\right)\right]$$

Since τ is arbitrary, we conclude that $v_\varphi(t+u,x) - v^{am}_{\widehat{\varphi}_u}(t,x) \geq \mathbb{E}\left[\Phi\left((B_s)_{s\leq t}\right)\right] > 0$ ∎

Remark 14 *1. For any $x > 0$, $t \to v_\varphi(t+u,x)$ is continuous and increasing for t big enough. Hence $\tilde{t}_u(x) = \sup\{t : v_\varphi(t+u,x) = \widehat{\varphi}_u(x)\}$ is finite. When $u < \beta$, $\forall t \leq \tilde{t}_u(x)$,*

$$\widehat{\varphi}_u(x) \leq v^{am}_{\widehat{\varphi}_u}(t,x) \leq v^{am}_{\widehat{\varphi}_u}(\tilde{t}_u(x),x) \leq v_\varphi\left(\tilde{t}_u(x)+u, x\right) \quad \textit{i.e.} \quad v^{am}_{\widehat{\varphi}_u}(t,x) = \widehat{\varphi}_u(x)$$

but $\exists T(x)$ such that for $t \geq T(x)$, $x \in]\xi_1(t+u), \xi_2(t+u)[$ and we cannot deduce $v^{am}_{\widehat{\varphi}_u}(t,x)$ from the price of the European option with payoff φ.

2. Let $u < \beta$ and $x^* = \sup\{x : (t,x) \in F_1 \cap F\}$ where F, F_1 are defined in the previous proof. Since F_1 and F are closed and $\lim_{t\to+\infty} \xi_1(t) = 0$, $\exists t^* \geq \beta - u$ such that $(t^*, x^*) \in F_1 \cap F$ i.e. $v_\varphi(t^*+u, x^*) = \widehat{\varphi}_u(x^*)$. Since $x^* = \sup\{x : (t,x) \in F_1 \cap F\}$, $\forall x \in]x^*, \xi_2(u)]$, $v_\varphi(u,x) = \widehat{\varphi}_u(x)$ and by continuity, $v_\varphi(u,x^*) = \widehat{\varphi}_u(x^*)$. Hence $\{t \geq 0,\ v_\varphi(t+u,x^*) = \widehat{\varphi}_u(x^*)\}$ contains at least two elements which is not surprising with regard to Remark 4.

4 A one-to-one property of the map $\varphi \mapsto \widehat{\varphi}$

In this section we shall show the following property:

Proposition 15 *Let φ_1 and φ_2 satisfy the assumptions of Theorem 3 and assume:*

For $i = 1, 2$ there are continuous curve $\widehat{t}_i$ with values in $[0, \infty[$ such that for every x

$$\inf_t v_{\varphi_i}(t,x) = v_{\varphi_i}\left(\widehat{t}_i(x), x\right)$$

Then $\widehat{\varphi}_1 = \widehat{\varphi}_2 \ \Rightarrow\ \varphi_1 = \varphi_2$.

We shall need the following which is a straightforward consequence of the analyticity w.r.t. t of the solution of the heat equation [4].

Lemma 16 *Under $(H0)$, for every $x \in \mathbb{R}_+^*$ the function $t \mapsto v_\varphi(t,x)$ is analytic from $]0,\infty[$ to $\mathbb{R}$.*

Proof of Proposition 15: Pick $x \in \mathbb{R}_+^*$. By Theorem 3, for $i = 1, 2$, the American price of $\widehat{\varphi}_i$ is given by the European price of φ_i for $t \geq \widehat{t}_i(x)$. Hence the European prices of φ_1 and φ_2 coincide for $t \geq \widehat{t}_1(x) \vee \widehat{t}_2(x)$. By the lemma this entails $v_{\varphi_1}(t,x) = v_{\varphi_2}(t,x)$ for $t > 0$ and also for $t = 0$ by continuity, which gives $\varphi_1(x) = \varphi_2(x)$. ∎

5 Conclusion

In this paper, for a fairly general class of payoffs φ, we deduce from the European price $v_\varphi(t,x)$ the American price of the claim with payoff $\widehat{\varphi}(x) = \inf_{t\geq 0} v_\varphi(t,x)$. We give examples of explicit computations. The characterization of the payoffs $\widehat{\varphi}$ obtained in this way remains an open question. A work devoted to design new approximations of the American Put price relying on our approach is in progress.

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