Yet Another Approximation of the American Put

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Abstract

In this paper, as an application of the theoretical result in [1], we exhibit a family of payoffs $\widehat{\varphi}_h(x)$ indexed by a measure h, for the American price of which an almost closed formula holds, which are very close to the Put payoff $(K-x)^+$ in the following sense: $\widehat{\varphi}_h(x) = (K-x)^+$ for $x \leq K^*$ and $x \geq K$ (K^* is the exercise price of the perpetual option), $\widehat{\varphi}'_h\left(K^*_+\right) = \widehat{\varphi}'_h(K_-) = -1$. The (almost explicit) free boundary $x \mapsto \widehat{t}_h(x)$ between the Exercice and Continuation Region shares with that of the American Put the following properties: $\lim_{K^-} \widehat{t}_h(x) = K$, $\lim_{K^{*+}} \widehat{t}_h(x) = \infty$, $\widehat{t}_h$ is analytic on $]K^*, K[$, and the equivalent in K^- of Barles&alii [2] and Lamberton [3] holds. Unfortunately, we can prove that there is no h such that $\widehat{\varphi}_h(x) = (K-x)^+$ everywhere. But by a numerical minimization of a discretized version of $\sup_x \left|\widehat{\varphi}_h(x) - (K-x)^+\right| = err(\alpha, h)$, we obtain a measure h^* depending only on $\alpha = \frac{2\rho}{\sigma^2}$ such that the quantity $err^*(\alpha) = err(\alpha, h^*)$ decreases with α and satisfies: $err^*(0.5) = 0.3\frac{K}{100}$, $err^*(\alpha) \leq 0.05\frac{K}{100}$ for $\alpha > 7$. The first interest of our result is to yield the corresponding sub- and super- very sharp hedging strategies, since there is a hedge ratio associated to our price. Also since the error is smoothed by the probability of crossing the free boundary, the corresponding errors regarding the price is much smaller, thus yielding good approximations of the Black-Scholes American Put price.

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Introduction

Consider the classical Black-Scholes model

$$\begin{aligned} dX_t^x &= \rho X_t^x dt + \sigma X_t^x dB_t \\ X_0^x &= x > 0 \\ \rho &\in \mathbb{R} \end{aligned} \tag{1}$$

where B is a standard Brownian motion, ρ the instantaneous interest rate and σ the volatility of X and denote by

$$\mathcal{A}f(x) = \frac{\sigma^2 x^2}{2} f''(x) + \rho x f'(x) - \rho f(x)$$

the corresponding infinitesimal generator. Given a continuous function $\psi : \mathbb{R}_+^* \to \mathbb{R}_+$ satisfying some growth assumptions, the price of the so-called American option with payoff ψ, time to maturity $t > 0$ and spot x is given by the expression

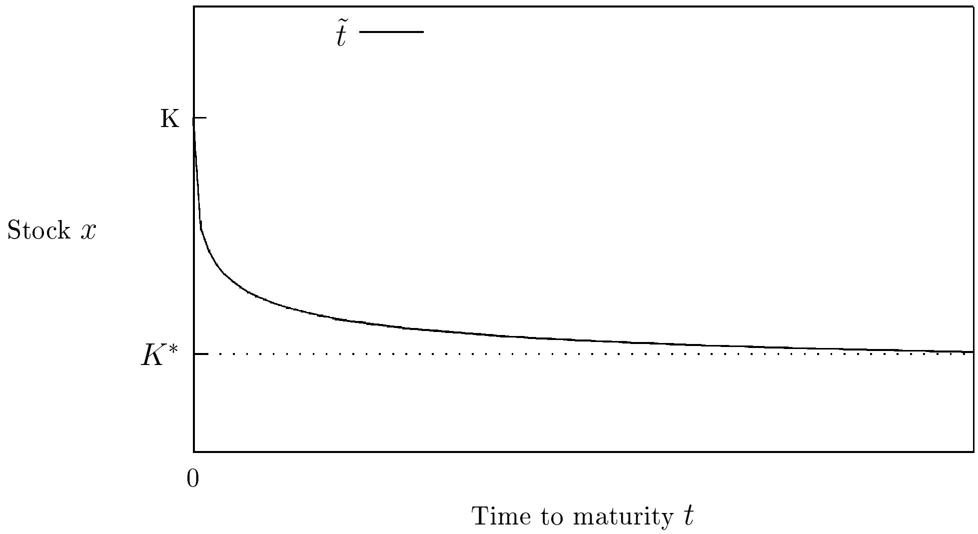
$$v_\psi^{am}(t,x) = \sup_{\tau \in \mathcal{T}(0,t)} \mathbb{E}\left[e^{-\rho\tau}\psi\left(X_\tau^x\right)\right] \tag{2}$$

where τ runs across the set of stopping times of the Brownian filtration such that $\tau \leq t$ almost surely. For $x > 0$, the function $t \to v_\psi^{am}(t,x)$ is non-decreasing. Moreover, it is greater than $\psi(x)$ and typically the space $]0,\infty[\times \mathbb{R}_+^*$ splits into two regions, the so-called Exercice region where by definition $v_\psi^{am} = \psi$ and its complement the Continuation region where $v_\psi^{am} > \psi$.

In this paper, we are interested in the price $v_{Put}^{am}(t,x)$ of the American Put option given by $\psi(x) = (K-x)^+$ where K is some positive constant (the strike of the option). In case $\rho \leq 0$, it is obvious by a convexity argument that the optimal stopping price is $\tau = t$ and $v_{Put}^{am}(t,x)$ is equal to the price of the European Put option. From now on, we suppose that $\rho > 0$. Even if there is no closed-form expression for $v_{Put}^{am}(t,x)$, its limit as $t \to +\infty$, the price of the so-called perpetual Put option, can be computed explicitly as:

$$\begin{aligned} v_{Put}^{am}(\infty,x) &= (K-K^*)\left(\frac{x}{K^*}\right)^{-\alpha} 1_{\{x \geq K^*\}} + (K-x)1_{\{x<K^*\}} \\ \text{where} \quad \alpha &= \frac{2\rho}{\sigma^2} \quad \text{and} \quad K^* = \frac{\alpha K}{1+\alpha}. \end{aligned} \tag{3}$$

K^* is called the perpetual strike. Moreover there is a continuous non-increasing function $\tilde{t}$:$]0,+\infty[\to[0,+\infty]$ with $\tilde{t}(x)=+\infty$ if $x\leq K^*$ and $\tilde{t}(x)=0$ if $x\geq K$ such that the Exercice region of the American Put option is given by $\{(t,x):\ 0<t\leq\tilde{t}(x)\}$. In the (t,x) plane the situation is thus the following:



The purpose of the paper is to construct an approximation of $v_{Put}^{am}(t,x)$ thanks to the following embedding result obtained in a previous work [1]: let $\varphi:\mathbb{R}_+^*\to\mathbb{R}_+$ be a continuous function such that $\sup_{x>0}\varphi(x)/(x+x^\alpha)<+\infty$ and $v_\varphi(t,x)=\mathbb{E}[e^{-\rho t}\varphi(X_t^x)]$ denote the price of the European option with payoff φ. If the function $x\to\widehat{\varphi}(x)=\inf_{t\geq0}v_\varphi(t,x)$ is continuous and if there is a continuous function $\widehat{t}$:$]0,+\infty[\to[0,+\infty]$ such that $\forall x>0,\ \widehat{\varphi}(x)=v_\varphi(\widehat{t}(x),x)$ (Convention : $v_\varphi(\infty,x)=\liminf_{t\to+\infty}v_\varphi(t,x)$), then the price of the American option with payoff $\widehat{\varphi}$ is embedded in the function $v_\varphi(t,x)$ in the following sense :

$$\forall(t,x)\in[0,+\infty[\times]0,+\infty[,\ v_{\widehat{\varphi}}^{am}(t,x)=v_\varphi(t\vee\widehat{t}(x),x).$$

As an easy consequence, the set $\{(t,x):\ 0<t\leq\widehat{t}(x)\}$ is included in the Exercice region of the American option.

The main drawback of the above result is that we do not know, at the moment, how to design a function φ such that $\widehat{\varphi}$ matches a given target payoff of interest. Even in the special Put case, despite many attempts, we could not find any European payoff φ with associated American payoff

$\widehat{\varphi}(x) = (K-x)^+$. Nevertheless we rely on the above theoretical result to design closed-form prices for a large class of payoffs very close to the Put payoff. This is done in three steps.
First we design (section 1) a family of European payoffs which verify very crude necessary conditions for $\widehat{\varphi}(x) = (K-x)^+$ to have any chance to hold. This is the main step, it relies on the parameterization of φ by a measure h related to $\mathcal{A}\varphi$. Then we focus on the Continuation region. Amid our family we find out necessary and sufficient conditions which grant that the equation $\inf_{t\geq 0} v_\varphi(t,x) = v_\varphi(\widehat{t}(x), x)$ defines a curve which displays the same features as the free boundary of the American Put (section 2).
Unfortunately, it is easy to see that for any function amid our family $\widehat{\varphi}(x) = (K-K^*)\left(\frac{x}{K^*}\right)^{-\alpha} 1_{\{x\geq K^*\}}$ below K^*, which is not satisfactory. The third step, which is easy making use of the fact that K^* is the perpetual Put strike, is to prove that the price of the American option with modified payoff $(K-x)^+ 1_{\{x\leq K^*\}} + \widehat{\varphi}(x) 1_{\{x>K^*\}}$, denoted by $\widehat{\varphi}_h$ to emphasize the dependence on the parameter h, and matching $(K-x)^+$ both for $x \geq K$ and for $x \leq K^*$ is still embedded in $v_\varphi(t,x)$: $v^{am}_{\widehat{\varphi}_h}(t,x) = (K-x)^+ 1_{\{x\leq K^*\}} + v_\varphi(t \vee \widehat{t}(x), x) 1_{\{x>K^*\}}$. This is done in section 3.

Since we show that $\widehat{\varphi}_h$ cannot be equal to the Put payoff everywhere (indeed $\widehat{\varphi}''_h(K^{*+}) > 0$), we believe that at this stage there is little to get from further calculations. The last stage is to select amid our family the point h^* so that, in some sense, $\widehat{\varphi}_{h^*}$ is the closest payoff to $(K-x)^+$. We choose the criterion

$$\sup_x \left| \widehat{\varphi}_h(x) - (K-x)^+ \right|$$

This is done in a numerical manner which is explained in detail in the last section (section 4): choosing φ in a peculiar low-dimensional subclass, we compute a discretized version of $\widehat{\varphi}$ and then minimize the above criterion. The numerical results seem very good.

1 A first set of tentative payoffs φ

Let us now look for a class of initial payoffs φ for which there is some hope that $\widehat{\varphi}(x) = (K-x)^+$ holds, at least for x between K^* and K.
Notice first that the European price of φ should match the American Put price in the Continuation region. In particular it should increase from 0 to $(K-K^*)\left(\frac{x}{K^*}\right)^{-\alpha}$ as t goes from 0 to ∞ for $x \geq K$. This gives at once $\varphi(x) = 0$

for $x \geq K$. Another condition is that the European price of φ decreases to $\widehat{\varphi}(x)$, for x between K^* and K, as t goes from 0 to $\widehat{t}(x)$ (the tentative free boundary). This should also hold for x below K^* with $\widehat{t}(x) = \infty$. Note that these conditions are necessary only if we restrict ourselves to the simple case of a single curve where $\inf_{t \geq 0} v_\varphi(t,x)$ is attained which splits the (t,x) plane in two regions where respectively $\partial_t v_\varphi \leq 0$ and $\partial_t v_\varphi \geq 0$. Thanks to the Black-Scholes PDE this gives that $\mathcal{A}\varphi(x)$ (defined in any reasonable sense) should be non-positive between 0 and K. Now a natural way to proceed is to parameterize φ by $\mathcal{A}\varphi$, or in other words to solve the ODE

$$\mathcal{A}\varphi = m.$$

The solutions of $\mathcal{A}\varphi = 0$ are the functions $x \to ax + bx^{-\alpha}$ for 2 reals (a,b). By a straightforward integration this gives

$$\varphi(x) = ax + bx^{-\alpha} - \frac{2}{\sigma^2} x^{-\alpha} \int_0^x y^\alpha \int_y^\infty \frac{m(dr)}{r^2} \tag{4}$$

or yet by Fubini's theorem, since m should be supported in $]0,K]$ to ensure $\varphi = 0$ above K:

$$\varphi(x) = ax + bx^{-\alpha} - \frac{2}{\sigma^2(\alpha+1)} x^{-\alpha} \int_0^K (r \wedge x)^{\alpha+1} \frac{m(dr)}{r^2} \tag{5}$$

as soon as the measure m satisfies $\int_0^K r^{\alpha-1}|m|(dr) < \infty$.
Now by the Lebesgue theorem, it is easy to see that $a = \lim_{x\to\infty} \frac{\varphi(x)}{x}$ which gives for us $a = 0$. Then $\varphi(x) = 0$ for $x \geq K$ gives the condition:

$$b = \frac{2}{\sigma^2(\alpha+1)} \int_0^K r^{\alpha-1} m(dr) \tag{6}$$

Observe next that since $\lim_{x\to\infty} \frac{\varphi(x)}{x} = a = 0$ and by Lebesgue Theorem $\lim_{x\to 0^+} \frac{\varphi(x)}{x^{-\alpha}} = b$, according to Appendix B, $\lim_{t\to\infty} v_\varphi(t,x) = ax + bx^{-\alpha} = bx^{-\alpha}$. This gives the value of b: $b = \frac{K-K^*}{K^{*-\alpha}}$
We have not yet used the fact that m should be non-positive on $]0,K[$. Obviously for (6) to hold, since b is positive, m should be of the form:

$$m(dr) = c\delta_K(dr) - 1_{]0,K[}(r) \frac{\sigma^2(\alpha+1)K^*}{2} h(dr)$$

where h is a positive measure on $]0,K[$ (we wrote the indicator function for clarity's sake. Also the factor $\frac{\sigma^2(\alpha+1)K^*}{2}$ before h will lead to easier calculations later on) and c a strictly positive number.
By the way, c is related to the left derivative of φ at K: by (4) $(\varphi(x)x^\alpha)' = -\frac{2}{\sigma^2}x^\alpha \int_x^K \frac{m(dr)}{r^2}$ whence by $\varphi(K)=0$:

$$c = \frac{-\sigma^2\varphi'(K_-)K^2}{2}$$

As soon as φ has a few regularity properties on the left of K, since $\widehat{t}(x)$ goes to 0 as x goes to K from below, $\widehat{\varphi}'(x)$ should go to $\varphi'(K_-)$. But $\widehat{\varphi}'(x)$ should be -1, so we get the value of c: $c = \frac{\sigma^2 K^2}{2}$.
The last point to check is that this is compatible with (6). This rewrites now:

$$K^* \int_0^K r^{\alpha-1} h\,(dr) = \frac{K^2}{(\alpha+1)} K^{\alpha-1} - (K-K^*)\,K^{*\alpha} = K^* \frac{K^\alpha - K^{*\alpha}}{\alpha}$$

In particular this is a positive quantity.
So far we have reached the following:

Lemma 1 *Let $\varphi(x)$ be a continuous payoff satisfying $\mathcal{A}\varphi = m$ where m is a measure on $]0,+\infty[$ such that $\int_0^{+\infty} r^{\alpha-1}|m|(dr) < +\infty$.*
Then the four conditions
(i) $\varphi(x) = 0$ for $x \geq K$
(ii) For every $x \geq K$, $v_\varphi(t,x) \to (K-K^*)(\frac{x}{K^*})^{-\alpha}$ as $t \to \infty$
(iii) In a weak sense $\mathcal{A}\varphi \leq 0$ below K
(iv) $\varphi'(K^-) = -1$
hold if and only if $m(dr) = \frac{\sigma^2K^2}{2}\delta_K(dr) - \frac{\sigma^2(\alpha+1)K^*}{2}h(dr)$ where h is a positive measure on $]0,K[$ such that $\int_0^K r^{\alpha-1}h(dr) = (K^\alpha - K^{*\alpha})/\alpha$ and

$$\varphi(x) = (K-K^*)(\frac{x}{K^*})^{-\alpha} - x^{-\alpha}\frac{(K\wedge x)^{\alpha+1}}{\alpha+1} + K^* \int_0^K x^{-\alpha}\frac{(r\wedge x)^{\alpha+1}}{r^2}h(dr) \quad (7)$$

An additional calculation (cf Appendix A) gives also:

Lemma 2 *The function φ in (7) is non-negative.*

1.1 Computing the corresponding price

>From now on we suppose that φ is given by (7). Let

$$\begin{aligned} e_r(x) &= x^{-\alpha}(r \wedge x)^{\alpha+1} \\ &= r^{\alpha+1}x^{-\alpha}1(x>r)+x1(x\leq r) \end{aligned}$$

Then after (7), since the function $x \mapsto x^{-\alpha}$ is invariant, also using $\frac{K}{K^*}=1+\frac{1}{\alpha}$:

$$\frac{v_\varphi(t,x)}{K^*}=\frac{\left(\frac{x}{K^*}\right)^{-\alpha}}{\alpha}-\frac{1}{\alpha K}v_{e_K}(t,x)+\int_0^K v_{e_r}(t,x)\,\frac{h(dr)}{r^2}$$

where

$$v_{e_r}(t,x)=e^{-\rho t}\mathbb{E}\left[e_r\left(x\exp\left(\left(\rho-\frac{\sigma^2}{2}\right)t+\sigma B_t\right)\right)\right]$$

which gives after straightforward calculations (cf Appendix C):

Lemma 3 *One has*

$$v_{e_r}(t,x)=r^{\alpha+1}x^{-\alpha}N\left(-\left(\frac{\ln\left(\frac{r}{x}\right)+\left(\frac{\alpha+1}{2}\right)\sigma^2t}{\sqrt{\sigma^2t}}\right)\right)+xN\left(\frac{\ln\left(\frac{r}{x}\right)-\left(\frac{\alpha+1}{2}\right)\sigma^2t}{\sqrt{\sigma^2t}}\right)$$

where $N(z)=\int_{-\infty}^z e^{-y^2/2}\frac{dy}{\sqrt{2\pi}}$ *denotes the cumulative distribution function of the Normal law.*

Setting $a=\ln(K^*)$, $b=\ln(K)$, $y=\ln(x)$, $u=\ln(r)$, also $\lambda=\frac{1}{\sigma^2t}$ and denoting the image of the measure $h(dr)$ by the function $r\to\ln(r)$ by $dh(e^u)$, we thus get

$$\begin{aligned} e^{-a}v_\varphi(\lambda,y) &= \frac{e^{\alpha(a-y)}}{\alpha}-\frac{e^{\alpha(b-y)}}{\alpha}N\left(-(b-y)\sqrt{\lambda}-\left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right) \\ &\quad -\frac{e^{(y-b)}}{\alpha}N\left((b-y)\sqrt{\lambda}-\left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right) \\ &\quad +e^{-\alpha y}\int_{-\infty}^b e^{\alpha u}\frac{dh(e^u)}{e^u}N\left(-(u-y)\sqrt{\lambda}-\left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right) \\ &\quad +e^{y}\int_{-\infty}^b e^{-u}\frac{dh(e^u)}{e^u}N\left((u-y)\sqrt{\lambda}-\left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right) \end{aligned}$$

In terms of the measure $\widetilde{h}(du)=\alpha e^{\frac{(\alpha-1)(u-b)}{2}}\frac{dh(e^u)}{e^u}$, we get

Lemma 4 *Let* $a = \ln(K^*)$, $b = \ln(K)$, $y = \ln(x)$, $u = \ln(r)$, $\lambda = \frac{1}{\sigma^2 t}$, *also*

$$\widetilde{h}\,(du) = \alpha e^{\frac{(\alpha-1)(u-b)}{2}} \frac{dh\,(e^u)}{e^u}$$

Then one has:

$$\begin{aligned}
\alpha e^{-a} v_\varphi\,(\lambda, y) &= e^{\alpha(a-y)} - e^{\alpha(b-y)} N\left(-\,(b-y)\sqrt{\lambda} - \left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right) \\
&\quad - e^{(y-b)} N\left((b-y)\sqrt{\lambda} - \left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right) \\
&\quad + e^{\frac{(\alpha-1)(b-y)}{2}} \int_{-\infty}^{b} e^{\frac{(\alpha+1)(u-y)}{2}} \widetilde{h}\,(du)\, N\left(-\,(u-y)\sqrt{\lambda} - \left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right) \\
&\quad + e^{\frac{(\alpha-1)(b-y)}{2}} \int_{-\infty}^{b} e^{-\frac{(\alpha+1)(u-y)}{2}} \widetilde{h}\,(du)\, N\left((u-y)\sqrt{\lambda} - \left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right)
\end{aligned}$$

2 Tentative φ's with good-looking theta-zero curve

As we are interested in $\widehat{t}(x)$ such that $\inf_{t\geq 0} v_\varphi(t,x) = v_\varphi(\widehat{t}(x), x)$, we are going to study the so-called theta-zero points solution of $\partial_t v_\varphi(t,x) = 0$. More precisely we look for conditions on the measure h which ensure that

$$\widehat{t}(x) \text{ is continuous, } \widehat{t}^{-1}(0) = [K, +\infty[, \; \widehat{t}^{-1}(+\infty) =]0, K^*]. \tag{8}$$

2.1 The theta-zero curve

Since the price of the European option with payoff φ satisfies the Black-Scholes partial differential equation $\partial_t v_\varphi(t,x) = \mathcal{A} v_\varphi(t,x)$ for $t, x > 0$, in order to find the theta-zero points, we compute $\mathcal{A} v_\varphi(t,x)$.

One main advantage of our parameterization of φ by $\mathcal{A}\varphi = m$ is the simplicity of the following computations. Indeed by the semi-group property $\mathcal{A} v_\varphi(t,x) = v_{\mathcal{A}\varphi}(t,x)$. Since $v_{\mathcal{A}\varphi}$ solves the Black-Scholes Partial Differential Equation

$$\forall t, x > 0, \; \partial_t v_{\mathcal{A}\varphi}(t,x) = \mathcal{A} v_{\mathcal{A}\varphi}(t,x), \;\; v_{\mathcal{A}\varphi}(0,.) = m,$$

by the Feynman-Kacs representation formula,

$$\forall t, x > 0,\ v_{\mathcal{A}\varphi}(t,x) = e^{-\rho t}\int_0^K p_t^X(x,r)m(dr)$$

where $p_t^X(x,r)$ is the transition density of the Black-Scholes process. If $n_{\alpha^2}(z) = e^{-\frac{z^2}{2\alpha^2}}/\sqrt{2\pi\alpha^2}$ denotes the Gaussian density, an easy calculation yields $p_t^X(x,r) = \frac{x}{r}n_{\sigma^2 t}\left(\ln\left(\frac{r}{x}\right) - \left(\rho - \frac{\sigma^2}{2}\right)t\right)$.
As a conclusion,

Lemma 5 *We have*

$$\mathcal{A}v_\varphi(t,x) = e^{-\rho t}\int_0^K n_{\sigma^2 t}\left(\ln\left(\frac{r}{x}\right) - \left(\rho - \frac{\sigma^2}{2}\right)t\right)\frac{m(dr)}{r}.$$

We recall that $m(dr) = \frac{\sigma^2 K^2}{2}\delta_K(dr) - \frac{\sigma^2(\alpha+1)K^*}{2}h(dr)$. Changing notations by setting

$$y = \ln(x),\ u = \ln(r),\ \lambda = \frac{1}{\sigma^2 t},\ a = \ln(K^*),\ b = \ln(K).$$

we obtain that

$$\partial_t v_\varphi(t,x) = \mathcal{A}v_\varphi(t,x) = C(\lambda,y)\,F(\lambda,y)$$

where $C(\lambda,y) = \sigma^2\sqrt{\lambda}e^{-(1+\alpha)^2/8\lambda}e^{(\alpha-1)(b-y)/2}e^b/(2\sqrt{2\pi}) > 0$ for $\lambda > 0$ and

$$F(\lambda,y) = e^{-\frac{\lambda}{2}(b-y)^2} - \int_{-\infty}^b e^{-\frac{\lambda}{2}(u-y)^2}\tilde{h}(du) \tag{9}$$

Thus we are interested in the solutions of

$$F(\lambda,y) = 0 \tag{10}$$

>From now on, we suppose that

$$\forall x < K,\ h(]x,K[) > 0 \ \text{ and } \ \int_0^K \ln^2(r)r^{\frac{\alpha-3}{2}}h(dr) < +\infty \tag{11}$$

Lemma 6 *The function F is C^1 on $[0,+\infty)\times\mathbb{R}$. Moreover, for $y \in \mathbb{R}$ the function $\lambda \geq 0 \to F(\lambda,y)$ vanishes at most twice. Lastly, $\forall y \geq b$ (resp $y < b$), $F(\lambda,y)$ is positive (resp. negative) for λ big enough.*

Proof. The integrability assumption in (11) is equivalent to the convergence of $\int_{-\infty}^{b} u^2\tilde{h}(du)$. By Lebesgue theorem, we easily deduce that F is C^1.
Equation (10) writes

$$-\frac{\lambda}{2}(b-y)^2 = \ln\left(\int_{-\infty}^{b} e^{-\frac{\lambda}{2}(u-y)^2}\tilde{h}(du)\right).$$

Hence for fixed $y \in \mathbb{R}$, the solutions are given by the intersection of a straight line and the Log-Laplace transform of a positive measure which is strictly convex under (11). We conclude that $\lambda \geq 0 \to F(\lambda, y)$ vanishes at most twice. The last assertion is a consequence of the first part of (11). ■

Let us now derive necessary conditions on h for (8) to hold.
If (8) holds then $t \to v_\varphi(t \vee \widehat{t}(x), x)$ is non-decreasing. As a consequence, when $x \in]K^*, K[$, $\partial_t v_\varphi(t,x) \geq 0$ for $t \geq \widehat{t}(x)$ i.e. when $y \in]a,b[$, $F(\lambda, y) \geq 0$ for λ positive and small. For $x \leq K^*$, $\widehat{t}(x) = +\infty$ i.e. $\inf_{t\geq 0} v_\varphi(t,x) = \liminf_{t\to+\infty} v_\varphi(t,x)$. Since $\lambda \geq 0 \to F(\lambda, y)$ vanishes at most twice, so does $t > 0 \to \partial_t v_\varphi(t,x)$. Hence when $x \leq K^*$, $\partial_t v_\varphi(t,x) < 0$ for t big enough i.e. when $y \leq a$, $F(\lambda, y) < 0$ for λ positive and small.
Since F is continuous, to get the previous sign conditions, we need $F(0,a) = 0$ e.g. $\tilde{h}$ is a probability measure

$$\int_{-\infty}^{b} \tilde{h}(du) = 1 \tag{12}$$

As $F(0,y)$ is independent of y, the sign conditions then imply respectively $\partial_\lambda F(0,y) \geq 0$ for $y \in]a,b[$ and $\partial_\lambda F(0,y) \leq 0$ for $y \leq a$. Since F is C^1, $\partial_\lambda F(a,0) = 0$ e.g.

$$\int_{-\infty}^{b} (u-a)^2\tilde{h}(du) = (b-a)^2 \tag{13}$$

The necessary conditions (12) and (13) will turn out to be sufficient for (8) to hold:

Proposition 7 *If* $\forall y < b$, $\widetilde{h}(]y,b[) > 0$ *and (12) and (13) hold then*

$$\begin{aligned}
\forall y \in\,]a,b[\,, \qquad & \exists! \lambda^*(y) > 0 \quad \textit{such that} \quad F(\lambda^*(y), y) = 0 \\
& F(\lambda, y) > 0 \quad \textit{for} \quad \lambda \in]0, \lambda^*(y)[\\
& F(\lambda, y) < 0 \quad \textit{for} \quad \lambda > \lambda^*(y) && (14) \\
\forall y \geq b, \qquad & \forall \lambda > 0,\ F(\lambda, y) > 0 && (15) \\
\forall y \leq a, \qquad & \forall \lambda > 0,\ F(\lambda, y) < 0 && (16)
\end{aligned}$$

Proof. By (12), $\forall y \in \mathbb{R},\ F(0,y)=0$. It is easy then to deduce (15) from (9).
Next, $\forall y \in \mathbb{R}$, writing $(u-y)^2 = (u-a)^2 + (a-y)^2 - 2(y-a)(u-a)$, developing $(b-y)^2$ in a similar way and using (12) and (13) we get

$$\partial_\lambda F(0,y) = \frac{1}{2}\int_{-\infty}^{b} (u-y)^2\,\tilde{h}(du) - \frac{1}{2}(b-y)^2 = (y-a)\int_{-\infty}^{b}(b-u)\tilde{h}(du).$$

Hence $\partial_\lambda F(0,y)$ is positive (resp. negative) for $y>a$ (resp. $y<a$), which implies that $F(\lambda,y)$ is positive (resp. negative) for λ positive and small when $y>a$ (resp. $y<a$). By Lemma 6, when $y<b$, $F(\lambda,y)$ is negative for λ big enough. Moreover, as $\lambda \to F(\lambda,y)$ vanishes at $\lambda=0$, this function vanishes at most for at most one $\lambda(y)>0$ and then $\partial_\lambda F(\lambda(y),y)\neq 0$. By the intermediate value property, we deduce (14) and (16) for $y<a$. As $F(0,a)=\partial_\lambda F(0,a)=0$, the function $F(\lambda,a)$ does not vanish for $\lambda>0$ and (16) also holds for $y=a$. ■

Setting $\lambda^*(y)=0$ for $y\le a$ and $\lambda^*(y)=+\infty$ for $y\ge b$, then $\forall y\in\mathbb{R},\ \widehat{\varphi}(e^y)=v_\varphi(\lambda^*(y),y)$. It is enough to check that $\widehat{\varphi}$ is continuous and that λ^* is continuous and non-decreasing to conclude that (8) holds. Let us now turn to a detailed study of λ^* and $\widehat{\varphi}$.

2.2 Behaviour of $\lambda^*(y)$ for $y \in]a,b[$

Proposition 8 *Under the assumptions of Proposition 7, the function λ^* is analytic and increasing from $]a,b[$ to $\mathbb{R}_+^*$ and satisfies*

$$\lim_{y\to a^+}\lambda^*(y)=0,\ \lim_{y\to b^-}\lambda^*(y)=\infty.$$

More precisely,

$$\lambda^*(y)(b-y)^2 \to_{y\to b^-} \infty. \tag{17}$$

If we suppose moreover that $d\tilde{h}$ is absolutely continuous in a neighborhood of b i.e. for some $b_*\in]a,b[$ $\tilde{h}(du)=\tilde{h}(u)du$ on $]b_*,b[$ and that $\lim_{u\to b^-}\tilde{h}(u)=\tilde{h}(b^-)>0$ exists, then

$$\lim_{y\to b^-}\frac{\ln(b-y)}{\lambda^*(y)(b-y)^2} = -\frac{1}{2}. \tag{18}$$

Lastly, the following equivalent holds for $\lambda^*(y)$ *as* $y \to a^+$ *:*

$$\lambda^*(y) \sim_{y\to a^+} 8(y-a)\int_{-\infty}^{b}(b-u)\widetilde{h}(du)/\Big(\int_{-\infty}^{b}(u-a)^4\widetilde{h}\,(du)-(b-a)^4\Big). \quad (19)$$

In case, $\int_{-\infty}^{b}(u-a)^4\tilde{h}(du) = +\infty$ *(*$\Leftrightarrow \int_0^K \ln^4(r) r^{\frac{\alpha-3}{2}} m(dr) = +\infty$*), (19) means that* $\lambda^*(y) = o(y-a)$*.*

Before coming to the proof of the proposition let us notice that (18) is equivalent to the equivalent of Barles&Alii [2] and Lamberton [3]:

Lemma 9 *Let* $\lambda^*(y) \to \infty$ *as* $y \to b^-$*. Then (18) holds if and only if*

$$\lim_{y\to b^-}\frac{\lambda^*(y)\,(b-y)^2}{\ln(\lambda^*(y))} = 1 \quad (20)$$

Proof. If (20) holds then $\ln(\lambda^*) + 2\ln(b-y) - \ln(\ln(\lambda^*)) \to 0$. By dividing by $\lambda^*(b-y)^2$, which is far from zero since it goes to infinity by (20) we get $\frac{\ln(\lambda^*)}{\lambda^*(b-y)^2} + 2\frac{\ln(b-y)}{\lambda^*(b-y)^2} - \frac{\ln(\ln(\lambda^*))}{\lambda^*(b-y)^2} \to 0$ which gives (18) since $\frac{\ln(\ln(\lambda^*)}{\ln(\lambda^*))} \to 0$.
Conversely we get from (18) $\ln(-\ln(b-y)) - \ln(\lambda^*) + 2\ln(b-y) \to -\ln(2)$ whence if (18) holds $\frac{\ln(-\ln(b-y))}{\lambda^*(b-y)^2} - \frac{\ln(\lambda^*)}{\lambda^*(b-y)^2} - 2\frac{\ln(b-y)}{\lambda^*(b-y)^2} \to 0$ then (20) since $\frac{\ln(-\ln(b-y))}{\ln(b-y)} \to 0$. ■

Let us now prove the proposition:
Proof. We first compute the first order derivatives of F :

$$\begin{aligned}
\partial_y F(\lambda, y) &= \lambda\left((b-y)e^{-\frac{\lambda}{2}(b-y)^2} - \int_{-\infty}^{b}(u-y)e^{-\frac{\lambda}{2}(u-y)^2}\tilde{h}(du)\right) \\
\partial_\lambda F(\lambda, y) &= -\frac{1}{2}(b-y)^2 e^{-\frac{\lambda}{2}(b-y)^2} + \frac{1}{2}\int_{-\infty}^{b}(u-y)^2 e^{-\frac{\lambda}{2}(u-y)^2}\tilde{h}(du).
\end{aligned}$$

Let $y \in]a,b[$. Applying Jensen inequality to the strictly convex function $z\ln(z)$ and the moment equality $F(\lambda^*(y), y) = 0$, we get $\partial_\lambda F(\lambda^*(y), y) < 0$. Moreover, using $F(\lambda^*(y), y) = 0$, we get

$$\partial_y F(\lambda^*(y), y) = \lambda^*(y)\int_{-\infty}^{b}(b-u)e^{-\frac{\lambda^*(y)}{2}(u-y)^2}\tilde{h}(du) > 0.$$

Now the price $v_\varphi(t,x)$ of the European option is analytic on $\mathbb{R}_+^* \times \mathbb{R}_+^*$, therefore $\partial_t v_\varphi(\lambda, y)$ is analytic on $\mathbb{R}_+^* \times \mathbb{R}$. Since for $y \in]a,b[$, $\lambda^*(y)$ is the unique $\lambda > 0$ solution of $\partial_t v_\varphi(\lambda, y) = 0$ and

$$\partial_\lambda \left(\partial_t v_\varphi(\lambda^*(y), y)\right) = C(\lambda^*(y), y)\partial_\lambda F(\lambda^*(y), y) < 0, \quad \partial_y \left(\partial_t v_\varphi(\lambda^*(y), y)\right) > 0,$$

by the implicit functions theorem for analytic functions λ^* is analytic with a positive derivative on $]a,b[$.
We deduce that $\lambda^*(y)$ has a limit when $y \to a^+$. Since F is continuous, $F(\lim_{y\to a^+} \lambda^*(y), a) = 0$. Now the unique $\lambda \geq 0$ such that $F(\lambda, a) = 0$ is 0. Hence $\lim_{y\to a^+} \lambda^*(y) = 0$. By a similar reasoning, we check that $\lim_{y\to b^-} \lambda^*(y) = +\infty$.
To precise the speed of convergence, we recall that $\lambda^*(y)$ is given by

$$e^{-\frac{\lambda^*(y)}{2}(b-y)^2} = \int_{-\infty}^{b} e^{-\frac{\lambda^*(y)}{2}(u-y)^2} \widetilde{h}(du). \tag{21}$$

As $y \to b^-$, $\lambda^*(y) \to +\infty$ and $\forall u < b$, $e^{-\frac{\lambda^*(y)}{2}(u-y)^2} \to 0$. Hence by Lebesgue theorem the right-hand-side of (21) goes to 0 and $\lambda^*(y)(b-y)^2 \to +\infty$.
Let us now turn to (18). By Lebesgue theorem,

$$e^{\frac{\lambda^*(y)}{2}(b-y)^2} \int_{-\infty}^{2y-b} e^{-\frac{\lambda^*(y)}{2}(u-y)^2} \widetilde{h}(du) = \int_{-\infty}^{2y-b} e^{-\frac{\lambda^*(y)}{2}(b-u)(2y-b-u)} \widetilde{h}(du) \to_{y\to b^-} 0.$$

We now suppose that $\tilde{h}(du)$ has a density $\tilde{h}$ on $]b_*, b[$ and that $\lim_{u\to b^-} \tilde{h}(u) = \tilde{h}(b^-) > 0$. Setting $u = y + \beta\,(b-y)$ we get from the above remark:

$$\begin{aligned} 1 \quad \sim_{y\to b^-} \quad & \int_{2y-b}^{b} e^{-\frac{\lambda^*(y)}{2}\left((u-y)^2-(b-y)^2\right)} \widetilde{h}(du) \\ = \quad & (b-y) \int_{-1}^{1} e^{-\frac{\lambda^*(y)}{2}(b-y)^2(\beta^2-1)} \widetilde{h}\left(y + \beta\,(b-y)\right) d\beta \\ \sim \quad & (b-y)\widetilde{h}(b^-) \int_{-1}^{1} e^{-\frac{\lambda^*(y)}{2}(b-y)^2(\beta^2-1)} d\beta \end{aligned}$$

Therefore, by the Laplace method,

$$\left(\frac{1}{b-y}\right)^{\frac{2}{\lambda^*(y)(b-y)^2}} \sim \left(\widetilde{h}(b^-) \int_{-1}^{1} e^{-\frac{\lambda^*(y)}{2}(b-y)^2(\beta^2-1)} d\beta\right)^{\frac{2}{\lambda^*(y)(b-y)^2}} \to \sup_{\beta \in]-1,1[} e^{-\left(\beta^2-1\right)} = e$$

which gives (18).

To precise the behaviour of $\lambda^*(y)$ as $y \to a^+$, we make Taylor expansions in (21):

$$\begin{aligned} 1 &- \frac{\lambda^*(y)}{2}(b-y)^2 + \frac{\lambda^*(y)^2}{8}(b-y)^4 + o(\lambda^*(y)^2) \\ &= \int_{-\infty}^{b} \left(1 - \frac{\lambda^*(y)}{2}(u-y)^2 + \frac{\lambda^*(y)^2}{4}(u-y)^4 \int_0^1 (1-\theta) e^{-\frac{\theta\lambda^*(y)}{2}(u-y)^2} d\theta \right) \widetilde{h}\,(du) \end{aligned}$$

which simplifies after (12) and (13), writing $(b-y)^2 = (b-a)^2 + (y-a)^2 + 2(b-a)(a-y)$, developing $(u-y)^2$ and also $(b-y)^4$ in a similar way, to

$$\begin{aligned} \lambda^*(y) \quad (y-a) \quad & \int_{-\infty}^{b} (b-u)\widetilde{h}(du) + \frac{\lambda^*(y)^2(b-a)^4}{8} + o(\lambda^*(y)^2) \\ = \quad & \lambda^*(y)^2 \int_0^1 \frac{1-\theta}{4} \left(\int_{-\infty}^{b} (u-y)^4 e^{-\frac{\theta\lambda^*(y)}{2}(u-y)^2} \tilde{h}(du) \right) d\theta \end{aligned}$$

In case $\int_{-\infty}^{b} (u-a)^4 \tilde{h}(du) < +\infty$ the r.h.s. is equivalent to

$$\lambda^*(y)^2 \int_{-\infty}^{b} (u-a)^4 \tilde{h}(du)/8.$$

Since $\widetilde{h}$ is not a Dirac mass, by Jensen inequality

$$\int_{-\infty}^{b} (u-a)^4 \widetilde{h}(du) > \left(\int_{-\infty}^{b} (u-a)^2 \widetilde{h}(du) \right)^2 = (b-a)^4 \quad \text{according to (13)}$$

and we deduce (19).
This assertion still holds in case $\int_{-\infty}^{b} (u-a)^4 \tilde{h}(du) = +\infty$: indeed by Fatou lemma

$$\int_0^1 \frac{1-\theta}{4} \left(\int_{-\infty}^{b} (u-y)^4 e^{-\frac{\theta\lambda^*(y)}{2}(u-y)^2} \tilde{h}(du) \right) d\theta \to +\infty.$$

■

2.3 The price along the theta-zero curve

The interesting price is obtained by setting $\lambda = \lambda^*(y)$:

Proposition 10 *Under the assumptions of Proposition 7, the payoff $\widehat{\varphi}$ is given for x between K^* and K (y between a and b) by*

$$
\begin{aligned}
\alpha e^{-a}\widehat{\varphi}\left(e^{y}\right) &= e^{\alpha(a-y)} - e^{\alpha(b-y)} N\left(-\left(b-y\right)\sqrt{\lambda} - \left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right) \\
&\quad - e^{(y-b)} N\left(\left(b-y\right)\sqrt{\lambda} - \left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right) \\
&\quad + e^{\frac{(\alpha-1)(b-y)}{2}} \int_{-\infty}^{b} e^{\frac{(\alpha+1)(u-y)}{2}} \widetilde{h}\left(du\right) N\left(-\left(u-y\right)\sqrt{\lambda} - \left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right) \\
&\quad + e^{\frac{(\alpha-1)(b-y)}{2}} \int_{-\infty}^{b} e^{-\frac{(\alpha+1)(u-y)}{2}} \widetilde{h}\left(du\right) N\left(\left(u-y\right)\sqrt{\lambda} - \left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right)
\end{aligned}
$$

where $\lambda = \lambda^*\left(y\right) > 0$ is given by $F(\lambda^*(y), y) = 0$.

2.4 Computation of $\widehat{\varphi}'$ for $K^* < x < K$:

By derivation of $\widehat{\varphi}(e^y)$ with respect to y (see Appendix D), we obtain :

Lemma 11 *For $y \in]a, b[$,*

$$
\begin{aligned}
e^{-a}\widehat{\varphi}'\left(e^{y}\right) &= -e^{-y}e^{\alpha(a-y)} + e^{-y}e^{\alpha(b-y)} N\left(-\left(b-y\right)\sqrt{\lambda} - \left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right) \\
&\quad - \frac{e^{-b}}{\alpha} N\left(\left(b-y\right)\sqrt{\lambda} - \left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right) \\
&\quad - e^{-(\alpha+1)y} e^{\frac{(\alpha-1)b}{2}} \int_{-\infty}^{b} e^{\frac{(\alpha+1)u}{2}} \widetilde{h}\left(du\right) N\left(-\left(u-y\right)\sqrt{\lambda} - \left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right) \\
&\quad + \frac{e^{\frac{(\alpha-1)b}{2}}}{\alpha} \int_{-\infty}^{b} e^{-\frac{(\alpha+1)u}{2}} \widetilde{h}\left(du\right) N\left(\left(u-y\right)\sqrt{\lambda} - \left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right) \qquad (22)
\end{aligned}
$$

where $\lambda = \lambda^*\left(y\right) > 0$ is given by $F(\lambda^*(y), y) = 0$.

2.5 Behaviour of $\widehat{\varphi}$ as $x \to K^{*+}$:

Proposition 12 *Under the assumptions of Proposition 7 ,*

$$\lim_{x \to K^{*+}} \widehat{\varphi}(x) = K - K^*.$$

Moreover, $\lim_{x\to K^{*+}} \widehat{\varphi}'(x) = -1$ *and*

$$\lim_{x\to K^{*+}} \frac{\widehat{\varphi}'(x)+1}{x-K^*} = \frac{\alpha+1}{K^*} > 0.$$

i.e. the behaviour of $\widehat{\varphi}(x)$ *when* $x \to K^{*+}$ *is similar to the one of the perpetual Put price and* $\widehat{\varphi}$ *cannot be equal to* $K-x$ *on* $[K^*, K]$.

Proof. We recall that $\lim_{y\to a^+} \lambda^*(y) = 0$. Hence, in the expression of $e^{-a}\widehat{\varphi}(e^y)$ given by Proposition 10, when $y \to a^+$, the first term has a limit equal to $1/\alpha$ and the second and third terms go to 0. The fourth and the fifth term also vanish according to Lebesgue theorem and the following upper-bounds: $\forall u \le b$, $\forall y \ge a$,

$$\begin{aligned} e^{\frac{(\alpha+1)(u-y)}{2}} \quad & N\left(-(u-y)\sqrt{\lambda} - \left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right) \\ \le \quad & e^{-\frac{(\alpha+1)^2}{8\lambda}} 1_{\{u-y\le -(\alpha+1)/4\lambda\}} + e^{\frac{(\alpha+1)(b-a)}{2}} N\left(-\frac{\alpha+1}{4\sqrt{\lambda}}\right) 1_{\{u-y\ge -(\alpha+1)/4\lambda\}} \\ \le \quad & e^{-\frac{(\alpha+1)^2}{8\lambda}} + e^{\frac{(\alpha+1)(b-a)}{2}} N\left(-\frac{\alpha+1}{4\sqrt{\lambda}}\right) \end{aligned}$$

$$\begin{aligned} e^{-\frac{(\alpha+1)(u-y)}{2}} \quad & N\left((u-y)\sqrt{\lambda} - \left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right) \\ \le \quad & e^{-\frac{(\alpha+1)(\alpha+1-2\sqrt{\lambda})}{4\lambda}} 1_{\{(u-y)\sqrt{\lambda} - \left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}} \ge -1\}} \\ + \quad & \frac{1}{\sqrt{2\pi}} e^{-\frac{(\alpha+1)(u-y)}{2}} e^{-\frac{1}{2}\left((u-y)\sqrt{\lambda} - \left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right)^2} 1_{\{(u-y)\sqrt{\lambda} - \left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}} \le -1\}} \\ \le \quad & e^{-\frac{(\alpha+1)(\alpha+1-2\sqrt{\lambda})}{4\lambda}} + e^{-\frac{(\alpha+1)^2}{8\lambda}} \end{aligned}$$

Hence $\lim_{x\to K^{*+}} \widehat{\varphi}(x) = e^a/\alpha = K^*/\alpha = K - K^*$.
Denoting by $T_i(y)$, $1 \le i \le 5$ the terms of the right-hand-side of (22), we have $T_1(a) = -e^{-a}$ and $T_1'(a) = (\alpha+1)e^{-a}$. We conclude the proof by checking that $\forall 2 \le i \le 5$, $\forall n \in \mathbb{N}$, $\lim_{y\to a^+} T_i(y)/(y-a)^n = 0$ thanks to (19) and the previous upper-bounds. ■

2.6 Behaviour of $\widehat{\varphi}$ as $x \to K^-$:

Proposition 13 *If the assumptions of Proposition 7 are satisfied and*

$$\int_0^K r^{\alpha-1} h(dr) = \frac{K^\alpha - K^{*\alpha}}{\alpha}$$

which is equivalent to

$$\int_{-\infty}^{b} e^{\frac{(\alpha+1)u}{2}} \tilde{h}(du) = e^{\frac{(1-\alpha)b}{2}} (e^{\alpha b} - e^{\alpha a}),$$

where

$$\widetilde{h}\,(du) = \alpha e^{\frac{(\alpha-1)(u-b)}{2}} \frac{dh\,(e^u)}{e^u}$$

then

$$\lim_{x \to K^-} \widehat{\varphi}(x) = 0 \quad \textit{and} \quad \lim_{x \to K^-} \widehat{\varphi}'(x) = -1.$$

Proof. Since $\lim_{y \to b^-} \sqrt{\lambda}(b-y) = +\infty$, taking the limit $y \to b^-$ in the expression of $e^{-a}\widehat{\varphi}(e^y)$ given by Proposition 10,

$$\begin{aligned}
\lim_{y \to b^-} e^{-a}\widehat{\varphi}(e^y) &= \frac{e^{\alpha(a-b)}}{\alpha} + 0 - \frac{1}{\alpha} + \frac{1}{\alpha} \int_{-\infty}^{b} e^{\frac{(\alpha+1)(u-b)}{2}} \tilde{h}(du) + 0 \\
&= \left(e^{\alpha(a-b)} - 1 + e^{-\alpha b}(e^{\alpha b} - e^{\alpha a})\right)/\alpha = 0
\end{aligned}$$

Taking the limit in (22), we obtain

$$\begin{aligned}
\lim_{y \to b^-} e^{-a}\widehat{\varphi}'(e^y) &= -e^{\alpha(a-b)}e^{-b} + 0 - \frac{e^{-b}}{\alpha} - e^{-(\alpha+1)b} e^{\frac{(\alpha-1)b}{2}} \int_{-\infty}^{b} e^{\frac{(\alpha+1)u}{2}} \widetilde{h}(du) + 0 \\
&= -e^{\alpha(a-b)}e^{-b} - \frac{e^{-b}}{\alpha} - e^{-(\alpha+1)b}(e^{\alpha b} - e^{\alpha a}) = -e^{-b}\left(1 + \frac{1}{\alpha}\right) \\
&= -e^{-a}.
\end{aligned}$$

■

Remark 14
- *In case $d\tilde{h}$ is absolutely continuous in a neighborhood of b with a density $\tilde{h}$ such that $\lim_{u \to b^-} \tilde{h}(u) = \tilde{h}(b^-) > 0$ exists, it is possible to prove that the second order derivative of $\widehat{\varphi}$ at K^- depends on $\tilde{h}(b^-)$:*

$$\lim_{x \to K^-} \frac{\widehat{\varphi}'(x) + 1}{x - K} = e^{-b} \lim_{y \to b^-} \frac{\widehat{\varphi}'(e^y) + 1}{y - b} = \frac{\alpha - \widetilde{h}(b^-)}{K}$$

- *Under the assumptions of the Proposition, we have $\varphi'(K^-) = -1 = \widehat{\varphi}'(K^-)$. If moreover, the above assumption on $d\tilde{h}$ is satisfied, we can check that $\varphi''(K^-) = \frac{\alpha - \widetilde{h}(b^-)}{K} = \widehat{\varphi}''(K^-)$. The equality of the first and second derivatives of φ and $\widehat{\varphi}$ at K^- is not surprising since for $y \in]a, b[$, $\widehat{\varphi}(e^y) = v_\varphi \left(\frac{1}{\sigma^2 \lambda^*(y)}, e^y\right)$ and $\frac{1}{\sigma^2 \lambda^*(y)} = o((b-y)^2)$ as $y \to b^-$.*

3 The main result

We are now ready to summarize all the properties of $\widehat{t}(x) = 1/(\sigma^2\lambda^*(\ln(x))$ and $\widehat{\varphi}$ and to apply the embedding result of [1]. First we state a theorem which is a direct application of [1], then a modification well-suited to the Put case.
Note that (12) and (13) rewrites into the two last conditions on h in the following theorem.

Theorem 15 *Assume that*

$$\varphi(x) = (K - K^*)(\frac{x}{K^*})^{-\alpha} - x^{-\alpha}\frac{(K \wedge x)^{\alpha+1}}{\alpha+1} + K^* \int_0^K x^{-\alpha}\frac{(r \wedge x)^{\alpha+1}}{r^2} h(dr)$$

where h is a positive measure on $]0, K[$ such that $\forall x < K,\ h(]x, K[) > 0$ and

$$\begin{aligned} \int_0^K r^{\alpha-1} h(dr) &= (K^\alpha - K^{*\alpha})/\alpha \\ \int_0^K r^{\frac{\alpha-3}{2}} h(dr) &= K^{\frac{\alpha-1}{2}}/\alpha \\ \int_0^K \ln^2(r/K^*) r^{\frac{\alpha-3}{2}} h(dr) &= K^{\frac{\alpha-1}{2}} \ln^2(K/K^*)/\alpha \end{aligned}$$

then $\widehat{\varphi}(x) = \inf_{t\geq 0} v_\varphi(t,x)$ is continuous equal to 0 for $x \geq K$, equal to $(K - K^*)\left(\frac{x}{K^*}\right)^{-\alpha}$ if $x \leq K^*$, satisfies $\widehat{\varphi}'(K^{*+}) = \widehat{\varphi}'(K^-) = -1$ and $\widehat{\varphi}''(K^{*+}) = (\alpha+1)/K^*$. Moreover $\widehat{\varphi}(x) = v_\varphi(\widehat{t}(x), x)$ where $\widehat{t}$ is continuous, non-increasing, analytic on $]K^*, K[$, equal to 0 for $x \geq K$ and to $+\infty$ for $x \leq K^*$. The price of the American option with payoff $\widehat{\varphi}$ is $v^{am}_{\widehat{\varphi}}(t,x) = v_\varphi(t \vee \widehat{t}(x), x)$.

Here is now the main result:

Theorem 16 *Under the assumptions of the previous theorem, the payoff $\widehat{\varphi}_h(x) = (K - x)^+ 1_{\{x \leq K^*\}} + \widehat{\varphi}(x) 1_{\{x > K^*\}}$ is continuous and its American price is given by*

$$(K - x)^+ 1_{\{x\leq K^*\}} + v_\varphi\left(t \vee \widehat{t}\,(x)\,, x\right) 1_{\{x>K^*\}}.$$

Proof. It is easily seen that $\widehat{\varphi}_h(x) = (K - x)^+ \leq (K - K^*)(x/K^*)^{-\alpha} = \widehat{\varphi}(x)$ for $x \leq K^*$, therefore the American price $v^{am}_{\widehat{\varphi}_h}(t,x)$ is smaller than

$v^{am}_{\widehat{\varphi}}(t,x)$. Now in the region $x > K^*$, the American price of $\widehat{\varphi}_h$ is greater than $v_\varphi\left(t \vee \widehat{t}(x), x\right)$: indeed the latter may be written as $\mathbb{E}[e^{-\rho\tau}\widehat{\varphi}_h(X^x_\tau)]$ where τ is the entrance time in the region $\left\{t \leq \widehat{t}(x)\right\}$ (convention $\tau = 0$ if $t \leq \widehat{t}(x)$) and $\widehat{\varphi}_h(X^x_\tau) = \widehat{\varphi}(X^x_\tau)$. Therefore $v^{am}_{\widehat{\varphi}_h}(t,x) = v_\varphi\left(t \vee \widehat{t}(x), x\right)$ for $x > K^*$ and also $x \geq K^*$ by continuity. In particular the line $x = K^*$ is contained in the Exercice region.

Take now a point (t,x) with $x < K^*$. By the optimal stopping representation of the American price, one has

$$v^{am}_{\widehat{\varphi}_h}(t,x) = \sup_{\tau < \tau^*} \mathbb{E}\left[e^{-\rho\tau}\widehat{\varphi}_h(X^x_\tau)\right]$$

where τ runs across the set of stopping times of the Brownian filtration less than the crossing time τ^* of the boundary $\{(0,x)\,,\, x < K^*\} \cup \{(t, K^*)\,,\, t \geq 0\}$. In this area $\widehat{\varphi}_h$ is equal to the Put payoff, therefore this quantity is less than the American price of the Put. But by definition of K^* we lie in the Exercice region of the American Put, so $v^{am}_{\widehat{\varphi}_h}(t,x) \leq (K-x)^+$ and on another hand $(K-x)^+ = \widehat{\varphi}_h(x) \leq v^{am}_{\widehat{\varphi}_h}(t,x)$. ∎

Remark 17 *The same result holds for any continuous payoff obtained by replacing $\widehat{\varphi}(x)$ under K^* by a continuous function $\psi(x)$ smaller than $(K - K^*)\left(\frac{x}{K^*}\right)^{-\alpha}$ with $\psi(K^*) = (K - K^*)$ and such that the region $\{x \leq K^*\}$ lies in the Exercice region of the modified payoff. For instance in case $k \leq K^*$ it is easy to check by comparison with the Put option that the region $\{x \leq K^*\}$ is included in the exercise region of the American Put-Spread option with payoff $(K-x)^+ - (k-x)^+ = (K-k) \wedge (K-x)^+$. Hence the price of the American option with modified payoff $\widehat{\varphi}_k(x) = (K-k) \wedge (K-x)^+ 1_{\{x \leq K^*\}} + \widehat{\varphi}(x) 1_{\{x > K^*\}}$ is*

$$(K-k) \wedge (K-x)^+ 1_{\{x \leq K^*\}} + v_\varphi(t \vee \widehat{t}(x), x) 1_{\{x > K^*\}}.$$

It is natural to wonder whether the payoff $\widehat{\varphi}_h$ is non-increasing like the Put payoff. The answer is positive at least for values of α of practical interest since :

Lemma 18 *There is a constant $\alpha_0 < 1/2$, such that when $\alpha \geq \alpha_0$, under the assumptions of Theorem 15, both $\widehat{\varphi}$ and $\widehat{\varphi}_h$ are non-increasing.*

The proof of this Lemma is postponed to Appendix E.

4 Discretization

In this section we solve a discretized version of the program:

$$\inf_{h\in H}\sup_{x}\left|\widehat{\varphi}_h(x)-(K-x)^+\right|$$

where H is a low-dimensional subspace of the set of measures h which verify the moment conditions of the theorems.

4.1 Normalization

For practical purposes, it would be interesting to get a measure h^* which depend on as few parameters as possible. It will certainly depend on α, but we can design an approximation which will work for every value of K in the following way: we normalize the situation so that $K^*=1$, (any other value would work!), therefore $K=k\overset{def}{=}1+\frac{1}{\alpha}$.
This does not matter in the following sense: to emphasize the dependence on the strike K, we denote by $v_{Put}^{am}(t,x,K)$ the American Put price for the maturity t and the underlying value x. If we manage to design an approximation such that, for a given value of t :

$$\sup_{x}\left|v_{Put}^{am}(t,x,k)-Approx(t,x,k)\right|<\varepsilon$$

then since obviously $v_{Put}^{am}(t,x,k)=\frac{K}{k}v_{Put}^{am}\left(t,\frac{k}{K}x,k\right)$, the approximation by $\frac{K}{k}Approx\left(t,\frac{k}{K},x,k\right)$ will satisfy

$$\sup_{x}\left|v_{Put}^{am}(t,x,K)-\frac{K}{k}Approx\left(t,\frac{k}{K}x,k\right)\right|<\frac{K}{k}\varepsilon$$

In other words, the error we face in term of a percentage of the strike K is given by $\frac{\varepsilon}{k}$.

>From now on we work thus with:

$$K^*=1,\ k\overset{def}{=}K=1+\frac{1}{\alpha},(K-k)\,K^{*-\alpha}=K-K^*=\frac{1}{\alpha}$$

and with the variables $y=\ln(x)$ and $\lambda=1/(\sigma^2t)$

4.2 Choice of a peculiar class of $\widetilde{h}$

We further restrict ourselves to a peculiar class of measures $\widetilde{h}$ which lead to easy implementation. Whatever the measure $\widetilde{h}$ at hand there is a priori two steps to obtain $v^{am}_{\widehat{\varphi}_h}(\lambda, y)$ for given values of $y \in]a = 0, b = \ln(k)[$ and $\lambda > 0$: first compute the value of the theta-zero curve i.e. find $\lambda^*(y) \in]0, +\infty[$ solving $F(\lambda^*(y), y) = 0$ then compute the price $v_\varphi(\lambda \wedge \lambda^*(y), y) = v^{am}_{\widehat{\varphi}_h}(\lambda, y)$. In general both steps require numerical procedures, a dichotomy to find the zero of the time derivative (there is exactly one for every $y \in]a, b[$ after the above calculations), next a numerical (one-dimensional) integration (with respect to $\widetilde{h}$) to get the price. In case $y \geq b$, only the second step is required since $\lambda^*(y) = +\infty$ and in case $y \leq 0$, $v^{am}_{\widehat{\varphi}_h} = (k - e^y)$.

We choose to work with a low-dimensional family of combination of point measures. This allows the direct computation of the price at the second step.

Notice that the condition $\widetilde{h}(]y, b[) > 0$ for $y < b$ is not satisfied yet: so we add a uniform measure $\varepsilon 1_{]0,b[} du$, for which it is easily seen that the corresponding contribution to the price may be computed explicitly. We have implemented the case of 3-points measures, which gives already astonishing results. Our family may be parametrized in the following way:

$$\begin{aligned}
\widetilde{h}(du) &= \varepsilon 1_{]0,b[} du \\
&\quad + \beta\, \delta_{\log(r_1)}(du) + \gamma \delta_{\log(r_2)}(du) \\
&\quad + (1 - \varepsilon b - \beta - \gamma)\, \delta_{\log(r_3)}(du)
\end{aligned}$$

with $\varepsilon > 0$, $\varepsilon b < 1$, $\beta > 0$, $\gamma > 0$ and $\beta + \gamma < 1 - \varepsilon b$.

By convention we choose $\log(r_1) < \log(r_2) < \log(r_3)$.

Remember that the support of $\widetilde{h}$ should lie below b, so we further set

$$\log(r_3) = \mu\, b$$

and also

$$\begin{aligned}
\log(r_1) &= x_1\, \mu\, b \\
\log(r_2) &= x_2\, \mu\, b
\end{aligned}$$

Therefore the parameter $(\varepsilon, \mu, x_1, x_2)$ should live in: $0 < \varepsilon < b$, $\mu \leq 1$, $x_1 \leq x_2 \leq 1$.

For a given value of $(\varepsilon, \mu, x_1, x_2)$ we compute the values of β and γ which fit the two remaining moment conditions:

$$\begin{aligned} \int_{-\infty}^{b} u^2 \widetilde{h}(du) &= b^2 \\ \int_{-\infty}^{b} e^{\frac{(\alpha+1)u}{2}} \widetilde{h}(du) &= \frac{e^{\alpha b}-1}{e^{\frac{(\alpha-1)b}{2}}} \end{aligned}$$

This translates in the 2x2 linear system

$$\begin{aligned} &(1-x_1^2)\beta + (1-x_2^2)\gamma = \epsilon b\left(\frac{1}{3\mu^2}-1\right) + 1 - \frac{1}{\mu^2} \\ &\left(1-e^{\frac{(\alpha+1)(x_1-1)\,\mu\, b}{2}}\right)\beta + \left(1-e^{\frac{(\alpha+1)(x_2-1)\,\mu\, b}{2}}\right)\gamma \\ &\qquad = \epsilon\left(\frac{2e^{-\frac{(\alpha+1)\,\mu\, b}{2}}}{(\alpha+1)}\left(e^{\frac{(\alpha+1)b}{2}}-1\right)-b\right) + 1 - e^{\frac{(\alpha+1)\,(1-\mu)\, b}{2}}\left(1-e^{-\alpha b}\right) \end{aligned}$$

which gives close-formula for β and γ. In case one of the conditions $\beta > 0$, $\gamma > 0$ and $\beta + \gamma < 1 - \varepsilon b$ is not satisfied the point $(\varepsilon, \mu, x_1, x_2)$ is rejected, otherwise we sample the range $]0, b[$ with n points, say $y_i = \frac{i}{n} b$ with $0 < i < n$ and for every y_i we proceed as follows.

4.3 Calculation of $\lambda^*(y)$

We find $\lambda^*(y_i)$ by a dichotomy algorithm making use of the closed formula for $F(\lambda, y)$. This is obviously very fast, altough a little care is required when y_i is near 0 or b to deal with possibly very high or small values of $\lambda^*(y_i)$.

4.4 Computation of the price

This is also very fast since no numerical integration is required. We make use of the standard approximation of the normal cumulative distribution which relies on the classical series expansion.

4.5 Selection of the optimal point

Then for a given value of $(\varepsilon, \mu, x_1, x_2)$ we compute the error quantity

$$err(\varepsilon, \mu, x_1, x_2) = \sup_i \left|\widehat{\varphi}(e^{y_i}) - (k - e^{y_i})^+\right|$$

and next after a clever or systematic scan of the domain we pick the point which minimizes this criteria, with a value $err^* = err\,(\varepsilon^*, \mu^*, x_1^*, x_2^*)$. The corresponding American payoff is denoted by $\widehat{\varphi}^*$.

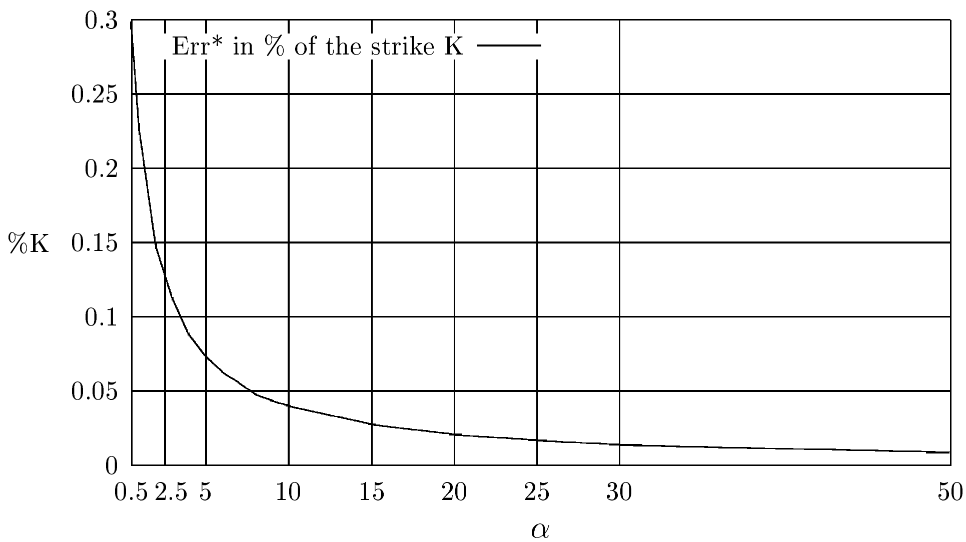
4.6 Archiving the results

The optimal point will depend on α. In practice we maintain an archive with 100 values of α equally sampled between 0.5 and 50.0 (for an annual interest rate of 5%, $\alpha = 0.4$ is $\sigma = 50\%$, $\alpha = 25.6$ is $\sigma = 6.25\%$). The computation of the archive is done once for all, the practical usage for the ambient value of α consists in picking up the closest value of the table or performing a linear interpolation since the optimal point, for our choice of the domain at least, depends "continuously" on α.

Therefore the computation time is that of the dichotomy (typically ten iterations…) and of the price, which is very fast.

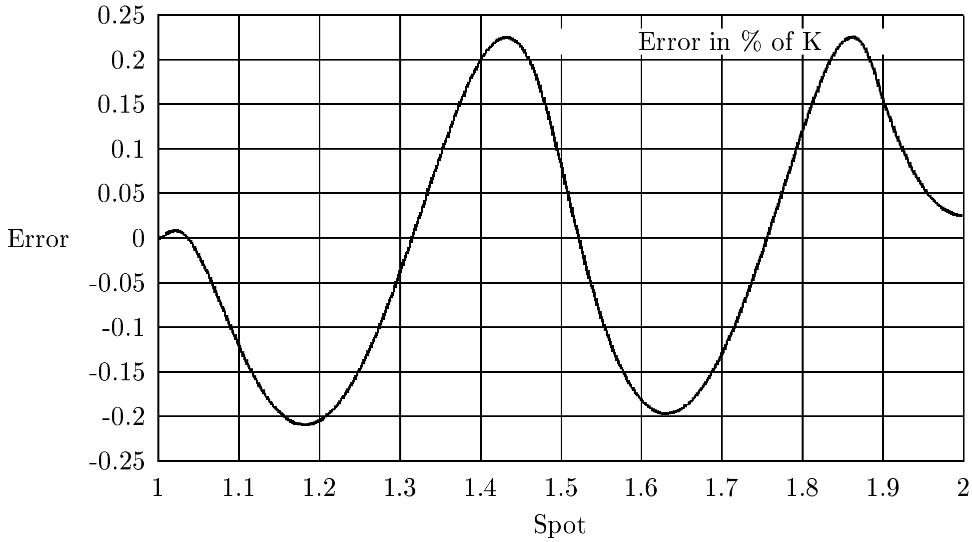
4.7 Numerical Results

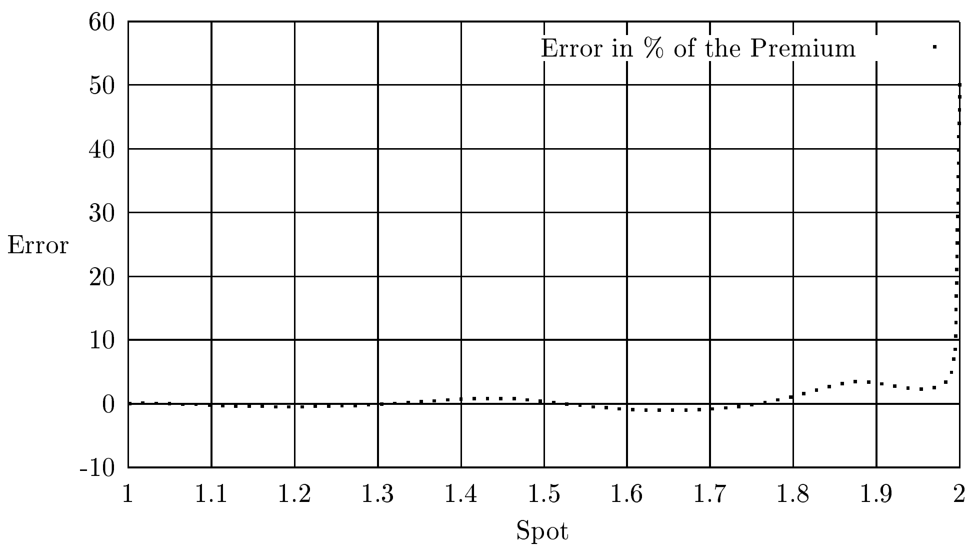
Let us first plot err^* as a function of α, expressed in percentage of the strike K :



The fact that this plot is decreasing corresponds to the fact that the size of the range $]K^*, K[$ increases as α decreases, whereas our family of approximating payoffs does not get richer as α decreases. It seems that at least for values of α not too small, this error is relevant in practice.

Here are now the difference $D(x) = \widehat{\varphi}^*(x) - (k-x)^+$ for $\alpha = 1$, in percentage of the strike $k = 2$ and next of the premium at maturity (i.e. $(k-x)^+$):





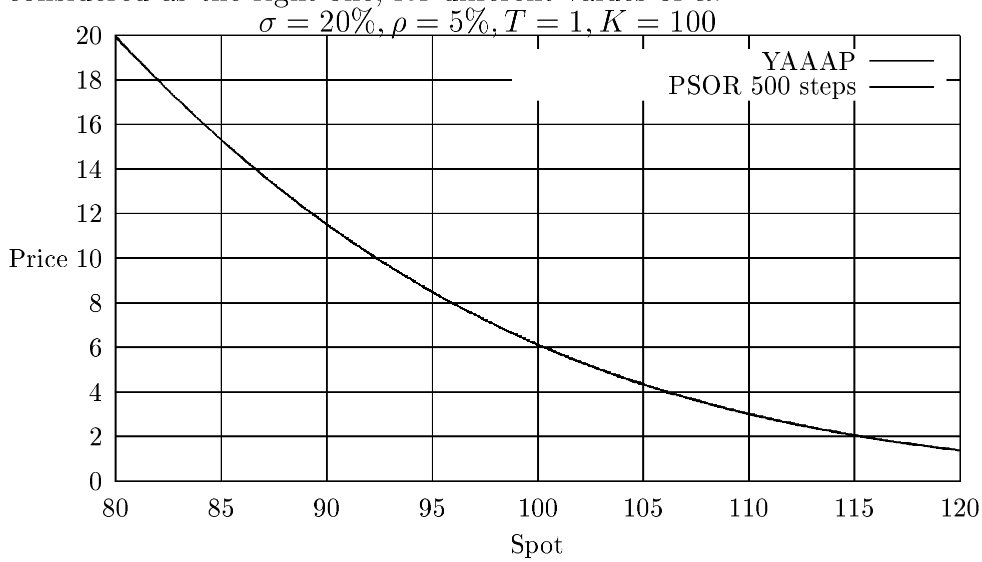
The price error will be much smaller since err^* is the maximal error over the underlying and since it will be smoothed by the probability law of the spot value at the time the free boundary is reached and reduced by the corresponding discounting factor. More precisely, if τ_{opt} and τ_* denote respectively the entrance times in the exercise regions of the American Put option and of the American option with payoff $\widehat{\varphi}^*$, then

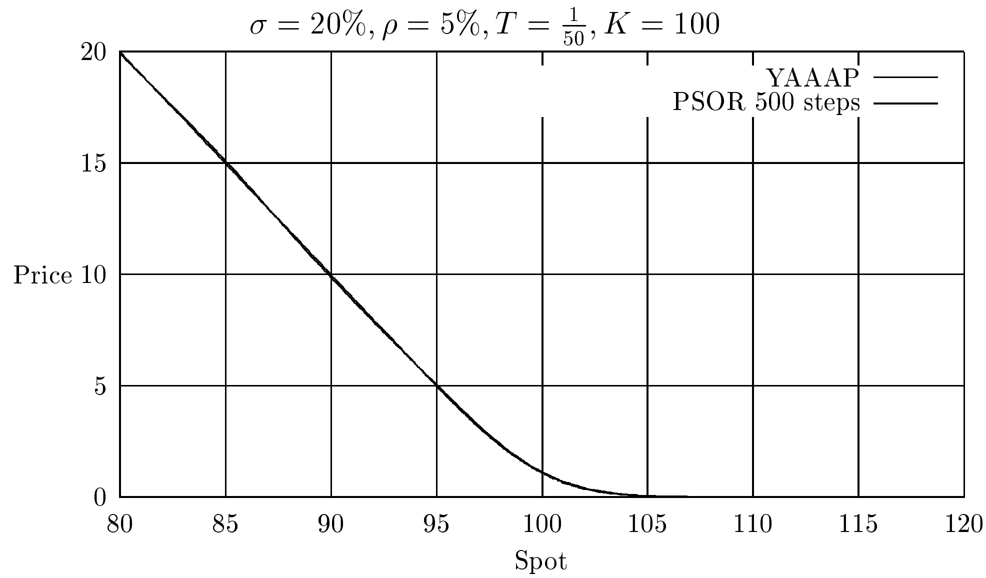
$$v^{am}_{Put}(t,x) = \mathbb{E}\left[e^{-\rho\tau_{opt}}(k - X^x_{\tau_{opt}})^+\right] \quad \text{and} \quad v^{am}_{\widehat{\varphi}^*}(t,x) = \mathbb{E}\left[e^{-\rho\tau_*}\widehat{\varphi}^*(X^x_{\tau_*})\right]$$

and as τ_{opt} and τ_* are optimal stopping times, we easily check that

$$v^{am}_{\widehat{\varphi}^*}(t,x) - \mathbb{E}\left[e^{-\rho\tau_*}D(X^x_{\tau_*})\right] \leq v^{am}_{Put}(t,x) \leq v^{am}_{\widehat{\varphi}^*}(t,x) + \mathbb{E}\left[e^{-\rho\tau_{opt}}(-D(X^x_{\tau_{opt}}))\right].$$

The larger the maturity, the more effective the smoothing of the error. The next plots show the comparison with a heavy finite-difference method (PSOR algorithm) with a large number of steps (500), so that the yielded price may be considered as the right one, for different values of α.





5 Conclusion: practical considerations

In this paper, we apply the theoretical result in [1] to the pricing of the American Put in the Black-Scholes model. We get a closed-formula for a payoff which is very close to the Put payoff. Let us insist on some remarkable features of our approximation: unlike many other kind of numerical approximation methods there is a hedge ratio associated to our price, which can be computed through the same type of almost-closed formula. Moreover, the YAAAP prices and deltas are the exact Black-Scholes American prices and deltas of a contingent claim the payoff of which matches the Put payoff below K^* and above K, is analytic within the range $]K^*, K[$, has the right first derivative -1 at K^*_+ and K_-, and lastly which deviates at most of err^* from the Put payoff within $]K^*, K[$. Therefore a safe way of making use of our approximation method is to trade the corresponding sub- and super-strategies with the YAAAP deltas and the selling price YAAAP price$+err^*$, buying price price-err^*, which leaving aside discrete-time hedging and model errors considerations will allways yield a non-negative Profit&Loss. Remember that err^* is less than 0.15% of the strike as soon as $\frac{2\rho}{\sigma^2}$ is greater than 2.

Because of the oscillating behavior of the difference $\widehat{\varphi}(x) - (K-x)^+$, in case of the trading of a portfolio of Puts spread across different strikes it is likely that the YAAAP prices may be used directly since the Profit&Loss for the different strikes will compensate eachother.

A Proof of lemma 2

Indeed by (4):

$$\varphi(x) = bx^{-\alpha} - \frac{2}{\sigma^2(\alpha+1)} x^{-\alpha} \int_0^K (r \wedge x)^{\alpha+1} \frac{m(dr)}{r^2}$$

where by (6) $b = \frac{2}{\sigma^2(\alpha+1)} \int_0^K r^{\alpha-1} m(dr)$. Therefore

$$\varphi(x) = \frac{2}{\sigma^2(\alpha+1)} x^{-\alpha} \int_0^K [r^{\alpha-1} - \frac{(r \wedge x)^{\alpha+1}}{r^2}] m(dr)$$

Now $m = \frac{\sigma^2 K^2}{2} \delta_K(dr) - 1_{]0,K[}(r) \frac{\sigma^2(\alpha+1)K^*}{2} h$, whence

$$\varphi(x) = \frac{x^{-\alpha}}{(\alpha+1)} [K^{\alpha+1} - (K \wedge x)^{\alpha+1}] - x^{-\alpha} K^* \int_0^K [r^{\alpha-1} - \frac{(r \wedge x)^{\alpha+1}}{r^2}] h(dr)$$

For $x < K$

$$\frac{x^\alpha \varphi(x)}{K^{\alpha+1} - (K \wedge x)^{\alpha+1}} = \frac{1}{(\alpha+1)} - K^* \int_0^K \frac{r^{\alpha+1} - (r \wedge x)^{\alpha+1}}{K^{\alpha+1} - (K \wedge x)^{\alpha+1}} \frac{h(dr)}{r^2}$$

Now $\frac{r^{\alpha+1} - (r \wedge x)^{\alpha+1}}{K^{\alpha+1} - (K \wedge x)^{\alpha+1}} \leq \frac{r^{\alpha+1}}{K^{\alpha+1}}$, plugging $\int_0^K r^{\alpha-1} h(dr) = (K^\alpha - K^{*\alpha})/\alpha$ we get

$$\frac{x^\alpha \varphi(x)}{K^{\alpha+1} - (K \wedge x)^{\alpha+1}} \geq \frac{1}{(\alpha+1)} (\frac{\alpha}{\alpha+1})^\alpha > 0$$

and φ is non-negative.

B Behaviour of the European price as the maturity goes to $+\infty$

We prove here the following:

Proposition 19 *Let $\varphi : \mathbb{R}_+^* \to \mathbb{R}$ a measurable function such that*

$$\sup_{x>0} |\varphi|(x)/(x + x^{\alpha}) < +\infty$$

If $a = \lim_{x\to\infty} \frac{\varphi(x)}{x}$ and $b = \lim_{x\to 0^+} \frac{\varphi(x)}{x^{-\alpha}}$ exist and are finite, then

$$\lim_{t\to\infty} v_{\varphi}(t,x) = ax + bx^{-\alpha}$$

Proof.

$$\begin{aligned} v_{\varphi}(t,x) &= e^{-\rho t}\mathbb{E}[\varphi(X_t^x)] \\ &= e^{-\rho t}\mathbb{E}\left[\frac{\varphi(X_t^x)}{X_t^x + (X_t^x)^{-\alpha}}\left(X_t^x + (X_t^x)^{-\alpha}\right)\right] \\ &= x\mathbb{E}\left[\frac{\varphi(X_t^x)}{X_t^x + (X_t^x)^{-\alpha}}\ e^{-\rho t}X_t^1\right] + x^{-\alpha}\mathbb{E}\left[\frac{\varphi(X_t^x)}{X_t^x + (X_t^x)^{-\alpha}}\ e^{-\rho t}\left(X_t^1\right)^{-\alpha}\right] \end{aligned}$$

Since $e^{-\rho t}X_t^1 = e^{\sigma B_t - \frac{\sigma^2}{2}t}$, by Girsanov theorem the first term is equal to $x\mathbb{E}^{\widetilde{P}}\left[\frac{\varphi(Y_t^x)}{Y_t^x + (Y_t^x)^{-\alpha}}\right]$ where $Y_t^x = xe^{\rho t + \sigma(B_t - \sigma t) + \sigma^2 t - \frac{\sigma^2}{2}t} = xe^{\rho t + \sigma\widetilde{B}_t + \frac{\sigma^2}{2}t}$ and $\widetilde{B}$ is a $\widetilde{P}$ Brownian motion. In particular $\widetilde{P}$ a.s., $e^{\rho t + \sigma\widetilde{B}_t + \frac{\sigma^2}{2}t} \to \infty$ as $t \to \infty$. Therefore by Lebesgue theorem $\lim_{t\to\infty}\mathbb{E}^{\widetilde{P}}\left[\frac{\varphi(Y_t^x)}{Y_t^x + (Y_t^x)^{-\alpha}}\right] = \lim_{y\to\infty}\frac{\varphi(y)}{y}$.

In the same way $e^{-\rho t}\left(X_t^1\right)^{-\alpha} = e^{-\alpha\sigma B_t - \frac{\alpha^2\sigma^2}{2}t}$ is a martingale and the second term is equal to

$$x^{-\alpha}\mathbb{E}^{\widetilde{P}}\left[\frac{\varphi\left((Z_t^x)^{-\frac{1}{\alpha}}\right)}{Z_t^x + (Z_t^x)^{-\frac{1}{\alpha}}}\right] \quad \text{where} \quad Z_t^x = x^{-\alpha}e^{\rho t + \alpha\sigma\widetilde{B}_t + \frac{\alpha^2\sigma^2}{2}t}.$$

Therefore it goes to $x^{-\alpha}\lim_{y\to\infty}\frac{\varphi\left(y^{-\frac{1}{\alpha}}\right)}{y + y^{-\frac{1}{\alpha}}} = x^{-\alpha}\lim_{y\to 0}\frac{\varphi(y)}{y^{-\alpha}}$. ■

C Proof of lemma 3

One has

$$\begin{aligned} e^{\rho t}v_{e_r}(t,x) &= r^{\alpha+1}x^{-\alpha}e^{-\alpha\left(\rho - \frac{\sigma^2}{2}\right)t}\mathbb{E}\left[\exp(-\alpha\sigma B_t)\,1_{\{\alpha\sigma B_t > \alpha l(x)\}}\right] \\ &\quad + xe^{\left(\rho - \frac{\sigma^2}{2}\right)t}\mathbb{E}\left[\exp(\sigma B_t)\,1_{\{\sigma B_t < l(x)\}}\right] \end{aligned}$$

where

$$l(x) = \ln\left(\frac{r}{x}\right) - \left(\rho - \frac{\sigma^2}{2}\right)t$$

Since $\alpha\frac{\sigma^2}{2} = \rho$ and $\alpha\rho = \frac{(\alpha\sigma)^2}{2}$,

$$\begin{aligned} v_{e_r}(t,x) &= r^{\alpha+1}x^{-\alpha}e^{\frac{(\alpha\sigma)^2}{2}t}\mathbb{E}\left[\exp(-\alpha\sigma B_t)\, 1_{\{\alpha\sigma B_t > \alpha l(x)\}}\right] \\ &\quad + xe^{-\frac{\sigma^2}{2}t}\mathbb{E}\left[\exp(\sigma B_t)\, 1_{\{\sigma B_t < l(x)\}}\right] \end{aligned}$$

Now,

$$\begin{aligned} e^{-\frac{\gamma^2}{2}t}\mathbb{E}\left[\exp(-\gamma B_t)\, 1_{\{\gamma B_t > \beta\}}\right] &= e^{-\frac{\gamma^2}{2}t}\int_\beta^\infty e^{-z}e^{-\frac{z^2}{2\gamma^2 t}}\frac{dz}{\sqrt{2\pi\gamma^2 t}} \\ &= \int_\beta^\infty e^{-\frac{\left(z+\gamma^2 t\right)^2}{2\gamma^2 t}}\frac{dz}{\sqrt{2\pi\gamma^2 t}} \\ &= N\left(-\left(\frac{\beta+\gamma^2 t}{\sqrt{\gamma^2 t}}\right)\right) \end{aligned}$$

In the same way

$$e^{-\frac{\gamma^2}{2}t}\mathbb{E}\left[\exp(\gamma B_t)\, 1_{\{\gamma B_t < \beta\}}\right] = N\left(\frac{\beta - \gamma^2 t}{\sqrt{\gamma^2 t}}\right)$$

Whence

$$v_{e_r}(t,x) = r^{\alpha+1}x^{-\alpha}N\left(-\left(\frac{\alpha\sigma l(x) + (\alpha\sigma)^2 t}{\sqrt{(\alpha\sigma)^2 t}}\right)\right) + xN\left(\frac{\sigma l(x) - \sigma^2 t}{\sqrt{\sigma^2 t}}\right)$$

where $l(x) = \ln\left(\frac{r}{x}\right) - \left(\rho - \frac{\sigma^2}{2}\right)t = \ln\left(\frac{r}{x}\right) - \left(\frac{\alpha-1}{2}\right)(\sigma^2 t)$ so that

$$\begin{aligned} v_{e_r}(t,x) &= r^{\alpha+1}x^{-\alpha}N\left(-\left(\frac{\alpha\ln\left(\frac{r}{x}\right) - \alpha\left(\frac{\alpha-1}{2}\right)(\sigma^2 t) + \alpha^2(\sigma^2 t)}{\sqrt{\alpha^2(\sigma^2 t)}}\right)\right) \\ &\quad + xN\left(\frac{\ln\left(\frac{r}{x}\right) - \left(\frac{\alpha-1}{2}\right)(\sigma^2 t) - (\sigma^2 t)}{\sqrt{\sigma^2 t}}\right) \end{aligned}$$

$$= r^{\alpha+1}x^{-\alpha}N\left(-\left(\frac{\ln\left(\frac{r}{x}\right)+\left(\frac{\alpha+1}{2}\right)\sigma^2 t}{\sqrt{\sigma^2 t}}\right)\right)$$
$$+xN\left(\frac{\ln\left(\frac{r}{x}\right)-\left(\frac{\alpha+1}{2}\right)\sigma^2 t}{\sqrt{\sigma^2 t}}\right)$$

D Computation of $\widehat{\varphi}'$ for $K^* < x < K$

For $y \in]a,b[$, $\widehat{\varphi}(e^y) = P\varphi(\lambda^*(y),y)$ is given in Proposition 10. Since $\partial_\lambda v_\varphi(\lambda^*(y),y) = -\frac{1}{(\sigma\lambda^*(y))^2}\partial_t v_\varphi(\lambda^*(y),y) = 0$, derivation with respect to y yields :

$$\begin{aligned}
e^{-a}\widehat{\varphi}'(e^y) &= -e^{-y}e^{\alpha(a-y)} + e^{-y}e^{\alpha(b-y)}N\left(-(b-y)\sqrt{\lambda}-\left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right) \\
&\quad -\frac{e^{-y}e^{\alpha(b-y)}}{\alpha}\sqrt{\lambda}N'\left(-(b-y)\sqrt{\lambda}-\left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right) \\
&\quad -\frac{e^{-b}}{\alpha}N\left((b-y)\sqrt{\lambda}-\left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right) \\
&\quad +\frac{e^{-b}}{\alpha}\sqrt{\lambda}N'\left((b-y)\sqrt{\lambda}-\left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right) \\
&\quad -e^{-(\alpha+1)y}e^{\frac{(\alpha-1)b}{2}}\int_{-\infty}^{b} e^{\frac{(\alpha+1)u}{2}}\widetilde{h}(du)\,N\left(-(u-y)\sqrt{\lambda}-\left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right) \\
&\quad +\frac{e^{-(\alpha+1)y}}{\alpha}e^{\frac{(\alpha-1)b}{2}}\int_{-\infty}^{b} e^{\frac{(\alpha+1)u}{2}}\widetilde{h}(du)\,\sqrt{\lambda}N'\left(-(u-y)\sqrt{\lambda}-\left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right) \\
&\quad +\frac{e^{\frac{(\alpha-1)b}{2}}}{\alpha}\int_{-\infty}^{b} e^{-\frac{(\alpha+1)u}{2}}\widetilde{h}(du)\,N\left((u-y)\sqrt{\lambda}-\left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right) \\
&\quad -\frac{e^{\frac{(\alpha-1)b}{2}}}{\alpha}\int_{-\infty}^{b} e^{-\frac{(\alpha+1)u}{2}}\widetilde{h}(du)\,\sqrt{\lambda}N'\left((u-y)\sqrt{\lambda}-\left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right)
\end{aligned}$$

where for notation simplicity λ stands for $\lambda^*(y)$.
Since

$$\sqrt{2\pi}N'\left(-(z-y)\sqrt{\lambda}-\left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right) = e^{\frac{(\alpha+1)(y-z)}{2}}e^{-\frac{1}{2}\left(\frac{\alpha+1}{2\sqrt{\lambda}}\right)^2}e^{-\frac{\lambda}{2}(z-y)^2},$$

using the definition of $\lambda^*(y)$, we obtain that the sum of the third and the seventh terms of the r.h.s. is nil. Similarly the sum of the fifth and the ninth terms is nil.

E Proof of lemma 18

If φ is non-increasing then $\forall t \geq 0$, $x \to v_\varphi(t,x)$ is non-increasing. Since $\widehat{\varphi}(x) = \inf_{t\geq 0} v_\varphi(t,x)$, the same property holds for $\widehat{\varphi}$ and for the modified payoff $\widehat{\varphi}_h$.
Therefore, we are going to study the monotony of φ. Let $x < K$. We recall that $(x^\alpha \varphi(x))' = -\frac{2}{\sigma^2} x^\alpha \int_x^K \frac{m(dr)}{r^2} = -x^\alpha + x^\alpha \alpha K \int_x^K \frac{h(dr)}{r^2}$. As

$$\varphi(x) = (K-K^*)\left(\frac{K^*}{x}\right)^\alpha - \frac{x}{\alpha+1} + K^*\left(x\int_x^K \frac{h(dr)}{r^2} + x^{-\alpha}\int_0^x r^{\alpha-1}h(dr)\right),$$

and $1/(\alpha+1) = (K-K^*)/K$, we deduce that

$$x^\alpha \varphi'(x) = (K-K^*)\left(x^\alpha \alpha \int_x^K \frac{h(dr)}{r^2} - \frac{x^\alpha}{K} - \alpha\frac{K^{*\alpha}}{x}\right) - \alpha\frac{K^*}{x}\int_0^x r^{\alpha-1}h(dr).$$

We upper-bound $\int_x^K \frac{h(dr)}{r^2}$ thanks to the second moment assumption on h :

$$\int_x^K \frac{h(dr)}{r^2} \leq x^{-\frac{\alpha+1}{2}} \int_0^K r^{\frac{\alpha-3}{2}} h(dr) = \frac{x^{-\frac{\alpha+1}{2}} K^{\frac{\alpha-1}{2}}}{\alpha}.$$

Combining this inequality with $x^\alpha/K + \alpha K^{*\alpha}/x \geq 2x^{\frac{\alpha-1}{2}}\sqrt{\alpha K^{*\alpha}/K}$ we obtain

$$x^\alpha \varphi'(x) \leq (K-K^*) x^{\frac{\alpha-1}{2}} K^{\frac{\alpha-1}{2}} \left(1 - 2\sqrt{\alpha (K^*/K)^\alpha}\right).$$

Hence φ is non-increasing as soon as $4\alpha\left(\frac{\alpha}{\alpha+1}\right)^\alpha \geq 1$. It is easy to check that the function $\alpha \in]0,+\infty[\to f(\alpha) = 4\alpha\left(\frac{\alpha}{\alpha+1}\right)^\alpha$ is increasing. Since $f(1/2) = \sqrt{4/3} > 1$ and $\lim_{\alpha\to 0} f(\alpha) = 0$, the equation $f(\alpha) = 1$ has a unique solution α_0. Moreover $\alpha_0 \leq 1/2$ and $\forall \alpha \geq \alpha_0$, φ is non-increasing.

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