Rate of convergence of the discrete time hedging strategy in a complete multidimensional model

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March 3, 2000

Abstract

The aim of this paper is to compute the quadratic error of a discrete time hedging strategy in a complete multidimensional model. This results generalizes that of [3] and [9]. More precisely, our basic assumption is that the asset prices satify the $d-$dimensional stochastic differential equation $dX_t^i = X_t^i(b^i(X_t)dt + \sigma_{i,j}(X_t)dW_t^j)X_t^i$. We wish to analyse the risk of this strategy w.r.t. n, the number of discrete times of re-balancinga and we show that the error decreases as $1/\sqrt{n}$ for any options with lipschitz payoff and $1/n^{\frac{1}{4}}$ for digital options.

KEY WORDS: Discrete time hedging, approximation of stochastic integral, rate of convergence, Malliavin calculus.

1 Introduction

Our model of the market is the following: let S_t^0 denote the price of a non risky asset. It is subject to the ordinary differential equation

$$dS_t^0 = rS_t^0 dt.$$

Here $(W_t)_{t\geq 0}$ stands for a $d-$dimensional Brownian motion, and X_t^i, $i = 1, \ldots, d$ for the d risky assets prices at time t. They fullfil the stochastic differential equation

$$\begin{cases} dX_t^i &= X_t^i \left(\mu^i(X_t)dt + \sum_{j=1}^{d} \sigma_{i,j}(X_t)dB_t^j \right), \; i = 1, \ldots, d, \\ X_0^i &= x^i. \end{cases}$$

Let ψ be the $\mathcal{C}^\infty$ diffeomorphism from $\mathbb{R}^d$ to ${\mathbb{R}_+^*}^d$ given by $\psi(x^1, \ldots, x^d) = (\exp(x^1), \ldots, \exp(x^d))$. The requirements on μ and σ are

(H1) For all $j = 1, \ldots, d$, let σ^j be the jth column of the matrix σ. If $\hat{\sigma}^j(x) = \sigma^j(\psi(x))$ and $\hat{\mu}(x) = \mu(\psi(x))$ then $\hat{\mu}$ and $\hat{\sigma}^j$ belongs to $\mathcal{C}_b^3(\mathbb{R}^d, \mathbb{R}^d)$.

(H2) The matrix $a = \sigma\sigma^*$ is uniformly elliptic, i.e. there exists $\sigma_0 > 0$ such that, for all $x \in \mathbb{R}^d$, we have

$$\sigma(x)\sigma^*(x) \geq \sigma_0^2 I_{\mathbb{R}^d \otimes \mathbb{R}^d}.$$

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The assumption **(H2)** now allow us to apply the Girsanov theorem. Consequently, we are able to find an equivalent probability such that the discounted prices are matingales. It is call the neutral risk probability and denoted by $\mathbb{P}$. Under it, the process

$$W_t = B_t - \int_0^t \sigma(X_u)^{-1}(rI - \mu(X_u))du,$$

is a Brownian motion and the assets prices satisfy

$$X_t^i = x^i + \int_0^t rX_s^i ds + \int_0^t \sum_{j=1}^d \sigma_{i,j}(X_s)dW_s^j, \quad i = 1, \dots, d. \tag{1}$$

In the following, we consider European vanilla options with payoff function $f \in L^2(X_T)$. Mathematically, the prices of these options are given by

$$h(f) = \mathbb{E}\left(\exp(-rT)f(X_T)\right).$$

If we set

$$u(t,x) \ = \ \mathbb{E}_x\left(e^{-r(T-t)}f(X_{T-t})\right), \tag{2}$$

note that $h(f)$ is equal to $u(0,x)$ and that u solves the Cauchy problem:

$$-\frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\sum_{i,j=1}^d a_{i,j}(x)x^i x^j \frac{\partial^2}{\partial x^i \partial x^j}u(t,x) + \quad r\sum_{i=1}^d x^i \frac{\partial}{\partial x^i}u(t,x) - ru(t,x) \tag{3}$$

$$\text{with } (t,x) \in [0,T) \times (0,\infty)^d$$

$$\lim_{t\to T} u(t,x) = f(x) \qquad \text{for } x \in (0,\infty)^d.$$

It is well known that

$$e^{-rT}f(X_T) = h(f) + \int_0^T \langle \xi_t \ , \ d\widetilde{X}_t \rangle,$$

where $\widetilde{X_t} = e^{-rt}X_t$ is the discounted price of the risky asset. Itô's formula implies that the delta hedging strategy ξ is given by

$$\xi_t^i = \frac{\partial u}{\partial x^i}(t, X_t) \quad i = 1, \dots, d.$$

In other words, to have a perfect hedging, the seller of the option must trade at each time $t \in [0,T]$ and hold ξ_t^i units of the asset X_t^i. In practice, this is impossible.
An alternative solution is to hedge only at discrete times. In fact, assume that the investor will trade at n fixed times in the period $[0,T]$. At each trading times defined by $t_k = kT/n$ ($k \in \{0, \dots, n\}$), the trader holds $\xi_{t_k}^i$ units of the asset X_t^i. Hence, at maturity the seller of the option will be left with the difference:

$$\begin{aligned}\Delta_n(f) \ &:= \ e^{-rT}f(X_T) - \left(u(0,x) + \int_0^T \sum_{i=1}^d \frac{\partial u}{\partial x^i}(\varphi(t), X_{\varphi(t)})d\widetilde{X_t^i}\right) \\ &= \ \left(\int_0^T \sum_{i=1}^d \frac{\partial u}{\partial x^i}(t, X_t)d\widetilde{X_t^i} - \int_0^T \sum_{i=1}^d \frac{\partial u}{\partial x^i}(\varphi(t), X_{\varphi(t)})d\widetilde{X_t^i}\right),\end{aligned}$$

where $\varphi(t) = \sup\{t_i \ | \ t_i \le t\}$.
Our purpose is to study the convergence of $\Delta_n(f)$ to 0 when n goes to infinity, in different assumptions on f.

2 Results

We will study the risk incurred by the trader in evaluating the variance of $\Delta_n(f)$ when the time number of re-balancing goes to infinity. It would be desirable to study other criterion of convergence, but the choice of the variance makes computations easier. Nevertheless, we also give a result about weak convergence.
Zhang ([9]) has found the rate of convergence for C^2 functions in a general model or for the European call and put in the Black and Scholes model. Here, we prove the exact rate of convergence in the generalized Black and Scholes model described by (1) in the two following cases

- Let f satisfy the assumption

  (H3) f is lipschitz. f belongs to $\mathcal{H}_1$ the space of Lipschitz functions which is a Banach space with the norm

$$\|f\|_1 = \sup_{(x,x')\in\mathbb{R}^d, x\neq x'} \frac{|f(x)-f(x')|}{|x-x'|} + \sup_{x\in\mathbb{R}^d} \frac{|f(x)|}{|x|+1}.$$

  The theorem below ensures that the quadratic error vanishes when n goes to infinity at rate $1/\sqrt{n}$.

  Theorem 2.1. *Under the assumptions (H1), (H2) and (H3),*

$$\mathbb{E}\Delta_n^2(f) = \frac{T}{2n}\mathbb{E}\left(\int_0^T e^{-2rt}\, tr\left(\left(a(X_t)I_{X_t}J^u(t,X_t)I_{X_t}\right)^2\right)dt\right) + o\left(\frac{1}{n}\right).$$

  where

$$J^u(t,x) = \left(\frac{\partial^2 u}{\partial x^i \partial x^j}(t,x)\right)_{i,j=1,\dots,d}.$$

- A European call digital option with strike $K > 0$ and maturity T is a contingent claim which pays 1 if the price of the underlying risky asset lies above K at maturity and which pays nothing otherwise. If the underlying asset is an index, i.e. a linear combination of the X_t^i, the rate of convergence is $1/n^{\frac{1}{4}}$. Mathematically, we assume that

  (H4) $f(x) = \mathbf{1}_{\sum_k \lambda^k x^k \geq K}$ where $\sum_k \lambda^k = 1$ and $\forall k \in \{1,\dots,d\},\ \ \lambda^k > 0$.

  Theorem 2.2. Case $\mathcal{C}_0$

  Let assumptions (H1), (H2) and (H4) hold. Then,

$$\mathbb{E}\Delta_n^2(f) = \sqrt{\frac{T}{n}}\, C_0 \mathcal{D} e^{-2rT} + o\left(\frac{1}{\sqrt{n}}\right)$$

  where $\mathcal{D}$ is a constant defined in equation (25) depending only on the density of $\log(X_T)$ and $\lambda^k,\ k=1,\dots,d$ and K.

  Remark 2.1. *Some remarks on these theorems:*

  - *The constant $\mathcal{D}$ represents the probability for the process $\sum_k \lambda^k X_t^k$ to be near the strike at maturity.*
  - *The theorem 2.2 is still true if f can be written as $f(x) = C\ \mathbf{1}_{\sum_k \lambda^k x^k \geq K} + g(x)$, for some constants C and K, and for some function g of class C^1_{pol}, e.g. for the digital put $\mathbf{1}_{\sum_k \lambda^k x^k \leq K}$.*

- *The assumption* **(H3)** *can be weakened: f can be Hölder with coefficient strictly over $2-\sqrt{2}$. But, we conjecture that the theorem 2.1 can be extended to all Hölder functions of coefficient strictly over $1/2$.*
- *The above results still hold under the historical probability.*

It is interest to study the weak convergence of $\Delta_n(f)$. By a result of Rootzen ([8]), it is painfully seen that:

Theorem 2.3. *Let X_t be a $d-$dimensional diffusion, which satisfies $dX_t^i = \sigma_{i,j} X_t^i dW_t^j$. Then if u is defined as in (2), it follows that*

$$\sqrt{n}\Delta_n(f) \to_d \hat{W}_\tau, \quad n \to \infty, \tag{4}$$

where $\tau = \frac{1}{2}\int_0^T tr\left(\left(a(X_t) I_{X_t} J^u(t, X_t) I_{X_t}\right)^2\right) dt$, and $\hat{W}$ is an extra Brownian motion independent of τ.

In the sequel, $\mathcal{S}_n$ is the set $\{0, \dots, n\}^d$. If α is a multi-index which belongs to $\mathcal{S}_n$, $|\alpha| = \sum_{i=1}^{d} \alpha_i$ and if $\alpha = \emptyset$, $|\alpha| = 0$. If F is a smooth function $\partial_\alpha^x F(t, x, y)$ means that the multi-index α concerns the derivation w.r.t. the coordinates x, t and y being fixed. $K(\cdot)$ will always stands for a positive, finite and non decreasing map, independent of n and which can change throughout the calculus. For two integers i and j, $\delta_{i,j}$ is 1 if $i = j$ and 0 if not. For any vector $z \in \mathbb{R}^d$, $z^{(i)} = (z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n)$.

3 A general decomposition of the error

We begin by studying

$$\mathbb{E}[\Delta_n(f)]^2 = \mathbb{E}\left(\int_0^T \sum_{i=1}^{d} \frac{\partial u}{\partial x^i}(t, X_t) d\widetilde{X}_t^i - \int_0^T \sum_{i=1}^{d} \frac{\partial u}{\partial x^i}(\varphi(t), X_{\varphi(t)}) d\widetilde{X}_t^i\right)^2.$$

Using the Itô formula and the stochastic differential equation (1), we obtain the following proposition whose proof is postponed in Appendix A.

Proposition 3.1. *Let assumptions (H1) and (H2) hold. Then, the quadractic error $\mathbb{E}[\Delta_n(f)]^2$ can be written as*

$$\mathbb{E}[\Delta_n(f)]^2 = A_1 + A_2 + A_3,$$

where

$$A_1 = \int_0^T dt \int_{\varphi(t)}^{t} ds\, e^{-2rs} \left[\sum_{i,k,\alpha,\beta=1}^{d} X_s^\alpha X_s^\beta X_s^i X_s^k a_{i,k}(X_s) a_{\alpha,\beta}(X_s) \right.$$
$$\left. \frac{\partial^2 u}{\partial x^i \partial x^\alpha}(s, X_s) \frac{\partial^2 u}{\partial x^j \partial x^\beta}(s, X_s) \right],$$

$$
\begin{aligned}
A_2 = \int_0^T dt \int_{\varphi(t)}^t ds\, e^{-2rs} \Bigg[\sum_{i,j,k,\alpha,\beta=1}^d & \Bigg(x^\alpha x^\beta x^i a_{\alpha,\beta}(x) \sigma_{i,j}(x) \\
& \frac{\partial}{\partial x^\beta}(x^k \sigma_{k,j}(x)) \left(D_u^k(s) \frac{\partial^2 u}{\partial x^i \partial x^\alpha}(s,x) + 2 D_u^i(s) \frac{\partial^2 u}{\partial x^k \partial x^\alpha}(s,x) \right) \Bigg) \\
& - \sum_{i,k,\alpha,\beta=1}^d x^i x^k a_{i,k}(x) \frac{\partial}{\partial x^i}(x^\alpha x^\beta a_{\alpha,\beta}(x)) D_u^k(s) \frac{\partial^2 u}{\partial x^\alpha \partial x^\beta}(s,x) \Bigg) \Bigg] \Bigg|_{x=X_s},
\end{aligned}
$$

and

$$
\begin{aligned}
A_3 = \int_0^T dt \int_{\varphi(t)}^t ds\, e^{-2rs} \Bigg[2 \sum_{i,k=1}^d D_u^k(s) D_u^i(s) x^i x^k a_{i,k}(x) + \sum_{i,k,j,\alpha,\beta=1}^d D_u^i(s) D_u^k(s) x^\alpha x^\beta \\
a_{\alpha,\beta}(x) \Big(\frac{1}{2} \frac{\partial}{\partial x^\alpha}(x^i \sigma_{i,j}(x)) \frac{\partial}{\partial x^\beta}(x^k \sigma_{k,j}(x)) + x^k \sigma_{k,j}(x) \frac{\partial^2}{\partial x^\alpha \partial x^\beta}(x^i \sigma_{i,j}(x)) \Big) \Bigg] \Bigg|_{x=X_s}.
\end{aligned}
$$

Remark 3.1. *It is immediate that $x^\alpha \frac{\partial}{\partial x^\alpha}(x^i \sigma_{i,j}(x)) = x^i x^\alpha \frac{\partial \sigma_{i,j}}{\partial x^\alpha}(x) + x^i \delta_{i,\alpha} \sigma_{i,j}(x)$. Consequently, in the above expression of A_2 and A_3, each term which includes a derivative of u w.r.t. x^l for an integer l, includes also x^l.*

Our next goal is to show that A_1 is the main term in the decomposition of $\mathbb{E}[\Delta_n(f)]^2$ whereas A_2 and A_3 are negligible. We set now $Y_t = \psi^{-1}(X_t)$. This process satisfies the following stochastic differential system

$$
Y_t^i = \log(x^i) + \int_0^t \left(r - \frac{1}{2} a_{i,i}(\psi(Y_s)) \right) ds + \sum_{j=1}^d \int_0^t \sigma_{i,j}(\psi(Y_s))\, dW_s^j, \quad i = 1, \ldots, d. \tag{5}
$$

We also put

$$
y = \psi^{-1}(x);\ s(y) = \sigma(\psi(y));\ S(y) = s(y)s(y)^*;\ b(y) = r - \frac{1}{2} a_{i,i}(\psi(y)).
$$

As σ, s is uniformly elliptic. Assumption **(H1)** yields that the functions b and s belongs to $\mathcal{C}_b^5(\mathbb{R}^d)$. Let us first prove the theorem 2.1.

4 Proof of theorem 2.1

4.1 Preliminary estimates

We first recall some properties of the transition density of elliptic process.

Proposition 4.1. (Friedman, [2], Chapter 6) *Under the assumptions $(H1)$ and $(H2)$, for $t > 0$, the process $Y_t(y)$ has a smooth transition density $p_t(y, \cdot)$ w.r.t. the Lebesgue measure on $\mathbb{R}^d$, which fullfil:*

- *$\forall t > 0$, $p_t(\cdot, \cdot)$ belongs to $\mathcal{C}^4(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R})$*
- *$\forall \alpha, \beta \in \mathcal{S}^4$ such as $|\alpha| + |\beta| \leq 4$, there exist a function $K(T)$ and a constant $c > 0$, such that:*

$$
\forall (t, y, y') \in (0, T] \times \mathbb{R}^d \times \mathbb{R}^d \qquad \left| \frac{\partial^{|\alpha|+|\beta|} p_t}{\partial y^\alpha \partial y'^\beta}(y, y') \right| \leq \frac{K(T)}{t^{\frac{|\alpha|+|\beta|+d}{2}}} e^{-c\frac{|y-y'|^2}{t}}. \tag{6}
$$

If $v(t,y)$ is the function defined on $[0,T]\times\mathbb{R}^d$ by $v(t,y)=u(t,\psi(y))$, it is the solution of the Cauchy problem

$$-\frac{\partial}{\partial t}v(t,y)=\frac{1}{2}\sum_{i,j=1}^{d}S_{i,j}(y)\frac{\partial^2}{\partial y^i\partial y^j}v(t,y)+\sum_{i=1}^{d}b^i(y)\frac{\partial v}{\partial y^i}(t,y)-rv(t,y) \qquad \text{with } (t,y)\in[0,T)\times\mathbb{R}^d \tag{7}$$

$$v(T,y)=f(\psi(y)) \qquad \text{for } y\in\mathbb{R}^d.$$

4.2 Main estimates in the case of Lipschitz function

Proposition 4.2. *Suppose (H1), (H2) and (H3) hold. Then, there exists $K(\cdot)$ such as for all integers $i,j\in\{1,\dots,d\}$, for all $(t,y)\in(0,T]\times\mathbb{R}^d$, it holds that*

$$\left|\frac{\partial v}{\partial y^i}(t,y)\right|\le K(T)e^{|y|}, \tag{8}$$

and,

$$\mathbb{E}_{y_0}\left(\frac{\partial^2 v}{\partial y^i\partial y^j}(t,Y_t)\right)^2\le\frac{K(T)e^{2|y_0|}}{\sqrt{T}\sqrt{T-t}}. \tag{9}$$

Proof. Our proof starts with the observation that $f(\psi(y))\partial_{y^i}\int_{\mathbb{R}^d}p_{T-t}(y,y')dy'=0$. From this and (6), it follows that

$$\begin{aligned}\left|\frac{\partial v}{\partial y^i}(t,y)\right| &= \left|\int_{\mathbb{R}^d}(f(\psi(y'))-f(\psi(y)))\frac{\partial p_{T-t}}{\partial y^i}(y,y')dy'\right|\\ &\le \frac{K(T)}{(T-t)^{\frac{d+1}{2}}}\int_{\mathbb{R}^d}|\psi(y')-\psi(y)|e^{-c\sum_k\frac{(y'^k-y^k)^2}{T-t}}dy'.\end{aligned}$$

Since $|\psi(y')-\psi(y)|\le|y'-y|e^{|y|}$, inequality (8) is proved. The same proof still goes for (9) when $t\le T/2$ (note that $\frac{1}{T-t}\le\frac{\sqrt{2}}{\sqrt{T(T-t)}}$ in this case). If $t\ge T/2$, starting as above, we get

$$\left|\frac{\partial^2 v}{\partial y^i\partial y^j}(t,y)\right|=\left|\int_{\mathbb{R}^d}(f(\psi(y'))-f(\psi(y)))\frac{\partial^2 p_{T-t}}{\partial y^i\partial y^j}(y,y')dy'\right|$$

Let us introduce the $\mathcal{C}^\infty$ functions f_η defined by

$$f_\eta(y)=\int_{\mathbb{R}^d}f(z+y)e^{-\frac{|z|^2}{2\eta}}\frac{dz}{(2\eta)^{\frac{d}{2}}}. \tag{10}$$

Here are some elementary properties of these functions. The proof of this lemma is postponed in appendix Appendix B.

Lemma 4.1. *The functions f_η are $\mathcal{C}^\infty$, and under (H3)*

- $\|f_\eta-f\|_1\le\sqrt{\eta}$,
- $\left\|\frac{\partial^2 f_\eta}{\partial y^i\partial y^j}\right\|_\infty\le\frac{1}{\eta^{\frac{1}{2}}}$

We divide the above expression of the second partial derivative of v in two terms:

$$\left|\frac{\partial^2 v}{\partial y^i \partial y^j}(t,y)\right| \leq \underbrace{\left|\int_{\mathbb{R}^d}((f-f_\eta)(\psi(y')) - (f-f_\eta)(\psi(y)))\frac{\partial^2 p_{T-t}}{\partial y^i \partial y^j}(y,y')dy'\right|}_{\gamma_1} + \underbrace{\left|\int_{\mathbb{R}^d} f_\eta(\psi(y'))\frac{\partial^2 p_{T-t}}{\partial y^i \partial y^j}(y,y')dy'\right|}_{\gamma 2}$$

Lemma 4.1 and estimate (6) implies

$$\gamma_1 \leq \|f_\eta - f\|_1 \int_{\mathbb{R}^d} \frac{|y'-y|}{(T-t)^{\frac{d+2}{2}}} e^{-c\frac{|y'-y|^2}{T-t}+|y|}dy' \leq K(T)e^{|y|}\frac{\eta^{\frac{1}{2}}}{(T-t)^{\frac{1}{2}}}.$$

To estimate the term γ_2, we write $v_\eta(t,y) = \mathbb{E}(f_\eta(Y_{T-t}))$. Since f_η is $\mathcal{C}^2$, we are now in position to differentiate underneath the expectation. Hence,

$$\gamma_2 \leq K(T)e^{|y|}\left\|\frac{\partial^2 f_\eta}{\partial y^i \partial y^j}\right\|_\infty \leq K(T)\frac{e^{|y|}}{\sqrt{\eta}}.$$

Therefore, it is easy to check that

$$\mathbb{E}_{y_0}\left(\frac{\partial^2 v}{\partial y^i \partial y^j}(t,Y_t)\right)^2 \leq K(T)e^{|y_0|}\left(\frac{\eta}{T-t}+\frac{1}{\eta}\right).$$

We choose of $\eta = \sqrt{T}\sqrt{T-t}$, and the lemma follows. $\square$

4.3 Terms A_1 to A_3

To prove that A_2 and A_3 are negligible w.r.t. the expected rate of convergence, we begin by proving

$$\forall i \in \{1,\dots,d\}, \quad \mathbb{E}\left[X_s^i D_u^i(s)\right]^2 \leq \frac{K(T)}{n(T-s)}. \tag{11}$$

Combining Itô's formula, applied between $\varphi(s)$ and s, inequalities (21), (22) and standard exponential estimates to upper bound $\mathbb{E}(|X_s|^p + |X_s|^{-p})$ gives (11). Thus, by (22) and again (11) we can assert that

$$|A_2| \leq \int_0^T dt \int_{\varphi(t)}^t ds \frac{K(T)}{(T-s)^{\frac{3}{4}}\sqrt{n}} = O\left(\frac{1}{n^{\frac{3}{2}}}\right).$$

To deal with A_3, applying (21) and (11) yields

$$|A_3| \leq \int_0^T dt \int_{\varphi(t)}^t ds \frac{K(T)}{n(T-s)} = O\left(\frac{\log(n)}{n^2}\right).$$

Having shown that A_2 and A_3 are negligable, we can now return to the study of A_1. To this end, consider the function g defined by

$$g(t) = e^{-2rt}\mathbb{E}_{y_0}\left[\sum_{i,k,\alpha,\beta=1}^d s_{i,k}(Y_t)s_{\alpha,\beta}(Y_t) \left(\frac{\partial^2 v}{\partial y^i \partial y^\alpha}(t,Y_t) - \delta_{i,\alpha}\frac{\partial v}{\partial y^i}(t,Y_t)\right)\left(\frac{\partial^2 v}{\partial y^j \partial y^\beta}(t,Y_t) - \delta_{j,\beta}\frac{\partial v}{\partial y^j}(t,Y_t)\right)\right].$$

We claim that g is integrable on $[0,T]$ and continous on $[0,T[$. Indeed, the first property is an immediae consequence of estimates (8) and (9). The second one is due to the proposition 4.1. The proof of theorem 2.1 is complete by showing the lemma below. $\square$

Lemma 4.2. *Let $g:\ [0,T]\mapsto\mathbb{R}$ be a continous function on $[0,T[$ and such that $\int_0^T|g(t)|dt<+\infty$, then*

$$\lim_{n\to+\infty} n\left(\int_0^T ds\int_{\varphi(s)}^s g(t)\,dt\right)=\frac{T}{2}\int_0^T g(t)dt.$$

Proof. The property is obvious when g is piece wise constant function. Therefore, since all piece wise continous functions are limits of a sequence of piece wise constant functions, the proof is complete. $\square$

5 Proof of theorem 2.2

5.1 Malliavin Calculus

We begin with general results on Malliavin Calculus. For a thorought treatment we refer the reader to Nualart ([1, 7]) or Ikeda-Watanabe ([4]). We will use Malliavin Calculus with the elliptic diffusion Y_t. This is well known and we refer to Kusuoka and Stroock ([5]) for more details and proofs.

Proposition 5.1. *Let $b(\cdot)$ a vector in $\mathbb{R}^d$ which belongs to $\mathcal{C}_b^\infty(\mathbb{R}^d,\mathbb{R}^d)$ and $\sigma(\cdot)$ a matrix $(d\times d)$ which belongs to $\mathcal{C}_b^\infty(\mathbb{R}^d,\mathbb{R}^d\otimes\mathbb{R}^d)$, we define the diffusion X_t which is solution of*

$$X_t=x+\int_0^t b(X_s)ds+\int_0^t\sigma(X_s)dW_s$$

where $(W_s)_{s\geq 0}$ is a $d-$dimensional Brownian motion, then for all $t>0$, $X_t\in\mathbb{D}^\infty$ and for all $p>1$, all $k\in\mathbb{N}^*$, there exists a non decreasing map $K(T)$ such that

$$\sup_{t\in[0,T]}\|X_t(x)\|_{k,p}\leq K(T)(1+\|x\|),\quad \sup_{t\in[0,T]}\|X_t(x)-x\|_{k,p}\leq\sqrt{T}. \tag{12}$$

Under the assumption of uniform ellipticity of σ, the Malliavin matrix of $X_t(x)$ denoted by $\gamma_t(x)$ is invertible a.s. and $\gamma_t^{-1}\in\cap_{p\geq 1}L^p$ spaces. Moreover, we have

$$\|\gamma_t^{-1}(x)\|_{L^q}\leq\frac{K(T)}{t^d}. \tag{13}$$

For all multi-index α of length positive, for all $p>1$, for all function f which belongs to $\mathcal{C}_b^{|\alpha|}(\mathbb{R}^d,\mathbb{R})$, and for all random variable $G\in\mathbb{D}^\infty$, there exist a random variable $H_\alpha(X_t,G)\in L^p$, a map $K(T)$ and integers k and m which only depend on $|\alpha|$ and p, such that

$$\mathbb{E}_x\left(\partial_\alpha f(X_t)G\right)=\mathbb{E}_x\left(f(X_t)H_\alpha(X_t,G)\right), \tag{14}$$

with

$$\left[\mathbb{E}_x\,|H_\alpha(X_t,G)|^p\right]^{\frac{1}{p}}\leq\frac{K(T)}{t^{\frac{|\alpha|}{2}}}\|G\|_{k,m}. \tag{15}$$

The proposition 4.1 and Malliavin Calculus will now enable us to control the derivative of v. To get precise control we need to show first the following proposition which compare the derivative of the density of the process Y_t w.r.t. forward variable with the one w.r.t. backward variable.

Proposition 5.2. *Under the assumptions (H1) and (H2), there exists $K(\cdot)$ such as for any multi-index $\alpha \in \mathcal{S}^4$ and $|\alpha| > 0$, one has*

$$\partial^y_\alpha p_t(y,y') - (-1)^{|\alpha|}\partial^{y'}_\alpha p_t(y,y') = g^\alpha_t(y,y') \tag{16}$$

with

$$|g^\alpha_t(y,y')| \leq \frac{K(T)}{t^{\frac{|\alpha|-1+d}{2}}} \exp\left(-c\frac{|y-y'|^2}{2}\right). \tag{17}$$

Proof. Let ρ_0 be a smooth and symmetric probability density function with a compact support in $(-1,1)$. For $\delta > 0$, and $\xi \in \mathbb{R}^d$, we define:

$$\rho_\delta(\xi) = \prod_{i=1}^d \frac{\rho_0\left(\frac{\xi^i}{\delta}\right)}{\delta}. \tag{18}$$

Since $p_t(\cdot,\cdot)$ is a smooth function,

$$\partial^{y'}_\alpha p_t(y,y') = \lim_{\delta\to 0} \int \rho_\delta(\xi)\partial^{y'}_\alpha p_t(y,y'+\xi)d\xi.$$

Thus, integrating by parts and noting that $\partial^{y'}_\alpha p_t(y,y'+\xi)$ vanish when ξ goes to infinity, one has

$$\partial^{y'}_\alpha p_t(y,y') = (-1)^{|\alpha|} \lim_{\delta\to 0} \mathbb{E}_y\left[(\partial_\alpha \rho_\delta)(Y_t - y')\right].$$

But $|\alpha| > 0$ now leads to

$$\rho_{\delta,y,y'}(\xi) = \prod_{i=1}^d \frac{1}{\delta}\left\{\rho_0\left(\frac{\xi^i}{\delta}\right)\mathbf{1}_{[y'^i - y^i > 0]} + \left[\rho_0\left(\frac{\xi^i}{\delta}\right) - 1\right]\mathbf{1}_{[y'^i - y^i \leq 0]}\right\}. \tag{19}$$

Hence,

$$\partial^{y'}_\alpha p_t(y,y') = (-1)^{|\alpha|} \lim_{\delta\to 0} \mathbb{E}_y\left[(\partial_\alpha \rho_{\delta,y,y'})(Y_t - y')\right].$$

ρ_δ is smooth, which implies that, for the term with a derivative w.r.t. y, it is allowed to differentiate underneath the expectation (see for instance [5]). From this, we deduce that

$$\partial^y_\alpha p_t(y,y') = \lim_{\delta\to 0}\left\{\underbrace{\mathbb{E}_y\left[\sum_{0<|\beta|\leq|\alpha|-1} (\partial_\gamma \rho_\delta)(Y_t - y')R(t,y)\right]}_{g_{t,\delta}(y,y')} + \mathbb{E}_y\left[\sum_{|\beta|=|\alpha|}(\partial_\beta\rho_\delta)(Y_t - y')\prod_{i=1}^d \frac{\partial^{\beta_i} Y_t}{\partial y^{\alpha_i}}\right]\right\},$$

where $R(t,y)$ is a polynomial function depending only on the flow of $Y_t(y)$ (remember that $Y_t(y)$ can be chosen as a C^4 diffeomorphism w.r.t. y). Therefore, combining the definition of $\rho_{\delta,y,y'}$ and with $|\alpha| > 0$, we have

$$\partial^y_\alpha p_t(y,y') - (-1)^{|\alpha|}\partial^{y'}_\alpha p_t(y,y') = \lim_{\delta\to 0} g_{t,\delta}(y,y') + \lim_{\delta\to 0}\mathbb{E}_y\left[\sum_{|\beta|=|\alpha|}(\partial_\beta \rho_{\delta,y,y'})(Y_t - y')\left(\prod_{i=1}^d \frac{\partial^{\beta_i}Y_t}{\partial y^{\alpha_i}} - \prod_{i=1}^d \delta_{\beta_i\alpha_i}\right)\right].$$

Define γ^0 as the multi-index of length d such that $\forall i \in \{1,\dots,d\}$ $\gamma^0_i = 1$, and set $\Phi_{\delta,y,y'}$ as

$$\Phi_{\delta,y,y'}(\xi) = \int_0^{\xi^1} \dots \int_0^{\xi^d} \rho_{\delta,y,y'}(\zeta) d\zeta, \tag{20}$$

the Malliavin integration by parts formula (14) yields

$$\begin{aligned}\partial^y_\alpha p_t(y,y') - (-1)^{|\alpha|}\partial^{y'}_\alpha p_t(y,y') = \lim_{\delta\to 0}\Bigg\{ \sum_{0<|\beta|\le|\alpha|-1} \mathbb{E}_y\left[\Phi_{\delta,y,y'}(Y_t - y')H_{\beta+\gamma^0}(Y_t, R(t,y))\right] \\ + \sum_{|\beta|=|\alpha|} \mathbb{E}_y\left[\Phi_{\delta,y,y'}(Y_t-y')H_{\beta+\gamma^0}\left(Y_t, \left(\prod_{i=1}^d \frac{\partial^{\beta_i} Y_t}{\partial y^{\alpha_i}} - \prod_{i=1}^d \delta_{\beta_i\alpha_i}\right)\right)\right]\Bigg\}.\end{aligned}$$

Thus, making δ tends to 0, we can rewrite the above expression as

$$\begin{aligned}&\partial^y_\alpha p_t(y,y') - (-1)^{|\alpha|}\partial^{y'}_\alpha p_t(y,y') \\ &\quad = g^\alpha_t(y,y') := \mathbb{E}_y\Bigg[\left(\prod_{i=1}^d \left\{\mathbf{1}_{[0,+\infty)}(Y^i_t - {y'}^i)\mathbf{1}_{[{y'}^i - y^i>0]} + \mathbf{1}_{(-\infty,0]}(Y^i_t - {y'}^i)\mathbf{1}_{[{y'}^i-y^i\le 0]}\right\}\right) \\ &\qquad \times \left(\sum_{0<|\beta|\le|\alpha|-1} H_{\beta+\gamma^0}(Y_t, R(t,y)) + \sum_{|\beta|=|\alpha|} H_{\beta+\gamma^0}\left(Y_t, \prod_{i=1}^d\left(\frac{\partial^{\beta_i}Y_t}{\partial y^{\alpha_i}} - \delta_{\beta_i\alpha_i}\right)\right)\right)\Bigg].\end{aligned}$$

Noting that $\frac{\partial^{\beta_i}Y_0}{\partial y^{\alpha_i}} = \delta_{\beta_i\alpha_i}$, and applying (12), (15) and standard computations for $\mathbb{P}[|Y_t - y| \ge |y-y'|]$, we obtain the expression below wich is the desired conclusion.

$$|g^\alpha_t(y,y')| \le \frac{K(T)}{t^{\frac{|\alpha|+d-1}{2}}}\exp\left(-c\frac{|y-y'|^2}{2t}\right).$$

□

5.2 Main estimates in digital case

Proposition 5.3. *Under the assumptions (H1) and (H2), there exists $K(\cdot)$ such that for all integers $i,j \in \{1,\dots,d\}$, for all $(t,y) \in (0,T]\times\mathbb{R}^d$, it holds that*

$$\left|\frac{\partial v}{\partial y^i}(t,y)\right| \le \frac{K(T)}{\sqrt{T-t}}, \tag{21}$$

and

$$\mathbb{E}_{y_0}\left[\frac{\partial^2 v}{\partial y^i \partial y^j}(t,Y_t)\right]^2 \le \frac{K(T)}{(T-t)^{\frac{3}{2}}\sqrt{T}}. \tag{22}$$

Proof. Inequality (21) is clear from (6) and from

$$\begin{aligned}\left|\frac{\partial v}{\partial y^i}(t,y)\right| &\le e^{-r(T-t)}\int_{\sum_k \lambda^k e^{{y'}^k}\ge K}\left|\frac{\partial p_{T-t}}{\partial y^i}(y,y')\right| dy' \\ &\le \frac{K(T)}{(T-t)^{\frac{d+1}{2}}}\int_{\mathbb{R}^d} e^{-c\frac{|y-y'|^2}{T-t}}dy'. \le \frac{K(T)}{\sqrt{T-t}}\end{aligned}$$

The same proof remains valid for the estimate (22), since $t \leq T/2$. We now turn to the case $t \geq T/2$. Proposition 5.2 gives

$$\frac{\partial^2 v}{\partial y^i \partial y^j}(t,y) = e^{-r(T-t)} \int_{\sum^k \lambda^k e^{y'^k} \geq K} \frac{\partial^2}{\partial y'^i \partial y'^j} p_{T-t}(y,y') dy' \\ + e^{-r(T-t)} \int_{\sum_k \lambda^k e^{y'^k} \geq K} g^{(i,j)}_{T-t}(y,y') dy'.$$

Using (17), the second term of the r.h.s. is upper bounded by $\frac{K(T)}{\sqrt{T-t}}$. For the first term, we integrate directly w.r.t. y^i and get

$$\frac{\partial^2 v}{\partial y^i \partial y^j}(t,y) = \underbrace{e^{-r(T-t)} \int_{\sum_{k\neq i} \lambda_k e^{y'^k} \leq K} \left.\frac{\partial p_{T-t}}{\partial y'^j}(y,y')\right|_{y'^i = \log\left(\frac{K - \sum_{k\neq i} \lambda^k e^{y'^k}}{\lambda_i}\right)} dy'^{(i)}}_{\Psi_{i,j}(t,y)} + \frac{K(T)}{\sqrt{T-t}}. \quad (23)$$

Therefore, from estimates (6) it follows that

$$\frac{\partial^2 v}{\partial y^i \partial y^j}(t,y) \leq \frac{K(T)}{\sqrt{T-t}} + \frac{K(T)}{T-t} \int_{\sum_{k\neq i} \lambda^k e^{\sqrt{T-t} z^k + y^k} \leq K} \\ \exp\left(-c \sum_{k\neq i} |z^k|^2 - \frac{c}{\sqrt{T-t}} \log^2 \left(\frac{K - \sum_{k\neq i} \lambda^k e^{\sqrt{T-t} z^k + y^k}}{\lambda_i e^{y_i}} \right) \right) dz^{(i)}.$$

Substituting Y_t into y in the above inequality and applying (6) and the Cauchy-Schwarz inequality, we see that

$$\mathbb{E}_{y_0}\left[\frac{\partial^2 v}{\partial y^i \partial y^j}(t,Y_t) \right]^2 \leq \frac{K(T)}{(T-t)^{\frac{3}{2}}} + \frac{K(T)}{(T-t)^2 t^d} \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^{d-1}} dz^{(i)} \mathbf{1}_{\sum_{k\neq i} \lambda^k e^{\sqrt{T-t} z^k + y^k} \leq K} \\ \exp\left(-c \sum_k \frac{(y^k - y_0^k)^2}{t} - c \sum_{k\neq i} |z^k|^2 - \frac{2c}{T-t} \left(\log\left(\frac{K - \sum_{k\neq i} \lambda^k e^{\sqrt{T-t} z^k + y^k}}{\lambda_i} \right) - y^i \right)^2 \right) dz^{(i)} \quad (24)$$

The change of variable $z^i = \frac{y^i - \log\left(\frac{K - \sum_{k\neq i} \lambda^k e^{\sqrt{T-t} z^k + y^k}}{\lambda_i e^{y_i}} \right)}{\sqrt{T-t}}$ now leads to

$$\mathbb{E}_{y_0}\left[\frac{\partial^2 v}{\partial y^i \partial y^j}(t,Y_t) \right]^2 \leq \frac{K(T)}{T-t} + \frac{K(T)}{(T-t)^{\frac{3}{2}} t^{\frac{d}{2}}} \int_{\mathbb{R}^{d-1}} dy^{(i)} \int_{\mathbb{R}} dz^i \int_{\mathbb{R}^{d-1}} dz^{(i)} \\ \mathbf{1}_{\sum_{k\neq i} \lambda^k e^{\sqrt{T-t} z^k + y^k} \leq K} \exp\left(-c \sum_{k\neq i} \frac{(y^k - y_0^k)^2}{t} - c \sum_{k\neq i} |z^k|^2 - c|y^i|^2 \right)$$

We replace z'^k by $\frac{y^k - y_0^k}{\sqrt{t}}$, and the inequality (22) follows (remember that $\frac{1}{\sqrt{t}} \leq \frac{\sqrt{2}}{\sqrt{T}}$ since $t \geq T/2$). □

5.3 Terms A_2 and A_3

We can proceed analogously to the proof of 2.1 (see section 4.3. The details are left to the reader.

5.4 Term A_1

To obtain the expansion result in theorem 2.2, we first state an analysis lemma, whose proof is given in Appendix C.

Lemma 5.1. *Let $g: [0,T] \mapsto \mathbb{R}$ be a measurable bounded function which is continuous in T.*

$$\int_0^T ds \int_{\varphi(s)}^s dt \frac{g(t)}{(T-t)^{\frac{3}{2}}} = C_0\, g(T) \left(\frac{T}{n}\right)^{1/2} + o\left(\frac{1}{\sqrt{n}}\right)$$

where $C_0 := \sum_{k=1}^{+\infty} \int_0^1 ds \int_0^s \frac{dt}{(k-t)^{\frac{3}{2}}} \in (0, +\infty)$.

The main idea of the proof is to apply the above lemma with the function g defined by

$$g(t) = (T-t)^{\frac{3}{2}} e^{-2rt} \mathbb{E}_{y_0}\left[\sum_{i,k,\alpha,\beta=1}^{d} s_{i,k}(Y_t) s_{\alpha,\beta}(Y_t) \left(\frac{\partial^2 v}{\partial y^i \partial y^\alpha}(t, Y_t) - \delta_{i,\alpha} \frac{\partial v}{\partial y^i}(t, Y_t) \right) \right.$$
$$\left. \left(\frac{\partial^2 v}{\partial y^j \partial y^\beta}(t, Y_t) - \delta_{j,\beta} \frac{\partial v}{\partial y^j}(t, Y_t) \right) \right]$$

We now intend to prove that g is bounded and has a limit in T (which enables to extend g as a continuous function in T), limit which will give the main term in the expansion of $\mathbb{E}(\Delta_n^2(f))$.

5.4.1 The function g is bounded

This follows from inequalities (21) and (22),

$$|g(t)| \leq K(T)(T-t)^{\frac{3}{2}} \left(\frac{1}{(T-t)^{\frac{3}{2}}} + \frac{1}{T-t} \right) \leq K(T).$$

5.4.2 Calculus of $\lim_{t \to T} g(t)$.

We use the equation (23), estimates (21), (22) and we get that

$$\left| g(t) - (T-t)^{\frac{3}{2}} e^{-2rT} \sum_{i,j,\alpha,\beta=1}^{d} \mathbb{E}_{y_0}\left[s_{i,\alpha}(Y_t) s_{j,\beta}(Y_t) \Psi_{i,\alpha}(t, Y_t) \Psi_{j,\beta}(t, Y_t) \right] \right| = o(\sqrt{T-t}).$$

Moreover, if we write the above approximation of g as in (24) (i.e. writing Ψ as an integral) and if we put $\nu(y^{(i,j)}, z, z'^{(j)}) = K - \sum_{k \neq i,j} \lambda^k e^{\sqrt{T-t} z'^k + y^k} + e^{\sqrt{T-t} z'^i + z^i} \left(K - \sum_{k \neq i,j} \lambda^k e^{\sqrt{T-t} z^k + y^k} \right)$

$$\mathbb{E}_{y_0}\left[\Psi_{i,\alpha}(t, Y_t) \Psi_{j,\beta}(t, Y_t) \right] \leq \frac{K(T)}{T-t} + \frac{K(T)}{(T-t)^{\frac{3}{2}} t^{\frac{d}{2}}} \int_{\mathbb{R}^{d-1}} dy^{(i)} \int_{\mathbb{R}} dz^i \int_{\mathbb{R}^{d-1}} dz^{(i)} \int_{\mathbb{R}^{d-1}} dz'^{(j)}$$
$$\mathbf{1}_{\nu(y^{(i,j)}, z, z'^{(j)}) - \lambda^j e^{\sqrt{T-t}(z^j + z'^i) + y^j + z^i} \geq 0} \exp\left(-c \sum_{k \neq i} \frac{(y^k - y_0^k)^2}{t} - c \sum_k |z^k|^2 - c \sum_{k \neq j} |z'^k|^2 \right)$$
$$\exp\left(-\frac{c}{T-t} \log^2 \left(\frac{\nu(y^{(i,j)}, z, z'^{(j)})}{\lambda^j e^{y^j}} - e^{\sqrt{T-t}(z^j + z'^i) + z^i} \right) \right),$$

we remark that if $i \neq j$ we can make the change of variable for y^j (on the set $\nu(y^{(i,j)}, z, z'^{(j)}) \geq \lambda^j e^{\sqrt{T-t}(z^j + z'^i) + y^j + z^i} > 0$)

$$z^j = \frac{1}{\sqrt{T-t}} \log \left(\frac{\nu(y^{(i,j)}, z, z'^{(j)})}{\lambda^j e^{y^j}} - e^{\sqrt{T-t}(z^j + z'^i) + z^i} \right)$$

and get

$$\begin{aligned}
&\mathbb{E}_{y_0}\left[\Psi_{i,\alpha}(t, Y_t)\Psi_{j,\beta}(t, Y_t)\right] \\
&\qquad \leq \frac{K(T)}{T-t} \int_{\mathbb{R}^{d-2}} dy^{(i,j)} \int_{\mathbb{R}^d} dz \int_{\mathbb{R}^d} dz' e^{-c\sum_k (|z^k|^2 + |z'^k|^2) - c\sum_{k \neq i,j} |y^k|^2} \leq \frac{K(T)}{T-t}.
\end{aligned}$$

Therefore,

$$\left| g(t) - \underbrace{(T-t)^{\frac{3}{2}} e^{-2rT} \mathbb{E}_{y_0} \sum_{i=1}^{d} \left[\sum_{j=1}^{d} s_{i,j}(Y_t) \Psi_{i,j}(t, Y_t) \right]^2}_{\hat{g}(t)} \right| = o(T-t).$$

We now intend to find a tractable expression for $\frac{\partial p_{T-t}}{\partial y'^j}(y, y')$ when t is near T. The approximation of a density of a multidimensional process when t is small has been a source of a lot of literature. Such approximation exists even if we consider hypoelliptic stochastic process on manifolds.
To obtain our expression we will proceed following a classical method set out by Leandre in [6]. Let $\bar{Y}_\theta^{T-t}$ be the solution of the stochastic differential equation

$$\bar{Y}_\theta^{T-t} = y + (T-t) \int_0^\theta b(\bar{Y}_\iota^{T-t}) d\iota + \sqrt{T-t} \int_0^\theta s(\bar{Y}_\iota^{T-t}) dW_\iota$$

Noting that $\bar{Y}_1^{T-t}$ has the same law as Y_{T-t} and following closely the proof described in Leandre [6], we put $\bar{Y}_\theta^{T-t} = \theta(z-y) + y + \sqrt{T-t}\hat{Y}_\theta^{T-t}$. Thanks to the Girsanov theorem $\hat{Y}_\theta^{T-t}$ has the same law as Λ_θ^{T-t} solution of

$$\begin{aligned}
d\Lambda_\theta^{t\,i}(y,z) &= \sum_{j=1}^{d} s_{i,j}(\theta(z-y) + y + \sqrt{t}\Lambda_\theta^t(y,z)) dW_\theta^j + \sqrt{t} b^i(\theta(z-y) + y + \sqrt{t}\Lambda_\theta^t(y,z)) d\theta \\
&+ \sum_{j=1}^{d} \Big(s_{i,j}(\theta(z-y) + y + \sqrt{t}\Lambda_\theta^t(y,z)) - s_{i,j}(\theta(z-y) + y) \Big) \frac{h_\theta^j(y,z)}{\sqrt{t}} d\theta \\
\Lambda_0^t(z,y) &= 0 \\
\text{and} \quad & h_\theta(y,z) = s^{-1}(\theta(z-y) + y)(z-y)
\end{aligned}$$

But, using the functions ρ_δ described by (18), one has

$$\frac{\partial p_{T-t}}{\partial y'^j}(y,z) = \frac{1}{(T-t)^{\frac{d+1}{2}}} \lim_{\delta \to 0} \mathbb{E}_y \left[(\partial_j \rho_\delta)(\bar{Y}_1^{T-t}) \right].$$

Therefore, the derivative of the density of $Y_{T-t}(y)$ is equal to

$$\begin{aligned}
\frac{\partial p_{T-t}}{\partial y'^j}(y,z) = \frac{1}{(T-t)^{\frac{d+1}{2}}} \exp &\left(- \frac{\int_0^1 (z-y)^* S^{-1}(\theta(z-y) + y)(z-y) d\theta}{2(T-t)} \right) \\
&\lim_{\delta \to 0} \mathbb{E} \left[(\partial_j \rho_\delta)(\Lambda_1^{T-t}(y,z)) e^{-\frac{1}{\sqrt{T-t}} \int_0^1 (z-y)^* s^{*-1}(\theta(z-y) + y) dW_\theta} \right],
\end{aligned}$$

We set $\forall(z,\chi)\in\mathbb{R}^{d-1}\times\mathbb{R}\ \ \eta(z^{(i)},\chi)=\log\left|\frac{K-\sum_{k\neq i}\lambda^k e^{z^k}}{\lambda^i e^{\chi}}\right|$, then $\hat{g}$ can be written as

$$\hat{g}(t)=\sum_{i=1}^{d}(T-t)^{\frac{3}{2}}e^{-2rT}\int_{\mathbb{R}^d}dy\ p_t(y_0,y)$$
$$\left\{\sum_{j=1}^{d}\int_{\sum_{k\neq i}\lambda^k e^{z^k}\leq K}dz^{(i)}s_{i,j}(y)\frac{\partial p_{T-t}}{\partial y'^j}(y,z)\Bigg|_{z^i=\eta(z^{(i)},y^i)}\right\}^2$$

Let y_t and z_t be deterministic vectors in $\mathbb{R}^d$. We suppose that there exists two $d-$dimensional vector χ and $\bar{y}$ such that $lim_{t\to T}(z_t-y_t-\chi\sqrt{T-t})=0$ and $lim_{t\to T}y_t=\bar{y}$, then

$$\lim_{t\to T}(T-t)^{\frac{d+1}{2}}\frac{\partial p_{T-t}}{\partial y'^j}(y_t,z_t)=\exp\left(-\frac{\chi^*S^{-1}(\bar{y})\chi}{2}\right)$$
$$\lim_{\delta\to 0,\ t\to T}\mathbb{E}\left[(\partial_j\rho_\delta)(\Lambda_1^{T-t}(y_t,z_t))e^{-\frac{1}{\sqrt{T-t}}\int_0^1(z_t-y_t)^*s^{*-1}(\theta(z_t-y_t)+y_t)dW_\theta}\right].$$

But Λ_1^0 is a Gaussian random vector equal to $s(\bar{y})W_1$ and since standard computations implies

$$\lim_{\delta\to 0}\mathbb{E}\left[(\partial_j\rho_\delta)(s(\bar{y})W_1)\exp\left(-\chi^*s^{*-1}(\bar{y})W_1\right)\right]=\frac{(S^{-1}(\bar{y})\chi)^j}{(2\pi)^{\frac{d}{2}}\sqrt{\det(S(\bar{y}))}},$$

it follows that

$$\lim_{t\to T}(T-t)^{\frac{d+1}{2}}\frac{\partial p_{T-t}}{\partial y'^j}(y_t,z_t)=\exp\left(-\frac{\chi^*S^{-1}(\bar{y})\chi}{2}\right)\frac{(S^{-1}(\bar{y})\chi)^j}{(2\pi)^{\frac{d}{2}}\sqrt{\det(S(\bar{y}))}}.$$

We will use this with $z_t^{(i)}=z^{(i)}\sqrt{T-t}+y^{(i)}$, $z_t^i=\log\left(\frac{K-\sum_{k\neq i}\lambda^k e^{z^k\sqrt{T-t}+y^k}}{\lambda^i}\right)$ and $y_t^i=\sqrt{T-t}y^i+\log\left(\frac{K-\sum_{k\neq i}\lambda^k e^{y^k}}{\lambda^i}\right)$.
Therefore,

$$\chi=\left(z^1,\ldots,z^{i-1},-y^i-\frac{\sum_{k\neq i}z^k\lambda^k e^{y^k}}{K-\sum_{k\neq i}z^k\lambda^k e^{y^k}},z^{i+1},\ldots,z^d\right)$$
$$\bar{y}=\left(y^1,\ldots,y^{i-1},\log\left(\frac{K-\sum_{k\neq i}\lambda^k e^{y^k}}{\lambda^i}\right),y^{i+1},\ldots,y^d\right)$$

And in conclusion, we find that

$$\lim_{t\to T}g(t)=e^{-2rT}\mathcal{D}:=e^{-2rT}\sum_{i=1}^{d}\int_{\sum_{k\neq i}\lambda^k e^{y^k}\leq K}dy\ p_T(y_0,\bar{y})$$
$$\left\{\int_{\mathbb{R}^{d-1}}\frac{dz^{(i)}}{(2\pi)^{\frac{d}{2}}\sqrt{\det(S(\bar{y}))}}\exp\left[-\frac{1}{2}\chi^*S^{-1}(\bar{y})\chi\right](S^{-1}(\bar{y})\chi)^i\right\}^2. \quad (25)$$

The upper bound (6) allow us to apply the Lebesgue dominated convergence theorem.

Appendix A Proof of Proposition 3.1

Proposition 3.1
Let (H1) and (H2) hold. $\mathbb{E}[\Delta_n(f)]^2$ can be written as

$$\mathbb{E}[\Delta_n(f)]^2 = A_1 + A_2 + A_3,$$

where

$$A_1 = \int_0^T dt \int_{\varphi(t)}^t ds\; e^{-2rs}\Bigg[\sum_{i,k,\alpha,\beta=1}^d X_s^\alpha X_s^\beta X_s^i X_s^k a_{i,k}(X_s) a_{\alpha,\beta}(X_s) \frac{\partial^2 u}{\partial x^i \partial x^\alpha}(s,X_s)\frac{\partial^2 u}{\partial x^j \partial x^\beta}(s,X_s)\Bigg],$$

$$A_2 = \int_0^T dt \int_{\varphi(t)}^t ds\; e^{-2rs}\Bigg[\sum_{i,j,k,\alpha,\beta=1}^d \Bigg(x^\alpha x^\beta x^i a_{\alpha,\beta}(x)\sigma_{i,j}(x) \frac{\partial}{\partial x^\beta}(x^k\sigma_{k,j}(x))\left(D_u^k(s)\frac{\partial^2 u}{\partial x^i\partial x^\alpha}(s,x) + 2D_u^i(s)\frac{\partial^2 u}{\partial x^k \partial x^\alpha}(s,x)\right)\Bigg) - \sum_{i,k,\alpha,\beta=1}^d x^i x^k a_{i,k}(x)\frac{\partial}{\partial x^i}(x^\alpha x^\beta a_{\alpha,\beta}(x)) D_u^k(s)\frac{\partial^2 u}{\partial x^\alpha \partial x^\beta}(s,x)\Bigg)\Bigg]\Bigg|_{x=X_s},$$

and

$$A_3 = \int_0^T dt \int_{\varphi(t)}^t ds\; e^{-2rs}\Bigg[2\sum_{i,k=1}^d D_u^k(s)D_u^i(s)x^i x^k a_{i,k}(x) + \sum_{i,k,j,\alpha,\beta=1}^d D_u^i(s)D_u^k(s)x^\alpha x^\beta a_{\alpha,\beta}(x)\Big(\frac{1}{2}\frac{\partial}{\partial x^\alpha}(x^i\sigma_{i,j}(x))\frac{\partial}{\partial x^\beta}(x^k\sigma_{k,j}(x)) + x^k\sigma_{k,j}(x)\frac{\partial^2}{\partial x^\alpha\partial x^\beta}(x^i\sigma_{i,j}(x))\Big)\Bigg]\Bigg|_{x=X_s}.$$

Proof. Set $D_u^i(t) = \frac{\partial u}{\partial x^i}(t,X_t) - \frac{\partial u}{\partial x^i}(\varphi(t), X_{\varphi(t)})$. Using the stochastic differential system whose X_t is the solution, and substituting $\widetilde{X_t^i}$ into the preceding, we obtain

$$\mathbb{E}[\Delta_n(f)]^2 = \mathbb{E}\left[\sum_{i,j=1}^d \int_0^T D_u^i(t)e^{-rt}X_t^i\sigma_{i,j}(X_t)dW_t^j\right]^2.$$

Since $(W_t)_{t\geq 0}$ is a $d-$dimensional Brownian motion, the W_t^j are independent. It leads to

$$\mathbb{E}[\Delta_n(f)]^2 = \sum_{j=1}^d \mathbb{E}\left[\int_0^T\left(\sum_{i=1}^d \underbrace{D_u^i(t)e^{-rt}X_t^i\sigma_{i,j}(X_t)}_{M_t^{i,j}}\right)^2 dt\right].$$

The Itô formula applied between $\varphi(t)$ and t, implies that

$$\left(\sum_{i=1}^d M_t^{i,j}\right)^2 = 2\int_{\varphi(t)}^t \sum_{i,k=1}^d M_s^{k,j}dM_s^{i,j} + \int_{\varphi(t)}^t \sum_{i,k=1}^d d\langle M_\cdot^{i,j}, M_\cdot^{k,j}\rangle_s.$$

With the above notations, a straightforward calculation leads to

$$e^{2rs}d\langle M_\cdot^{i,j}, M_\cdot^{k,j}\rangle_s = \frac{1}{2}\sum_{\alpha,\beta=1}^{d}\Bigg(x^\alpha x^\beta a_{\alpha,\beta}(x)\Bigg(\frac{\partial^2 u}{\partial x^i \partial x^\alpha}(s,x)x^i\sigma_{i,j}(x)$$

$$+ D_u^i(s)\frac{\partial}{\partial x^\alpha}(x^i\sigma_{i,j}(x))\Bigg)\times\left(\frac{\partial^2 u}{\partial x^k \partial x^\beta}(s,x)x^k\sigma_{k,j}(x) + D_u^k(s)\frac{\partial}{\partial x^\beta}(x^k\sigma_{k,j}(x))\right)\Bigg)\Bigg|_{x=X_s} ds,$$

and

$$e^{rs}dM_s^{i,j} = \left(-rD_u^i(s)X_s^i\sigma_{i,j}(X_s) + \frac{\partial^2 u}{\partial x^i \partial t}(s,X_s)X_s^i\sigma_{i,j}(X_s)\right)ds$$

$$+ \sum_{\alpha=1}^{d}\left(rx^\alpha\left(\frac{\partial^2 u}{\partial x^i\partial x^\alpha}(s,x)x^i\sigma_{i,j}(x) + D_u^i(s)\frac{\partial}{\partial x^\alpha}(x^i\sigma_{i,j}(x))\right)\right)\Bigg|_{x=X_s} ds$$

$$+ \sum_{\alpha,\beta=1}^{d}\left(x^\alpha\sigma_{\alpha,\beta}(x)\left(\frac{\partial^2 u}{\partial x^i\partial x^\alpha}(s,x)x^i\sigma_{i,j}(x) + D_u^i(s)\frac{\partial}{\partial x^\alpha}(x^i\sigma_{i,j}(x))\right)\right)\Bigg|_{x=X_s} dW_s^\beta$$

$$+ \frac{1}{2}\sum_{\alpha,\beta=1}^{d}\Bigg(x^\alpha x^\beta a_{\alpha,\beta}(x)\Bigg(\frac{\partial^3 u}{\partial x^i\partial x^\alpha\partial x^\beta}(s,x)x^i\sigma_{i,j}(x) + \frac{\partial^2 u}{\partial x^i\partial x^\alpha}(s,x)\frac{\partial}{\partial x^\beta}(x^i\sigma_{i,j}(x))$$

$$+ \frac{\partial^2 u}{\partial x^i\partial x^\beta}(s,x)\frac{\partial}{\partial x^\alpha}(x^i\sigma_{i,j}(x)) + D_u^i(s)\frac{\partial^2}{\partial x^\alpha\partial x^\beta}(x^i\sigma_{i,j}(x))\Bigg)\Bigg)\Bigg|_{x=X_s} ds.$$

Combining the second order derivative of u w.r.t. t and x^i with (3) gives

$$-\frac{\partial^2}{\partial t\partial x^i}u(t,x) = \frac{1}{2}\sum_{\alpha,\beta=1}^{d} a_{\alpha,\beta}(x)x^\alpha x^\beta\frac{\partial^3 u}{\partial x^\alpha\partial x^\beta\partial x^i}(t,x)$$

$$+ \frac{1}{2}\sum_{\alpha,\beta=1}^{d}\frac{\partial}{\partial x^i}(a_{\alpha,\beta}(x)x^\alpha x^\beta)\frac{\partial^2 u}{\partial x^\alpha\partial x^\beta}(t,x) + r\sum_{\alpha=1}^{d}x^\alpha\frac{\partial^2 u}{\partial x^i\partial x^\alpha}(t,x)$$

Finally, noting that $a_{i,k} = \sum_{j=1}^{d}\sigma_{i,j}\sigma_{k,j}$, the proposition is proved. □

Appendix B Proof of lemma 4.1

Lemma 4.1. The functions f_η are $\mathcal{C}^\infty$, and under (H3)

- $\|f_\eta - f\|_1 \le \sqrt{\eta}$,
- $\left\|\frac{\partial^2 f_\eta}{\partial y^i\partial y^j}\right\|_\infty \le \frac{1}{\eta^{\frac{1}{2}}}$

Proof. It is clear that $\|f - f_\eta\|_\infty \le \sqrt{\eta}$. Thus,

$$\sup_{y\in\mathbb{R}^d}\frac{|f(y) - f_\eta(y)|}{|y|+1} \le \sqrt{\eta}.$$

Let us put $\omega(\eta, z, y, y') = f(z\sqrt{\eta} + y) - f(y) - f(z\sqrt{\eta} + y') + f(y')$. Hence, obviously

$$|f_\eta(y') - f(y') - f_\eta(y) + f(y)| \le C\left(\int_{\mathbb{R}^d}|\omega(\eta,z,y,y')|^2 e^{-|z|^2}dz\right)^{\frac{1}{2}}$$

$$\left(\int_{\mathbb{R}^d}|\omega(\eta,z,y,y')|^2 e^{-|z|^2}dz\right)^{\frac{1}{2}}. \quad (26)$$

The Lipschitz property of f gives

$$|\omega(\eta, z, y, y')| \leq 2\sqrt{\eta}|z| \quad and \quad |\omega(\eta, z, y, y')| \leq 2|y - y'|.$$

Using the first (resp. second) estimate for the first (resp. second) integral of (26) yields the first statement of the lemma. For the second assertion, the proof is the same as for (8). $\square$

Appendix C Proof of lemma 5.1

Lemma 5.1. Let $g : [0,T] \mapsto \mathbb{R}$ be a measurable bounded function which is continuous in T.

$$\int_0^T ds \int_{\varphi(s)}^s dt \frac{g(t)}{(T-t)^{\frac{3}{2}}} = C_0 \, g(T) \left(\frac{T}{n}\right)^{1/2} + o\left(\frac{1}{\sqrt{n}}\right) \tag{27}$$

where $C_0 := \sum_{k=1}^{+\infty} \int_0^1 ds \int_0^s \frac{dt}{(k-t)^{\frac{3}{2}}} \in (0, +\infty)$.

Proof. 1. Suppose first that g is constant.
We can assume $g \equiv 1$ e.g. . A simple change of variables leads to

$$\int_0^T ds \int_{\varphi(s)}^s \frac{dt}{(T-t)^{\frac{3}{2}}} = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} ds \int_{t_k}^s \frac{dt}{(T-t)^{\frac{3}{2}}} = \left(\frac{T}{n}\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} \int_0^1 ds \int_0^s \frac{dt}{(k-t)^{\frac{3}{2}}}\right).$$

The series above is convergent because its terms decrease like $n^{-3/2}$: we denote by C_0 its limit. This completes the proof of (27) in that case. *2. Suppose now that g is a bounded measurable function, continuous in T.*
There's no restriction to assume that $g(T) = 0$, up to replacing g by $g - g(T)$ and applying the first case. The proof of (27) now consists in showing that

$$\lim_n \left(\left(\frac{n}{T}\right)^{1/2} \int_0^T ds \int_{\varphi(s)}^s dt \frac{g(t)}{(T-t)^{\frac{3}{2}}}\right) = 0. \tag{28}$$

Fix $\delta > 0$. Since g is continuous in T, there exists $\eta > 0$ such that $\forall t \in [T-\eta, T]$, $|g(t)| \leq \frac{\delta}{C_0}$. Thus, we deduce that

$$\left|\int_{T-\eta}^T ds \int_{\varphi(s)}^s dt \frac{g(t)}{(T-t)^{\frac{3}{2}}}\right| \leq \delta \left(\frac{T}{n}\right)^{1/2},$$

and for $0 \leq s \leq T - \eta$, since $T - s \geq \eta$, we obtain that

$$\left|\int_0^{T-\eta} ds \int_{\varphi(s)}^s dt \frac{g(t)}{(T-t)^{\frac{3}{2}}}\right| \leq \frac{T^2}{n} \frac{\|g\|_\infty}{\eta^{\frac{3}{2}}}.$$

Therefore, for n large enough,

$$\left|\left(\frac{n}{T}\right)^{1/2} \int_0^T ds \int_{\varphi(s)}^s dt \frac{g(t)}{(T-t)^{\frac{3}{2}}}\right| \leq 2\delta,$$

which completes the proof of (28) and consequently (27), when $g(T) = 0$. $\square$

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