Probabilistic characteristics method for a 1D scalar conservation law

B.Jourdain*

October 2, 2000

Abstract

In this paper, we are interested in approximating the solution of a 1D inviscid scalar conservation law starting from an initial condition with bounded variation thanks to a system of interacting diffusions. We modify the system of signed particles associated with the parabolic equation obtained from addition of a viscous term to this equation (see [3][4][6]) by killing couples of particles with opposite sign that merge. The sample-paths of the corresponding reordered particles can be seen as probabilistic characteristics along which the approximate solution is constant. This enables us to prove that when the viscosity vanishes as the initial numbers of particles goes to $+\infty$, the approximate solution converges to the unique entropy solution of the inviscid conservation law. We illustrate this convergence by numerical results.

In this paper, we are interested in giving a probabilistic particle approximation of the entropy solution of the scalar conservation law

$$\partial_t u + \partial_x A(u) = 0, \qquad u(0,x) = u_0(x). \tag{0.1}$$

where A is a C^1 function and the initial condition u_0 is a function with bounded variation i.e. there are a bounded signed measure m and a real constant a such that dx a.e., $u_0(x) = a + \int_{-\infty}^{x} m(dy)$. Uniqueness does not hold for weak solutions of this equation. But according to Kruzkhov theorem, there is a unique entropy solution u bounded and belonging to $C([0,+\infty), L^1_{loc}(\mathbb{R}))$ characterized by the entropy inequalities : $\forall c \in \mathbb{R}$, for any positive C^∞ function g with compact support on $[0,+\infty) \times \mathbb{R}$,

$$\int_0^{+\infty} \int_{\mathbb{R}} (|u-c|\partial_t g + \text{sgn}(u-c)(A(u)-A(c))\partial_x g)(t,x)dxdt + \int_{\mathbb{R}} |u_0(x)-c|g(0,x)dx \geq 0. \tag{0.2}$$

Taking $c > \|u\|_\infty$ and $c < -\|u\|_\infty$ in (0.2), one easily checks that the entropy solution is a weak solution.

Let $|m|$ and $\|m\|$ denote respectively the total variation of the measure m and its total mass. As the entropy solution $u(t,x)$ of (0.1) is equal to $a + \|m\| v(t,x)$ where v is the entropy solution of $\partial_t v + \partial_x f(v) = 0$ for initial data $v_0(x) = (u_0(x)-a)/\|m\|$, $f(v) = A(a + \|m\| v)/\|m\|$, it is not restrictive to assume from now on that $a = 0$ and $\|m\| = 1$ i.e. $|m|$ is a probability measure.

It is well-known that the solution u_σ of the viscous scalar conservation law

$$\partial_t u_\sigma = \frac{\sigma^2}{2}\partial_{xx} u_\sigma - \partial_x A(u_\sigma), \qquad u_\sigma(0,x) = H * m(x) \tag{0.3}$$

*ENPC-CERMICS, 6-8 av Blaise Pascal, Cité Descartes, Champs sur Marne, 77455 Marne la Vallé Cedex 2, France - e-mail : jourdain@cermics.enpc.fr

where $\sigma > 0$ converges to the entropy solution of (0.1) in the vanishing viscosity limit $\sigma \to 0$. In [6], following the approach developped by Bossy and Talay [3] [4] in case of the viscous Burgers equation ($A(u) = u^2/2$), we introduce the parabolic problem satisfied by $w = \partial_x u_\sigma$ in order to construct a probabilistic particle approximation of u_σ :

$$\partial_t w = \frac{\sigma^2}{2}\partial_{xx} w - \partial_x(A'(u_\sigma)w),\ w(0,.) = m,\ u_\sigma(t,x) = \int_{-\infty}^{x} w(t,y)dy$$

which writes

$$\partial_t w = \frac{\sigma^2}{2}\partial_{xx} - \partial_x(A'(H * w)w),\ w(0,.) = m \tag{0.4}$$

where $(H * w)(t,x) = \int_{-\infty}^{x} w(t,y)dy$ denotes the spatial convolution of $w(t,.)$ with the Heaviside function $H(y) = 1_{\{y \geq 0\}}$. A weak solution of this equation is obtained thanks to the unique solution $P \in \mathcal{P}(C([0,+\infty),\mathbb{R}))$ of the nonlinear martingale problem (PM^σ) starting at m:

Definition 0.1 *From now on, h is a density of m with respect to $|m|$ with values in $\{-1,1\}$. With any probability measure Q on $C([0,+\infty),\mathbb{R})$ we associate the bounded signed measure $\tilde{Q}$ defined by $d\tilde{Q}/dQ = h(X_0)$ where $(X_t)_{t\geq 0}$ denotes the canonical process on $C([0,+\infty),\mathbb{R})$. The times marginals of Q and $\tilde{Q}$ are respectively denoted by $(Q_t)_{t\geq 0}$ and $(\tilde{Q}_t)_{t\geq 0}$. We say that Q solves the martingale problem (PM^σ) starting at m if $Q_0 = |m|$ and*

$$\forall \phi \in C_b^2(\mathbb{R}),\ M_t^\phi = \phi(X_t) - \phi(X_0) - \int_0^t \frac{\sigma^2}{2}\phi''(X_s) + A'(H * \tilde{Q}_s(X_s))\phi'(X_s)ds \quad \text{is a } Q\text{-martingale.}$$

Indeed, by the constancy of the expectation of the P martingale $h(X_0)M_t^\phi$, one easily checks that $t \to \tilde{P}_t$ is a weak solution of (0.4). As a consequence, the function $u_\sigma(t,x)$ is equal to $H * \tilde{P}_t(x)$. That is why we are induced to approximate $u_\sigma(t,x)$ by the cumulative distribution function

$$U_\sigma^n(t,x) = H * \tilde{\mu}_t^n(x) = \frac{1}{n}\sum_{i=1}^{n} H(x - X_t^i)h(X_0^i)$$

with $\mu^n = \frac{1}{n}\sum_{i=1}^{n}\delta_{X^i}$ denoting the empirical measure of the particle system defined by the stochastic differential equation

$$X_t^i = X_0^i + \sigma B_t^i + \int_0^t A'(H * \tilde{\mu}_s^n(X_s^i))ds,\ i \leq n$$

where $(B^1, \ldots, B^n)$ a $\mathbb{R}^n$-valued Brownian motion independent of the initial variables X_0^i, $1 \leq i \leq n$ I.I.D. with law $|m| \in \mathcal{P}(\mathbb{R})$. Ever in [6], we show that as $n \to +\infty$, the empirical measures μ^n (considered as a $\mathcal{P}(C([0,+\infty),\mathbb{R}))$ random variables) converge in distribution to the constant P (such a result is called propagation of chaos : see [11]) which implies the convergence of U_σ^n to u_σ. Since u_σ converges to the entropy solution u of (0.1) as $\sigma \to 0$, it is natural to wonder whether $U_{\sigma_n}^n$ converges to u as $n \to +\infty$ when $\lim_{n\to+\infty}\sigma_n = 0$. This paper is dedicated to this problem. According to the numerical results given in [2], the answer is likely to be positive.

In case m is a probability measure, there is no signed weights and $U_{\sigma_n}^n(t,x) = \frac{1}{n}\sum_{i=1}^{n} 1_{\{X_t^i \leq x\}}$. To prove that $U_{\sigma_n}^n$ converges to the entropy solution of (0.1), we want to compute the left-hand-side of the entropy inequalities (0.2) with $U_{\sigma_n}^n$ and $c_n = [cn]/n$ ($[x]$ denotes the integral part of x) replacing u and c. That is why we are interested in $|U_{\sigma_n}^n(t,x) - c_n|$. Let $(Y_t^1, \ldots, Y_t^n)$

denote the increasing reordering of $(X_t^1, \dots, X_t^n)$. The function $x \to |U_{\sigma_n}^n(t,x) - c_n| - |c_n|$ is the cumulative distribution function of the signed measure $\frac{1}{n}\sum_{j=1}^n (1_{\{j>[cn]\}} - 1_{\{j\leq[cn]\}})\delta_{Y_t^j}$. Of course, it is also the cumulative distribution function of a linear combination of $\delta_{X_t^i}, 1 \leq i \leq n$ but the corresponding coefficients are not constant in time as previously. That is why the reordered system $(Y^1, \dots, Y^n)$ is very interesting to compute the approximate left-hand-side of (0.2). Moreover this system has a very simple interpretation. By the occupation times formula, a.s., dt a.e., the positions $X_t^1, \dots, X_t^n$ are distinct and $U_{\sigma_n}^n(t, Y_t^i) = i/n$. Therefore the curves $t \to Y_t^i$ can be seen as probabilistic characteristics along which the approximate solution is dt a.e. constant. One can check that $(Y^1, \dots, Y^n)$ is a diffusion with diffusion matrix σ_n times the identity and constant drift coefficient $(A'(1/n), \dots, A'(1))$ normally reflected at the boundary of the closed convex set $D_n = \{y = (y_1, \dots, y_n) \in \mathbb{R}^n,\ y_1 \leq y_2 \leq \dots \leq y_n\}$. The deterministic characteristics associated with the scalar conservation law (0.1) for the initial data $u_0(x)$ are given by $y(t) = y + A'(u_0(y))t$. For $t \geq \inf_{x\neq y} |x-y|/|A'(u_0(x)) - A'(u_0(y))|$ they may intersect. The small Brownian perturbation that is added to define the probabilistic characteristics allows to introduce reflexion which prevents strict crossings with $Y_s^i > Y_s^{i+1}$. If we set $\phi(t,x) = \int_{-\infty}^x g(t,y)dy$ where g is the nonnegative test function in (0.2), compute $d\phi(t, Y_t^i)$ by Itô's formula, sum over i the obtained result multiplied by $(1_{\{i>[cn]\}} - 1_{\{i\leq[cn]\}})$, make integrations by parts in the spatial integrals, we get that the left-hand-side of (0.2) with $U_{\sigma_n}^n$ and c_n replacing u and c is equal to the contribution of the local time term giving the reflexion plus a remainder which vanishes as $n \to +\infty$. One remarkable feature is that the contribution of the local time which prevents strict crossings of our probabilistic characteristics is positive and gives the entropy inequality in the limit $n \to +\infty$.

When m is a signed measure, the situation is more complicated. Because of the possibility of crossings of couples of particles (X^i, X^j) with opposite signs $h(X_0^i) = -h(X_0^j)$, $x \to |U_{\sigma_n}^n(t,x) - c_n| - c_n$ is no longer the cumulative distribution function of a linear combination of $\delta_{Y_t^i}$, $1 \leq i \leq n$ with coefficients constant in time. That is why the computation of the approximate left-hand-side of the entropy inequality (0.2) is not easier with the reordered system $(Y^1, \dots, Y^n)$ than with the original one. To overcome this difficulty, we can define directly $(Y^1, \dots Y^n)$ as a diffusion normally reflected at the boundary of D_n with diffusion matrix σ_n times the identity and drift coefficient $\left(A'(\frac{1}{n}h(Y_0^1)), A'(\frac{1}{n}(h(Y_0^1) + h(Y_0^2))), \dots, A'(\frac{1}{n}\sum_{i=1}^n h(Y_0^i))\right)$ where the initial vector $(Y_0^1, \dots, Y_0^n)$ is distributed according to the law of the increasing reordering of n independent variables with law $|m|$. But when we compute the left-hand-side of (0.2) with u replaced by the new approximate solution $\frac{1}{n}\sum_{i=1}^n h(Y_0^i)H(x - Y_t^i)$, the contribution of the local time on hyperplanes $y^i = y^{i+1}$ such that $h(Y_0^i) = -h(Y_0^{i+1})$ has the wrong sign.
In fact, the right approach consists in modifying the dynamics of the original particle system $(X^1, \dots, X^n)$ by killing the couples of particles with opposite sign that merge. This modification is in fact very natural : this causes the variation of the approximate solution $x \to U_\sigma^n(t,x)$ to decrease with t, which is a transcription of the same property satisfied by $x \to u_\sigma(t,x)$. In the first section of the paper, we construct the modified particle system and prove that for fixed $\sigma > 0$, the approximate solution of (0.3) based on the surviving particles still converges to the exact solution u_σ as the initial number of particles n goes to $+\infty$. In the second section, by considering the increasing reordering of the modified system, we prove that when σ depends on n and converges to 0 as $n \to +\infty$, this approximate solution converges to the entropy solution of (0.1). If we assume that m is a probability measure, since all particles share the same sign, there is no killing and we get back to the much simpler situation described previously. That is why we obtain stronger convergence results, such as a propagation of chaos result for the reordered system. The last section is dedicated to an example of numerical simulation of the modified system with decreasing number of particles.

To conclude this introduction, we mention the approximation of the solution of (0.1) by interacting processes with jumps introduced by Perthame and Pulvirenti [9] (see also [5]). The principle is radically different : the system of interacting particles is associated with a nonlinear kinetic equation from which the scalar conservation law can be recovered when a relaxation parameter λ goes to $+\infty$. This approach is not limited to space dimension 1 as the one presented here. But the convergence result is for fixed relaxation parameter $\lambda > 0$ e.g. λ does not go do $+\infty$ with the number of particles. Moreover, the initial data of (0.1) is not only assumed to have a bounded variation but also to be nonnegative and integrable.

1 Modification of the particle system associated with the viscous conservation law

The modification of the system of diffusing particles consists in killing the couples of particles with opposite sign that merge. Before giving a precise construction, we explain why such an annihilation procedure is naturally associated with the martingale problem (PM^σ).

Lemma 1.1 *For any signed measure m with $\|m\| = 1$ and for any $\sigma > 0$, the solution P of the martingale problem (PM^σ) starting at m is such that the total mass $\|\tilde{P}_t\|$ of $\tilde{P}_t$ is non-increasing.*

Proof : This proof is based on the Markov property.
According to the Jordan-Hahn decomposition, $\forall s \geq 0$ there exist two Borel subsets of $\mathbb{R}$ denoted by C_s^+ and C_s^- such that $C_s^+ \cup C_s^- = \mathbb{R}$, $C_s^+ \cap C_s^- = \emptyset$ and $\|\tilde{P}_s\| = \tilde{P}_s(C_s^+) - \tilde{P}_s(C_s^-)$.
Let $0 \leq t_1 \leq t_2$.

$$\|\tilde{P}_{t_2}\| = \mathbb{E}^P\left(\left(1_{C_{t_2}^+}(X_{t_2}) - 1_{C_{t_2}^-}(X_{t_2})\right) h(X_0)\right) = \mathbb{E}^P\left(\mathbb{E}^P\left(1_{C_{t_2}^+}(X_{t_2}) - 1_{C_{t_2}^-}(X_{t_2}) \middle| \mathcal{G}_{t_1}\right) h(X_0)\right)$$

where $(\mathcal{G}_t)_{t\geq 0}$ denotes the canonical filtration on $\mathcal{C}([0,+\infty),\mathbb{R})$.
The drift coefficient $b(s,x) = A'(H * \tilde{P}_s(x))$ is bounded whereas the diffusion coefficient is a strictly positive constant. Combining Theorems 6.2.2, 6.3.4 and 6.4.3 [10], we obtain that if $Q^{t_1,x}$ denotes the solution of the martingale problem $Q_0 = \delta_x$,

$$\forall \phi \in C_b^2(\mathbb{R}),\ \phi(X_t) - \phi(X_0) - \int_0^t \frac{\sigma^2}{2}\phi''(X_s) + b(t_1+s, X_s)\phi'(X_s)ds \ \text{ is a } Q \text{ martingale}$$

then P a.s., $Q_{t_2-t_1}^{t_1,X_{t_1}}$ is a regular conditional probability distribution of X_{t_2} given $\mathcal{G}_{t_1}$. Hence

$$\begin{aligned}
\|\tilde{P}_{t_2}\| &= \int_{\mathbb{R}} Q_{t_2-t_1}^{t_1,x}(C_{t_2}^+) - Q_{t_2-t_1}^{t_1,x}(C_{t_2}^-)\tilde{P}_{t_1}(dx) \\
&\leq \int_{C_{t_1}^+} Q_{t_2-t_1}^{t_1,x}(C_{t_2}^+)\tilde{P}_{t_1}(dx) - \int_{C_{t_1}^-} Q_{t_2-t_1}^{t_1,x}(C_{t_2}^-)\tilde{P}_{t_1}(dx) \\
&\leq \tilde{P}_{t_1}(C_{t_1}^+) - \tilde{P}_{t_1}(C_{t_1}^-) \leq \|\tilde{P}_{t_1}\|.
\end{aligned}$$

■

This monotonicity property is linked to the intersection of sample-paths with opposite sign. The discretized version of this phenomenom is the murder of the couples of particles with opposite

sign that merge.
The precise construction of the particle system is based on Girsanov theorem. On a filtered probability space $(\Omega, \mathcal{F}, \mathbb{Q}, (\mathcal{F}_t)_{t\geq 0})$ let $X_0^1, \dots, X_0^n$ be $\mathcal{F}_0$ measurable variables I.I.D. according to $|m|$ and $(W^1, \dots, W^n)$ a n-dimensional $(\mathcal{F}_t)$ Brownian motion. The first time when two particles with opposite sign merge is

$$\tau_1 = \inf\{s > 0,\ \exists i, j \in [1, n] \text{ with } h(X_0^i) = -h(X_0^j) \text{ such that } X_0^i + \sigma W_s^i = X_0^j + \sigma W_s^j\}.$$

When $n^+ = \mathrm{Card}(\{i \in [1, n],\ h(X_0^i) = 1\})$ and $n^- = \mathrm{Card}(\{i \in [1, n],\ h(X_0^i) = -1\})$ are both positive, then respectively by recurrence of straights lines and polarity of points for the two-dimensional Brownian motion, $\mathbb{Q}$ a.s., $\tau_1 < +\infty$ and

$$I^1 = \{i \in [1, n], \exists j \in [1, n],\ h(X_0^i) = -h(X_0^j) \text{ and } X_0^i + \sigma W_{\tau_1}^i = X_0^j + \sigma W_{\tau_1}^j\}$$

contains two elements. If $n^+ \geq 2$ and $n^- \geq 2$, then $\mathbb{Q}$ a.s.

$$\tau_2 = \inf\{s > \tau_1,\ \exists i, j \in [1, n] \setminus I^1 \text{ with } h(X_0^i) = -h(X_0^j),\ X_0^i + \sigma W_s^i = X_0^j + \sigma W_s^j\} < +\infty$$

and $I^2 = \{i \in [1, n] \setminus I^1, \exists j \in [1, n] \setminus I^1,\ h(X_0^i) = -h(X_0^j)$ and $X_0^i + \sigma W_{\tau_2}^i = X_0^j + \sigma W_{\tau_2}^j\}$ contains two elements. Inductively, we obtain that $\mathbb{Q}$ a.s. $0 < \tau_1 < \tau_2 < \ldots < \tau_{n^+\wedge n^-} < +\infty$, where

$$\tau_k = \inf\{s > \tau_{k-1},\ \exists i, j \in [1, n] \setminus (I^1 \cup \ldots \cup I^{k-1}) \text{ with } h(X_0^i) = -h(X_0^j),\ X_0^i + \sigma W_s^i = X_0^j + \sigma W_s^j\}$$

(convention: $\tau_0 = 0$) and $I^k = \{i \in [1, n] \setminus (I^1 \cup \ldots \cup I^{k-1}), \exists j \in [1, n] \setminus (I^1 \cup \ldots \cup I^{k-1}),\ h(X_0^i) = -h(X_0^j)$ and $X_0^i + \sigma W_{\tau_k}^i = X_0^j + \sigma W_{\tau_k}^j\}$ contains two elements. At time τ_k, we kill the pair of particles with opposite sign which have just merged. More precisely, for convenience we freeze their position : $\forall 1 \leq k \leq n^+ \wedge n^-,\ \forall i \in I^k,\ \forall t \geq 0,\ X_t^i = X_0^i + \sigma W_{t\wedge\tau_k}^i$. After time $\tau_{n^+\wedge n^-}$, either there is no remaining particle (case $n^+ = n^- = n/2$) or all the remaining particles share the same sign and keep moving according to the corresponding coordinates of the Brownian motion : $\forall i \in [1, n] \setminus (I^1 \cup \ldots \cup I^{n^+\wedge n^-}),\ \forall t \geq 0,\ X_t^i = X_0^i + \sigma W_t^i$.
Let $I_t = \emptyset$ if $0 \leq t < \tau_1$, $= \bigcup_{l=1}^k I^l$ if $\tau_k \leq t < \tau_{k+1}$ for $1 \leq k \leq n^+ \wedge n^-$ (convention $\tau_{n^+\wedge n^-+1} = +\infty$) denote the set of indexes of particles killed at time t. The approximate solution is constructed thanks to the surviving particles :

$$U_\sigma^n(t, x) = \frac{1}{n} \sum_{i \notin I_t} h(X_0^i) H(x - X_t^i).$$

We denote by $\mu^n = \frac{1}{n} \sum_{i=1}^n \delta_{X^i}$ the empirical measure of the system. According to definition 0.1, $\tilde{\mu}^n = \frac{1}{n} \sum_{i=1}^n h(X_0^i) \delta_{X^i}$. Since the indexes in I_t correspond to couples of particles with the same position but opposite sign, as their position is frozen after the time when they merge, we have

$$\tilde{\mu}_t^n = \frac{1}{n} \sum_{i \notin I_t} h(X_0^i) \delta_{X_t^i} \quad \text{and} \quad U_\sigma^n(t, x) = H * \tilde{\mu}_t^n(x). \tag{1.1}$$

By Girsanov theorem, if $\mathbb{P} \in \mathcal{P}(C([0, +\infty), \mathbb{R}))$ is defined by

$$\left.\frac{d\mathbb{P}}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = \exp\left(\frac{1}{\sigma} \sum_{i=1}^n \int_0^t A'(U_\sigma^n(s, X_s^i)) dB_s^i - \frac{1}{2\sigma^2} \sum_{i=1}^n \int_0^t A'(U_\sigma^n(s, X_s^i))^2 ds\right)$$

then for $B_t^i = W_t^i - \frac{1}{\sigma} \int_0^t A'(U_\sigma^n(s, X_s^i)) ds$, $(B^1, \dots, B^n)$ is a $\mathbb{P}$ n-dimensional Brownian motion. Moreover the particle system $(X_t^1, \dots, X_t^n)$ solves

$$X_t^i = X_0^i + \int_0^t 1_{\{i \notin I_s\}} \left(\sigma dB_s^i + A'(U_\sigma^n(s, X_s^i)) ds\right),\ 1 \leq i \leq n. \tag{1.2}$$

For notational simplicity, we do not emphasize the dependence of $\mathbb{P}$ on n. The probability measures $\mathbb{P}$ and $\mathbb{Q}$ are not necessarily equivalent on $\mathcal{F}$. As a consequence, it is possible that $\mathbb{P}(\tau_k < +\infty) < 1$ for some $k \in [1, n^+ \wedge n^-]$. Nethertheless, since $\mathbb{P}$ and $\mathbb{Q}$ are equivalent on $\mathcal{F}_t$ for any $t \in [0, +\infty)$, defining $k_{max} = \max\{k \leq n^+ \wedge n^- : \tau_k < +\infty\}$ (convention $\max \emptyset = 0$), $\mathbb{P}$ a.s. $0 < \tau_1 < \ldots < \tau_{k_{max}} < +\infty$ and $\forall k \in [1, k_{max}]$, I_k contains two elements.

To state the convergence result of the approximate solution $U_\sigma^n(t,x) = \frac{1}{n}\sum_{i \notin I_t} h(X_0^i) H(x - X_t^i) = \frac{1}{n}\sum_{i=1}^n h(X_0^i) H(x - X_t^i)$ to the solution u_σ of (0.3), we introduce the weighted space

$$L^1_{1/(1+x^2)} = \left\{ f : \mathbb{R} \to \mathbb{R} : \; |||f||| \stackrel{\text{def}}{=} \int_{\mathbb{R}} \frac{|f(x)|}{1+x^2} dx < +\infty \right\}.$$

For any $1 \leq i \leq n$, the continuity of $t \to X_t^i$ implies that $H(x - X_t^i) \in C([0,+\infty), L^1_{1/(1+x^2)})$. Hence $U_\sigma^n \in C([0,+\infty), L^1_{1/(1+x^2)})$ by linearity.

Theorem 1.2 *The viscous conservation law* (0.3) *has a unique bounded weak solution* u_σ. *Moreover* u_σ *belongs to* $L^1_{1/(1+x^2)}$ *and the approximate solution* U_σ^n *converges to it in the following sense :*

$$\forall T > 0, \lim_{n \to +\infty} \mathbb{E} \sup_{t \leq T} |||U_\sigma^n(t,x) - u_\sigma(t,x)||| = 0$$

where $\mathbb{E}$ *denotes the expectation with respect to the probability measure* $\mathbb{P}$.

Let π_σ^n denote the image of $\mathbb{P}$ by $\mu^n = \frac{1}{n}\sum_{i=1}^n \delta_{X^i}$. We are going to take advantage of the equality $U_\sigma^n(t,x) = H * \tilde{\mu}_t^n(x)$ to study properties of the sequence $(\pi_\sigma^n)_n$ in order to prove the Theorem.

Lemma 1.3 *The sequence* $(\pi_\sigma^n)_n$ *is tight*

Proof : Since μ^n is the empirical measure of the exchangeable processes $(X^1, \ldots, X^n)$, according to [11], the tightness of $(\pi_\sigma^n)_n$ is equivalent to the tightness of the distributions of the processes X^1. Let $0 \leq s \leq t \leq T$ and $1 \leq i \leq n$,

$$|X_t^1 - X_s^1| \leq \sigma \sup_{r \in [s,t]} |B_r^1 - B_s^1| + \int_s^t |A'(U_\sigma^n(r, X_r^1))| dr.$$

Remarking that A' is bounded on $[-1,1]$ and applying Burkholder-Davis-Gundy inequality, we obtain

$$\mathbb{E}((X_t^1 - X_s^1)^4) \leq C_T (t-s)^2 \tag{1.3}$$

where the constant C_T does not depend on n and is non-decreasing in σ. As for any $n \geq 1$, X_0^1 is distributed according to $|m|/\|m\|$, by Kolmogorov criterion, we conclude that both sequences are tight. ∎

Proposition 1.4 *Any weak limit* π_σ^∞ *of the tight sequence* $(\pi_\sigma^n)_n$ *gives full measure to*

$$\{Q \in \mathcal{P}(C([0,+\infty), \mathbb{R})) \text{ such that } H * \tilde{Q}_s(x) \text{ solves (0.3) weakly}\}.$$

To prove the Proposition, we have to deal with the possible lack of regularity of the density h. We approximate $h(x)$ by functions of the form $(1 - Cd(x,F)) \vee -1$ where $C > 0$ and $d(x,F)$ is the distance from x to some closed set F included in $\{x : h(x) = 1\}$. By the regularity of the probability measure $|m|$, $|m|(\{x : h(x) = 1\} \setminus F)$ can be chosen arbitrarily small. We deduce that :

Lemma 1.5 *For any $\epsilon > 0$, there is a Lipschitz continuous function h^ϵ with values in $[-1,1]$ such that $|m|(\{x : \; h(x) \neq h^\epsilon(x)\}) \leq \epsilon$.*

Proof of Proposition 1.4 : Let π_σ^∞ denote the limit point of a weakly converging subsequence of $(\pi_\sigma^n)_n$ that we still index by n for simplicity, g be a C^∞ function with compact support on $[0,+\infty)\times\mathbb{R}$ and $\phi(t,x) = \int_{-\infty}^x g(t,y)dy$. Computing $\phi(t, X_t^i)$ by Itô's formula and (1.2), summing over i the obtained equality multiplied by $h(X_0^i)$, we obtain

$$
\begin{aligned}
< \tilde{\mu}_t^n, \phi(t,.) > - < \tilde{\mu}_0^n, \phi(0,.) > - \int_0^t < \tilde{\mu}_s^n, \partial_s\phi(s,.) + \frac{\sigma^2}{2}\partial_{xx}\phi(s,.) + A'(U_\sigma^n(s,.))\partial_x\phi(s,.) > ds \\
= \frac{\sigma}{n}\sum_{i=1}^n \int_0^t 1_{\{i\notin I_s\}}\partial_x\phi(s,X_s^i)dB_s^i.
\end{aligned}
$$

The right-hand-side converges to 0 in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ as $n \to +\infty$. So does the left-hand-side which is transformed by spatial integrations by parts into

$$
\begin{aligned}
&\tilde{\mu}_t^n(\mathbb{R})\int_{\mathbb{R}} g(t,y)dy - \int_{\mathbb{R}} g(t,y)H * \tilde{\mu}_t^n(y)dy - \tilde{\mu}_0^n(\mathbb{R})\int_{\mathbb{R}} g(0,y)dy + \int_{\mathbb{R}} g(0,y)H * \tilde{\mu}_0^n(y)dy \\
&- \int_0^t \tilde{\mu}_s^n(\mathbb{R})\int_{\mathbb{R}} \partial_s g(s,y)dyds + \int_0^t\int_{\mathbb{R}} H * \tilde{\mu}_s^n(y)(\partial_s + \frac{\sigma^2}{2}\partial_{xx})g(s,y)dyds \\
&+ \int_0^t\int_{\mathbb{R}} \partial_x g(s,y)\int_{-\infty}^y A'(U_\sigma^n(s,z))\tilde{\mu}_s^n(dz)dyds
\end{aligned}
$$

As $\tilde{\mu}_s^n(\mathbb{R})$ does not depend on s, the sum of the first, the third and the fifth term is nil. It is an easy consequence of the occupation times formula that $\mathbb{P}$ *a.s.*, ds *a.e.*, $\forall i \neq j \in [1,n] \setminus I_s$, $X_s^i \neq X_s^j$. When this property is satisfied, according to (1.1),

$$
\begin{aligned}
&\left| A(U_\sigma^n(s,y)) - A(0) - \int_{-\infty}^y A'(U_\sigma^n(s,z))\tilde{\mu}_s^n(dz) \right| = \\
&\left| \sum_{\substack{i\notin I_s \\ X_s^i \leq y}} A\left(\sum_{j\notin I_s} 1_{\{X_s^j \leq X_s^i\}}\frac{h(X_0^j)}{n}\right) - A\left(\sum_{j\notin I_s} 1_{\{X_s^j < X_s^i\}}\frac{h(X_0^j)}{n}\right) - \frac{h(X_0^i)}{n}A'\left(\sum_{j\notin I_s} 1_{\{X_s^j \leq X_s^i\}}\frac{h(X_0^j)}{n}\right) \right| \\
&\leq \sup_{\substack{x,z\in[-1,1] \\ |x-z|\leq 1/n}} |A'(x) - A'(z)| \longrightarrow_{n\to+\infty} 0.
\end{aligned}
$$

We conclude that for the bounded function $F : \mathcal{P}(C([0,+\infty), \mathbb{R})) \to \mathbb{R}$:

$$
\begin{aligned}
F(Q) = &\int_{\mathbb{R}} g(0,y)H * \tilde{Q}_0(y)dy - \int_{\mathbb{R}} g(t,y)H * \tilde{Q}_t(y)dy \\
&+ \int_0^t\int_{\mathbb{R}} H * \tilde{Q}_s(y)(\partial_s + \frac{\sigma^2}{2}\partial_{xx})g(s,y) + A(H * \tilde{Q}_s(y))\partial_x g(s,y)dyds
\end{aligned}
$$

$\mathbb{E}|F(\mu^n)|$ converges to zero as $n \to +\infty$. In spite of the weak convergence of π_σ^n to π_σ^∞, we cannot deduce immediately that $\mathbb{E}^{\pi_\sigma^\infty}|F(Q)| = 0$ since because of the density h, the function F is not necessarily continuous. That is why we define a continuous function F^ϵ by replacing $H * \tilde{Q}_s(x)$ by $< Q, H(x - X_s)h^\epsilon(X_0) >$ in the definition of F to upper-bound $\mathbb{E}^{\pi_\sigma^\infty}|F(Q)|$.

$$\mathbb{E}^{\pi_\sigma^\infty}|F(Q)| \leq \mathbb{E}^{\pi_\sigma^\infty}(|F - F^\epsilon|(Q)) + |(\mathbb{E}^{\pi_\sigma^\infty} - \mathbb{E}^{\pi_\sigma^n})|F^\epsilon(Q)|| + \mathbb{E}^{\pi_\sigma^n}(|F - F^\epsilon|(Q)) + \mathbb{E}^{\pi_\sigma^n}|F(Q)|$$

As F^ϵ is a continuous and bounded function, for fixed $\epsilon > 0$, the second term of the right-hand-side converges to 0 as $n \to +\infty$. As the initial variables $(X_0^1, \ldots, X_0^n)$ are I.I.D. according to $|m|/\|m\|$, using Lemma 1.5 we obtain $\forall n \geq 1$, $\forall (s, x) \in [0, +\infty) \times \mathbb{R}$,

$$\mathbb{E}^{\pi_\sigma^n}|H * \tilde{Q}_s(x) - < Q, H(x - X_s)h^\epsilon(X_0) > | \leq \frac{1}{n}\sum_{i=1}^n \mathbb{E}(|h - h^\epsilon|(X_0^i)) = \mathbb{E}(|h - h^\epsilon|(X_0^1)) \leq \epsilon.$$

With the uniform continuity of the function A' on $[-1, 1]$, we deduce that $\mathbb{E}^{\pi_\sigma^n}(|F - F^\epsilon|(Q))$ converges to 0 uniformly in n as $\epsilon \to 0$. Remarking that π_σ^∞ a.s., $Q_0 = |m|$, we check that $\mathbb{E}^{\pi_\sigma^\infty}(|F - F^\epsilon|(Q))$ also converges to 0. Hence $\mathbb{E}^{\pi_\sigma^\infty}|F(Q)| = 0$. Taking t, c, g in denumerate dense sets and then taking limits, we deduce that π_σ^∞ a.s., for any test function g, $F(Q) = 0$ i.e. π_σ^∞ a.s. $H * \tilde{Q}_s(x)$ is a weak solution of (0.3). ∎

We are now ready to conclude the Proof of Theorem 1.2

Proof of Theorem 1.2 : Proposition 1.4 ensures existence of bounded weak solutions of (0.3). If u is such a solution, then by a good choice of test functions one obtains the following integral representation :

$$dx \text{ a.e.}, \quad u(t, x) = G_t^\sigma * (H * m)(x) - \int_0^t \left(\partial_x G_{t-s}^\sigma * A(u(s, .))\right)(x)ds$$

where $G_t^\sigma(x) = \exp(-x^2/2\sigma^2 t)/\sigma\sqrt{2\pi t}$ denotes the heat kernel. Uniqueness of bounded weak solutions is easily derived (see [6] for instance). From now on, u_σ denotes the unique bounded weak solution of (0.3). Again according to Proposition 2.2, there exists $Q \in \mathcal{P}(C([0, +\infty), \mathbb{R})$ such that $u_\sigma(s, x)$ is equal to $H * \tilde{Q}_s(x)$. Since $\forall t \geq 0$, $s \to H * \tilde{Q}_s(x) =< Q, h(X_0)H(x - X_s) >$ is continuous at t as soon as $Q_t(\{x\}) = 0$ (condition satisfied dx a.e.), we deduce that the function u_σ belongs to $C([0, +\infty), L^1_{1/(1+x^2)})$.

Let $T > 0$. We want to prove that 0 is the only limit point of $(\mathbb{E}\sup_{t \in [0,T]} |||U_\sigma^n(t, x) - u_\sigma(t, x)|||)_n$. For any subsequence, according to Lemma 1.3, we can extract from the corresponding subsequence of $(\pi_\sigma^n)_n$ a further subsequence converging weakly to π_σ^∞, that we still index by n for simplicity. Since $U_\sigma^n(t, x) = H * \tilde{\mu}_t^n(x)$, it is sufficient to show that $\lim_n \mathbb{E}^{\pi_\sigma^n} \sup_{t \leq T} |||H * \tilde{Q}_t(x) - u(t, x)||| = 0$. The function $Q \to \sup_{t \leq T} |||H * \tilde{Q}_t(x) - u(t, x)|||$ is not necessarily continuous. That is why, for $\epsilon > 0$, we introduce $H^\epsilon(x) = 1_{\{x>0\}} + \frac{x+\epsilon}{\epsilon}1_{\{-\epsilon \leq x \leq 0\}}$ and h^ϵ as in Lemma 1.5 which are Lipschitz continuous approximations of the functions H and h. Using Proposition 2.2, we get

$$\begin{aligned}\mathbb{E}^{\pi_\sigma^n} \sup_{t \in [0,T]} |||H * \tilde{Q}_t(x) - u(t, x)||| &\leq (\mathbb{E}^{\pi_\sigma^n} - \mathbb{E}^{\pi_\sigma^\infty}) \sup_{t \in [0,T]} ||| < Q, H^\epsilon(x - X_t)h^\epsilon(X_0) > -u(t, x)||| \\ &+ (\mathbb{E}^{\pi_\sigma^n} + \mathbb{E}^{\pi_\sigma^\infty}) \sup_{t \in [0,T]} ||| < Q, H^\epsilon(x - X_t)h^\epsilon(X_0) - H(x - X_t)h(X_0) > |||. \quad (1.4)\end{aligned}$$

The functions $Q \in \mathcal{P}(C([0, +\infty), \mathbb{R})) \to < Q, H^\epsilon(x - X_t)h^\epsilon(X_0) >$ indexed by $(t, x) \in [0, T] \times \mathbb{R}$ are equicontinuous and bounded by 1. We deduce that $Q \to \sup_{t \in [0,T]} ||| < Q, H^\epsilon(x -$

$X_t)h^\epsilon(X_0) > -u(t,x)|||$ is continuous and bounded. Hence for fixed ϵ, the first term of the right-hand-side of (1.4) converges to 0 as $n \to +\infty$.

$$
\begin{aligned}
||| < Q, H^\epsilon(x-X_t)h^\epsilon(X_0) - H(x-X_t)h(X_0) > ||| &\leq ||| < Q, |h^\epsilon - h|(X_0) > ||| + |||Q_t((x-\epsilon, x])||| \\
&= \pi| < Q, |h^\epsilon - h|(X_0) > | + \int_{\mathbb{R}} \left(\int_y^{y+\epsilon} \frac{dx}{1+x^2} \right) Q_t(dy) \\
&\leq \pi| < Q, |h^\epsilon - h|(X_0) > | + 2\arctan\left(\frac{\epsilon}{2}\right)
\end{aligned}
$$

As the variables $(X_0^1, \dots, X_0^n)$ are I.I.D. according to $|m|$, π_σ^∞ a.s., $Q_0 = |m|$. With Lemma 1.5, we obtain that the second term of the right-hand-side of (1.4) converges to 0 uniformly in n as $\epsilon \to 0$. ∎

2 Convergence of the approximate solution to the entropy solution of (0.1)

2.1 The convergence result

Let $(\sigma_n)_n$ be a sequence of positive numbers such that $\lim_{n\to+\infty} \sigma_n = 0$ and $(X^1, \dots, X^n)$ and $\mathbb{P}$ be defined like previously with σ_n replacing σ. We are interested in the asymptotic behaviour of $U_{\sigma_n}^n(t,x) = \frac{1}{n}\sum_{i=1}^n h(X_0^i)H(x - X_t^i)$ as $n \to +\infty$. Considering Theorem 1.2 and the convergence of the solution u_σ of the viscous conservation law (0.3) to the unique entropy solution of (0.1) as $\sigma \to 0$, our main result is not surprising.

Theorem 2.1 *If $(\sigma_n)_n$ is a sequence of positive numbers such that $\lim_{n\to+\infty} \sigma_n = 0$, then the approximate solution $U_{\sigma_n}^n(t,x)$ converges to the unique entropy solution $u(t,x)$ of (0.1) with initial data $u_0(x) = H * m(x)$ in $C([0,+\infty), L^1_{1/(1+x^2)})$. More precisely,*

$$\forall T > 0, \lim_{n\to+\infty} \mathbb{E} \sup_{t\leq T} |||U_{\sigma_n}^n(t,x) - u(t,x)||| = 0.$$

Let $\pi_{\sigma_n}^n$ denote the image of $\mathbb{P}$ by the empirical measure $\mu^n = \frac{1}{n}\sum_{i=1}^n \delta_{X^i}$. Since the sequence $(\sigma_n)_n$ is bounded, by an easy adaptation of the proof of Lemma 1.3, we check that the sequence $(\pi_{\sigma_n}^n)_n$ is tight. The proof of Theorem 2.1 is the same as the one of Theorem 1.2 as soon as we check that the following Proposition analogous to Proposition 1.4 holds :

Proposition 2.2 *Any weak limit π_0^∞ of the tight sequence $(\pi_{\sigma_n}^n)_n$ gives full measure to*

$$\{Q \in \mathcal{P}(C([0,+\infty), \mathbb{R})) \text{ such that the entropy solution of (0.1) is equal to } H * \tilde{Q}_s(x)\}.$$

Before introducing reordered particles in the general case in order to prove this Proposition, we first suppose that m is a probability measure. In this much simpler case, since all particles are positive there is no killing and the definition of the system of reordered particles is quite simple. Moreover, we deduce from Proposition 2.2 a propagation of chaos for this system.

2.2 Propagation of chaos for the reordered system in case m is a probability measure

By Kruzkhov uniqueness result for entropy solutions of (0.1), there is no more than one mapping $P(t) \in C([0,+\infty),\mathcal{P}(\mathbb{R}))$ such that the entropy solution $u(s,x)$ of (0.1) is equal to $(H * P(s))(x)$. Combining the tightness of the distributions of the empirical measures μ^n, the continuity of the mapping $Q \in \mathcal{P}(C([0,+\infty),\mathbb{R})) \to (t \to Q_t) \in C([0,+\infty),\mathcal{P}(\mathbb{R}))$ and Proposition 2.2, we deduce the following convergence result for the flow of time-marginals $t \to \mu^n_t$.

Corollary 2.3 *The variables $t \to \mu^n_t \in C([0,+\infty),\mathcal{P}(\mathbb{R}))$ converge in distribution to the unique mapping $P(t) \in C([0,+\infty),\mathcal{P}(\mathbb{R}))$ such that the entropy solution $u(s,x)$ of (0.1) is equal to $(H * P(s))(x)$.*

This convergence is weaker than a classical propagation of chaos result i.e. the convergence in distribution of the empirical measures μ^n considered as $\mathcal{P}(C([0,+\infty),\mathbb{R}))$-valued random variables to a constant P. Here the natural candidate for the limit is a probability measure $P \in \mathcal{P}(C([0,+\infty),\mathbb{R}))$ such that $H * P_s(x)$ is equal to the entropy solution $u(s,x)$ of (0.1) and P a.s., $\forall t \geq 0$, $X_t = X_0 + \int_0^t A'(H * P_s(X_s))ds$. We would like to prove uniqueness of probability measures satisfying both these properties and to check that any weak limit π^∞_0 of the sequence $(\pi^n_{\sigma_n})_n$ is concentrated on such probability measures. Because of the possible discontinuities of the entropy solution $u(t,x)$, we cannot prove these results.
Nethertheless, we are able to prove a propagation of chaos on the sample-path space for the reordered particle system $(Y^1,\ldots,Y^n)$ which is defined as follows : for any $t \geq 0$, $Y^1_t \leq Y^2_t \leq \ldots \leq Y^n_t$ is the increasing reordering (order statistics) of $(X^1_t,\ldots,X^n_t)$. By an easy adaptation of the proof given in [7] for particle systems associated with the porous medium equation, we check that $(Y^1,\ldots,Y^n)$ is a diffusion normally reflected at the boundary of the closed convex set $D_n = \{y = (y^1,\ldots,y^n) \in \mathbb{R}^n,\ y^1 \leq y^2 \leq \ldots \leq y^n\}$. More precisely, for $1 \leq j \leq n$,

$$Y^j_t = Y^j_0 + \sigma_n \beta^j_t + \int_0^t A'(U^n_{\sigma_n}(s, Y^j_s))ds + \int_0^t (\gamma^j_s - \gamma^{j+1}_s) d|V|_s$$

where $\beta^j_t = \int_0^t \sum_{i=1}^n 1_{\{Y^j_s = X^i_s\}} dB^i_s$, $\gamma^1_s = \gamma^{n+1}_s = 0$, $(\int_0^t (\gamma^j_s - \gamma^{j+1}_s) d|V|_s)_{1\leq j\leq n}$ is a continuous process with finite variation $|V|_t$ and $d|V|_s$ a.e. $\forall 2 \leq j \leq n$, $\gamma^j_s \geq 0$ and $\gamma^j_s(Y^j_s - Y^{j-1}_s) = 0$. By the occupation times formula $\mathbb{P}$ a.s., ds a.e. the positions $X^1_s,\ldots,X^n_s$ are distinct. As a consequence $\forall 1 \leq i,j \leq n$, $< \beta^i, \beta^j >_t = 1_{\{i=j\}} t$ and $(\beta^1,\ldots,\beta^n)$ is a n-dimensional Brownian motion. Moreover ds a.e., $\forall 1 \leq j \leq n$, $U^n_{\sigma_n}(s, Y^j_s) = j/n$ i.e. the reordered sample-paths are stochastic characteristics along which the approximate solution is ds a.e. constant.
Let $\eta^n = \frac{1}{n}\sum_{i=1}^n \delta_{Y^i}$ denote the corresponding empirical measure. Even if $\forall s \geq 0$, $\eta^n_s = \mu^n_s$, in general $\eta^n \neq \mu^n$. For $Q \in \mathcal{P}(C([0,+\infty),\mathbb{R}))$ let $G^Q_t : x \in [0,1] \to \inf\{y :\ H * Q_t(y) \geq x\}$ denote the pseudo-inverse of the cumulative distribution function of the marginal Q_t. The Lebesgue measure on $[0,1]$ is denoted by λ. We recall that $Q_t = \lambda \circ (G^Q_t)^{-1}$.

Theorem 2.4 *The empirical measures $\eta^n \in \mathcal{P}(C([0,+\infty),\mathbb{R}))$ of the reordered particle systems converge in distribution to the unique P element of*

$$\mathcal{A} = \{Q \in \mathcal{P}(C([0,+\infty),\mathbb{R})) :\ \forall k \in \mathbb{N}^*,\ \forall 0 \leq t_1 < t_2 < \ldots < t_k,\ Q_{t_1,\ldots,t_k} = \lambda \circ (G^Q_{t_1},\ldots,G^Q_{t_k})^{-1}\}$$

and such that $\forall t \geq 0$, $P_t = P(t)$.

Proof : Since the finite-dimensional marginals $Q_{t_1,\ldots,t_k}$ of $Q \in \mathcal{A}$ are determined by its one-dimensional marginals Q_t, there is no more than one probability measure $P \in \mathcal{A}$ such that $\forall t \geq 0,\ P_t = P(t)$.
We have to check that the distribution $\bar{\pi}^n$ of the empirical measures η_n converge weakly to a probability measure concentrated on $\{Q \in \mathcal{A} :\ \forall t \geq 0,\ Q_t = P(t)\}$. According to Sznitman [11], the tightness of the sequence $(\bar{\pi}^n)_n$ is equivalent to the tightness of the sequence $(\frac{1}{n}\sum_{j=1}^n \mathbb{P} \circ (Y^j)^{-1})_n$. We easily check that $\forall n \geq 1,\ \frac{1}{n}\sum_{j=1}^n \mathbb{P} \circ (Y_0^j)^{-1} = m$. Moreover, if $y_1 \leq y_2 \leq \ldots \leq y_n$ (resp. $y'_1 \leq y'_2 \leq \ldots \leq y'_n$) denote the increasing reordering of $(x^1, \ldots, x^n) \in \mathbb{R}^n$ (resp. $(x'_1, \ldots, x'_n)$), $\sum_{i=1}^n (y'_i - y_i)^4 \leq \sum_{i=1}^n (x'_i - x_i)^4$: this inequality can be checked by an easy computation for $n = 2$ and then generalized by induction. Hence

$$\forall T > 0,\ \forall s, t \in [0,T],\ \frac{1}{n}\sum_{j=1}^n \mathbb{E}((Y_t^j - Y_s^j)^4) \leq \frac{1}{n}\sum_{i=1}^n \mathbb{E}((X_t^i - X_s^i)^4) \leq C_T(t-s)^2.$$

By Kolmogorov criterion, we conclude that both sequences are tight.

Let $\bar{\pi}^\infty$ denote the limit of a convergent subsequence of $(\bar{\pi}^n)_n$ that we still index by n for simplicity. Since $\forall t,\ \eta_t^n = \mu_t^n$ and $Q \in \mathcal{P}(C([0,+\infty),\mathbb{R})) \to (t \to Q_t) \in C([0,+\infty),\mathcal{P}(\mathbb{R}))$ is continuous, by Corollary 2.3, we obtain that $\bar{\pi}^\infty(\{Q :\ \forall t \geq 0,\ Q_t = P(t)\}) = 1$. As $\mathcal{A}$ is closed (see Lemma 2.5 below), $\bar{\pi}^\infty(\mathcal{A}) \geq \limsup_n \bar{\pi}^n(\mathcal{A})$.
We easily check that for $0 \leq t_1 < t_2 < \ldots < t_k$,

$$\forall 1 \leq i \leq n,\ \forall x \in ((i-1)/n, i/n],\ (G_{t_1}^{\eta^n}, \ldots, G_{t_k}^{\eta^n})(x) = (Y_{t_1}^i, \ldots, Y_{t_k}^i).$$

Hence $\bar{\pi}^n(\mathcal{A}) = 1$ which concludes the proof. ∎

Lemma 2.5 *The set $\mathcal{A}$ is closed for the weak convergence topology. Moreover it is equal to $\tilde{\mathcal{A}} = \{Q \in \mathcal{P}(C([0,+\infty),\mathbb{R})) :\ \forall x \in [0,1],\ Q(\inf_{s \geq 0} H * Q_s(X_s) \leq x) \leq x\}$.*

Proof : Suppose that $(Q^n)_n \in \mathcal{A}$ converges weakly to Q. Let $t_1 < t_2 < \ldots < t_k$. According to Billingsley [1](proof of Theorem 25.6 p.343), $\forall 1 \leq i \leq k$, $\lambda(dx)$ a.e. $G_{t_i}^{Q^n}(x) \to G_{t_i}^Q(x)$. Hence $\lambda(dx)$ a.e. $(G_{t_1}^{Q^n}, \ldots, G_{t_k}^{Q^n})(x) \to (G_{t_1}^Q, \ldots, G_{t_k}^Q)(x)$. Since $Q_{t_1,\ldots,t_k}^n = \lambda \circ (G_{t_1}^{Q^n}, \ldots, G_{t_k}^{Q^n})^{-1}$ converges weakly to $Q_{t_1,\ldots,t_k}$, we deduce that $Q_{t_1,\ldots,t_k} = \lambda \circ (G_{t_1}^Q, \ldots, G_{t_k}^Q)^{-1}$. Hence $\mathcal{A}$ is closed.

For $Q \in \mathcal{P}(C([0,+\infty),\mathbb{R}))$, because of the weak continuity of $s \to Q_s$, $\inf_{s \geq 0} H * Q_s(X_s) = \inf_{q \in \mathbb{Q}_+} H * Q_q(X_q)$ and $X \to \inf_{s \geq 0} H * Q_s(X_s)$ is measurable.
Let $Q \in \mathcal{A}$, $(q_i)_{i \in \mathbb{N}^*}$ denote the elements of $\mathbb{Q}_+$ and $x \in [0,1]$. Since $H * Q_t(G_t^Q(y)) \geq y$,

$$\begin{aligned} Q\bigg(&\min(H * Q_{q_1}(X_{q_1}), \ldots, H * Q_{q_k}(X_{q_k})) \leq x\bigg) \\ &= \lambda\bigg(y :\ \min(H * Q_{q_1}(G_{q_1}^Q(y)), \ldots, H * Q_{q_k}(G_{q_k}^Q(y))) \leq x\bigg) \leq \lambda(y :\ y \leq x) = x. \end{aligned}$$

Taking the limit $k \to +\infty$, we deduce $Q(\inf_{q \in \mathbb{Q}_+} H * Q_q(X_q) \leq x) \leq x$. We easily conclude that $Q \in \tilde{\mathcal{A}}$.
Let $Q \in \tilde{\mathcal{A}}$, $t_1 < t_2 < \ldots < t_k$, $x \in \mathbb{R}$ and $1 \leq i \leq k$. As $\{G_t^Q(y) \leq x\} = \{y \leq H * Q_t(x)\}$,

$$Q(\{G_{t_i}^Q(\min_{j=1}^k H * Q_{t_j}(X_{t_j})) \leq x\}) = Q(\{\min_{j=1}^k H * Q_{t_j}(X_{t_j}) \leq H * Q_{t_i}(x)\}) \leq H * Q_{t_i}(x)$$

Moreover since $G_t^Q(H*Q_t(y)) \leq y$, the converse inequality holds :

$$Q(\{G_{t_i}^Q(\min_{j=1}^k H*Q_{t_j}(X_{t_j})) \leq x\}) \geq Q(\{G_{t_i}^Q(H*Q_{t_i}(X_{t_i})) \leq x\}) \geq Q(X_{t_i} \leq x) = H*Q_{t_i}(x).$$

Hence if

$$\Gamma^Q_{t_1,\dots,t_k} : x \in [0,1] \to \inf\{y : \ Q(\min_{j=1}^k H*Q_{t_j}(X_{t_j}) \leq y) \geq x\},$$

$Q_{t_1,\dots,t_k} = \lambda \circ ((G^Q_{t_1},\dots,G^Q_{t_k}) \circ \Gamma^Q_{t_1,\dots,t_k})^{-1}$. Since $Q \in \tilde{\mathcal{A}}$, $\forall y \in [0,1]$, $Q(\min_{j=1}^k H*Q_{t_j}(X_{t_j}) \leq y) \leq y$, which implies $\Gamma^Q_{t_1,\dots,t_k}(x) \geq x$. As $Q_{t_i} = \lambda \circ (G^Q_{t_i})^{-1}$ we deduce that $\lambda(dx)$ a.e., $G^Q_{t_i}(x) = G^Q_{t_i}(\Gamma^Q_{t_1,\dots,t_k}(x))$. Hence $\lambda(dx)$ a.e., $(G^Q_{t_1},\dots,G^Q_{t_k})(x) = (G^Q_{t_1},\dots,G^Q_{t_k})(\Gamma^Q_{t_1,\dots,t_k}(x))$ and $Q_{t_1,\dots,t_k} = \lambda \circ (G^Q_{t_1},\dots,G^Q_{t_k})^{-1}$. We conclude that $\tilde{\mathcal{A}} \subset \mathcal{A}$. ∎

Remark 2.6 *If the entropy solution $(t,x) \to u(t,x) = H*P_t(x)$ of (0.1) is continuous, then for any $t \geq 0$, the probability measure P_t does not weight points and $\forall x \in [0,1]$, $P(H*P_t(X_t) \leq x) = x$. Since $P \in \tilde{\mathcal{A}}$ and $H*P_t(X_t) \geq \inf_{s\geq 0} H*P_s(X_s)$, we deduce that $P(H*P_t(X_t) = \inf_{s\geq 0} H*P_s(X_s)) = 1$. By the continuity of $t \to H*P_t(X_t)$, we conclude that P a.s., $t \to H*P_t(X_t)$ is constant. Hence the sample-paths $t \to X_t$ are stochastic characteristics along which the entropy solution is constant.*
On the other hand, when a shock i.e. a discontinuity curve appears at time $t_0 > 0$ and position x_0 for the entropy solution, $P_{t_0}(\{x_0\}) = P(\{X_{t_0} = x_0\}) > 0$ and for P almost all the sample-paths such that $X_{t_0} = x_0$, $t \to H*P_t(X_t)$ is constant on $[0,t_0)$ and presents a strictly positive jump at time t_0.

Remark 2.7 *For any bounded monotone initial data $u_0(x)$, Kunik [8] gives an explicit representation formula for the entropy solution of (0.1). When $u_0(x)$ is the cumulative distribution function of a probability measure, the solution is given by $u = \partial_x v$ where $v(t,x) = \sup_{s\in[0,1]}(xs - tA(s) - I(s))$ and I is a primitive of the pseudo-inverse of u_0: $x \to \inf\{y : \ u_0(y) \geq x\}$.*

2.3 System of reordered particles and probabilistic characteristics

In the general case, because of the murder of the couples of particles with opposite sign that merge, the description of the reordered system is more complicated that when m is a probability measure. We recall that in the construction of the particle system $(X^1,\dots,X^n)$, $\tau_1 < \tau_2 < \dots < \tau_{k_{max}}$ denote the successive times when couples of surviving particle with opposite sign merge and are killed. For $t \in [0,\tau_1]$ let $Y_t^1 \leq Y_t^2 \leq \dots \leq Y_t^n$ denote the increasing reordering of $(X_t^1,\dots,X_t^n)$. Again by an easy adaptation of the proof given in [7], we check that on $[0,\tau_1]$, $(Y^1,\dots,Y^n)$ is a diffusion normally reflected at the boundary of the closed convex set $D_n = \{y = (y^1,\dots,y^n) \in \mathbb{R}^n, \ y^1 \leq y^2 \leq \dots \leq y^n\}$. More precisely, for $t \leq \tau_1$ and $1 \leq j \leq n$,

$$Y_t^j = Y_0^j + \sigma_n \beta_t^j + \int_0^t A'(U^n_{\sigma_n}(s,Y_s^j))ds + \int_0^t (\gamma_s^j - \gamma_s^{j+1})d|V|_s \tag{2.1}$$

where $\beta_t^j = \int_0^t \sum_{i=1}^n 1_{\{Y_s^j = X_s^i\}} dB_s^i$, $\gamma_s^1 = \gamma_s^{n+1} = 0$, $(\int_0^t (\gamma_s^j - \gamma_s^{j+1})d|V|_s)_{1\leq j\leq n}$ is a continuous process with finite variation $|V|_t$ and $d|V|_s$ a.e. $\forall 2 \leq j \leq n$, $\gamma_s^j \geq 0$ and $\gamma_s^j(Y_s^j - Y_s^{j-1}) = 0$.
We easily check that

$$\tau_1 = \inf\{t \geq 0, \ \exists 2 \leq l \leq n, \ Y_t^l = Y_t^{l-1} \ \text{ and } \ h(Y_0^l) \neq h(Y_0^{l-1})\},$$

that there is a unique such index l denoted by l_1 and that l_1 and l_1-1 are the reordered indexes of the first pair of killed particles i.e. with original indexes in I_1. After time τ_1, we freeze Y^{l_1} and Y^{l_1-1} i.e. $\forall t\geq\tau_1,\ Y^{l_1}_t=Y^{l_1-1}_t=Y^{l_1}_{\tau_1}$ and for $l=l_1,l_1-1$, we set $\forall t\geq\tau_1,\ \beta^l_t=\beta^l_{\tau_1}+\sum_{i\in I^1}1_{\{h(Y^l_0)=h(X^i_0)\}}(B^i_t-B^i_{\tau_1})$. We list the indexes of the surviving reordered particles thanks to the increasing function $\varphi_1:[1,n-2]\to[1,n]\setminus\{l_1,l_1-1\}$.

For $t\in[\tau_1,\tau_2]$, we define $Y^{\varphi_1(1)}_t\leq\ldots\leq Y^{\varphi_1(n-2)}_t$ as the increasing reordering of the surviving particles $(X^i_t)_{i\notin I_1}$. Therefore for $t\in[\tau_1,\tau_2]$, $(Y^{\varphi_1(1)}_t,\ldots,Y^{\varphi_1(n-2)}_t)$ is a diffusion normally reflected at the boundary of $D_{n-2}:\forall 1\leq l\leq n-2,\ \forall t\in[\tau_1,\tau_2]$,

$$Y^{\varphi_1(l)}_t=Y^{\varphi_1(l)}_{\tau_1}+\sigma_n\beta^{\varphi_1(l)}_t+\int_0^t A'(U^n_{\sigma_n}(s,Y^{\varphi_1(l)}_s))ds+\int_0^t(\gamma^l_s-\gamma^{l+1}_s)d|V|_s \tag{2.2}$$

where $\beta^{\varphi_1(l)}_t=\beta^{\varphi_1(l)}_{\tau_1}+\int_{\tau_1}^t\sum_{i\notin I^1}1_{\{Y^{\varphi_1(l)}_s=X^i_s\}}dB^i_s$, $\gamma^1_s=\gamma^{n-1}_s=0$, $(\int_0^t(\gamma^l_s-\gamma^{l+1}_s)d|V|_s)_{1\leq j\leq n-2}$ is a continuous process with finite variation $|V|_t$ and $d|V|_s$ a.e. $\forall 2\leq l\leq n-2$, $\gamma^l_s\geq 0$ and $\gamma^l_s(Y^{\varphi_1(l)}_s-Y^{\varphi_1(l-1)}_s)=0$. Moreover,

$$\tau_2=\inf\{t\geq\tau_1,\ \exists 2\leq l\leq n-2,\ Y^{\varphi_1(l)}_t=Y^{\varphi_1(l-1)}_t\ \text{ and }\ h(Y^{\varphi_1(l)}_0)\neq h(Y^{\varphi_1(l-1)}_0)\},$$

and there is a unique such index l that we denote by l_2. The reordered indexes of the second pair of killed particles i.e. with original indexes in I_2 are $\varphi_1(l_2)$ and $\varphi_1(l_2-1)$. After time τ_2, we freeze their positions : $\forall t\geq\tau_2,\ Y^{\varphi_1(l_2)}_t=Y^{\varphi_1(l_2-1)}_t=Y^{\varphi_1(l_2)}_{\tau_2}$ and for $l=l_2,l_2-1$, we set $\forall t\geq\tau_2,\ \beta^{\varphi_1(l)}_t=\beta^{\varphi_1(l)}_{\tau_2}+\sum_{i\in I^2}1_{\{h(Y^{\varphi_1(l)}_0)=h(X^i_0)\}}(B^i_t-B^i_{\tau_2})$. We list the indexes of the surviving reordered particles thanks to the increasing function $\varphi_2:[1,n-4]\to[1,n]\setminus\{l_1,l_1-1,\varphi_1(l_2),\varphi_1(l_2-1)\}$.

Now supposing inductively that for some $k\leq k_{max}-1$ we have defined the reordered system up to time τ_k, the functions $\varphi_1,\ldots,\varphi_k$, the indexes $l_1,\ldots,l_k$. Then we freeze $Y^{\varphi_{k-1}(l_k)}_t=Y^{\varphi_{k-1}(l_k-1)}_t=Y^{\varphi_{k-1}(l_k)}_{\tau_k}$ for $t\geq\tau_k$ and for $l=l_k,l_k-1$, we set $\forall t\geq\tau_k,\ \beta^{\varphi_{k-1}(l)}_t=\beta^{\varphi_{k-1}(l)}_{\tau_k}+\sum_{i\in I^k}1_{\{h(Y^{\varphi_{k-1}(l)}_0)=h(X^i_0)\}}(B^i_t-B^i_{\tau_k})$. For $t\in[\tau_k,\tau_{k+1}]$, we define $Y^{\varphi_k(1)}_t\leq\ldots\leq Y^{\varphi_k(n-2k)}_t$ as the increasing reordering of $(X^i_t)_{i\notin I_1\cup\ldots\cup I_k}$ and we set $\beta^{\varphi_k(l)}_t=\beta^{\varphi_k(l)}_{\tau_k}+\int_{\tau_k}^t\sum_{i\notin I^1\cup\ldots\cup I^k}1_{\{Y^{\varphi_k(l)}_s=X^i_s\}}dB^i_s$. The index l_{k+1} is defined as the unique $l\in[2,n-2k]$ such that $Y^{\varphi_k(l)}_{\tau_{k+1}}=Y^{\varphi_k(l-1)}_{\tau_{k+1}}$ and $h(Y^{\varphi_k(l)}_0)\neq h(Y^{\varphi_k(l-1)}_0)$ and we list the indexes of the $n-2(k+1)$ surviving particles thanks to the increasing function $\varphi_k:[1,n-2(k+1)]\to[1,n]\setminus\{l_1,l_1-1,\varphi_1(l_2),\varphi_1(l_2-1),\ldots,\varphi_k(l_{k+1}),\varphi_k(l_{k+1}-1)\}$. This way, the reordered system is defined up to time $\tau_{k_{max}}$.
For $t\geq\tau_{k_{max}}$, $Y^{\varphi_{k_{max}}(1)}_t\leq\ldots\leq Y^{\varphi_{k_{max}}(n-2k_{max})}_t$ is defined as the increasing reordering of $(X^i_t)_{i\notin I_1\cup\ldots\cup I_{k_{max}}}$ and $\beta^{\varphi_{k_{max}}(l)}_t=\beta^{\varphi_{k_{max}}(l)}_{\tau_{k_{max}}}+\int_{\tau_{k_{max}}}^t\sum_{i\notin I^1\cup\ldots\cup I^{k_{max}}}1_{\{Y^{\varphi_{k_{max}}(l)}_s=X^i_s\}}dB^i_s$.

Let $N_t=n-2\sum_{k=1}^{k_{max}}1_{\{\tau_k\leq t\}}$, $J_t=\bigcup_{k:\tau_k\leq t}\{\varphi_{k-1}(l_k),\varphi_{k-1}(l_k-1)\}$ (convention : φ_0 is the identity function) and by a slight abuse of notations, $\varphi_t:l\in[1,N_t]\to\sum_{k=0}^{k_{max}}1_{[\tau_k,\tau_{k+1})}(t)\varphi_k(l)\in[1,n]\setminus J_t$ (convention : $\tau_0=0,\ \tau_{k_{max}+1}=+\infty$) denote respectively the number of particles surviving at time t, the indexes of the particles killed before time t and the original index of the l-th surviving particle. To simplify notations, we set $h_j=h(Y^j_0)$ and $U(j)=\frac{1}{n}\sum_{i=1}^j h_i$.

Proposition 2.8 *Each reordered particle is a probabilistic characteristic along which the approximate solution $U^n_{\sigma_n}(s,.)$ is ds a.e. constant up to the time when the particle is killed. More precisely, for ds a.e. $s\geq 0$, $\forall j\in[1,n]\setminus J_s$, $U^n_{\sigma_n}(s,Y^j_s)=U(j)=\frac{1}{n}\sum_{i=1}^j h_i$. Moreover the* *dynamics of the reordered system is given by :*

$$\forall 1 \leq j \leq n,\ dY_t^j = 1_{\{j \notin J_t\}} \Big[\sigma_n d\beta_t^j + A'(U(j))dt + (\gamma_t^{\varphi_t^{-1}(j)} - \gamma_t^{\varphi_t^{-1}(j)+1}) d|V|_t \Big]. \tag{2.3}$$

where $\beta = (\beta^1, \ldots, \beta^n)$ is a $\mathbb{P}$ Brownian motion and $\mathbb{P}$ a.s., $d|V|_t$ a.e. $\gamma_t^1 = \gamma_t^{N_t+1} = 0$ and for $l \in [2, N_t]$, $\gamma_t^l = 0$ if $h_{\varphi_t(l)} \neq h_{\varphi_t(l-1)}$ and $\gamma_t^l \geq 0$, $\gamma_t^l(Y_t^{\varphi_t(l)} - Y_t^{\varphi_t(l-1)}) = 0$ otherwise.

Proof : By construction $Y_t^{\varphi_t(1)} \leq \ldots \leq Y_t^{\varphi_t(N_t)}$ is the increasing reordering of $(X_t^i)_{i \notin I_t}$. Since couples of particles with opposite sign that merge are killed,

$$\{(X_t^i, h(X_0^i)), i \notin I_t\} = \{(Y_t^{\varphi_t(l)}, h_{\varphi_t(l)}),\ 1 \leq l \leq N_t\} = \{(Y_t^j, h_j),\ j \notin J_t\}.$$

According to (1.1), we deduce that $\tilde{\mu}_t^n = \frac{1}{n} \sum_{j \notin J_t} h_j \delta_{Y_t^j} = \frac{1}{n} \sum_{l=1}^{N_t} h_{\varphi_t(l)} \delta_{Y_t^{\varphi_t(l)}}$.
Hence the approximate solution writes

$$U_{\sigma_n}^n(t,x) = \frac{1}{n} \sum_{l=1}^{N_t} h_{\varphi_t(l)} 1_{\{Y_t^{\varphi_t(l)} \leq x\}}. \tag{2.4}$$

By the occupation times formula, a.s. for dt a.e. $t \geq 0$ the positions $(X_t^i)_{i \notin I_t}$ are distinct and as a consequence $Y_t^{\varphi_t(1)} < Y_t^{\varphi_t(2)} < \ldots < Y_t^{\varphi_t(N_t)}$. Hence dt a.e., $\forall j \notin J_t$, $U_{\sigma_n}^n(t, Y_t^j) = \frac{1}{n} \sum_{l=1}^{\varphi_t^{-1}(j)} h_{\varphi_t(l)} = \frac{1}{n} \sum_{i=1}^{j} h_j - \frac{1}{n} \sum_{i=1, i \in J_t}^{j} h_i$. Since the indexes in $[1,j] \cap J_t$ correspond to couples of killed particles with opposite sign, the second summation in the right-hand-side is nil and $U_{\sigma_n}^n(t, Y_t^j) = U(j)$.

Equation (2.3) is obtained by setting $l = \varphi_t^{-1}(j)$ in the successive equations similar to (2.1) and (2.2) and using the result we have just proved. Since ds a.e. the positions $(X_s^i)_{i \notin I_s}$ are distinct, $\forall 1 \leq i, j \leq n$, $< \beta^j \beta^i >_t = 1_{\{i=j\}} t$ and β is a n-dimensional Brownian motion.
By definition of the particle system, $\forall 0 \leq k \leq k_{max}$, $\forall t \in [\tau_k, \tau_{k+1})$, $\gamma_t^1 = \gamma_t^{n+1-2k} = 0$ and for $d|V|_t$ a.e. $t \in [\tau_k, \tau_k + 1)$, $\forall 2 \leq l \leq n - 2k$, $\gamma_t^l \geq 0$ and $\gamma_t^l(Y_t^{\varphi_k(l)} - Y_t^{\varphi_k(l-1)}) = 0$. As the stopping time τ_{k+1} is the first time after τ_k when two surviving particles with opposite sign merge, for $l \in [2, n-2k]$ if $h_{\varphi_k(l)} \neq h_{\varphi_k(l-1)}$, then $\forall t \in [\tau_k, \tau_{k+1})$, $Y_t^{\varphi_k(l)} - Y_t^{\varphi_k(l-1)} > 0$ which combined with the previous property yields that for $d|V|_t$ a.e. $t \in [\tau_k, \tau_k + 1)$, $\gamma_t^l = 0$. Since a property holding $\forall k$, for $d|V|_t$ a.e. $t \in [\tau_k, \tau_{k+1})$, holds for $d|V|_t$ a.e. $t \geq 0$, the proof is completed. ■

2.4 Proof of Proposition 2.2

For $c \in \mathbb{R}$, let $c_n = [cn]/n$ where $[x]$ denotes the integral part of x. The entropy inequalities (0.2) are based on the functions $|u - c|$ and $\text{sgn}(u - c)(A(u) - A(c))$. That is why, we are interested in the approximation $|U_{\sigma_n}^n(t,x) - c_n|$ of the first one. According to (2.4), the function $x \to |U_{\sigma_n}^n(t,x) - c_n| - |c_n|$ is the cumulative distribution function of the signed measure

$$\nu_t^{n,c} = \frac{1}{n} \sum_{l=1}^{N_t} \left(\text{sgn}(\frac{1}{n} \sum_{i=1}^{l} h_{\varphi_t(i)} - c_n) h_{\varphi_t(l)} - 1_{\{\frac{1}{n} \sum_{i=1}^{l} h_{\varphi_t(i)} = c_n\}} \right) \delta_{Y_t^{\varphi_t(l)}} \quad \text{(convention: sgn}(0) = 0).$$

The next Lemma gives a much simpler expression of this measure.

Lemma 2.9 *Let for $1 \leq j \leq n$, $w_j = sgn(U(j) - c_n)h_j - 1_{\{U(j)=c_n\}}$.*

1. $\forall l \in [1, N_t]$, $U(\varphi_t(l)) = \frac{1}{n}\sum_{i=1}^{l} h_{\varphi_t(i)}$.
2. If for some $l \in [2, N_t]$, $w_{\varphi_t(l-1)} = 1$ and $w_{\varphi_t(l)} = -1$, then $h_{\varphi_t(l-1)} \neq h_{\varphi_t(l)}$.
3. If for some $l \in [2, N_t]$, $h_{\varphi_t(l-1)} \neq h_{\varphi_t(l)}$ then $w_{\varphi_t(l-1)} \neq w_{\varphi_t(l)}$.
4.

$$\forall t \geq 0,\ \nu_t^{n,c} = \frac{1}{n}\sum_{l=1}^{N_t} w_{\varphi_t(l)}\delta_{Y_t^{\varphi_t(l)}} = \frac{1}{n}\sum_{j=1}^{n} w_j \delta_{Y_t^j}.$$

Proof : 1. For $l \in [1, N_t]$, $U(\varphi_t(l)) = \frac{1}{n}\sum_{j=1}^{\varphi_t(l)} h_j = \frac{1}{n}\sum_{\substack{j=1 \\ j\in J_t}}^{\varphi_t(l)} h_j + \frac{1}{n}\sum_{\substack{j=1 \\ j\notin J_t}}^{\varphi_t(l)} h_j$. Since the indexes in $[1, \varphi_t(l)] \cap J_t$ correspond to couples of particles with opposite sign, the first summation in the right-hand-side is nil. Setting $i = \varphi_t^{-1}(j)$ in the second summation, we obtain $U(\varphi_t(l)) = \frac{1}{n}\sum_{i=1}^{l} h_{\varphi_t(i)}$

2. Let $l \in [2, N_t]$ be such that $w_{\varphi_t(l-1)} = 1$ and $w_{\varphi_t(l)} = -1$. Necessarily $U(\varphi_t(l-1)) \neq c_n$.
• In case $U(\varphi_t(l)) \neq c_n$ since according to 1., $U(\varphi_t(l)) = U(\varphi_t(l-1)) + h_{\varphi_t(l)}/n$, $\text{sgn}(U(\varphi_t(l-1)) - c_n) = \text{sgn}(U(\varphi_t(l)) - c_n)$. By the definition of the weights w_j, we deduce that $h_{\varphi_t(l-1)} \neq h_{\varphi_t(l)}$.
• In case $U_{\varphi_t(l)} = c_n$, then according to 1., $U(\varphi_t(l-1)) + h_{\varphi_t(l)}/n = c_n$.
Hence $h_{\varphi_t(l)} = -\text{sgn}(U(\varphi_t(l-1)) - c_n)$. Multiplying both sides by $h_{\varphi_t(l-1)}$, we get $h_{\varphi_t(l-1)}h_{\varphi_t(l)} = -w_{\varphi_t(l-1)} = -1$.

3. • In case $U(\varphi_t(l-1)) \neq c_n$ and $U(\varphi_t(l)) \neq c_n$, according to 1., $\text{sgn}(U(\varphi_t(l-1)) - c_n) = \text{sgn}(U(\varphi_t(l)) - c_n)$ and $w_{\varphi_t(l-1)} \neq w_{\varphi_t(l)}$.
• In case $U(\varphi_t(l-1)) = c_n$, $w_{\varphi_t(l-1)} = -1$ whereas $w_{\varphi_t(l)} = \text{sgn}(h_{\varphi_t(l)}/n)h_{\varphi_t(l)} = +1$.
• In case $U(\varphi_t(l)) = c_n$, $w_{\varphi_t(l)} = -1$ whereas $\text{sgn}(U(\varphi_t(l-1)) - c_n) = -h_{\varphi_t(l)}$ whence multiplying both sides by $h_{\varphi_t(l-1)}$, we get $w_{\varphi_t(l-1)} = -h_{\varphi_t(l-1)}h_{\varphi_t(l)} = 1$.

4. Combining the definition of $\nu_t^{n,c}$ and 1., we obtain that $\nu_t^{n,c} = \frac{1}{n}\sum_{l=1}^{N_t} w_{\varphi_t(l)}\delta_{Y_t^{\varphi_t(l)}}$. According to 3., the couples of particles that merge and are killed at successive times $\tau_1 < \ldots < \tau_{k_{max}}$ have opposite weights w. Since their positions are frozen afterwards, $\forall t \geq 0$, $\sum_{j\in J_t} w_j \delta_{Y_t^j}$ is the nil measure and

$$\frac{1}{n}\sum_{j=1}^{n} w_j \delta_{Y_t^j} = \frac{1}{n}\sum_{j\in J_t} w_j \delta_{Y_t^j} + \frac{1}{n}\sum_{l=1}^{N_t} w_{\varphi_t(l)}\delta_{Y_t^{\varphi_t(l)}} = \nu_t^{n,c}.$$

■

We are now ready to prove Proposition 2.2. Let π_0^∞ denote the limit point of a weakly converging subsequence of $(\pi_{\sigma_n}^n)_n$ that we still index by n for simplicity, g be a non-negative C^∞ function with compact support on $[0, +\infty) \times \mathbb{R}$ and $\phi(t, x) = \int_{-\infty}^{x} g(t, y)dy$. According to Lemma 2.9, computing $\phi(t, Y_t^j)$ thanks to (2.3), summing the obtained result multiplied by w_j over $1 \leq j \leq n$, we get

$$
\begin{aligned}
0 =& - < \nu_t^{n,c}, \phi(t,.) > + < \nu_0^{n,c}, \phi(0,.) > + \int_0^t < \nu_s^{n,c}, \partial_s \phi(s,.) > + < \xi_s^{n,c}, \partial_x \phi(s,.) > ds \\
&+ \frac{\sigma_n^2}{2n} \int_0^t \sum_{j \notin J_s} w_j \partial_{xx} \phi(s, Y_s^j) ds + \int_0^t \frac{\sigma_n}{n} \sum_{j \notin J_s} w_j \partial_x \phi(s, Y_s^j) d\beta_s^j \\
&+ \int_0^t \frac{1}{n} \sum_{j \notin J_s} w_j (\gamma_s^{\varphi_s^{-1}(j)} - \gamma_s^{\varphi_s^{-1}(j+1)}) \partial_x \phi(s, Y_s^j) d|V|_s
\end{aligned}
\tag{2.5}
$$

where

$$
\xi_s^{n,c} = \frac{1}{n} \sum_{j \notin J_s} w_j A'(U(j)) \delta_{Y_s^j} = \frac{1}{n} \sum_{l=1}^{N_s} w_{\varphi_s(l)} A'(U(\varphi_s(l))) \delta_{Y_s^{\varphi_s(l)}}. \tag{2.6}
$$

Denoting respectively by T_n^1, T_n^2 and T_n^3 the sum of the three first terms, the sum of the fourth and the fifth terms and the last term of the r.h.s., (2.5) writes $T_n^1 + T_n^2 + T_n^3 = 0$. Clearly, $\lim_{n \to +\infty} \mathbb{E}|T_n^2| = 0$.

$$
\begin{aligned}
nT_n^3 =& \int_0^t \sum_{l=2}^{N_s} w_{\varphi_s(l)} 1_{\{w_{\varphi_s(l)} = w_{\varphi_s(l-1)}\}} \gamma_s^l (\partial_x \phi(s, Y_s^{\varphi_s(l)}) - \partial_x \phi(s, Y_s^{\varphi_s(l-1)})) d|V|_s \\
&+ \int_0^t \sum_{l=2}^{N_s} 1_{\{w_{\varphi_s(l)} = 1, w_{\varphi_s(l-1)} = -1\}} \gamma_s^l (\partial_x \phi(s, Y_s^{\varphi_s(l)}) + \partial_x \phi(s, Y_s^{\varphi_s(l-1)})) d|V|_s \\
&- \int_0^t \sum_{l=2}^{N_s} 1_{\{w_{\varphi_s(l)} = -1, w_{\varphi_s(l-1)} = 1\}} \gamma_s^l (\partial_x \phi(s, Y_s^{\varphi_s(l)}) + \partial_x \phi(s, Y_s^{\varphi_s(l-1)})) d|V|_s
\end{aligned}
\tag{2.7}
$$

According to Proposition 2.8, the first term of the r.h.s. is nil. Combining assertion 2. in Lemma 2.9 and Proposition 2.8, we check that the third term is also nil. Since $\partial_x \phi = g \geq 0$, T_n^3 is non-negative. Therefore to conclude, it is enough to check that for the bounded function $F : \mathcal{P}(C([0,+\infty), \mathbb{R})) \to \mathbb{R}$ defined by

$$
\begin{aligned}
F(Q) =& - \int_{\mathbb{R}} g(t,y) |H * \tilde{Q}_t(y) - c| dy + \int_{\mathbb{R}} g(0,y) |H * \tilde{Q}_0(y) - c| dy \\
&+ \int_0^t \int_{\mathbb{R}} |H * \tilde{Q}_s(y) - c| \partial_s g(s,y) + \mathrm{sgn}(H * \tilde{Q}_s(y) - c)(A(H * \tilde{Q}_s(y)) - A(c)) \partial_x g(s,y) dy ds,
\end{aligned}
$$

$\lim_{n \to +\infty} \mathbb{E}|F(\mu^n) + T_n^1| = 0$. Indeed supposing this convergence, since $F(\mu^n) = F(\mu^n) + T_n^1 + T_n^2 + T_n^3$, we have $\mathbb{E}(F(\mu^n)^-) \leq \mathbb{E}(|F(\mu^n) + T_n^1| + |T_n^2| + (T_n^3)^-) \to_{n \to +\infty} 0$. Approximating F by continuous functions like in the proof of Proposition 1.4, we deduce from the weak convergence of $\pi_{\sigma_n}^n$ to π_0^∞ that $\mathbb{E}^{\pi_0^\infty}(F(Q)^-) = 0$. Taking t, c, g in denumerate dense sets and then taking limits, we deduce that π_0^∞ a.s., for any positive test function g, $\forall c \in \mathbb{R}$, $\forall t \geq 0$, $F(Q) \geq 0$ i.e. π_0^∞ a.s. $H * \tilde{Q}_s(x)$ is the entropy solution of (0.1).

Let us prove that the variables $F(\mu^n) + T_n^1$ converge to zero. Since $x \to |U_{\sigma_n}^n(t,x) - c_n| - |c_n|$ is the cumulative distribution function of the signed measure $\nu_t^{n,c}$, computing the brackets $<,>$

in T_n^1 by the integration by parts formula, we get

$$\begin{aligned}T_n^1 &= -|U^n_{\sigma_n}(t,+\infty)-c_n|\int_{\mathbb{R}} g(t,y)dy + \int_{\mathbb{R}} g(t,y)|U^n_{\sigma_n}(t,y)-c_n|dy\\ &+ |U^n_{\sigma_n}(0,+\infty)-c_n|\int_{\mathbb{R}} g(0,y)dy - \int_{\mathbb{R}} g(0,y)|U^n_{\sigma_n}(0,y)-c_n|dy\\ &+ \int_0^t |U^n_{\sigma_n}(s,+\infty)-c_n|\int_{\mathbb{R}}\partial_s g(s,y)dyds - \int_0^t\int_{\mathbb{R}}\partial_s g(s,y)|U^n_{\sigma_n}(s,y)-c_n|dyds\\ &- \int_0^t\int_{\mathbb{R}}\partial_x g(s,y)(H*\xi^{n,c}_s(y)-\mathrm{sgn}(c_n)(A(0)-A(c_n)))dyds\end{aligned}$$

As $U^n_{\sigma_n}(s,+\infty)=\tilde{\mu}^n_s(\mathbb{R})$ does not depend on s, the sum of the first, the third and the fifth terms of the r.h.s. is nil.

We set $N_s(y) = \max\{l\in[1,N_s],\ Y_s^{\varphi_s(l)}\le y\}$. By Lemma 2.9 1., if $U(\varphi_s(l))=c_n$ then $\mathrm{sgn}(U(\varphi_s(l-1))-c_n) = -h_{\varphi_s(l)}$ and $w_{\varphi_s(l)} = -1 = -h_{\varphi_s(l)}\mathrm{sgn}(U(\varphi_s(l-1))-c_n)$. Hence by (2.6),

$$H*\xi^{n,c}_s(y) = \frac{1}{n}\sum_{l=1}^{N_s(y)}\left(\mathrm{sgn}(U(\varphi_s(l))-c_n)+1_{\{U(\varphi_s(l))=c_n\}}\mathrm{sgn}(U(\varphi_s(l-1))-c_n)\right)h_{\varphi_s(l)}A'(U(\varphi_s(l))).$$

Moreover according to (2.4), $U^n_{\sigma_n}(s,y)=\frac{1}{n}\sum_{l=1}^{N_s(y)}h_{\varphi_s(l)}$ and with the convention $U(\varphi_s(0))=0$,

$$\begin{aligned}\mathrm{sgn}(U^n_{\sigma_n}(s,y)-c_n)(A(U^n_{\sigma_n}(s,y))-A(c_n)) &= \mathrm{sgn}(0-c_n)(A(0)-A(c_n)))\\ &+\sum_{l=1}^{N_s(y)}\Big[\mathrm{sgn}(U(\varphi_s(l))-c_n)(A(U(\varphi_s(l)))-A(U(\varphi_s(l-1))))\\ &\quad + 1_{\{U_{\varphi_s(l)}=c_n\}}\mathrm{sgn}(U(\varphi_s(l-1))-c_n)(A(U(\varphi_s(l)))-A(U(\varphi_s(l-1))))\Big].\end{aligned}$$

Therefore

$$\begin{aligned}&\left|H*\xi^{n,c}_s(y)-\mathrm{sgn}(c_n)(A(0)-A(c_n)))-\mathrm{sgn}(U^n_{\sigma_n}(s,y)-c_n)(A(U^n_{\sigma_n}(s,y))-A(c_n))\right|\\ &\qquad\le\sum_{l=1}^{N_s(y)}|A(U(\varphi_s(l)))-A(U(\varphi_s(l-1)))-A'(U(\varphi_s(l)))h_{\varphi_s(l)}/n|.\end{aligned}$$

Since by Lemma 2.9 1., $U(\varphi_s(l))=U(\varphi_s(l-1))+h_{\varphi_s(l)}/n$, the right-hand-side is smaller than $\sup_{\substack{x,y\in[-1,1]\\ |x-y|\le\frac{1}{n}}}|A'(x)-A'(y)|$. As the support of g is compact, we deduce that the random variables

$$\begin{aligned}&\Big|T_n^1-\int_{\mathbb{R}} g(t,y)|U^n_{\sigma_n}(t,y)-c_n|dy+\int_{\mathbb{R}} g(0,y)|U^n_{\sigma_n}(0,y)-c_n|dy\\ &+\int_0^t\int_{\mathbb{R}}|U^n_{\sigma_n}(s,y)-c_n|\partial_s g(s,y)+\mathrm{sgn}(U^n_{\sigma_n}(s,y)-c_n)(A(U^n_{\sigma_n}(s,y))-A(c_n))\partial_x g(s,y)dyds\Big|\end{aligned}$$

converge uniformly to 0 as $n\to+\infty$. Since $\forall x\in\mathbb{R},\ ||x-c_n|-|x-c||\le|c_n-c|\le\frac{1}{n}$,

$$|\mathrm{sgn}(x-c)(A(x)-A(c))-\mathrm{sgn}(x-c_n)(A(x)-A(c_n))|\le\sup_{y\in[c_n,c]}(|2A(y)-A(c)-A(c_n)|),$$

and according to (1.1), $\forall(s,y)\in[0,+\infty)\times\mathbb{R},\ U^n_{\sigma_n}(s,y)=H*\tilde{\mu}^n_s(y)$, the variables $|F(\mu^n)+T_n^1|$ also converge uniformly to 0.

Remark 2.10 *It should be noted that we obtain the entropy inequalities because T_n^2 is non-negative i.e. thanks to the local time term which prevents strict crossings of the surviving characteristics Y_s^j, $j \notin J_s$ which share the same sign. Moreover, it is necessary to kill couples of particles with opposite sign that merge so that the non-positive third term of the right-hand-side of* (2.7) *vanishes.*

3 Numerical example

As a numerical benchmark, we consider the Burgers equation ($A(u) = u^2/2$) with initial data $u_0(x) = \frac{1}{4}\left(1_{[-3,-2]}(x) - 1_{[2,3]}(x)\right)$ which is the cumulative distribution function of the signed measure $m = \frac{1}{4}(\delta_{-3} - \delta_{-2} - \delta_2 + \delta_3)$. The corresponding entropy solution is given by

$$u(t,x) = \frac{1}{t}\Bigg[\min\left(x+3, \frac{t}{4}\right) 1_{[-3,\min(-2+\frac{t}{8},-3+\sqrt{\frac{t}{2}},0)]}(x) + \max\left(x-3, -\frac{t}{4}\right) 1_{[\max(2-\frac{t}{8},3-\sqrt{\frac{t}{2}},0),3]}(x)\Bigg].$$

We easily check that the L^1 norm (resp. variation) of $x \to u(t,x)$ is equal to $1/2$ if $t \leq 18$ and $9/t$ if $t \geq 18$ (resp. 1 if $t \leq 8$, $2\sqrt{2/t}$ if $8 \leq t \leq 18$ and $12/t$ if $t \geq 18$). We simulate the system (1.2) for $n = 4000$ particles and viscosity coefficient $\sigma = 0.001$. The initialization is deterministic : for $1 \leq i \leq 1000$, $X_0^i = -3$ and $h(X_0^i) = 1$, for $1001 \leq i \leq 2000$, $X_0^i = -2$ and $h(X_0^i) = -1$, for $2001 \leq i \leq 3000$, $h(X_0^i) = -1$ and for $3001 \leq i \leq 4000$, $X_0^i = 3$ and $h(X_0^i) = 1$. This way, there is no initialization error i.e. the approximate solution at time 0 $U(0,x) = \frac{1}{n}\sum_{i=1}^n h(X_0^i)H(x - X_0^i)$ is equal to $u_0(x)$. The system is discretized in time thanks to the Euler scheme with time step $\Delta t = 0.4$. If at time $k\Delta t$, the set of indexes of killed particles is $I_{k\Delta t}$ and the positions of the $N_{k\Delta t}$ remaining particles are $(X_{k\Delta t}^i)_{i \notin I_{k\Delta t}}$, the approximate solution at time $k\Delta t$ and the positions of the particles at the next time step are given by

$$\begin{cases} U(k\Delta t, x) = \frac{1}{n}\sum_{i \notin I_{k\Delta t}} h(X_0^i)H(x - X_{k\Delta t}^i) \\ \forall i \notin I_{k\Delta t},\ X_{(k+1)\Delta t}^i = X_{k\Delta t}^i + \sigma(B_{(k+1)\Delta t}^i - B_{k\Delta t}^i) + A'(U(k\Delta t, X_{k\Delta t}^i))\Delta t. \end{cases}$$

Then the couples of particles with opposite sign which are closer than $s = 0.005$ are killed i.e. their indexes are added to $I_{k\Delta t}$ to obtain $I_{(k+1)\Delta t}$.

In figure 1, we compare the exact solution $u(t,.)$ and the approximate solution $U(t,.)$ at times $t = 4, 8, 16$ and 40. We can only distinguish very slight differences. The number of surviving particles $N_{k\Delta t}$ is decreasing with k : indeed $N_4 = 4000$, $N_8 = 3984$, $N_{16} = 2836$ and $N_{40} = 1192$ is smaller than 30% of N_0. In table 1, we give the evolution of the expectation of the L^1 norm of the error with respect to time. This expectation is estimated from 20 runs of the particle system. The width of the corresponding Confidence Interval at 95% is also precised. For each run, at time $k\Delta t$, the L^1 norm of the error is computed thanks to the increasing reordering $(Y_{k\Delta t}^{\varphi_{k\Delta t}(l)})_{1 \leq l \leq N_{k\Delta t}}$ of the surviving particles $(X_{k\Delta t}^i)_{i \notin I_{k\Delta t}}$ by the following formula

$$\sum_{l=1}^{N_{k\Delta t}-1} \frac{1}{2}(Y_{k\Delta t}^{\varphi_{k\Delta t}(l+1)} - Y_{k\Delta t}^{\varphi_{k\Delta t}(l)})\left(|u-U|(k\Delta t, Y_{k\Delta t}^{\varphi_{k\Delta t}(l+1)}) + |u-U|(k\Delta t, Y_{k\Delta t}^{\varphi_{k\Delta t}(l)})\right).$$

The expectation of the L^1 norm of the error remains small in comparison with the L^1 norm of the explicit solution (approximately 1%). We also compare the expectation of the variation of the approximate solution which is given by $N_{k\Delta t}/n$ (the width of corresponding confidence interval at 95% is nether greater than 0.0005) with the variation of the explicit solution. They are very close. This result is not surprising because we kill couples of particles of opposite sign that merge to mimic the decreasing property of the variation of the explicit solution.

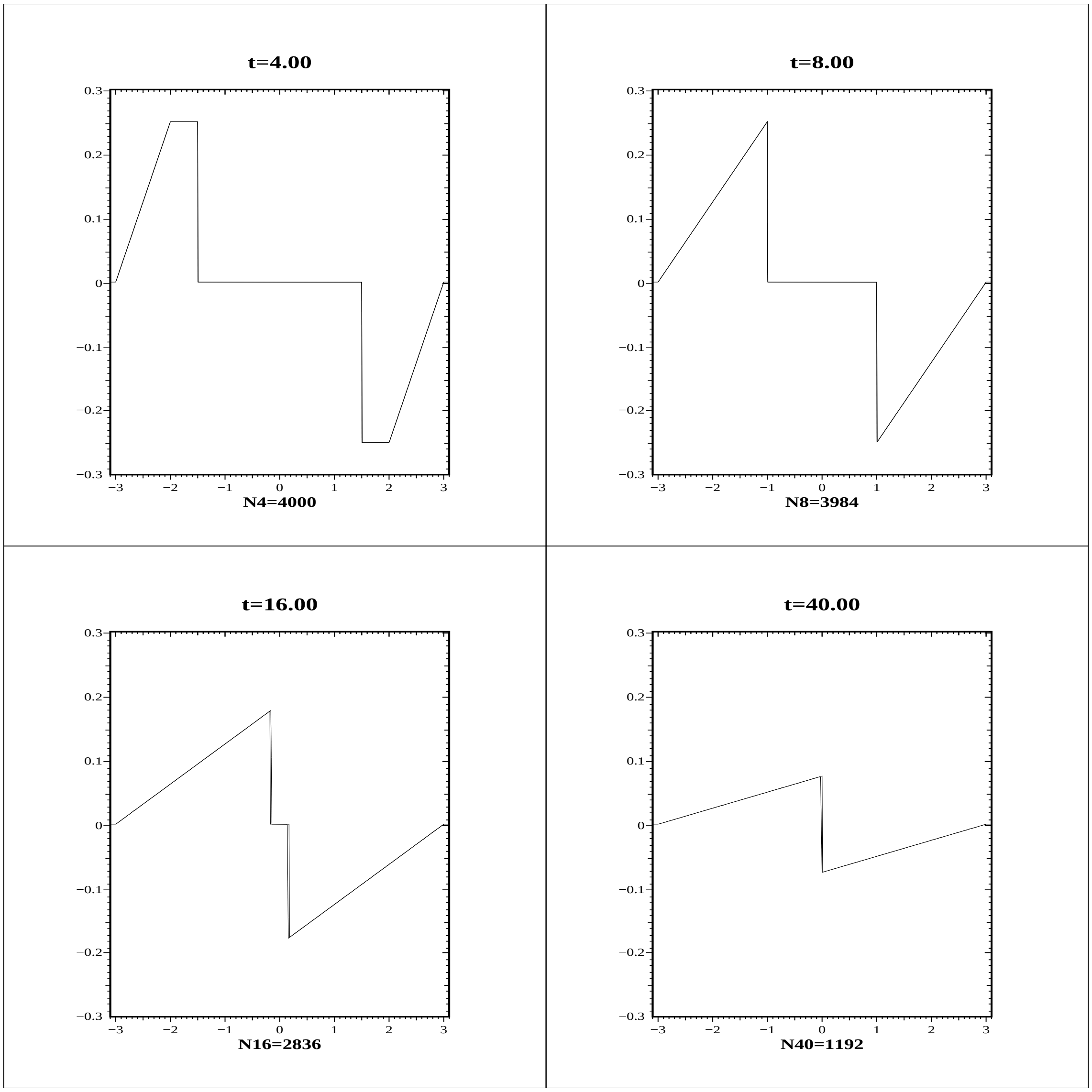

Figure 1: Comparison of $U(t,x)$ and $u(t,x)$

time t	4	8	12	16	20	28	40
$\|u(t,.)\|_1$	0.5	0.5	0.5	0.5	0.45	0.321	0.225
$\mathbb{E}\|U(t,.)-u(t,.)\|_1$	0.0015	0.0018	0.0063	0.0081	0.0039	0.0030	0.0035
width of C.I. at 95 %	2.5e-5	2.3e-5	2.7e-5	4.8e-5	7.8e-5	7.8e-5	3e-4
variation $u(t,.)$	1	1	0.816	0.707	0.6	0.429	0.3
$\mathbb{E}(N_t)/n$	1	0.995	0.816	0.709	0.595	0.425	0.298

Table 1: Evolution of the L^1 norm of the error with respect to t

Conclusion

In this paper we proved the convergence of a stochastic particles approximation of the entropy solution of (0.1) as the initial number of particles goes to $+\infty$. In case the initial data u_0 is monotonic, the system of interacting particles is the same as the one introduced by Bossy and Talay [3] [4] for the Burgers equation ($A(u) = u^2/2$). But otherwise, we have modified the dynamics by killing the couples of particles with opposite sign that merge. This mimics the decreasing property of the variation of the entropy solution $x \to u(t,x)$ with repect to t. To obtain an effective numerical procedure, it is necessary to discretize the particle system in time. Our results can be seen as a preliminary step in the study of the convergence rate of the approximate solution based on the time-discretized system with respect to the time step Δt, the number of particles n and the parameter s governing the murders introduced in the numerical example. From a numerical point of view, killing of particles is interesting because the computational effort needed to compute the successive positions of the particles decreases in time with the number of surviving particles. In return additional effort is needed to deal with the murders.

We should also mention a very convenient feature of the particle system with killing : if the approximate solution defined as the cumulative distribution function of the weighted empirical measure is non-negative (resp. non-positive) at time 0, it remains non-negative (resp. non-positive) afterwards. This feature can be exploited to generalize the convergence results for the particle approximation of the solution of the porous medium equation given in [7] : using a system with killing, we could deal with any non-negative initial data with bounded variation and not only monotonic ones. Indeed the diffusion coefficient of each particle which is a fractional power of the approximate solution would remain well-defined.

An interesting question is whether killing of couples of particles with opposite sign can be generalized in space dimension $d \geq 2$. Because points are polar for the d-dimensional Brownian motion, the particles are not likely to merge and it is not sensible to study a system with killing in continuous time. But it is still possible to contemplate killing couples with opposite sign which are closer than some critical distance s after discretization in time.

References

[1] P. Billingsley. *Probability and Measure.* John Wiley and Sons, 1986.

[2] M. Bossy, L. Fezoui, and S. Piperno. Comparison of a stochastic particle method and a finite volume deterministic method applied to Burgers equation. *Monte Carlo Methods Appl.*, 3(2):113–140, 1997.

[3] M. Bossy and D. Talay. Convergence rate for the approximation of the limit law of weakly interacting particles: Application to the Burgers equation. *Annals of Applied Prob.*, 6(3):818–861, 1996.

[4] M. Bossy and D. Talay. A stochastic particle method for the McKean-Vlasov and the Burgers equation. *Math. Comp.*, 66(217):157–192, 1997.

[5] B. Jourdain. Propagation trajectorielle du chaos pour les lois de conservation scalaires. In *Séminaire de Probabilités XXXII*, pages 215–230. Springer, 1998.

[6] B. Jourdain. Diffusion processes associated with nonlinear evolution equations for signed measures. *Methodology and Computing in Applied Probability*, 2(1):69–91, 2000.

[7] B. Jourdain. Probabilistic approximation for a porous medium equation. *Stochastic Processes and their Applications*, 89(1):81–99, 2000.

[8] M. Kunik. A solution formula for a non-convex scalar hyperbolic conservation law with monotone initial data. *Math. Methods in the Appl. Scien.*, 16:895–902, 1993.

[9] B. Perthame and M. Pulvirenti. On some large systems of random particles which approximate scalar conservation laws. *Asymt. Anal.*, 10(3):263–278, 1995.

[10] D.W. Stroock and S.R.S. Varadhan. *Multidimensional Diffusion Processes*. Springer, 1997.

[11] A.S. Sznitman. Topics in propagation of chaos. In *Ecole d'été de probabilités de Saint-Flour XIX - 1989, Lect. Notes in Math. 1464*. Springer-Verlag, 1991.