Error Propagation and Sensitivity in Financial Calculus

Classical models and European options

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Abstract

We study the sensitivity of pricing and hedging formulas of simple financial models to changes of parameters and, that is the main focus, to perturbations of the underlying Brownian motion. The method we apply is not specific to finance, it is a general setting of the so-called Malliavin calculus based on the theory of Dirichlet forms. This approach to error calculus is explained here both intuitively and mathematically.

1 Introduction

Once a model is chosen and used to price contingent claims and to hedge a position, the main question remains at time t : what is our exposure to changes in the market ? This risk assessment is usually done in terms of sensitivity of the portfolios to variations of the financial quantities and parameters of the model. For example the volatility, constant in first approximation, seems to vary proportionally to itself in second approximation, as it were erroneous with a constant relative error.

As long as these sensitivity computations concern the role of finite dimensional quantities, the classical differential calculus can be performed either in a deterministic framework or almost surely (path by path) in the random case.

Now what allows the theory of Dirichlet forms is to take in account a perturbation of the stochastic process (e.g. the Brownian motion) itself with which the model is mathematically constructed.

This derivation of a random variable with respect to the sample path of an underlying process is the central idea of the so-called Malliavin calculus, cf. [M], but the theory of Dirichlet forms gives a general form to this kind of computation in a close connection with the interpretation in terms of errors.

We take the case of financial models as an example to expose this method. This explains the choice of simple models. The first part (section 2) is devoted to intuitive ideas and to the formal definitions. The

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following part (section 3) concerns the infinite dimensional error structures on the Wiener space. Then the Black-Scholes case is examined and, in the last part (section 5), a diffusion model is studied. Finally we mention some ways of research.

2 Error calculus based on Dirichlet forms

This part begins with a short historical and intuitive introduction. It is not here for cultural reasons but as the simplest way to help the reader in improving his personal idea of the tools developed later. Next, we give the mathematical framework that we will apply to financial models and explain its main properties. Finally a comparison with other approaches is commented for clarity.

2.1 Error calculus à la Gauss

Twelve years after his argument showing the importance of the normal law as probability law for the errors (*Theoria motus corporum coelestium* 1809), Gauss was interested in the propagation of errors (*Theoria Combinationis* 1821). He has to be considered as the founder of error calculus. Given a quantity $U = F(V_1, V_2, \ldots)$ function of other erroneous quantities $V_1, V_2, \ldots$ he states the problem of computing the quadratic error to fear on U knowing the quadratic errors $\sigma_1^2, \sigma_2^2, \ldots$ on $V_1, V_2, \ldots,$ these errors being supposed small and independent. His answer is the following formula

$$\sigma_U^2 = (\frac{\partial F}{\partial V_1})^2 \sigma_1^2 + (\frac{\partial F}{\partial V_2})^2 \sigma_2^2 + \tag{1}$$

he gives also the covariance between the error on F and the error of an other function of the V_i's.

Formula (1) possesses a property which makes it highly better, in several questions, than other formulas used here and there in textbooks during the 19th and 20th centuries. It is a *coherence* property. With a formula such that

$$\sigma_U = |\frac{\partial F}{\partial V_1}| \sigma_1 + |\frac{\partial F}{\partial V_2}| \sigma_2 + \ldots \tag{2}$$

errors can depend on the manner the function F is written : in dimension 2 already composing an injective linear map with its inverse leads with formula (2) to the fact that the identity map increases the errors what is hardly acceptable.

This doesn't happen in Gauss' calculus. Introducing the operator

$$L = \frac{1}{2}\sigma_1^2 \frac{\partial^2}{\partial V_1^2} + \frac{1}{2}\sigma_2^2 \frac{\partial^2}{\partial V_2^2} + \ldots$$

and supposing the functions smooth, we remark that formula (1) can be written

$$\sigma_U^2 = LF^2 - 2FLF$$

and the coherence of this calculus comes from the coherence of the transport of a differential operator by a function : if L is such an operator, u and v injective regular maps, denoting the operator $\varphi \to L(\varphi \circ u) \circ u^{-1}$ by $\theta_u L$ we have $\theta_{v\circ u} L = \theta_v(\theta_u L)$.

The errors on $V_1, V_2, \ldots$ are not necessarily supposed to be independent nor constant, they can depend on $V_1, V_2, \ldots$: Let be given a field of symmetric positive matrices $(\sigma_{ij}(v_1, v_2, \ldots))$ on $\mathbb{R}^d$ representing the conditional variances and covariances on $V_1, V_2, \ldots$ given the values $v_1, v_2, \ldots$ of $V_1, V_2, \ldots$ then the error on $U = F(V_1, V_2, \ldots)$ is

$$\sigma_F^2 = \sum_{ij} \frac{\partial F}{\partial V_i}(v_1, v_2, \ldots) \frac{\partial F}{\partial V_j}(v_1, v_2, \ldots) \sigma_{ij}(v_1, v_2, \ldots) \tag{3}$$

which depends solely on F as mapping, provided F be suitably regular.

2.2 Extension tool using Dirichlet forms

The error calculus of Gauss has the limitation that it has no mean of extension. If the error on (V_1, V_2, V_3) is known it gives the error on any differentiable function of (V_1, V_2, V_3) but that's all.

Now, in the usual probabilistic situations where a sequence of quantities $X_1, X_2, \ldots, X_n, \ldots$ is given and where the errors are known on the regular functions of a finite number of them, we would like to deduce the error on a function of an infinite number of the X_i's or at least on some such functions.

It is actually possible to reinforce this error calculus giving it a powerful extension tool and preserving the coherence property. In addition, it will give us the comfortable possibility to handle Lipschitz functions as well.

For this we come back to the idea that the erroneous quantities are themselves random, as Gauss had supposed for his proof of the 'law of errors', say defined on $(\Omega, \mathcal{A}, \mathbb{P})$. The quadratic error on a random variable X is then itself a random variable that we will denote by $\Gamma[X]$. Intuitively we still suppose the errors are infinitely small although this doesn't appear in the notation. It is as we had an infinitely small unit to measure errors fixed in the whole problem. The extension tool is the following, we assume that if $X_n \to X$ in $L^2(\Omega, \mathcal{A}, \mathbb{P})$ and if the error $\Gamma[X_m - X_n]$ on $X_m - X_n$ can be made as small as we want in $L^1(\Omega, \mathcal{A}, \mathbb{P})$ for m, n large enough, then the error $\Gamma[X_n - X]$ on $X_n - X$ goes to zero in L^1.

It is a *reinforced coherence principle* since this means that the error on a random variable X is attached to X and that furthermore if the sequence of pairs (X_n, error on X_n) converges in a suitable sense, it converges necessarily to (X, error on X).

This can be axiomatized as follows : we call *error structure* a probability space equipped with a local Dirichlet form possessing a *carré du champ*. Thus *an error structure is a term*

$$(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$$

where $(\Omega, \mathcal{A}, \mathbb{P})$ *is a probability space, satisfying the four properties :*

1.) $\mathbb{D}$ *is a dense subvectorspace of* $L^2(\Omega, \mathcal{A}, \mathbb{P})$

2.) Γ *is a positive symmetric bilinear map from* $\mathbb{D} \times \mathbb{D}$ *into* $L^1(\mathbb{P})$ *fulfilling the functional calculus of class* $\mathcal{C}^1 \cap Lip$*, what means that if* $u \in \mathbb{D}^m$ *and* $v \in \mathbb{D}^n$ *for* F *and* G *of class* $\mathcal{C}^1$ *and Lipschitz from* $\mathbb{R}^m$ [*resp.* $\mathbb{R}^n$] *into* $\mathbb{R}$*, one has* $F \circ u \in \mathbb{D}$ *and* $G \circ v \in \mathbb{D}$ *and*

$$\Gamma[F \circ u, G \circ v] = \sum_{i,j} F_i'(u) G_j'(v) \Gamma[u_i, v_j] \quad \mathbb{P}\text{-}p.s..$$

3.) the bilinear form $\mathcal{E}[f,g] = \mathbb{E}\Gamma[f,g]$ *is closed, i.e.* $\mathbb{D}$ *is complete under the norm* $\| \, . \, \|_{\mathbb{D}} = (\| \, . \, \|^2_{L^2(\mathbb{P})} + \mathcal{E}[\, . \,, \, . \,])^{\frac{1}{2}}$*.*

4.) $1 \in \mathbb{D}$ *and* $\Gamma[1,1] = 0$*.*

We always write $\mathcal{E}[f]$ for $\mathcal{E}[f,f]$ and $\Gamma[f]$ for $\Gamma[f,f]$.

With this definition, the form $\mathcal{E}$ defined at point 3.) is a *Dirichlet form*. This notion has been introduced by A. Beurling and J. Deny as a tool in potential theory, cf. [B-D] [F] [S], and received a probabilistic interpretation in terms of symmetric Markov processes by M. L. Silverstein and M. Fukushima, cf. [F-O-T] [M-R]. The operator Γ is the carré du champ or squared field operator associated with $\mathcal{E}$, it has been studied by several authors in more general context, cf. [D-M] [B-H]. Here we refer to Γ as the *quadratic error operator* of the error structure. Its intuitive meaning is the conditional variance of the error.

First examples. (a) A simple example of error structure is the term

$$(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu, H^1(m), \gamma)$$

where m is the normal law $N(0,1)$ and

$$H^1(m) = \{f \in L^2(m) : f' \text{ in the sense of distributions } \in L^2(m)\}$$

with $\gamma[f] = f'^2$ for $f \in H^1(m)$. This structure is associated to the real valued Ornstein-Uhlenbeck process.

(b) Let D be a connected open set of finite volume in $\mathbb{R}^d$, λ_d be the Lebesgue measure, let us take $(\Omega, \mathcal{A}, \mathbb{P}) = (D, \mathcal{B}(D), \frac{1}{\lambda_d(D)}\lambda_d)$ and

$$\Gamma[u,v] = \sum_{ij} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} a_{ij} \qquad \text{for } u, v \in \mathcal{C}_K^\infty(D)$$

where the a_{ij}'s are maps from D into $\mathbb{R}$ such that

$$a_{ij} \in L^2_{loc}(D),\ a_{ij} = a_{ji},\ \frac{\partial a_{ij}}{\partial x_k} \in L^2_{loc}(D),\ \sum_{ij} a_{ij}(x)\xi_i\xi_j \geq 0\ \forall \xi \in \mathbb{R}^d\ \forall x \in D.$$

Then it can be shown that the form $\mathcal{E}[u,v] = \mathbb{E}\Gamma[u,v]$ with $u,v \in \mathcal{C}^\infty_K(D)$ is closable, cf. [F-O-T] [M-R], i.e. there exists an extension of Γ to a subvectorspace $\mathbb{D}$ of L^2, $\mathbb{D} \supset \mathcal{C}^\infty_K(D)$ such that $(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$ be an error structure.

2.3 First order and second order calculus

The following remark, although very simple, is important to understand the role of the error calculus à la Gauss that will used in the sequel in the extended form allowed by Dirichlet forms.

Let us start with a quantity x with a small centred error Y, on which acts a non-linear regular function f. Thus we have at the beginning a random variable that we can write $x+\varepsilon Y$, it has no bias (centred at the true value x) and its variance is $\varepsilon^2\sigma_Y^2$: thus $\text{bias}_0 = 0$, $\text{variance}_0 = \varepsilon^2\sigma_Y^2$.

After having applied the function f, using Taylor formula shows that the error is no more centred. The bias has the same order of magnitude as the variance :

$$\begin{aligned}
\text{bias}_1 &= \mathbb{E}[f(x+\varepsilon Y) - f(x)] = \varepsilon^2\sigma_Y^2\tfrac{1}{2}f''(x) + \varepsilon^3 0(1) \\
\text{variance}_1 &= \mathbb{E}[(f(x+\varepsilon Y) - f(x))^2] - (\text{bias}_1)^2 \\
&= \varepsilon^2\sigma_Y^2 f'^2(x) + \varepsilon^3 0(1)
\end{aligned}$$

then applying a new regular non-linear function g gives us a recurrence formula :

$$\begin{aligned}
\text{bias}_2 &= \text{bias}_1 g'(f(x)) + \tfrac{1}{2}\text{variance}_1 g''(f(x)) + \varepsilon^3 0(1) \\
\text{variance}_2 &= \text{variance}_1 g'^2(f(x)) + \varepsilon^3 0(1)
\end{aligned}$$

which could be easily extended to applications from $\mathbb{R}^p$ to $\mathbb{R}^q$ (for the general formulas on the bias and the variance of the error under regular mappings see [B-H] chapter I paragraph 6 corollaries 6.1.3 and 6.1.4).

We see that *the calculus on the bias is a second order calculus involving the variance. Instead, the calculus on the variances is a first order calculus not involving the bias.*

Thus, the error calculus on the variances appears to be necessarily the first step in an analysis of errors propagation based on differential methods and supposing small errors.

2.4 Comparison of approaches

Before looking at the infinite dimensional examples needed in finance, let us try to give an outlook over the different approaches to error calculus.

Table 1: Main classes of error calculi

<table>
<tr><td>deterministic approach</td><td colspan="3">probabilistic approaches</td></tr>
<tr><td rowspan="2">Sensitivity calculus: derivation with respect to the parameters of the model</td><td colspan="2">Extended error calculus using Dirichlet forms</td><td rowspan="2">Probability theory</td></tr>
<tr><td>first order calculus only dealing with variances</td><td>second order calculus with variances and bias</td></tr>
<tr><td colspan="3">infinitesimal errors</td><td>finite errors</td></tr>
</table>

At the extreme right-hand side of the table we have the usual probability calculus in which the errors are random variables. The knowledge of the joint laws of the quantities and their errors is supposed to be yielded by statistical methods. The errors are finite, the propagation of the errors needs computation of image probability laws.

At the extreme left-hand side the sensitivity calculus consists of computing derivatives with respect to parameters, including Gateaux or Fréchet derivatives in functional spaces to get the sensitivity with respect to a functional data.

Between these two purely probabilistic and purely deterministic approaches lies the extended error calculus based on Dirichlet forms. It supposes the errors are infinitely small but takes in account some features of the probabilistic approach allowing to put the computations and the arguments inside a powerful mathematical theory: the theory of Dirichlet forms. In the same framework can be performed either a first order calculus on variances which is simple and significant enough for most applications or a second order calculus dealing with both variance and bias which is more complicated and in close connection with the so-called stochastic differential geometry. We cannot here go further to explain this connection in more details. We will rather describe the main properties this calculus receives from Dirichlet forms theory.

Let us add just a comment. On one hand the error calculus based on Dirichlet forms can be seen as a special case of more general mathematical theories of differential calculus in metric or abstract spaces, see e.g. [A-K] [W]. On the other hand it can be enriched and made more precise by geometric additional hypotheses, see e.g. [M] [C-M] [St]. We believe that the level of axiomatisation of *error structures* is the best adapted to error analysis for stochastic models.

2.5 The method

The construction of an error structure on a stochastic model can be done in two steps

1) If there are, as usually, deterministic parameters which can be erroneous or with respect to which a sensitivity is wished, these parameters have to be randomized with *a priori* laws.

2) errors operators must be chosen to act on random quantities (initially random or randomized parameters) in order to describe errors, in such a way that we obtain mathematically an error structure as defined above.

As we will see further, the choice of the *a priori* law is not so crucial as it could be thought because the computations are done almost surely.

Several properties of error structures make it easier such a construction.

1) The operation of *taking the image* of an error structure by a mapping is quite natural and gives an error structure as soon as the mapping, even non injective, satisfies some rather weak conditions. In particular if $(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$ is an error structure and if X is a random variable with values in $\mathbb{R}^d$ whose components are in $\mathbb{D}$, $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mathbb{P}_X, \mathbb{D}_X, \Gamma_X)$ is an error structure where $\mathbb{P}_X$ is the law of X,

$$\begin{aligned} \mathbb{D}_X &= \{f \in L^2(\mathbb{P}_X) : f \circ X \in \mathbb{D}\} \\ \Gamma_X[f] &= \mathbb{E}[\Gamma[f \circ X] | X = x], \quad f \in \mathbb{D} \end{aligned}$$

2) If $f \in \mathbb{D}$ and F is Lipschitz from $\mathbb{R}$ to $\mathbb{R}$ then $F \circ f \in \mathbb{D}$ and $\Gamma[F \circ f] \leq \Gamma[f]$. For example the structure of example (a) $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu, H^1(\mu), \gamma)$ possesses an image by the map $x \to |\sin\sqrt{1+|x|}|$ which is an error structure on $[0,1]$. Such a use of non injective functions is tricky in the deterministic sensitivity calculus. More generally if F is a contraction from $\mathbb{R}^d$ into $\mathbb{R}$ in the following sense

$$|F(x) - F(y)| \leq \sum_{in1}^{d} |x_i - y_i|$$

then for $f_1, f_2, \ldots, f_d \in \mathbb{D}$ one has $F(f_1, f_2, \ldots, f_d) \in \mathbb{D}$ and

$$\Gamma[F(f_1, f_2, \ldots, f_d)]^{\frac{1}{2}} \leq \sum_{i=1}^{d} \Gamma[f_i]^{\frac{1}{2}}.$$

This property allows to consider more general images with values in metric spaces as soon as a suitable density property is preserved, see [B-H] chapter V paragraph 1.3 p 197.

3) The *product* of two or countably many error structures is an error structure. It is the mathematical expression of the independence of the random variables and the non-correlation of the errors. By this way error structures on infinite dimensional spaces are easily obtained, e.g. on the Wiener space, as we will see in the next part, or on the general Poisson space or other spaces of stochastic processes, see [B-H][M-R][B1].

For later reference we give the following statement (see [B-H] and [M-R] for more general cases).

Theorem 2.1 *Product structures*

Let $S_n = (\Omega_n, \mathcal{F}_n, m_n, \mathbb{D}_n, \Gamma_n)$, $n \geq 0$ be error structures.

The term $S = (\Omega, \mathcal{F}, m, \mathbb{D}, \Gamma)$ defined below is an error structure denoted $S = \prod_{n=1}^{\infty} S_n$ and called the product structure *of the S_n:*

$$(\Omega, \mathcal{F}, m) = (\prod_{n=0}^{\infty} \Omega_n, \bigotimes_{n=0}^{\infty} \mathcal{F}_n, \prod_{n=0}^{\infty} m_n)$$

$$\begin{aligned} \mathbb{D} = \{f \in L^2(m) : \ & \forall n, \text{ for } m\text{-a.e. } \omega = (\omega_0, \omega_1, \ldots) \\ & \text{the function } x \to f(\omega_0, \ldots, \omega_{n-1}, x, \omega_{n+1}, \ldots) \in \mathbb{D}_n \\ & \text{and } \int \textstyle\sum_n \Gamma_n[f]\, dm < +\infty\} \end{aligned}$$

and for $f \in \mathbb{D}$ $\quad \Gamma[f] = \sum_n \Gamma_n[f]$.

Thanks these properties, is possible the construction of a variety of error structures on a given probabilistic model. Now for a rational treatment of a practical case these error hypotheses should be obtained by statistical methods. This is connected with the *Fisher information theory*, see [B2] for the main ideas. Anyhow, these statistical methods are not yet sufficiently studied to be exposed here, especially in the infinite dimensional case we have to use in finance. Thus we limit ourselves to error computations with *a priori* errors chosen the most likely we can. We consider it is significant already.

3 Error structures on the Wiener space

Let us first recall the classical construction of the Brownian motion thanks the Wiener integral.

3.1 The Wiener space as Gaussian product space

Since we aim here at applications to simple financial models we will consider only the case where a given measured space $(E, \mathcal{E}, \mu)$ is given which is either $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), dt)$ or $([0,1], \mathcal{B}([0,1]), dt)$ and a one-dimensional Brownian motion, because products are easily done as we have seen just above (for the abstract Wiener space setting see [B-H]).

Let (χ_n) be an orthonormal basis of $L^2(E, \mathcal{E}, \mu)$ and let (g_n) be a sequence of i.i.d. reduced Gaussian variables defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. To each $f \in L^2(E, \mathcal{E}, \mu)$ we associate $I(f) \in L^2(\Omega, \mathcal{A}, \mathbb{P})$ by

$$I(f) = \sum_n < f, \chi_n > g_n.$$

then I is an isometric homomorphism from the Hilbert space $L^2(E, \mathcal{E}, \mu)$ into the Hilbert space $L^2(\Omega, \mathcal{A}, \mathbb{P})$. If f and g are orthogonal in $L^2(E, \mathcal{E}, \mu)$, $I(f)$ and $I(g)$ are independent Gaussian random variables and putting

$$B_t = \sum_n < 1_{[0,t]}, \chi_n > g_n \qquad (t \in [0,1] \text{ or } t \in \mathbb{R}_+) \tag{4}$$

defines a Gaussian stochastic process which is easily shown to be a standard Brownian motion. By extending the case where f is a step function, the random variable $I(f)$ is denoted by

$$\int f(s)\, dB_s$$

and defines the Wiener integral of f.

In this construction we can suppose the space $(\Omega, \mathcal{A}, \mathbb{P})$ be a product space :

$$(\Omega, \mathcal{A}, \mathbb{P}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), m)^{\mathbb{N}} \qquad m = N(0,1)$$

and the g_n's be the coordinate maps. Thus $\omega = (\omega_0, \ldots, \omega_n, \ldots)$ and $g_n(\omega) = \omega_n$. By the theorem on products of error structures, as soon as errors structures are defined on the factor spaces $(\mathbb{R}, \mathcal{B}(\mathbb{R}), m, \mathbf{d}_n, \gamma_n)$ this defines an error structure

$$(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma) = \prod_{n=0}^{\infty}((\mathbb{R}, \mathcal{B}(\mathbb{R}), m, \mathbf{d}_n, \gamma_n)$$

whose domain $\mathbb{D}$ is explicitely given in theorem 1.

3.2 The Ornstein-Uhlenbeck structure

Let us take for each factor the one-dimensional structure of example (a) i.e.

$$\begin{aligned} \gamma_n[f] = &\ f'^2 \\ \mathbf{d}_n = &\ H^1(m) \qquad m = N(0,1) \end{aligned}$$

hence the associated form is

$$\epsilon_n(f) = \int f'^2\, dm$$

The structure $(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$ is the infinite dimensional Ornstein-Uhlenbeck structure. Let $f \in L^2(\mathbb{R}_+)$ then

$$I(f) = \int f(s)\, dB_s = \sum_n < f, \chi_n > g_n$$

and by the theorem 1 we have

$$\begin{aligned} \Gamma[g_n] = &\ 1 \\ \Gamma[g_m, g_n] = &\ 0 \quad \text{if } m \neq n \end{aligned}$$

and $I(f) \in \mathbb{D}$ with

$$\Gamma[I(f)] = \sum_n < f, \chi_n >^2 \Gamma[g_n] = \|f\|^2_{L^2(\mathbb{R}_+}.$$

This property $\Gamma[\int f(s)\ dB_s] = \|f\|^2_{L^2(\mathbb{R}_+)}$ caracterizes the Ornstein-Uhlenbeck error structure on $(\Omega, \mathcal{A}, \mathbb{P}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), m)^{\mathbb{N}}$ because it follows that if $F \in \mathcal{C}^1 \cap Lip(\mathbb{R}^k)$

$$\Gamma[F(\int f_1 dB, \ldots, \int f_k dB)] = \sum_{i,j=1}^k F'_i(\int f_1 dB, \ldots) F'_j(\int f_1 dB, \ldots) \Gamma[\int f_i dB, \int f_j dB]$$

and the random variables

$$F(\int f_1 dB, \ldots, \int f_k dB)$$

for $F \in \mathcal{C}^1 \cap Lip(\mathbb{R}^k)$ and $f_i \in L^2(\mathbb{R}_+)$ are a dense subspace of $L^2(\mathbb{P})$ since containing cylindrical functions of class $\mathcal{C}^1 \cap Lip$.

It can be shown that a rather large class of random variables obtained by stochastic calculus are in the domain $\mathbb{D}$, especially the solutions of stochastic differential equations with Lipschitz coefficients, cf. [B-H] chapter IV.

3.3 Other structures on the Wiener space

Variants of the preceding construction yield other error structures. For example let us consider the case $(E, \mathcal{E}, \mu) = ([0,1], \mathcal{B}([0,1]), dt)$ and let (χ_n) be the following basis of $L^2([0,1], dt)$

$$\begin{aligned} \chi_n(t) &= \sqrt{2}\cos(nt) && \text{if } n > 0 \\ \chi_0(t) &= 1 \\ \chi_n(t) &= \sqrt{2}\sin(nt) && \text{if } n < 0 \end{aligned}$$

and for the error structure let us take

$$(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma) = \prod_{n\in\mathbb{Z}} (\mathbb{R}, \mathcal{B}(\mathbb{R}), m, H^1(m), \gamma_n)$$

where $\gamma_n[u] = (2\pi n)^{2q} u'^2$ for $u \in H^1(m)$, q being a fixed natural number. Denoting as before the coordinate maps by g_n, we have $\Gamma[g_n] = (2\pi n)^{2q}$ and $\Gamma[g_m, g_n] = 0$ if $m \neq n$. For $f \in L^2([0,1], dt)$ whose Fourier series representation is $f = \sum_{n\in\mathbb{Z}} \hat{f}_n \chi_n$ we obtain

$$\Gamma[\int_0^1 f(s)\ dB_s]\Gamma[\sum_{n\in\mathbb{Z}} \hat{f}_n \chi_n] = \sum_{n\in\mathbb{Z}} \hat{f}_n^{\,2} (2\pi n)^{2q}$$

from which it is easily shown that for $f \in L^2([0,1], dt)$, $\int f\ dB$ belongs to $\mathbb{D}$ if and only if the q-th derivative $f^{(q)}$ in the sens of distributions belongs to $L^2([0,1], dt)$ and then

$$\Gamma[\int f\ dB] = \int f^{(q)2}(s)\ ds.$$

As explained for the Ornstein-Uhlenbeck case, the above formula determines uniquely the error structure on $(\Omega, \mathcal{A}, \mathbb{P})$.

More generally this can be extended in connection with the so-called second quantization. Let p_t be a strongly continuous contraction semi-group on $L^2([0,1], dt)$ with generator $(a, \mathcal{D}a)$, let us consider the associated closed positive quadratic form $(\varepsilon, \mathcal{D}(\sqrt{-a}))$ defined by $\varepsilon[f] = \|\sqrt{-a}f\|^2$, then the structure on the Wiener space induced by the formula

$$\Gamma[\int_0^\infty f(s)\, dB_s] = \varepsilon[f], \qquad f \in \mathcal{D}(\sqrt{(-a)})$$

is closable and thus defines an error structure, see [B-R]. It is worth noting that the semi-group p_t is not supposed to act positively on positive functions (i.e. the form ε is not necessarily Dirichlet) and the form ε does not need to be local.

3.4 The gradient operator and the derivative

Let us come back to the Ornstein-Uhlenbeck structure for simplicity.

$$(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)(\mathbb{R}, \mathcal{B}(\mathbb{R}), m, H^1(m), \gamma)^{\mathbb{N}}$$

with $m = N(0,1)$ and $\gamma[u] = u'^2$. The same ideas extend to more general structures, in fact to any error structure whose space $\mathbb{D}$ is separable, see [B-H] chapter V exercise 5.9.

Let $\omega = (\omega_1, \ldots, \omega_n, \ldots)$ and let g_n be the coordinate maps as before. By theorem 1 we know that if $U \in \mathbb{D}$ then $\frac{\partial U}{\partial \omega_n}(\omega_1, \ldots, \omega_{n-1}, ., \omega_{n+1}, \ldots)$ exists in the sense of distributions for almost every $\omega_1, \ldots, \omega_{n-1}, \omega_{n+1}, \ldots$ and $\sum_n (\frac{\partial U}{\partial \omega_n})^2 \in L^1(\mathbb{P})$.

Thus we can define the gradient operator D on $\mathbb{D}$ with values in $L^2(\mathbb{P}, H)$ with $H = L^2(\mathbb{R}_+)$ by

Definition 3.1

$$DU = \sum_n \frac{\partial U}{\partial \omega_n} \chi_n(t).$$

Proposition 3.2 *D is a continuous application from $\mathbb{D}$ into $L^2(\mathbb{P}, H)$ such that*

1) $\forall U, V \in \mathbb{D} \quad < DU, DV >_H = \Gamma[U, V]$

2) $\forall F \in \mathcal{C}^1 \cap Lip(\mathbb{R}^d)$, $\forall X \in \mathbb{D}$

$$D(F \circ X) = \sum_{i=1}^d F_i' \circ X.DX_i \quad \mathbb{P}-a.s.$$

Proof : The continuity comes from the equalities

$$\|DU\|_{L^2(\mathbb{P},H)} = \| \, \|DU\|_H \|_{L^2(\mathbb{P})} = \sqrt{\mathbb{E}\|DU\|_H^2} = \sqrt{\mathcal{E}[U]}$$

where $\mathcal{E}[.]$ is the form associated with Γ (definition of an error structure item 3).

Then immediately

$$< DU, DV >_H = \sum_n \frac{\partial U}{\partial \omega_n} \frac{\partial V}{\partial \omega_n} = \Gamma[U, V]$$

and for F and X as in the statement

$$\begin{aligned}&\|D(F\circ X)-\textstyle\sum_i F'_i\circ X DX_i\|^2_H\\ &=\Gamma[F\circ X]-2\textstyle\sum_i F'_i\circ X\Gamma[F\circ X,X_i]\quad+\textstyle\sum_{i,j}F'_i\circ XF'_j\circ X\Gamma[X_i,X_j]\end{aligned}$$

which is zero by the functional calculus satisfied by Γ (definition of an error structure item 2). ∎

The operator D satisfies the following properties

. $\forall h\in L^2(\mathbb{R}_+)\quad D[\int_0^\infty h(s)\,dB_s]=h$

. $\mathbb{D}\cap L^\infty$ is an algebra and if $U,V\in\mathbb{D}\cap L^\infty$

$$D(UV)=DU.V+U.DV$$

. with suitable hypotheses on the adapted process H_t

$$D[\int_0^\infty H_s\,dB_s](t)=H_t+\int_0^\infty(DH_s)(t)\,dB_s.$$

For properties and use of the gradient operator we refer to the book of D. Nualart [N], in particular for the so-called Clark formula :

Let $U\in\mathbb{D}$ and $\mathcal{F}_t=\sigma(B_s,s\le t)$

$$U=\mathbb{E}U+\int_0^\infty\mathbb{E}[DU(t)|\mathcal{F}_t]\,dB_t$$

cf. [N] chapter I p42.

Now a slight variant of the gradient operator, the notion of 'derivative', is useful when computing errors on solutions of stochastic differential equations thanks the tool of Ito's formula (this notion was theoretically introduced by e.g. [F-laP]).

Definition 3.3 *Let $(\hat{B}_t)_{t\ge0}$ be an auxiliary independent Brownian motion. For $U\in\mathbb{D}$ the derivative $U^\#$ is a random variable depending on ω and $\hat{\omega}$ defined by*

$$U^\#=\int_0^\infty(DU)(\omega,t)\,d\hat{B}_t.$$

>From the properties of the gradient one gets

. $\Gamma[U]=\hat{\mathbb{E}}[U^{\#2}]$

. For $F\in\mathcal{C}^1\cap Lip\qquad(F\circ U)^\#=F'\circ U.U^\#$

. With suitable hypotheses on the adapted process H_t

$$(\int H_s\,dB_s)^\#=\int H_s^\#\,dB_s+\int H_s\,d\hat{B}_s$$

see [B-H] chapter III paragraph 2.

Let us mention two lemmas that we will use in the sequel. Their proofs are straightforward. They concern the weighted Ornstein-Uhlenbeck case :

$$\Gamma[\int_0^\infty f(s)\,dB_s]=\int_0^\infty\alpha(t)f^2(t)\,dt\qquad f\in\mathcal{D}(\mathbb{R}_+)\tag{5}$$

where α is non negative. For this structure if $f \in L^2(\mathbb{R}_+, (\alpha+1)dt)$

$$\begin{aligned} D(\textstyle\int f(s)\, dB_s)(t) &= \sqrt{\alpha(t)} f(t) \\ (\textstyle\int f(s)\, dB_s)^{\#} &= \textstyle\int_0^\infty \sqrt{\alpha(t)} f(t)\, d\hat{B}_t. \end{aligned}$$

Lemma 3.4 *The conditional expectation operators $\mathbb{E}[\,.\,|\mathcal{F}_t]$ are orthogonal projectors in $\mathbb{D}$ on errors sub-structures (closed sub-vector-spaces of $\mathbb{D}$ stable by Lipschitz functions).*

Lemma 3.5 *Under the same hypotheses, let Γ_t be defined from Γ by*

$$\Gamma_t[(\textstyle\int f(s)\, dB_s)] = \Gamma[(\textstyle\int 1_{[0,t]} f(s)\, dB_s)]$$

and let $U \to U^{\#_t}$ the derivation operator associated with Γ_t, then for $U \in \mathbb{D}$:

$$(\mathbb{E}[U|\mathcal{F}_t])^{\#} = \mathbb{E}[U^{\#_t}|\mathcal{F}_t].$$

4 Error calculus on the Black-Scholes model

Let us recall for completeness the main features of the Black-Scholes model, for the financial theory and formulas we refer to [L-L] [D-J] [El-K].

The interest rate for the bond is constant, the asset $(S_t)_{t\geq 0}$ is modelised as the solution of the equation

$$dS_t = S_t(\mu dt + \sigma dB_t)$$

where (B_t) is a Brownian motion.

For a European option of the form $f(S_T)$, T fixed deterministic time, the value at time $t \in [0,T]$ of the option is $V_t = F(t, S_t, \sigma, r)$ with

$$F(t,x,\sigma,r) = e^{-r(T-t)} \int_{\mathbb{R}} f(xe^{(r-\frac{\sigma^2}{2})(T-t)+\sigma y\sqrt{T-t}}) \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy. \tag{6}$$

If f is Borel with linear growth, the function F is $\mathcal{C}^1$ in $t \in [0,T[$, $\mathcal{C}^2$ and Lipschitz in $x \in]0,\infty[$.

Let us put

$$\begin{aligned} \text{delta}_t &= \frac{\partial F}{\partial x}(t, S_t, \sigma, r) \\ \text{gamma}_t &= \frac{\partial^2 F}{\partial x^2}(t, S_t, \sigma, r) \end{aligned}$$

F satisfies the equation

$$\frac{\partial F}{\partial t} + \frac{\sigma^2 x^2}{2}\frac{\partial^2 F}{\partial x^2} + rx\frac{\partial F}{\partial x} - rF = 0. \tag{7}$$

We shall evaluate the errors on significant quantities of the model supposing

a) an error on $(B_t)_{t\geq 0}$ represented by the Ornstein-Uhlenbeck error structure with the multiplicative constant e_B

b) errors on the initial value S_0, oĹn the volatility σ, on the rate r, which are 'constant relative errors' in the sense of physicists :

$$\begin{aligned}\Gamma[\phi(S_0)] &= \phi'^2(S_0)\, S_0^2\, e_0 \\ \Gamma[\psi(\sigma)] &= \psi'^2(\sigma)\, \sigma^2\, e_1 \\ \Gamma[\xi(r)] &= \xi'^2(r)\, r^2\, e_1\end{aligned}$$

c) we chose *a priori* laws on S_0, on σ, on r which are for example exponential laws or lognormal laws, etc.

d) and we suppose $(B_t)_{t\geq 0}$ and the randomized quantities are independent and their errors uncorrelated.

In other words the error on a regular function

$$F((B_t)_{t\geq 0}, S_0, \sigma, r)$$

will be represented by the product error structure i.e.

$$\Gamma[F((B_t)_{t\geq 0}, S_0, \sigma, r)]\Gamma_{OU}[F(., S_0, \sigma, r)]e_B + F'^2_{S_0} S_0^2 e_0 + F'^2_\sigma \sigma^2 e_1 + F'^2_r r^2 e_2$$

where e_B, e_0, e_1, e_2 are positive constants and Γ_{OU} the Ornstein-Uhlenbeck quadratic error operator.

Actually, the theory tells us that hedging and pricing formulas do not involve the drift coefficient μ. So we may take $\mu = r$, i.e. we work under the probability $\mathbb{P}$ such that $\tilde{S}_t = e^{-rt} S_t$, the discounted stock price, is a martingale. since $S_t = S_0 e^{\sigma B_t + (r - \frac{\sigma^2}{2})t}$ we have

$$\Gamma[S_t] = S_t^2\{\sigma^2 t e_B + e_0 + (B_t - \sigma t)^2 \sigma^2 e_1 + t^2 e_2\}$$

The relative standard deviation of the error $\frac{\sqrt{\Gamma[S_t]}}{S_t}$ writes

$$\frac{\sqrt{\Gamma[S_t]}}{S_t} = \{\sigma^2 t e_B + e_0 + [\log \frac{S_t}{S_0} - (\frac{\sigma^2}{2} + r)t]^2 e_1 + t^2 e_2\}^{\frac{1}{2}}.$$

We see that the part of the error coming from S_0 does not depend on t, that one from (B_t) is proportional to $\sqrt{t}$, that ones from σ and r are of the order of t.

4.1 European options

Let us consider an option of the form $f(S_T)$ where f is Lipschitz.

By the independence hypothesis, the errors on B, S_0, σ, r can be managed separately. Let us denote Γ_B, Γ_0, Γ_σ, Γ_r the corresponding quadratic operators.

a) *Error the value of the option*

The value of the option is $V_t = F(t, S_t, \sigma, r)$ with F given by (6)

a1)*Error due to B.*

B being present only in S_t, we have $\Gamma_B[V_t] = (\frac{\partial F}{\partial x}(S_t, \sigma, r))^2 \Gamma_B[S_t]$ so

$$\begin{aligned} \Gamma_B[V_t] = &\ \text{delta}_t{}^2\ \Gamma_B[S_t] \\ \Gamma_B[V_s, V_t] = &\ \text{delta}_s \text{delta}_t\ \Gamma_B[S_s, S_t] \end{aligned} \tag{8}$$

with $\Gamma_B[S_s, S_t] = S_s S_t \sigma^2\, s \wedge t$.

Proposition 4.1 *If f is Lipschitz, V_t is in $\mathbb{D}_B$ and when $t \uparrow T$*

$$V_t = F(t, S_t, \sigma, r) \to f(S_T) \quad \textit{in } \mathbb{D}_B \textit{ and } \mathbb{P} - a.s.$$

$$\Gamma_B[V_t] = (delta_t)^2 \Gamma_B[S_t] \to f'^2(S_T)\Gamma_B[S_T] \quad \textit{in } L^1 \textit{ and } \mathbb{P} - a.s.$$

Proof : Let us suppose first $f \in \mathcal{C}^1 \cap Lip$. By the relation

$$V_t = \mathbb{E}[e^{-r(T-t)} f(S_T) | \mathcal{F}_t]$$

it follows that $V_t \to f(S_T)$ in $L^p \quad 1 \le p < \infty$ and a.s.

A computation that we shall do in a more general framework later, and that we do not repete here, gives

$$V_t^\# = e^{-r(T-t)} \mathbb{E}[f'(S_T) S_T | \mathcal{F}_t] \sigma \hat{B}_t$$

thus

$$V_t^\# \to f'(S_T) S_T \sigma \hat{B}_T \text{ in } L^2(\mathbb{P}, L^2(\hat{\Omega}, \hat{\mathbb{P}}))$$

and thanks $f(S_T)^\# = f'(S_T) S_T \sigma \hat{B}_T$ we ontain

$$V_t \to f(S_T) \quad \text{in } \mathbb{D}_B \text{ and } \mathbb{P} - a.s.$$

and

$$\Gamma_B[V_t] = e^{-2r(T-t)} (\mathbb{E}[f'(S_T) S_T | \mathcal{F}_t])^2 \sigma^2 t \to f'^2(S_T) \Gamma_B[S_T]$$

in L^1 and $\mathbb{P}$-a.s.

The case f only Lipschitz comes from a special property of the one-dimentional functional calculus in error structures (see [B-H] chapter III prop. 2.1.5) making the preceding argument remains valid. ∎

a2)*Error due to σ.*

We suppose here $f \in \mathcal{C}^1 \cap Lip$. As $V_t = F(t, S_t, \sigma, r)$

$$\Gamma_\sigma[V_t] = \{F'_x((t, S_t, \sigma, r) \frac{\partial S_t}{\partial \sigma} + F'_\sigma((t, S_t, \sigma, r)\}^2\ \sigma^2 e_1$$

and the computation can be done using the integral representation (6), puting $\tilde{S}_t = e^{-rt} S_t$ and

$$\begin{aligned} \tilde{F}(t, x) = &\ e^{-rt} F(t, S_t, \sigma, r) \\ \tilde{V}_t = e^{-rt} V_t = &\ \tilde{F}(t, \tilde{S}_t) \end{aligned}$$

and remarking that by (6) we have

$$\frac{\partial \tilde{F}}{\partial \sigma}(t,x) = -2\sqrt{T-t}\frac{\partial \tilde{F}}{\partial t}(t,x)$$

we have using the differential equation (7)

$$\Gamma_\sigma[V_t] = \{\sqrt{T-t}\sigma^2 S_t^2 \text{gamma}_t + S_t(B_t - \sigma t)\text{delta}_t\}^2 \, \sigma^2 \, e_1 \tag{9}$$

a3)*Error due to* r.

We have similarly

$$\Gamma_r[V_t] = \{F'_x((t, S_t, \sigma, r)\frac{\partial S_t}{\partial r} + F'_r((t, S_t, \sigma, r)\}^2 \; r^2 e_2$$

and we obtain

$$\Gamma_r[V_t] = \{T\text{delta}_t - (T-t)V_t\}^2 \; r^2 \, e_2 \tag{10}$$

a3)*Error coming from the exercise price.*

In the case of a call or a put, it is possible to evaluate the sensitivity due to the exercise price. One can use the classical explicit formulas or remark that if we denote F_{call} the function F in the case $f(S_T) = (S_T - K)^+$ then

$$\Gamma_K[V_t] = \{\frac{\partial F_{\text{call}}}{\partial K}(t, S_t)\}^2 \Gamma_K[I_K]$$

where I_K is the identity function of K, we have $\frac{\partial V_T}{\partial K} = -1_{\{S_T \geq K\}}$ and $\frac{\partial F_{\text{call}}}{\partial K}(t, S_t)$ is, up to the sign, the value at time t of the digital option $1_{\{S_T \geq K\}}$ for which the integral representation (6) applies.

b) *Error on the hedging portfolio*

Here we limit ourselves to the error due to (B_t). We suppose f and f' in $\mathcal{C}^1 \cap Lip$. The hedging equation is

$$e^{-rt}F(t, S_t, \sigma, r) = F(0, S_0, \sigma, r) + \int_0^t H_s \, d\tilde{S}_s$$

where the adapted process H_t is the quantity of stock in the portfolio :

$$H_t = \text{delta}_t = \frac{\partial F}{\partial x}(t, S_t, \sigma, r) = e^{-r(T-t)}\mathbb{E}[f'(S_T)S_T|\mathcal{F}_t]\frac{1}{S_t}.$$

By the same method as for V_t we obtain

$$\begin{aligned} \Gamma_B[H_t] &= \; (\text{gamma}_t)^2 \Gamma_B[S_t] \\ \Gamma_B[H_s, H_t] &= \; \text{gamma}_s \text{gamma}_t \Gamma_B[S_s, S_t] \end{aligned} \tag{11}$$

Proposition 4.2 *If* $f, f' \in \mathcal{C}^1 \cap Lip$, *then* $H_t \in \mathbb{D}$ *and as* $t \uparrow T$

$$\begin{aligned} H_t \to & \quad f'(S_T) \quad \text{ in } \mathbb{D}_B \text{ and a.s.} \\ \Gamma_B[H_t] \to & \quad f''^2(S_T)\Gamma_B[S_T] \quad \text{ in } L^1(\mathbb{P}) \text{ and a.s.} \end{aligned}$$

c) *More general errors on* (B_t)

These results show that the quantities delta_t and gamma_t introduced by practitioners have a direct sense as sensitivity of the value V_t and of the hedging H_t to a perturbation of the stock coming from an error on the Brownian motion.

Some relations still hold if we consider more general error structures on the Wiener space. Let us consider, as mentioned above, a structure induced by a closed positive quadratic form ε on $L^2(\mathbb{R}_+, dt)$ with

$$\Gamma_B[\int f\ dB] = \varepsilon[f]$$

for f in the domain of ε with, for example,

$$\begin{array}{ll} \text{a)} & \varepsilon[f] = \int_0^\infty \alpha(t) f^2(t)\ dt \\ \text{b)} & \varepsilon[f] = \int_0^\infty \int_0^\infty (f(s) - f(t))^2 \beta(s)\beta(t)\ dsdt \end{array}$$

then the formulas

$$\begin{aligned} \Gamma_B[V_t] = &\ (\text{delta}_t)^2\ \Gamma_B[S_t] \\ \Gamma_B[H_t] = &\ (\text{gamma}_t)^2 \Gamma_B[S_t] \end{aligned}$$

remain valid as soon as $S_t \in \mathbb{D}$ i.e.
in case a) if $\alpha \in L^1_{\text{loc}}(\mathbb{R}_+, dt)$ and $\Gamma_B[S_t] = S_t^2 \sigma^2 \int_0^t \alpha(s) ds$
in case b) if $\beta \in L^1(\mathbb{R}_+, dt)$ and $\Gamma_B[S_t] = S_t^2 \sigma^2 2 \int_t^\infty \beta(s) ds \int_\infty^t \beta(s) ds$.

Now in the case

$$\text{c)} \qquad \varepsilon[f] = \int_0^\infty (\sum_{i=1}^d a_i(s) f^{(i)}(s))^2\ ds \qquad a_1 \neq 0$$

we do not have anymore $1_{[0,t]} \in \text{dom}(\varepsilon)$, hence B_t doesn't belong to $\mathbb{D}$. Such error structures seem to be more convenient to modelize errors on processes with finite variation. For example in a model such that

$$dS_t = S_t \sigma(S_t) dB_t + S_t R(t) dt$$

we could modelize the rate $R(t)$ by

$$R(t) = \varphi(\int_0^t \tilde{B}_s ds)$$

where $\tilde{B}_t$ is an independent Brownian motion defined on $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mathbb{P}})$ and φ a regular function. If on $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mathbb{P}})$ we consider the error structure $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mathbb{P}}, \tilde{\mathbb{D}}, \tilde{\Gamma})$ satisfying

$$\tilde{\Gamma}[\int_0^\infty f\, d\tilde{B}] = \int_0^\infty f'^2(s)\, ds \qquad f \in H^1(\mathbb{R}_+, dt)$$

we have $R(t) \in \tilde{\mathbb{D}}$ and

$$\tilde{\Gamma}[R_s, R_t] = \varphi'(\int_0^s \tilde{B}_u\, du) \varphi'(\int_0^t \tilde{B}_v\, dv) s \wedge t.$$

4.2 American options

Here we will just install the problem.

To ways are a priori possible. In the case of an option of the form $f(S_T)$, the price of the option at time t can be obtained by solving the following system of partial differential inequalities

$$\left\{\begin{array}{ll} \frac{\partial u}{\partial t} + Au - ru \leq 0, \quad u \geq f & \text{in } [0,T] \times \mathbb{R} \\ (\frac{\partial u}{\partial t} + Au - ru)(f-u) = 0 & \text{in } [0,T] \times \mathbb{R} \\ u(T,x) = f(x) \text{ in } \mathbb{R} & \end{array}\right. \tag{12}$$

where $A = \frac{\sigma^2}{2}x^2\frac{\partial^2}{\partial x} + rx\frac{\partial}{\partial x}$ and taking $V_t = u(t, S_t)$. Then errors depend on the analytic problem of the regularity of the function u with respect to x, σ, r.

An other way consists to use the stochastic form of the optimal stopping problem, maximazing an expectation over the stopping times with values in $[0,T]$. This second way seems less convenient than the first to deal with errors.

We return now to the European case for diffusion models which can be managed, as we will see, by stochastic calculus.

5 Diffusion models

We will display the method in the case of a complete market, the probability being a martingale measure and for a simple one-dimensional diffusion model.

The stock is supposed to be the solution of the equation

$$dX_t = X_t\sigma(t, X_t)\, dB_t + X_t r(t)\, dt.$$

We limit the study to the error due to (B_t) which is a weighted Ornstein-Uhlenbeck structure:

$$\Gamma[\int_0^\infty h(s)\, dB_s] = \int_0^\infty \alpha(s)h^2(s)\, ds$$

α positive and bounded. The rate is deterministic, the function $\sigma(t,x)$ will be supposed bounded with bounded derivative in x uniformly for $t \in [0,T]$.

Let $f(X_T)$ be a European option. Its value at time t is

$$V_t = \mathbb{E}[\exp(-\int_t^T r(s)ds)f(X_T)|\mathcal{F}_t]$$

the hedging portfolio is given by the adapted process H_t which satisfies

$$\tilde{V}_t = \exp(-\int_0^t r(s)ds)V_t = V_0 + \int_0^t H_s\, d\tilde{X}_s \tag{13}$$

where $\tilde{X}_s = \exp(-\int_0^t r(s)ds)X_t$.

We proceed as follows: from the equation

$$X_t = X_0 + \int_0^t X_s\sigma(s, X_s)dB_s + \int_0^t r(s)X_s ds$$

we obtain

$$X_u^\# = \int_0^u (\sigma(X_s)+X_s\sigma'_x(X_s))X_s^\# \, dB_s + \int_0^u \sqrt{\alpha(s)} X_s\sigma(X_s)\, d\hat{B}_s + \int_0^u r(s)X_s^\# ds$$

this equation is solved by putting

$$\begin{aligned} K_s = &\ \ \sigma(s,X_s) + X_s\sigma'_x(s,X_s) \\ M_u = &\ \ \exp\left\{\int_0^u K_s\, dB_s - \tfrac{1}{2}\int_0^u K_s^2 ds + \int_0^u r(s)ds\right\} \end{aligned}$$

and remarking that

$$X_u^\# = M_u \int_0^u \frac{\sqrt{\alpha(s)}X_s\sigma(X_s)}{M_s}\, d\hat{B}_s.$$

a) Let us first suppose $f \in \mathcal{C}^1 \cap Lip$ and let us define $Y = \exp(-\int_t^T r(s)ds)f(X_T)$. To compute $(\mathbb{E}[Y|\mathcal{F}_t])^\#$ we will apply the lemma 2 of section 3:

$$Y^{\#_t} = \exp(-\textstyle\int_t^T r(s)ds)f'(X_T)X_T^{\#_t}$$

and

$$\begin{aligned} (\mathbb{E}[Y|\mathcal{F}_t])^\# &= \exp(-\textstyle\int_t^T r(s)ds)\mathbb{E}[f'(X_T)X_T^{\#_t}|\mathcal{F}_t] \\ &= \exp(-\textstyle\int_t^T r(s)ds)\mathbb{E}[f'(X_T)M_T|\mathcal{F}_t]\displaystyle\int_0^t \frac{\sqrt{\alpha(s)}X_s\sigma(X_s)}{M_s}\, d\hat{B}_s \end{aligned}$$

and applying the lemma 2 gives

$$\begin{aligned} \Gamma[V_t] = &\ \ \Gamma[\mathbb{E}[Y|\mathcal{F}_t]] \\ = &\ \ \exp(-2\textstyle\int_t^T r(s)ds)(\mathbb{E}[f'(X_T)M_T|\mathcal{F}_t])^2 \int_0^t \frac{\alpha(s)X_s^2\sigma^2(X_s)}{M_s^2}\, ds \end{aligned} \tag{14}$$

this yields also the cross error of V_s and V_t which is usefull to compute errors on random variables such that $\int_0^T h(s)dV_s$

$$\begin{aligned} \Gamma[V_s,V_t] = &\ \ \exp(-\textstyle\int_s^T r(s)ds - \int_t^T r(s)ds) \\ &\ \ \mathbb{E}[f'(X_T)M_T|\mathcal{F}_s]\mathbb{E}[f'(X_T)M_T|\mathcal{F}_t] \\ &\ \ \textstyle\int_0^{s\wedge t} \frac{\alpha(u)X_u^2\sigma^2(X_u)}{M_u^2}\, du. \end{aligned} \tag{15}$$

With our hypotheses as $t \uparrow T$

$$\Gamma[V_t] \to f'^2(X_T)M_T^2 \int_0^T \frac{\alpha(s)X_s^2\sigma^2(X_s)}{M_s^2}\, ds = f'^2(X_T)\Gamma[X_T]$$

in $L^1(\mathbb{P})$ and a.s.

b) Now to deal with H_t, let us remark first that H_t is easily obtained by the Clark formula, see [N]. Formula 13 gives

$$H_t\exp(-\textstyle\int_0^t r(s)ds)X_t\sigma(X_t) = D_{ad}[\exp(-\int_0^T r(s)ds)f(X_T)]$$

where D_{ad} is the adapted gradient defined by

$$D_{ad}[Z](t) = \mathbb{E}[DZ(t)|\mathcal{F}_t].$$

Since

$$D[\exp(-\int_0^T r(s)ds)f(X_T)] = \exp(-\int_0^T r(s)ds)f'(X_T)(DX_T)(t)$$

we have by the computation done for V_t

$$D[\exp(-\int_0^T r(s)ds)f(X_T)] = \exp(-\int_0^T r(s)ds)\mathbb{E}[f'(X_T)M_T|\mathcal{F}_t]\frac{X_t\sigma(X_t)}{M_t}.$$

Thus

$$H_t = \exp(-\int_t^T r(s)ds)\mathbb{E}[f'(X_T)M_T|\mathcal{F}_t]\frac{1}{M_t}.$$

Now supposing f and $f' \in \mathcal{C}^1 \cap Lip$ we apply the same method as for obtaining $\Gamma[V_t]$ which leads to

$$\begin{aligned} \Gamma[H_t] = \ & \exp(-2\int_t^T r(s)ds) \\ & \left(\mathbb{E}[\tfrac{M_T}{M_t}(f''(X_T)M_T + f'(X_T)Z_t^T|\mathcal{F}_t]\right)^2 \\ & \int_0^t \tfrac{\alpha(u)X_u^2\sigma^2(X_u)}{M_u^2}du \end{aligned} \tag{16}$$

$$\begin{aligned} \text{with} \quad Z_t^T = \ & \int_t^T L_s dB_s - \int_t^T K_s L_s M_s ds \\ \text{and} \quad K_s = \ & \sigma(X_s) + X_s\sigma'(X_s) \\ L_s = \ & 2\sigma'(X_s) + X_s\sigma''(X_s). \end{aligned}$$

If we introduce the following notation which extends the Black-Scholes case

$$\begin{aligned} \text{delta}_t = \ & H_t = \exp(-\int_t^T r(s)ds)\mathbb{E}[f'(X_T)M_T|\mathcal{F}_t]\tfrac{1}{M_t} \\ \text{gamma}_t = \ & \exp(-\int_t^T r(s)ds)\mathbb{E}[\tfrac{M_T^2}{M_t^2}(f''(X_T) + \tfrac{M_T}{M_t^2}f'(X_T)Z_t^T|\mathcal{F}_t] \end{aligned}$$

we can summarize the formulas of this diffusion case by

$$\begin{aligned} V_t^{\#} = \ & \text{delta}_t X_t^{\#} \\ H_t^{\#} = \ & \text{gamma}_t X_t^{\#} \\ \Gamma[V_t] = \ & \text{delta}_t^2 \Gamma[X_t] \\ \Gamma[V_s, V_t] = \ & \text{delta}_s \text{delta}_t \Gamma[X_s, X_t] \\ \\ \Gamma[H_t] = \ & \text{gamma}_t^2 \Gamma[X_t] \\ \Gamma[H_s, H_t] = \ & \text{gamma}_s \text{gamma}_t \Gamma[X_s, X_t] \\ \Gamma[V_s, H_t] = \ & \text{delta}_s \text{gamma}_t \Gamma[X_s, X_t] \\ \\ \Gamma[X_t] = \ & M_t^2 \int_0^t \tfrac{\alpha(u)X_u^2\sigma^2(X_u)}{M_u^2}du \\ \Gamma[X_s, X_t] = \ & M_s M_t \int_0^{s\wedge t} \tfrac{\alpha(u)X_u^2\sigma^2(X_u)}{M_u^2}du \end{aligned}.$$

Of course the principle of the method applies to more general cases.

6 Conclusion

a) Let us sketch shortly what would be the second order calculus with variances and bias mentionned above in the table 1 of section 2. On an error structure $(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$ the bias of the error on a random variable X (i.e. the conditional expectation of the error) is represented by the generator A of the semi-group canonically associated with the error structure, see [B-H]. It has a domain $\mathcal{D}A$ smaller than $\mathbb{D}$. The functional calculus on A follows the following rules: for all $F \in \mathcal{C}^2(\mathbb{R}^d)$ with bounded first and second derivates, $\forall f_i \in \mathcal{D}A$, $i = 1, \ldots, d$, $F(f_1, \ldots, f_d) \in \mathcal{D}A$ and

$$A[F(f)] = \sum_{i=1}^{d} F_i'(f) A f_i + \frac{1}{2} \sum_{i,j=1}^{d} F_{ij}''(f) \Gamma[f_i, f_j]$$

On the Black-Scholes model, concerning solely the error due to (B_t), we obtain :

$$\begin{aligned} A_B[S_t] = &\ S_t \sigma e_B (B_t + \tfrac{1}{2}\sigma t) \\ \Gamma_B[S_t] = &\ S_t^2 \sigma^2 e_B t \\ A_B[V_t] = &\ \mathrm{delta}_t A_B[S_t] + \tfrac{1}{2}\mathrm{gamma}_t \Gamma_B[S_t]. \end{aligned}$$

b) What we have done could be relatively easily extended to d-dimensional models and to cross errors of several pricings and hedging portfolios, also to incomplete models with continuous underlying processes, see [K] part 2. The direction of improvement needing really new research seem at present to be the following:

- The means of obtaining Γ from statistics and the connection with the Fisher information theory, see [B2], especially to get Γ on the Wiener space.

- To study the errors and the sensitivity of American options to variations of the Brownian motion used in the modeling of the stock price.

- The extension of the method to models with jumps in connection with stochastic calculus of variation on the general Poisson space, see [B-G-J] [B1] [P].

References

[A-K] Ambrosio, L., Kirchheim, B., Currents in metric spaces, preprint, (to appear).

[B-D] Beurling, A., Deny, J.: Espaces de Dirichlet, I. le cas élémentaire. Acta Math. 99, 203-224 (1958); Dirichlet spaces. Proc. Nat. Acad. Sci. U.S.A. 45, 206-215 (1959)

[B-G-J] Bichteler, K., Gravereaux, J. B., Jacod, J.: *Malliavin calculus with jumps.* Gordon and Breach, 1987

[B-R] Bogachev, V. I., Roeckner, M.: Mehler formula and capacities for infinite dimensional Ornstein-Uhlenbeck processes with general linear drift. Osaka J. Math. 32, 237-274 (1995)

[B1] Bouleau, N. : Construction of Dirichlet structures. In: *Potential theory ICPT 1994*, de Gruyter, 1995.

[B2] Bouleau, N. : Calcul d'erreur complet lipschitzien et formes de Dirichlet, J. Math. pures et appl. (2001)

[B-H] Bouleau, N., Hirsch, F.: *Dirichlet forms and analysis on Wiener space* de Gruyter, 1991.

[B-L] Bouleau, N., Lamberton, D.: Residual risks and hedging strategies in Markovian markets. Stochast. Process. Appl. 33, 131-150 (1989)

[C-M] Cruzeiro, A., Malliavin, P.: Renormailized differential geometry on path space, structural equation, curvature. J. Funct. Anal. 139, 119-181 (1996)

[D-J] Dana, R.-A.,Jeanblanc-Picqué, M.: *Marchés financiers en temps continu*, Economica, 1994

[D-M] Dellacherie, Cl., Meyer, P. A.: *Probabilités et potentiel* Hermann 1987

[Eb-K] Eberlein, E., Keller, U.: Hyperbolic distributions in finance. Bernoulli, 1, 281-299 (1995)

[El-K] Elliott, R. J., Kopp, P. E., *Mathematics of financial markets*, Springer, 1999

[F-laP] Feyel, D., la Pradelle, A. de: Espaces de Sobolev Gaussiens. Ann. Inst. Fourier, 39-4, 875-908 (1989)

[F] Fukushima, M.: *Dirichlet forms and Markov processes* Noth-Holland-Kodansha, 1980

[F-O-T] Fukushima, M., Oshima, Y., Takeda, M.: *Dirichlet forms and Markov processes*de Gruyter, 1980

[K] Karatzas, I.: *Lectures on mathematics of finance*. CRM Monograph Series, AMS, 1997

[L-L] Lamberton, D., Lapeyre, B.: *Introduction au calcul stochastic appliqué à la finance*, Ellipses, 1997

[M-R] Ma, Z., Roeckner, M.:D *Dirichlet forms*, Springer, 1992

[M] Malliavin, P.: *Stochastic analysis*, Springer, 1997.

[N] Nualart, N.: *The Malliavin calculus and related topics*, Springer 1995

[P] Privault, N.: Inégalités de Meyer sur l'espace de Poisson. C. R. Acad. Sci. Paris, sI, t.318, 559, 1994

[S] Silverstein, M. L.: *Symmetric Markov processes*, L. N. in Math. vol 426, Springer, 1974

[St] Stroock, D.: *An introduction to the Analysis of paths on a Riemannian Manifold*. A. M. S. Surveys and Monographs, Vol. 74, AMS Providence, 2000

[W] Weaver, N. Lipschitz algebras and derivation of von Neumann algebras, J. Funct. Anal. 139, 261-300, (1996); Lipschitz algebras and derivation II, Exterior differenciation, J. Funct. Anal. 178, 64-112 (2000)