Probabilistic interpretation and particle method for vortex equations with Neumann's boundary condition

Benjamin Jourdain[1] and Sylvie Méléard[2]

June 26, 2002

Abstract

We are interested in proving the convergence of Monte-Carlo approximations for vortex equations in bounded domains of R^2 with Neumann's condition on the boundary. This work is the first step to justify theorically some numerical algorithms for Navier-Stokes equations in bounded domains with no-slip conditions.

We prove that the vortex equation has a unique solution in an appropriate space and can be interpreted in a probabilistic point of view through a nonlinear reflected process with space-time random births on the boundary of the domain.

Next, we approximate the solution w of this vortex equation by the weighted empirical measure of interacting diffusive particles with normal reflecting boundary conditions and space-time random births on the boundary. The weights are related to the initial data and to the Neumann condition. We can deduce from this result a simple stochastic particle algorithm to approximate w.

Key words: Vortex equation on a bounded domain; Monte-Carlo approximation; Interacting particle systems with reflection; space-time random births

MSC 1991: 60K35 76D05

1 Introduction

We are interested in proving the convergence of Monte-Carlo approximations for vortex equations in bounded domains of R^2 with Neumann's condition on the boundary. This work is the first step to justify theorically some numerical algorithms for Navier-Stokes equations in bounded domains with no-slip conditions, as proposed by Chorin or Cottet. To our knowledge, there was no proof of convergence of such particle methods, even for the deterministic ones, and even in the simplified case we consider in this paper.

[1]ENPC-CERMICS, 6-8 av Blaise Pascal, Cité Descartes, Champs sur Marne, 77455 Marne-la-Vallée Cedex 2, France, jourdain@cermics.enpc.fr

[2]Université Paris 10, MODALX, 200 av. de la République, 92000 Nanterre et Laboratoire de Probabilités et modèles aléatoires (UMR 7599), 4 place Jussieu, 75252 Paris cedex, sylm@ccr.jussieu.fr

We consider the Navier-Stokes equation which describes the evolution of the velocity field of an incompressible viscous fluid in a bounded domain Θ of $\mathbb{R}^2$ satisfying the no-slip boundary condition:

$$\partial_t u(t,x) + (u.\nabla)u(t,x) = \nu\Delta u(t,x) - \nabla p \quad \text{in } \Theta;$$
$$\nabla.u(t,x) = 0 \quad \text{in } \Theta \ ; \ u(x,t) = (0,0) \text{ for } x \in \partial\Theta,$$

where p is the pressure and $\nu > 0$ the viscosity coefficient. The aim we pursue is to obtain a probabilistic interpretation of this equation which enables us to construct an efficient Monte-Carlo particle method for the simulation of the solutions. Another probabilistic approach based on branching processes has already been developped by Benachour, Roynette, Vallois [1], generalized in dimension 3 by Giet [7]. But even if the authors propose some particle approximations, the convergence of the method is not shown and the particle systems they describe are not for use in practice. Our purpose is to construct some easily simulable particle systems, behaving as diffusion processes reflected on the boundary, with space-time random births located at the boundary and to prove rigorously the propagation of chaos of the laws of these processes to a probability measure associated with the solution of the Navier-Stokes equation.

One associates classically with the two dimensional Navier-Stokes equation in the whole plane the simpler vortex equation satisfied by the curl of the velocity. This equation behaves as a McKean-Vlasov equation and the famous vortex simulation algorithm due to Chorin [4] comes from its interaction structure, since the velocity can be written as the convolution of the vorticity by the Biot and Savart kernel. The probabilistic approach, firstly introduced by Marchioro-Pulvirenti [14], has been developed in Méléard [16], [17]. The particle systems are naturally defined and the propagation of chaos proved.

In a bounded domain, a similar approach would consist in replacing the Biot and Savart kernel by the orthogonal gradient K of the Green function of the Dirichlet problem in the domain. But one then only obtains the nullity of the normal component of the velocity on the boundary. To obtain in addition the nullity of the tangential component, we are inspired by Cottet [5], who proves that by adding a nonlinear Neumann condition to the vortex equation, one obtains an admissible vorticity field in the sense that the associated velocity satisfies *a posteriori* the no-slip condition.

This nonlinear Neumann condition is really hard to take into account, and in this paper, we deal with a fixed Neumann condition and consider the equation

$$\partial_t w(t,x) + \nabla.(wKw)(t,x) = \nu\Delta w(t,x) \ \text{ in } \Theta;$$
$$\partial_n w = \nabla w.n = g \text{ on } \partial\Theta$$

where the function g is fixed. Our aim is in particular to show that this Neumann condition is represented at the level of stochastic processes by space-time random births located at the boundary of the domain.

More generally, in this paper, we prove the existence and uniqueness of a solution of such a vortex equation with Neumann's boundary condition in an appropriate space for which we have obtained energy *a priori* estimates. Then we associate with the solution of the equation a nonlinear diffusive and reflected process, with space-time random births at the boundary (with law managed by the function g). We construct interacting normally reflected particle systems with space-time random birth at the boundary and prove the propagation of chaos to the law of the nonlinear process associated with the vortex equation. We are inspired by the paper of Snitman [21], which concerns the behaviour of interacting and reflected McKean-Vlasov particle systems living in a bounded domain. Some additional difficulties appear here, due to the singular interacting kernel K and to the space-time random births. Moreover, the interaction is mean-field but appears through the weighted empirical measure, the weights being related to the initial condition and to g.

Acknowledgements : We thank Tony Lelièvre for numerous discussions concerning the variational approach for evolution equations.

2 The model

Let $T > 0$. We are interested in the following equation :

$$\begin{aligned} &\partial_t w(t,x) + \nabla.(wKw)(t,x) = \nu \Delta w(t,x) \quad \text{in }]0,T] \times \Theta; \\ &w(0,x) = w_0(x) \text{ in } \Theta; \\ &\partial_n w = \nabla w.n = g \text{ on }]0,T] \times \partial\Theta \end{aligned} \tag{2.1}$$

where $n(x)$ denotes the outward normal to $\partial\Theta$ at the point x and $Kw(t,x) = \int_\Theta K(x,y)w(t,y)dy$. The kernel $K(x,y)$ is equal to $\nabla_x^\perp G(x,y) = (-\partial_{x_2} G(x,y), \partial_{x_1} G(x,y))$ where $G(x,y)$ is the fundamental solution of the Poisson equation

$$\Delta_x G(x,y) = \delta_y(x), \; x \in \Theta; \tag{2.2}$$

$$G(x,y) = 0, \; x \in \partial\Theta \tag{2.3}$$

Let us remark the important properties of the kernel K:

$$\forall (x,y) \in (\bar{\Theta})^2 \text{ with } x \neq y, \nabla_x.K(x,y) = 0 \; ; \; \forall x \in \partial\Theta, \forall y \in \bar{\Theta}, \; K(x,y).n(x) = 0 \tag{2.4}$$

In all the following, we will moreover assume

Hypotheses (H):

The domain Θ of $I\!R^2$ is bounded, simply connected and of class $\mathcal{C}^4$.

$$w_0 \in L^2(\Theta) \quad ; \quad g(t,x) \in L^2_t([0,T], L^2_x(\partial\Theta, d\sigma)). \tag{2.5}$$

where $d\sigma(x)$ denotes the surface measure on the boundary.

Thanks to the assumptions made on Θ, the following properties hold for the Green function G and the kernel $K = (K_1, K_2)$:

Lemma 2.1 $\exists C_0 > 0$, $\forall x \neq y \in \bar{\Theta}$,

$$|G(x,y)| \leq C_0(1 + |\ln|x-y||), \quad |K(x,y)| \leq \frac{C_0}{|x-y|}$$

$$|\nabla_x K_i(x,y)| + |\nabla_y K_i(x,y)| \leq \frac{C_0}{|x-y|^2} \quad \textit{for} \;\; i = 1,2.$$

Proof. For $y = (y_1, y_2) \in I\!R^2$, let $y^\perp = (-y_2, y_1)$ and $y^* = y/|y|^2$ if $y \neq (0,0)$.
In case Θ is the unit disk $B(0,1)$ of $I\!R^2$, one has the following explicit expression for the Green function (see [8] p.19)

$$G_0(x,y) = \frac{1}{2\pi} \ln\left(\frac{|x-y|}{|y||x-y^*|}\right). \tag{2.6}$$

We remark that

$$\forall x, y \in \bar{B}(0,1),\ |x-y^*| \geq |y||x-y^*| = \sqrt{|x-y|^2 + (|x|^2-1)(|y|^2-1))} \geq |x-y|. \tag{2.7}$$

As a consequence,

$$|2\pi G_0(x,y)| \leq -\ln|x-y| 1_{\{|x-y|\leq 1\}} + \ln(|y||x-y^*|) 1_{\{|y||x-y^*|\geq 1\}}.$$

As $|y||x-y^*| = |x|y| - y/|y|| \leq 2$, we conclude that $|2\pi G_0(x,y)| \leq |\ln|x-y|| + \ln(2)$.
We also deduce from (2.7) the bound on the corresponding kernel

$$K(x,y) = \frac{1}{2\pi}\left(\frac{(x-y)^\perp}{|x-y|^2} - \frac{(x-y^*)^\perp}{|x-y^*|^2}\right) = \frac{1}{2\pi}\left(((x-y)^*)^\perp - ((x-y^*)^*)^\perp\right).$$

To estimate ∇K_i, we combine (2.7) and the fact that each term of the jacobian matrix of $z \to z^*$ is bounded by $1/|z|^2$.
When Θ is a general bounded and simply connected domain of class $\mathcal{C}^3$, according to [19], there is a conformal mapping from $B(0,1)$ onto Θ which extends to a one-to-one $\mathcal{C}^2$ mapping from $\bar{B}(0,1)$ to $\bar{\Theta}$ denoted by f and such that Df, $(Df)^{-1}$ and $D^2 f$ are bounded on $\bar{B}(0,1)$. Since the Green function for Θ is given by

$$G(x,y) = G_0(f^{-1}(x), f^{-1}(y)),$$

the estimations on G, K and ∇K_i follow from those obtained for the unit disk and the just mentionned properties of f. $\square$

We are interested in weak solutions of (2.1) defined in the following sense

Definition 2.2 *We say that $w : [0,T] \times \Theta \to I\!R$ is a weak solution of (2.1) if :*

(i) *$w \in L^\infty_t(L^2_x) \cap L^2_t(H^1_x)$ where $L^\infty_t(L^2_x)$ and $L^2_t(H^1_x)$ stand respectively for $L^\infty([0,T], L^2(\Theta))$ and $L^2([0,T], H^1(\Theta))$ (and $H^1(\Theta)$ is the Sobolev space consisting of functions which belong together with their first order distribution derivatives to $L^2(\Theta)$).*

(ii) *for any $v \in H^1(\Theta)$, $\frac{d}{dt}\int_\Theta w_t v + \nu \int_\Theta \nabla w_t.\nabla v = \int_\Theta w_t K w_t.\nabla v + \nu \int_{\partial\Theta} g_t v d\sigma$ holds in $\mathcal{D}'(]0,T[)$*

(iii) *$w(0,.) = w_0$.*

Remark 2.3 *The variational formulation (ii) is well defined. Indeed, by the trace theory (see [3] pp. 196-197), $\forall v \in H^1(\Theta)$, $\|v_{|\partial\Theta}\|_{H^{1/2}(\partial\Theta)} \le C\|v\|_{H^1}$. Hence*

$$\left| \int_{\partial\Theta} g_t v d\sigma \right| \le C\|g_t\|_{L^2(\partial\Theta)}\|v\|_{H^1}, \tag{2.8}$$

and by **(H)**, *the second term of r.h.s. belongs to L^2_t. In addition, according to the Lemma 2.4 below, $\forall v \in H^1(\Theta)$, $|\int_\Theta w_t K w_t.\nabla v| \le C\|w_t\|_{L^2}\|w_t\|_{H^1}\|v\|_{H^1}$ and by (i), the first term of the r.h.s. also belongs to L^2_t. So does the second term of the l.h.s..*
The above inequalities together with $|\int_\Theta \nabla w_t.\nabla v| \le \|w_t\|_{H^1}\|v\|_{H^1}$ ensure that if w satisfies (i) and (ii), then the distribution derivative $\partial_t w$ belongs to $L^2_t(H^{1'}_x)$ where $H^{1'}_x$ denotes the dual space of $H^1(\Theta)$. Applying Lemma 1.2 p.261 [23] with $H = L^2(\Theta)$ and $V = H^1(\Theta)$, we deduce that w has a representative in $\mathcal{C}([0,T], L^2(\Theta))$ that we still denote by w.
Moreover, since according to [3] p.195,

$$\forall u \in H^1(\Theta),\ \|u\|_{L^4} \le C\|u\|^{1/2}_{L^2}\|u\|^{1/2}_{H^1} \tag{2.9}$$

any weak solution of (2.1) belongs to $L^4_t(L^4_x)$.

Before stating the existence of a unique weak solution to (2.1), we are going to check the following Lemma which prepares the study of the nonlinear term in (2.1).

Lemma 2.4

$$\forall 2 < p \le +\infty,\ \exists C > 0,\ \forall w \in L^p(\Theta),\ Kw \in \mathcal{C}(\bar\Theta) \text{ and } \|Kw\|_{L^\infty} \le C\|w\|_{L^p} \tag{2.10}$$

$$\exists C > 0,\ \forall w \in L^2(\Theta),\ \|Kw\|_{L^2} \le C\|w\|_{L^2} \tag{2.11}$$

$$\exists C > 0,\ \forall u \in L^2(\Theta), \forall v, w \in H^1(\Theta),\ \left|\int_\Theta uKw.\nabla v\right| \le C\|u\|_{L^2}\|w\|_{H^1}\|v\|_{H^1} \quad (2.12)$$

$$\forall u, v, w \in H^1(\Theta),\ \int_\Theta vKw.\nabla v = 0 \textit{ and } \int_\Theta uKw.\nabla v = -\int_\Theta vKw.\nabla u \qquad (2.13)$$

Proof. For $\alpha > 0$, let $K_\alpha(x,y) = \mathbf{1}_{\{|x-y|>\alpha\}}K(x,y)$. By Lebesgue's theorem and using the continuity of K away from the diagonal, we obtain the continuity of $x \in \bar\Theta \mapsto K_\alpha(x,.) \in L^q_y$, for each $q \ge 1$. When in addition $q < 2$, according to Lemma 2.1, $K_\alpha(x,.)$ converges to $K(x,.)$ in L^q_y uniformly on $\bar\Theta$, when α tends to 0. We deduce that $K(x,.)$ is continuous in L^q_y and obtain (2.10) by Hölder inequality.
Let $w \in L^2(\Theta)$. Using Lemma 2.1 and Cauchy-Schwarz inequality, we get

$$\|Kw\|^2_{L^2} \le \int_\Theta \left(\int_\Theta \frac{C_0}{|x-y|}dy\right)\left(\int_\Theta \frac{C_0 w^2(y)}{|x-y|}dy\right)dx \le \left(\sup_{x\in\bar\Theta}\int_\Theta \frac{C_0}{|x-y|}dy\right)^2 \|w\|^2_{L^2}.$$

Combining for $2 < p < +\infty$, (2.10) and the Sobolev inequality $\|w\|_{L^p} \le C\|w\|_{H^1}$ ([3] p.165), we get

$$\|Kw\|_{L^\infty} \le C\|w\|_{H^1} \qquad (2.14)$$

and conclude that (2.12) holds by Cauchy-Schwarz inequality.
We deduce that $v, w \in H^1(\Theta) \to \int_\Theta vKw.\nabla v$ is continuous. Since according to [3] p.162, the restrictions to Θ of $\mathcal{C}^\infty$ functions with compact support on $\mathbb{R}^2$ are dense in $H^1(\Theta)$, it is enough to check the first equality in (2.13) for smooth v, w. For $\alpha > 0$, let $G_\alpha w(x) = \int_\Theta \mathbf{1}_{\{|x-y|>\alpha\}}G(x,y)w(y)dy$. By Lemma 2.1, $K_\alpha w(x) = \nabla^\perp G_\alpha w(x)$ and $G_\alpha w$ and $K_\alpha w$ converge uniformly on $\bar\Theta$ respectively to Gw and Kw. Since Kw is continuous, we deduce that $Kw = \nabla^\perp Gw$ and $\nabla.Kw = 0$. The boundary condition : $\forall x \in \partial\Theta$, $Gw(x) = 0$ implies that the tangential derivative of Gw vanishes on $\partial\Theta$ i.e. $\forall x \in \partial\Theta$, $Kw(x).n(x) = 0$. Using Green's formula we deduce,

$$\int_\Theta vKw.\nabla v = \frac{1}{2}\int_\Theta Kw.\nabla v^2 = \frac{1}{2}\int_{\partial\Theta} v^2 Kw.n d\sigma - \frac{1}{2}\int_\Theta v^2\nabla.Kw = 0.$$

The second equality in (2.13) is deduced by polarization. □

We are now ready to prove

Theorem 2.5 *Under hypotheses* **(H)**, *equation (2.1) has a unique solution w in the sense of Definition 2.2. In addition, $w \in C([0,T], L^2_x) \cap L^4_t(L^4_x)$.*

The last assertion is a consequence of Remark 2.3. We are going to prove existence by the Galerkin method. Let us first check

Uniqueness : The proof is similar to the one made for the 2d Navier-Stokes equation (see for instance [23] p. 294).

Let v and w denote two solutions and $\tilde{w} = v - w$. As $\tilde{w} \in L^2_t(H^1_x)$ and by Remark 2.3, $\partial_t \tilde{w} \in L^2_t(H^{1'}_x)$, according to [23] p.261 $\frac{d}{dt}\|\tilde{w}(t)\|^2_{L^2} = 2 < \partial_t \tilde{w}_t, \tilde{w}_t >$ holds in the distribution sense on $[0,T]$. The right-hand-side is integrable. Using Definition 2.2 (ii) and (iii), we deduce that for $t \in [0,T]$:

$$\frac{1}{2}\|\tilde{w}(t)\|^2_{L^2} + \nu \int_0^t \|\nabla \tilde{w}_s\|^2_{L^2} ds = \int_0^t \int_\Theta (v_s K v_s - w_s K w_s).\nabla \tilde{w}_s ds. \tag{2.15}$$

Using (2.13) and then (2.12) and Young's inequality, we have

$$\begin{aligned} \left| \int_\Theta (v_s K v_s \right. &- \left. w_s K w_s).\nabla \tilde{w}_s \right| = \left| \int_\Theta \tilde{w}_s K v_s.\nabla \tilde{w}_s + w_s.K\tilde{w}_s.\nabla \tilde{w}_s \right| = \left| 0 - \int_\Theta \tilde{w}_s K \tilde{w}_s.\nabla w_s \right| \\ &\leq C\|\tilde{w}_s\|_{H^1}\|w_s\|_{H^1}\|\tilde{w}_s\|_{L^2} \leq \nu(\|\nabla \tilde{w}(s)\|^2_{L^2} + \|\tilde{w}(s)\|^2_{L^2}) + \frac{C^2}{4\nu}\|w_s\|^2_{H^1}\|\tilde{w}_s\|^2_{L^2}. \end{aligned}$$

Inserting this bound in (2.15), we obtain

$$\forall t \in [0,T],\ \|\tilde{w}(t)\|^2_{L^2} \leq 2\int_0^t \left(\nu + \frac{C^2}{4\nu}\|w_s\|^2_{H^1}\right)\|\tilde{w}_s\|^2_{L^2} ds.$$

Since $s \to \|w_s\|^2_{H^1}$ is integrable, by Gronwall's lemma, $\forall t \in [0,T],\ \|v_t - w_t\|_{L^2} = 0$.

Existence : We first derive an *a priori* estimate which will also hold at the discrete level. Let w be a weak solution of (2.1). As above,

$$\frac{1}{2}\frac{d}{dt}\|w_t\|^2_{L^2} + \nu\|\nabla w_t\|^2_{L^2} = \int_\Theta w_t K w_t.\nabla w_t + \nu \int_{\partial\Theta} g_t w_t d\sigma.$$

According to (2.13), the first term of the r.h.s. is nil. Using moreover (2.8) and Young's inequality, we deduce

$$\frac{1}{2}\frac{d}{dt}\|w_t\|^2_{L^2} + \nu\|\nabla w_t\|^2_{L^2} \leq \frac{\nu}{2}(\|\nabla w_t\|^2_{L^2} + \|w_t\|^2_{L^2}) + C\|g_t\|^2_{L^2(\partial\Theta)} \tag{2.16}$$

Removing the terms involving $\|\nabla w_t\|^2_{L^2}$, we upper-bound $\|w\|^2_{L^\infty_t(L^2_x)}$ by Gronwall's lemma. Inserting this bound in (2.16), we conclude that

$$\|w\|^2_{L^\infty_t(L^2_x)} + \|\nabla w\|^2_{L^2_t(L^2_x)} \leq C_T(\|w_0\|^2_{L^2} + \|g\|^2_{L^2_t(L^2_x(\partial\Theta))}). \tag{2.17}$$

We now employ the so-called Galerkin method. Let $(v_k)_{k\in\mathbb{N}^*}$ denote a Hilbertian basis of $H^1(\Theta)$ and $n \in \mathbb{N}^*$. We want to find $t \in [0,T] \to \Lambda(t) = (\lambda_1(t), ..., \lambda_n(t))$ such that $w^n_t = \sum_{k=1}^n \lambda_k(t) v_k$ satisfies the following approximate problem : w^n_0 is the orthogonal projection in the sense of the L^2 scalar product of w_0 onto $\text{span}(v_1, ..., v_n)$ and

$$\forall 1 \leq k \leq n,\ \frac{d}{dt}\int_\Theta w^n_t v_k + \nu \int_\Theta \nabla w^n_t.\nabla v_k = \int_\Theta w^n_t K w^n_t.\nabla v_k + \nu \int_{\partial\Theta} g_t v_k d\sigma. \tag{2.18}$$

Denoting $A_{jk} = \int_\Theta v_j v_k$, $B_{j,k} = \int_\Theta \nabla v_j.\nabla v_k$, $C_{i,j,k} = \int_\Theta v_i K v_j.\nabla v_k$, $\Gamma(t) = (\gamma_1(t), ..., \gamma_n(t))$

with $\gamma_k(t) = \int_{\partial\Theta} g_t v_k d\sigma$, we obtain that this approximate problem writes :

$$\frac{d}{dt}\Lambda(t) = A^{-1}(-\nu B.\Lambda(t) + \Lambda(t)^* C\Lambda(t) + \nu\Gamma(t)).$$

By a standard fixed-point approach, we obtain existence of a local in time solution $\Lambda(t)$ to this O.D.E.. Thanks to the *a priori* estimate (2.17) which holds for the corresponding w_t^n, and prevents explosion for $\Lambda(t)$, we can iterate this fixed-point approach to extend $\Lambda(t)$ on $[0,T]$.

We next want to take the limit $n \to +\infty$. According to the *a priori* estimate (2.17), the sequence $(w^n)_{n\in\mathbb{N}^*}$ is bounded in $L_t^\infty(L_x^2)$ and $L_t^2(H_x^1)$. Reasoning like in Remark 2.3, we check that $(\partial_t w^n)_{n\in\mathbb{N}^*}$ is bounded in $L_t^2(H_x^{1'})$. Using the compacity result stated in Theorem 2.3 p.276 [23], we deduce that we may extract a subsequence that converges to a limit w weakly* in $L_t^\infty(H_x^1)$, weakly in $L_t^2(H_x^1)$ and strongly in $L_t^2(L_x^2)$. This subsequence is still denoted by (w^n) for notational simplicity. The weak convergences are enough to take the limit in the linear terms of (2.18) (see [23] pp.257-260). We are only going to check the convergence of the nonlinear term. Let $v \in H^1(\Theta)$. Since by (2.11), $\|K(w_n - w)_t.\nabla v\|_{L^1} \leq C\|w_t^n - w_t\|_{L^2}\|v\|_{H^1}$, $Kw^n.\nabla v$ converges to $Kw.\nabla v$ in $L_t^2(L_x^1)$. Combining the *a priori* estimate for w^n and (2.14), we obtain that the sequence $Kw^n.\nabla v$ is bounded in $L_t^2(L_x^2)$. Therefore $Kw^n.\nabla v$ converges to $Kw.\nabla v$ weakly in $L_t^2(L_x^2)$. With the strong convergence of w^n to w in $L_t^2(L_x^2)$ we easily deduce that

$$\forall\psi \in \mathcal{D}(]0,T[),\ \int_0^T \psi'(t)\int_\Theta w_t^n K w_t^n.\nabla v\ dt \to \int_0^T \psi'(t)\int_\Theta w_t K w_t.\nabla v\ dt.$$

Hence w satisfies (ii) in Definition 2.2. Since by standard arguments ([23] pp.257-260) (iii) also holds, we conclude that w is a weak solution of (2.1).

In order to give a probabilistic interpretation to the obtained weak solution of (2.1), we introduce the semi-group $P_t^\nu(x,y)$ associated with $\sqrt{2\nu}$ times the Brownian motion normally reflected on the boundary and prove the following mild representation

Proposition 2.6 *Let w denote the weak solution of (2.1) given by Theorem 2.5. Then $\forall t \in [0,T]$, dx a.e. in Θ,*

$$w_t(x) = P_t^\nu w_0(y) + \int_0^t \nabla P_{t-s}^\nu.(w_s K w_s)(x)ds + \nu\int_0^t\int_{\partial\Theta} P_{t-s}^\nu(y,x)g(s,y)d\sigma(y)ds \tag{2.19}$$

where $\nabla P_{t-s}^\nu.(w_s K w_s)(x) = \int_\Theta \nabla_y P_{t-s}^\nu(y,x).w_s(y)Kw_s(y)dy$.

Proof. Let $t \in]0,T]$ and φ be a smooth function on $\bar{\Theta}$ with a vanishing normal derivative at the boundary : $\partial_n\varphi(x) = 0$ for $x \in \partial\Theta$. According to [12] Theorem 5.3 p.320, the boundary value problem

$$\partial_s \psi + \nu \Delta \psi = 0 \text{ on } [0,t] \times \Theta$$
$$\partial_n \psi = 0 \text{ on } [0,t] \times \partial\Theta$$
$$\psi(t,.) = \varphi(.) \text{ on } \Theta$$

admits a classical solution $\psi(s,x)$ which is $\mathcal{C}^{1,2}$ on $[0,t] \times \bar{\Theta}$. By the Feynman-Kac approach, this solution has the following representation : $\psi(s,x) = P^\nu_{t-s}\varphi(x)$. Clearly $\psi \in L^\infty([0,t], H^1(\Theta))$ and $\partial_s \psi \in L^2([0,t], (H^1)'_x(\Theta))$. By [23], Lemma 1.2 p. 261, we deduce that in $\mathcal{D}'(]0,t[)$,

$$\frac{d}{ds}\int_\Theta w_s\psi(s,.) = \int_\Theta w_s\partial_s\psi(s,.) - \nu\int_\Theta \nabla w_s.\nabla\psi(s,.) + \int_\Theta w_sKw_s.\nabla\psi(s,.) + \nu\int_{\partial\Theta} g_s\psi(s,.)d\sigma.$$

By the equation satisfied by ψ, the sum of the two first terms of the r.h.s. is nil. Hence

$$\begin{aligned}\int_\Theta w_t(x)\varphi(x)dx &= \int_\Theta w_0(x)\psi(0,x)dx + \int_0^t\int_\Theta w_sKw_s(x).\nabla\psi(s,x)dxds \\ &+ \nu\int_0^t\int_{\partial\Theta}\psi(s,x)g(s,x)d\sigma(x)ds.\end{aligned}$$

By the symmetry of P^ν and hypotheses **(H)**, $\int_0^t\int_{\partial\Theta}\int_\Theta P^\nu_{t-s}(x,y)|\varphi(y)|dy|g(s,x)|d\sigma(x)ds \leq \sup|\varphi|\|g\|_{L^1_t(L^1_x(\partial\Theta))} < +\infty$. Hence, by Fubini's theorem the last term of the r.h.s. is equal to $\nu\int_\Theta\varphi(x)\int_0^t\int_{\partial\Theta}P^\nu_{t-s}(y,x)g(s,y)d\sigma(y)dsdx$. We conclude the proof by applying similarly Fubini's theorem to the other terms of the r.h.s. and remarking that the derived equality holds for any smooth function φ with vanishing normal derivative.
To justify the use of Fubini's theorem in the second term, we need the following estimations given by [20] (a.13) and (a.14) p.600 :

$$\forall x \in \bar{\Theta},\ \forall y \in \bar{\Theta},\ |\nabla_x P^\nu_t(x,y)| \leq C_1/t^{3/2} \text{ and } \|\nabla_x P^\nu_t(x,y)\|_{L^1_y(\Theta)} \leq C_1/\sqrt{t}. \tag{2.20}$$

Indeed the first one ensures that $\nabla\psi(s,x) = \int_\Theta \nabla_x P^\nu_{t-s}(x,y)\varphi(y)dy$. By the second one and (2.11),

$$\int_0^t\int_\Theta |w_sKw_s|(x)\int_\Theta|\nabla_x P^\nu_{t-s}(x,y)||\varphi(y)|dydxds \leq C\sup|\varphi|\|w\|^2_{L^\infty_t(L^2_x)}\int_0^t(t-s)^{-1/2}ds.$$

$\square$

3 The probabilistic interpretation of the vortex equation on a bounded domain with a Neumann boundary condition

We are in a McKean-Vlasov context, and the interpretation of the vortex equation as a Fokker-Planck equation allows us to define naturally a nonlinear martingale problem (See for example Méléard [15]).

Here the difficulty is the treatment of the term due to the Neumann condition involving the function g. We essentially follow Fernandez-Méléard [6] and prove that this term is related to space-time random births located at the boundary in the probabilistic interpretation. Our situation is harder than the one of [6] since we are in a bounded domain instead of the whole space and the diffusion processes are reflected on the boundary. There are also births inside the domain at time 0 and the functions w_0 and g are not probability densities.

We follow Jourdain [11] to treat the last difficulty.

Let $\|w_0\|_1 = \int_\Theta |w_0|$ and $\|g\|_1 = \int_{[0,T]\times\partial\Theta} |g| d\sigma dt$. To govern the times and positions of births we introduce on $[0,T] \times \bar{\Theta}$ the probability measure

$$P_0(dt, dx) = \mathbf{1}_{\{x\in\Theta\}}\delta_{\{0\}}(dt)\frac{|w_0(x)|}{\|w_0\|_1 + \nu\|g\|_1}dx + \mathbf{1}_{\{x\in\partial\Theta\}}\frac{\nu|g(t,x)|}{\|w_0\|_1 + \nu\|g\|_1}dt d\sigma(x) \tag{3.1}$$

which does not weight $]0,T] \times \Theta$. To take into account the effect of the sign and mass of w_0 and g, we also consider for $t \in [0,T]$ and $x \in \bar{\Theta}$ the measurable function

$$h(t,x) = \mathbf{1}_{\{t=0,x\in\Theta\}}\frac{w_0(x)}{|w_0(x)|}(\|w_0\|_1 + \nu\|g\|_1) + \mathbf{1}_{\{x\in\partial\Theta\}}\frac{g(t,x)}{|g(t,x)|}(\|w_0\|_1 + \nu\|g\|_1) \tag{3.2}$$

with values in $\{-(\|w_0\|_1 + \nu\|g\|_1), 0, \|w_0\|_1 + \nu\|g\|_1\}$. Let us remark that if φ a bounded measurable function on $[0,T] \times \bar{\Theta}$, then

$$\int_{[0,T]\times\bar{\Theta}} \varphi(t,x)h(t,x)P_0(dt,dx) = \int_\Theta \varphi(0,x)w_0(x)dx + \nu\int_{[0,T]\times\partial\Theta} \varphi(t,x)g(t,x)dt d\sigma(x) \tag{3.3}$$

Let $(\tau, (X_t)_{t\leq T}, (k_t)_{t\leq T})$ denote the canonical process on $[0,T]\times\mathcal{C}([0,T],\bar{\Theta})\times\mathcal{C}([0,T],\mathbb{R}^2)$. For a probability measure Q on this space, we define the family $(\tilde{Q}_t)_{t\in[0,T]}$ of signed measures on $\bar{\Theta}$ by

$$\forall B \in \mathcal{B}(\bar{\Theta}), \quad \tilde{Q}_t(B) = E^Q(h(\tau, X_0)\mathbf{1}_{\{\tau\leq t\}}\mathbf{1}_B(X_t)), \tag{3.4}$$

(One associates with each sample path a signed weight depending on the initial datas).

It is easy to check that for each $t \in [0,T]$, the signed measure $\tilde{Q}_t$ is bounded with a total mass less than $\|w_0\|_1 + \nu\|g\|_1$.

We are now going to give a probabilistic interpretation to the vortex equation, seen as a Fokker-Planck equation, in terms of a martingale problem. This interpretation is inspired from Sznitman [21] and Bossy-Jourdain [2] for the reflected contribution and from Fernandez-Méléard [6] for the space-time random birth contribution.
Let us first define the probability space in which the solutions of the martingale problem we are interested in will live :

Definition 3.1 *Let $T > 0$. We denote by $\mathcal{P}_T$ the space of probability measures Q on $[0,T] \times \mathcal{C}([0,T], \bar{\Theta}) \times \mathcal{C}([0,T], I\!R^2)$ such that for each $t \in [0,T]$, the signed measure $\tilde{Q}_t$ has a density $\tilde{q}_t$ with respect to the Lebesgue measure on Θ and that $t \in [0,T] \to \tilde{q}_t \in L^\infty_t(L^2_x) \cap L^2_t(H^1_x)$.*

By adapting Meyer [18] p.194, one can prove that there exists a measurable version $(s,x) \to \tilde{q}(s,x)$ of the densities of the flow of signed measures $(\tilde{Q}_s)$.

Definition 3.2 *The probability measure $P \in \mathcal{P}_T$ is solution of the nonlinear martingale problem $(\mathcal{M}_T)$ if*

1) $P \circ (\tau, X_0, k_0)^{-1} = P_0 \otimes \delta_{(0,0)}$

2) for each $\phi \in \mathcal{C}^2_b(I\!R^2)$,

$$M^\phi_t = \phi(X_t + k_t) - \phi(X_0) - \int_0^t \mathbf{1}_{\{\tau \le s\}} \Big(K\tilde{p}_s(X_s).\nabla\phi(X_s + k_s) + \nu\Delta\phi(X_s + k_s)\Big) ds$$

is a P-martingale, for the filtration $\mathcal{F}_t = \sigma(\tau, (X_s, k_s), s \le t)$ ($\tilde{p}(s,x)$ denotes a measurable version of the densities of the flow $(\tilde{P}_s)$).

3) P a.s., $\forall t \in [0,T]$, $\int_0^t d|k|_s < +\infty$, $|k|_t = \int_0^t \mathbf{1}_{\{X_s \in \partial\Theta\}} \mathbf{1}_{\{\tau \le s\}} d|k|_s$, and $k_t = \int_0^t n(X_s) d|k|_s$.

The following Lemma states the link between problem $(\mathcal{M}_T)$ and the vortex equation (2.1).

Lemma 3.3 *If $P \in \mathcal{P}_T$ solves $\mathcal{M}_T$ then $\tilde{p}$ is a weak solution of (2.1).*

Proof. By Definition 3.2 1), (3.4) and (3.3), $\tilde{p}_0 = w_0$.

According to Definition 3.2 2), $\beta_t = X_t - X_0 - \int_0^t \mathbf{1}_{\{\tau \le s\}} K\tilde{p}_s(X_s) ds + k_t$ is a P continuous martingale with bracket $<\beta>_t = 2\nu(t-\tau)^+ I_2$ where I_2 denotes the 2×2 identity matrix which implies that $\beta_t = 0$ for $t \in [0,\tau]$. Using moreover Definition 3.2 3), we deduce that $X_t = X_0$ for $t \in [0,\tau]$.
Hence for $\psi \in \mathcal{C}^{1,2}([0,T] \times \bar{\Theta})$

$$\int_0^T \partial_s\psi(s, X_s) ds + \psi(0, X_0) = \psi(\tau, X_0) + \int_0^T \mathbf{1}_{\{\tau \le s\}} \partial_s\psi(s, X_s) ds,$$

If moreover $\forall (s,x) \in [0,T] \times \partial\Theta$, $\partial_n \psi(s,x) = 0$, by Itô's formula, we deduce that

$$\begin{aligned}\psi(T, X_t) &= \psi(\tau, X_0) + \int_0^T \nabla\psi(s, X_s).d\beta_s \\ &+ \int_0^T \mathbf{1}_{\{\tau \leq s\}} (\partial_s \psi(s, X_s) + K\tilde{p}_s(X_s).\nabla\psi(s, X_s) + \nu\Delta\psi(s, X_s))ds\end{aligned}$$

Multiplying by the $\mathcal{F}_0$-measurable variable $h(\tau, X_0)$, taking expectations and using the definition of $\tilde{p}$ and (3.3), we deduce that

$$\begin{aligned}\int_{\bar{\Theta}} \psi(T,x)\tilde{p}(T,x)dx &= \int_{\bar{\Theta}} \psi(0,x)w_0(x)dx + \nu \int_0^T \int_{\partial\Theta} \psi(s,x)g(s,x)d\sigma(x)ds \\ &+ \int_0^T \int_{\bar{\Theta}} (\partial_s\psi(s,x) + K\tilde{p}_s(x).\nabla\psi(s,x) + \nu\Delta\psi(s,x))\tilde{p}(s,x)dxds,\end{aligned}$$

For the choice $\psi(s,x) = \varphi(s)v(x)$ where v is a $\mathcal{C}^2$ function on $\bar{\Theta}$ such that $\partial_n v = 0$ on $\partial\Theta$ and $\varphi \in \mathcal{D}(]0,T[)$, we obtain

$$\int_0^T \left(\varphi'(s) \int_{\bar{\Theta}} \tilde{p}_s v + \varphi(s) \left(\int_{\Theta} \tilde{p}_s K\tilde{p}_s.\nabla v + \nu \int_{\Theta} \tilde{p}_s \Delta v + \nu \int_{\partial\Theta} g_s v d\sigma \right)\right) ds = 0.$$

As $P \in \mathcal{P}_T$, $\tilde{p} \in L_t^2(H_x^1)$. By Green's formula for functions in $H^1(\Theta)$ ([3] p.197) and since $\partial_n v$ vanishes on the boundary, ds a.e. in $[0,T]$, $\int_\Theta \tilde{p}_s \Delta v = -\int_\Theta \nabla\tilde{p}_s.\nabla v$.
Since Θ is $\mathcal{C}^4$, adapting [3] pp.192-193 to diagonalize the Neumann Laplacian, one obtains a Hibertian basis of $H^1(\Theta)$ consisting in $\mathcal{C}^2(\bar{\Theta})$-functions with a vanishing normal derivative. Therefore such functions are dense in H^1 and we conclude that $\tilde{p}$ satisfies Definition 2.2 (ii). $\square$

Theorem 3.4 *Under hypotheses* **(H)**, *there exists a unique solution P to the martingale problem $(\mathcal{M}_T)$. In addition, the corresponding $\tilde{p}$ is a weak solution of (2.1) and satisfies the mild equation (2.19).*

Proof. 1) Uniqueness

Let P^1 and P^2 be two solutions of $(\mathcal{M}_T)$. Then according to Lemma 3.3, $\tilde{p}^1$ and $\tilde{p}^2$ are weak solutions of (2.1). According to Theorem 2.5, $\tilde{p}_1 = \tilde{p}_2 = w$. Hence P^1 and P^2 both solve the martingale problem defined like $(\mathcal{M}_T)$ but with known drift coefficient Kw_s replacing $K\tilde{p}_s$ in Definition 3.2 2). Since $w \in L_t^4(L_x^4)$, by (2.10), $\|Kw_s\|_{L_x^\infty} \in L_t^4$.
Let Γ denote the first marginal of the probability measure P_0 on $[0,T] \times \bar{\Theta}$ and for $i = 1,2$ and $u \in [0,T]$, $p^i(u,.)$ be a regular conditional probability on $[0,T] \times \mathcal{C}([0,T],\bar{\Theta}) \times \mathcal{C}([0,T],I\!R^2)$ endowed with P^i given $\tau = u$.

Then $d\Gamma(u)$ a.e., $p^i(u,.)$ a.s., $\tau = u$, Definition 3.2 3) is satisfied and $p^i(u,.) \circ (X_0, k_0)^{-1}$

is equal to

$$\mathbf{1}_{\{u=0\}}\frac{|w_0(x)|dx}{\|w_0\|_1}\otimes\delta_{(0,0)}+\mathbf{1}_{\{u>0\}}\frac{|g(u,x)|d\sigma(x)}{\int_{\partial\Theta}|g(u,y)|d\sigma(y)}\otimes\delta_{(0,0)} \tag{3.5}$$

and $\forall\phi\in C_b^2(I\!R^2)$,

$$\phi(X_t+k_t)-\phi(X_0)-\int_0^t\mathbf{1}_{\{u\leq s\}}\Big(Kw_s(X_s).\nabla\phi(X_s+k_s)+\nu\Delta\phi(X_s+k_s)\Big)ds$$

is a $p^i(u,.)$-martingale.
Reasoning like in the proof of Lemma 3.3, we obtain that $d\Gamma(u)$ a.e., $p^i(u,.)$ a.s., $X_t = X_0$ and $k_t = (0,0)$ for $t\in[0,u]$. With (3.5), we deduce that $d\Gamma(u)$ a.e., $p^1(u,.)\circ(X_u,k_u)^{-1} = p^2(u,.)\circ(X_u,k_u)^{-1}$ and that for $i=1,2$, $p^i(u,.)$ is equal to the image of $p^i(u,.)\circ((X_{t+u},k_{t+u})_{t\in[0,T-u]})^{-1}$ by the mapping

$$(X_t,k_t)_{t\geq 0}\in\mathcal{C}([0,T-u],\bar{\Theta}\times I\!R^2)\to(X_{(t-u)^+},k_{(t-u)^+})_{t\in[0,T]}\in\mathcal{C}([0,T],\bar{\Theta}\times I\!R^2).$$

Moreover $d\Gamma(u)$ a.e. ,

$$W_t=\frac{1}{\sqrt{2\nu}}\left(X_{t+u}-X_u-\int_u^{t+u}Kw_s(X_s)ds+k_{t+u}\right)$$

is a $p^i(u,.)$ Brownian motion. Since $s\to\|Kw_s\|_{L^\infty}$ is square integrable, combining trajectorial uniqueness for the Brownian motion normally reflected at the boundary of Θ (see [13]), Girsanov's theorem and the equality $p^1(u,.)\circ(X_u,k_u)^{-1}=p^2(u,.)\circ(X_u,k_u)^{-1}$ which holds $d\Gamma(u)$ a.e., we deduce that

$$d\Gamma(u)\ \text{ a.e., }\ p^1(u,.)\circ((X_{t+u},k_{t+u})_{t\in[0,T-u]})^{-1}=p^2(u,.)\circ((X_{t+u},k_{t+u})_{t\in[0,T-u]})^{-1}.$$

Hence $d\Gamma(u)$ p.p. $p^1(u,.)=p^2(u,.)$ and $P^1=P^2$.

2) Existence. Let w be the solution of the vortex equation given by Theorem 2.5. We recall that $\|Kw_s\|_{L^\infty}\in L_t^4$. We construct a solution to the linear martingale problem defined like $(\mathcal{M}_T)$ but with known drift coefficient $Kw_s(.)$ replacing $K\tilde{p}_s$ in Definition 3.2 2) and we check that this probability measure solves $(\mathcal{M}_T)$.
Let (τ,X_0) be a random variable with law P_0 independent from $(W_t)_{t\in[0,T]}$ a two-dimensional Brownian motion. Existence and trajectorial uniqueness hold for the stochastic differential equation with normal reflection

$$X_t=X_0+\sqrt{2\nu}\int_0^t\mathbf{1}_{\{\tau\leq s\}}dW_s-k_t$$
$$|k|_t=\int_0^t\mathbf{1}_{\{X_s\in\partial\Theta\}}\mathbf{1}_{\{\tau\leq s\}}d|k|_s\ ;\ k_t=\int_0^t n(X_s)d|k|_s.$$

Moreover $\forall t \in [0,T]$, X_t admits

$$x \to \frac{1}{\|w_0\|_1 + \nu\|g\|_{L^1([0,t]\times\partial\Theta)}} \left(|w_0|P_t^\nu(x) + \nu \int_0^t \int_{\partial\Theta} |g|(s,y)P_{t-s}^\nu(y,x)\sigma(dy)ds\right)$$

as a density w.r.t. the Lebesgue measure on $\bar\Theta$. Since $\|Kw_s\|_{L^\infty}$ is square integrable, by Girsanov's theorem we deduce that the martingale problem defined like $(\mathcal{M}_T)$ but with known drift coefficient Kw_s replacing $K\tilde p_s$ admits a solution P such that $\forall t \in [0,T]$, the measure $\tilde P_t$ has a density. Let $\tilde p$ denote a measurable version of the densities.
We set $t \in [0,T]$. Reasoning like in the proof of Lemma 3.3, we obtain that for $\psi \in \mathcal{C}^{1,2}([0,t]\times\bar\Theta)$ such that $\forall (s,x) \in [0,t]\times\partial\Theta$, $\partial_n\psi(s,x) = 0$,

$$\begin{aligned}\int_{\bar\Theta} \psi(t,x)\tilde p(t,x)dx &= \int_{\bar\Theta}\psi(0,x)w_0(x)dx + \nu\int_0^t\int_{\partial\Theta}\psi(s,x)g(s,x)d\sigma(x)ds \\ &+ \int_0^t\int_{\bar\Theta}(\partial_s\psi(s,x) + Kw_s(x).\nabla\psi(s,x) + \nu\Delta\psi(s,x))\tilde p(s,x)dxds.\end{aligned}$$

Choosing $\psi(s,x) = P_{t-s}^\nu\varphi(x)$ like in the proof of Proposition 2.6 and remarking that because of (2.20) and the uniform in time bound $\|\tilde p_t\|_{L^1} \le \|w_0\|_1 + \nu\|g\|_1$,

$$\int_0^t\int_{\Theta^2}|\nabla_x P_{t-s}^\nu(x,y)||\varphi(y)||\tilde p_s(x)||Kw_s(x)|dxdyds \le C\int_0^t \frac{\|Kw_s\|_{L^\infty}ds}{\sqrt{t-s}} < +\infty,$$

we deduce by Fubini's theorem that

$$dx\ a.e.,\ \tilde p_t(x) = P_t^\nu w_0(x) + \int_0^t \nabla P_{t-s}^\nu.(\tilde p_s K w_s)(x)ds + \nu\int_{(0,t]\times\partial\Theta} P_{t-s}^\nu(x,y)g(s,y)d\sigma(y)ds.$$

Now, using the mild equation (2.19) satisfied by w and (2.20), we obtain

$$\exists C > 0,\ \forall t \in [0,T],\ \|\tilde p_t - w_t\|_{L^1} \le C\int_0^t \|\tilde p_s - w_s\|_{L^1}\frac{\|Kw_s\|_{L^\infty}}{\sqrt{t-s}}ds. \tag{3.6}$$

By iterating this bound, then using Hölder's inequality, we obtain

$$\begin{aligned}\|\tilde p_t - w_t\|_{L^1} &\le C\int_0^t \|\tilde p_s - w_s\|_{L^1}\|Kw_s\|_{L^\infty}\int_s^t \frac{\|Kw_u\|_{L^\infty}\ du}{\sqrt{t-u}\sqrt{u-s}}ds \\ &\le C\int_0^t \|\tilde p_s - w_s\|_{L^1}\|Kw_s\|_{L^\infty}\|Kw\|_{L_t^4(L_x^\infty)}\left(\int_s^t((t-u)(u-s))^{-2/3}du\right)^{3/4}ds.\end{aligned}$$

Hence (3.6) holds with $(t-s)^{-1/2}$ replaced by $(t-s)^{-1/4}$ in the r.h.s. After the next iteration we obtain that (3.6) holds with $(t-s)^{-1/2}$ replaced by 1 and conclude by Gronwall's lemma that $\forall t \in [0,T]$, $\tilde p_t = w_t$. $\square$

4 Stochastic Approximations of the solution of the vortex equation

4.1 The case of a cutoff kernel

As in Méléard [17], we introduce a cutoff kernel K_ε preserving the properties (2.4). More precisely we consider an increasing $\mathcal{C}^2$-function η from $I\!R_+$ to $I\!R_+$, such that $\eta(x) = x$ for $x \leq \frac{1}{2}$ and $\eta(x) = 1$ for $x \geq 1$. For $\varepsilon \leq 1$, we set

$$G_\varepsilon(x,y) = \eta\left(\frac{|x-y|^3}{\varepsilon^3}\right) G(x,y) \tag{4.1}$$

and

$$\begin{aligned} K_\varepsilon(x,y) &= \nabla_x^\perp G_\varepsilon(x,y) \\ &= \eta\left(\frac{|x-y|^3}{\varepsilon^3}\right) K(x,y) + \eta'\left(\frac{|x-y|^3}{\varepsilon^3}\right) \frac{3(x-y)^\perp |x-y|}{\varepsilon^3} G(x,y). \end{aligned} \tag{4.2}$$

The following Lemma states usefull properties of this cutoff kernel :

Lemma 4.1 *1)*

$$\begin{aligned} \nabla_x . K_\varepsilon(x,y) = 0 \quad &; \quad K_\varepsilon(x,y)\cdot n(x) = 0 \textit{ for } x \in \partial\Theta, \\ K_\varepsilon(x,y) &= K(x,y) \quad \textit{if } |x-y| \geq \varepsilon \\ \forall x,y \in \bar{\Theta}\ ,\ |K_\varepsilon(x,y)| &\leq \frac{C(1+|\ln|x-y||)}{|x-y|} \end{aligned} \tag{4.3}$$

where C does not depend on ε.

2) $\sup_{x\in\bar{\Theta}} \|K(x,.) - K_\varepsilon(x,.)\|_{L^p_y}$ *tends to 0 as ε tends to 0 as soon as $p < 2$.*

3) For ε sufficiently small, the kernel K_ε is bounded by $M_\varepsilon \leq \frac{C|\ln\varepsilon|}{\varepsilon}$ and Lipschitz continuous in both variables with constant $L_\varepsilon \leq \frac{C|\ln\varepsilon|}{\varepsilon^2}$ where C does not depend on ε.

Proof. The two first properties in 1) are obvious and 2) is an easy consequence of (4.3).

By the estimate of K given in Lemma 2.1 and the above definition of η, the norm of first term of the r.h.s. of (4.2) is smaller than $C_0(\frac{1}{|x-y|} \wedge \sup_{r\in[0,\varepsilon 2^{-1/3}]} \frac{r^2}{\epsilon^3}) \leq C_0(\frac{1}{|x-y|} \wedge \frac{1}{\epsilon})$. By the estimate of G in Lemma 2.1 and since $\eta'(x) = 0$ for $x > 1$, the second term of the r.h.s. of (4.2) is smaller than $3C_0\|\eta'\|_\infty$ times

$$\left(\frac{1+|\ln|x-y||}{|x-y|} \wedge \sup_{r\in[0,\varepsilon]} \frac{r^2(1+|\ln(r)|)}{\varepsilon^3}\right) \leq \left(\frac{1+\ln|x-y|}{|x-y|} \wedge \frac{1+|\ln(\varepsilon)|}{\varepsilon}\right)$$

as $\varepsilon \leq 1$. We deduce both (4.3) and the upper-bound in $C|\ln(\varepsilon)|/\varepsilon$. To prove that K_ε is Lipschitz continuous, we use in a similar way Lemma 2.1 combined with the definition of

η to check that the gradient of each coordinate of K_ε w.r.t. either x or y is bounded by $C|\ln(\varepsilon)|/\varepsilon^2$ (the contribution of the first term of the r.h.s. of (4.2) is C/ε^2 whereas the one of the second term is $C|\ln(\varepsilon)|/\varepsilon^2$). $\square$

With a slight adaptation of Sznitman [21] to take into account the random births on the boundary, we obtain the existence and pathwise uniqueness of the following interacting particle systems.

Definition 4.2 *Consider a sequence $(B^i)_{i\in\mathbb{N}}$ of independent Brownian motions on $\mathbb{R}^2$ and a sequence of independent variables $(\tau^i, Z_0^i)_{i\in\mathbb{N}}$ with values in $[0,T]\times\bar{\Theta}$ distributed according to P_0, and independent of the Brownian motions. For a fixed ε, for each $n\in\mathbb{N}^*$, and $1\leq i\leq n$, let us consider the interacting processes defined by*

$$\begin{aligned}
&Z_t^{in,\varepsilon}\in\bar{\Theta},\forall t\in[0,T]\\
&Z_t^{in,\varepsilon}=Z_0^i+\sqrt{2\nu}\int_0^t \mathbf{1}_{\{\tau^i\leq s\}}dB_s^i+\int_0^t \mathbf{1}_{\{\tau^i\leq s\}}K_\varepsilon\tilde{\mu}_s^{n,\varepsilon}(Z_s^{in,\varepsilon})ds-k_t^{in,\varepsilon}\ ;\\
&|k^{in,\varepsilon}|_t=\int_0^t \mathbf{1}_{\{Z_s^{in,\varepsilon}\in\partial\Theta\}}\mathbf{1}_{\{\tau^i\leq s\}}d|Z^{in,\varepsilon}|_s\ ;\ k_t^{in,\varepsilon}=\int_0^t n(Z_s^{in,\varepsilon})d|k^{in,\varepsilon}|_s
\end{aligned}\tag{4.4}$$

where $\tilde{\mu}_s^{n,\varepsilon}=\frac{1}{n}\sum_{j=1}^n h(\tau^j,Z_0^j)\mathbf{1}_{\{\tau^j\leq s\}}\delta_{Z_s^{jn,\varepsilon}}$ is the weighted empirical measure of the system at time s and $K_\varepsilon\tilde{\mu}_s^{n,\varepsilon}(z)=\frac{1}{n}\sum_{j=1}^n h(\tau^j,Z_0^j)\mathbf{1}_{\{\tau^j\leq s\}}K_\varepsilon(z,Z_s^{jn,\varepsilon})$.

Let us remark that the particles either have birth at time 0 inside the domain and evolve as diffusive particles with normal reflecting boundary conditions, or have birth at a random time on the boundary of the domain, and evolve after birth as the other ones. Moreover, all particles, as soon as they are born, interact together following a mean field depending on the parameter ε.

Again according to [21], we also get the existence and pathwise uniqueness of the limit processes (when n tends to infinity and ε is fixed), coupled with the interacting processes, as follows.

Definition 4.3 *We define $\bar{Z}^{i,\varepsilon}$ by*

$$\begin{aligned}
&\bar{Z}_t^{i,\varepsilon}\in\bar{\Theta},\forall t\in[0,T]\\
&\bar{Z}_t^{i,\varepsilon}=Z_0^i+\sqrt{2\nu}\int_0^t \mathbf{1}_{\{\tau^i\leq s\}}dB_s^i+\int_0^t \mathbf{1}_{\{\tau^i\leq s\}}K_\varepsilon\tilde{Q}_s^\varepsilon(\bar{Z}_s^{i,\varepsilon})ds-\bar{k}_t^{i,\varepsilon}\ ;\\
&|\bar{k}^{i,\varepsilon}|_t=\int_0^t \mathbf{1}_{\{\bar{Z}_s^{i,\varepsilon}\in\partial\Theta\}}\mathbf{1}_{\{\tau^i\leq s\}}d|\bar{k}^{i,\varepsilon}|_s\ ;\ \bar{k}_t^{i,\varepsilon}=\int_0^t n(\bar{Z}_s^{i,\varepsilon})d|\bar{k}^{i,\varepsilon}|_s
\end{aligned}\tag{4.5}$$

where Q^ε is the common law of $(\tau^i,\bar{Z}^{i,\varepsilon},\bar{k}^{i,\varepsilon})$, and $\tilde{Q}_s^\varepsilon$ is defined from Q^ε by (3.4).

Sznitman also proves a propagation of chaos result, but without precise estimates on the rate of convergence. In order to get such estimates, we denote by H a $\mathcal{C}_b^2(\bar{\Theta})$-extension of the distance-function $d(.,\partial\Theta)$ (defined on a restriction to Θ of a neighbourhood of $\partial\Theta$). The function H satisfies (see [8])

$$\nabla H = -n \text{ on } \partial\Theta. \tag{4.6}$$

We also recall that the domain Θ (since $\mathcal{C}^4$) satisfies the uniform "exterior sphere" condition:

$$\exists C_{sp} \geq 0 \;, \forall x \in \partial\Theta \;,\; \forall x' \in \bar{\Theta} \;,\; C_{sp}|x-x'|^2 + n(x).(x-x') \geq 0. \tag{4.7}$$

Proposition 4.4 *For $t \leq T$, for each $i \in \{1,...,n\}$,*

$$\begin{aligned} E(\sup_{s\leq t}|Z_s^{in,\varepsilon} - \bar{Z}_s^{i,\varepsilon}|^2) &\leq 2d(\Theta)\sqrt{\frac{A_\varepsilon}{n}}\exp(K_H(1+(\|w_0\|_1+\nu\|g\|_1)(M_\varepsilon/2+L_\varepsilon)t)) \\ E(\sup_{s\leq t}|k_s^{in,\varepsilon} - \bar{k}_s^{i,\varepsilon}|) &\leq E(\sup_{s\leq t}|Z_s^{in,\varepsilon} - \bar{Z}_s^{i,\varepsilon}|) + 2(\|w_0\|_1+\nu\|g\|_1)t\left(L_\varepsilon E(\sup_{s\leq t}|Z_s^{in,\varepsilon} - \bar{Z}_s^{i,\varepsilon}|) + \frac{M_\varepsilon}{\sqrt{n}}\right) \end{aligned}$$

where K_H is a constant which depends only on upper-bounds of the function H and its derivatives and $A_\varepsilon = \frac{4(\|w_0\|_1+\nu\|g\|_1)^2 M_\varepsilon^2}{2+(\|w_0\|_1+\nu\|g\|_1)(M_\varepsilon+2L_\varepsilon))}$.

Remark 4.5 *The convergence rate in the number n of particles given above is not optimal: indeed one can check that $E(\sup_{s\leq t}|Z_s^{in,\varepsilon} - \bar{Z}_s^{i,\varepsilon}|^4)$ is smaller than*

$$\frac{16(\|w_0\|_1+\|\nu g\|_1)^4 M_\varepsilon^4 t}{n^2(1+(2+(\|w_0\|_1+\nu\|g\|_1)^2(M_\varepsilon^2+4L_\varepsilon^2))t)}\exp(K_H(t+(2+(\|w_0\|_1+\nu\|g\|_1)^2(M_\varepsilon^2+4L_\varepsilon^2))t^2)).$$

But in the next section, we are going to let $\varepsilon = \varepsilon_n$ depend on n and converge to 0 in such a way that $E(\sup_{s\leq t}|Z_s^{in,\varepsilon_n} - \bar{Z}_s^{i,\varepsilon_n}|^2) \to 0$. The estimation given in the proposition allows a quicker (but still very slow) convergence of ε_n to 0 than the previous one.

Proof. We compare the two processes $Z^{in,\varepsilon}$ and $\bar{Z}^{i,\varepsilon}$. We denote for simplicity Z, k, $\bar{Z}$ and $\bar{k}$ instead of $Z^{in,\varepsilon}$, $k^{in,\varepsilon}$, $\bar{Z}^{i,\varepsilon}$ and $\bar{k}^{i,\varepsilon}$, $h_t = H(Z_t)$, $\bar{h}_t = H(\bar{Z}_t)$, $h'_t = \nabla H(Z_t)$, $\bar{h}'_t = \nabla H(\bar{Z}_t)$, $h''_t = \Delta H(Z_t)$, $\bar{h}''_t = \Delta H(\bar{Z}_t)$, $b_t = K_\varepsilon \tilde{\mu}_s^{n,\varepsilon}(Z_t)$ and $\bar{b}_t = K_\varepsilon \tilde{Q}_t^\varepsilon(\bar{Z}_t)$. Computing $d\exp(-2C_{sp}(h_t+\bar{h}_t))|Z_t-\bar{Z}_t|^2$ by Itô's formula, we get

$$\begin{aligned} & 1_{\{\tau_i\leq t\}}\exp(-2C_{sp}(h_t+\bar{h}_t)) \times \Big[2(Z_t-\bar{Z}_t).(d\bar{k}_t - dk_t) - 2C_{sp}|Z_t-\bar{Z}_t|^2(d|k|_t + d|\bar{k}|_t) \\ - \;& 2C_{sp}|Z_t-\bar{Z}_t|^2\left(\sqrt{2\nu}(h'_t+\bar{h}'_t)dB_t^i + \left\{h'_t b_t + \bar{h}'_t\bar{b}_t + \nu(-2C_{sp}|h'_t+\bar{h}'_t|^2 + h''_t + \bar{h}''_t)\right\}dt\right) \\ + \;& 2(Z_t-\bar{Z}_t).(b_t-\bar{b}_t)dt\Big] \end{aligned} \tag{4.8}$$

Because of the "exterior sphere" condition, the local time terms of the first line have a non-positive contribution after integration over time. We deduce that for K_H a constant which can be computed and depends only on upper-bounds of the function H and its derivatives,

$$\begin{aligned} E(|Z_t^{in,\varepsilon} - \bar{Z}_t^{i,\varepsilon}|^2) \quad &\leq \quad K_H \bigg((1 + M_\varepsilon(\|w_0\|_1 + \nu\|g\|_1)) \int_0^t E(|Z_s^{in,\varepsilon} - \bar{Z}_s^{i,\varepsilon}|^2) ds \\ &+ \int_0^t E(|Z_s^{in,\varepsilon} - \bar{Z}_s^{i,\varepsilon}||K_\varepsilon \tilde{\mu}_s^{n,\varepsilon}(Z_s^{in,\varepsilon}) - K_\varepsilon \tilde{Q}_s^\varepsilon(\bar{Z}_s^{i,\varepsilon})|) ds \bigg) \end{aligned} \tag{4.9}$$

Using the Lipschitz continuity of K_ε, the boundedness of h and the exchangeability of the processes $(Z^{in,\varepsilon}, \bar{Z}^{i,\varepsilon})$, $1 \leq i \leq n$, we obtain

$$\begin{aligned} E \quad (\quad & |Z_s^{in,\varepsilon} - \bar{Z}_s^{i,\varepsilon}||K_\varepsilon \tilde{\mu}_s^{n,\varepsilon}(Z_s^{in,\varepsilon}) - K_\varepsilon \tilde{Q}_s^\varepsilon(\bar{Z}_s^{i,\varepsilon})|) \\ \leq \quad & (\|w_0\|_1 + \nu\|g\|_1) L_\varepsilon E(|Z_s^{in,\varepsilon} - \bar{Z}_s^{i,\varepsilon}|(|Z_s^{in,\varepsilon} - \bar{Z}_s^{i,\varepsilon}| + \frac{1}{n}\sum_{j=1}^n |Z_s^{jn,\varepsilon} - \bar{Z}_s^{j,\varepsilon}|)) \\ & + E(|Z_s^{in,\varepsilon} - \bar{Z}_s^{i,\varepsilon}||\frac{1}{n}\sum_{j=1}^n h(\tau_j, Z_0^j)\mathbf{1}_{\{\tau^j \leq s\}} K_\varepsilon(\bar{Z}_s^{i,\varepsilon}, \bar{Z}_s^{j,\varepsilon}) - K_\varepsilon \tilde{Q}_s^\varepsilon(\bar{Z}_s^{i,\varepsilon})|) \\ \leq \quad & (1 + 2(\|w_0\|_1 + \nu\|g\|_1) L_\varepsilon) E(|Z_s^{in,\varepsilon} - \bar{Z}_s^{i,\varepsilon}|^2) \\ & + E(|\frac{1}{n}\sum_{j=1}^n h(\tau_j, Z_0^j)\mathbf{1}_{\{\tau^j \leq s\}} K_\varepsilon(\bar{Z}_s^{i,\varepsilon}, \bar{Z}_s^{j,\varepsilon}) - K_\varepsilon \tilde{Q}_s^\varepsilon(\bar{Z}_s^{i,\varepsilon})|^2) \end{aligned}$$

After expansion of $E(|\frac{1}{n}\sum_{j=1}^n h(\tau_j, Z_0^j)\mathbf{1}_{\{\tau^j \leq s\}} K_\varepsilon(\bar{Z}_s^{i,\varepsilon}, \bar{Z}_s^{j,\varepsilon}) - K_\varepsilon \tilde{Q}_s^\varepsilon(\bar{Z}_s^{i,\varepsilon})|^2)$, many terms disappear by independence of the variables which are centered conditionnally to $\bar{Z}^{i,\varepsilon}$ and it only remains n bounded terms. We deduce that

$$\begin{aligned} E(|Z_t^{in,\varepsilon} - \bar{Z}_t^{i,\varepsilon}|^2) \leq K_H \quad \bigg(\quad & (2 + (\|w_0\|_1 + \nu\|g\|_1)(M_\varepsilon + 2L_\varepsilon)) \int_0^t E(|Z_s^{in,\varepsilon} - \bar{Z}_s^{i,\varepsilon}|^2) ds \\ & + \frac{4(\|w_0\|_1 + \nu\|g\|_1)^2 M_\varepsilon^2 t}{n} \bigg) \end{aligned} \tag{4.10}$$

Using Gronwall's Lemma, we obtain that both sides of (4.9) and (4.10) are smaller than

$$f(t) = \frac{4(\|w_0\|_1 + \|\nu g\|_1)^2 M_\varepsilon^2}{n(2 + (\|w_0\|_1 + \nu\|g\|_1)(M_\varepsilon + 2L_\varepsilon))} \exp(K_H(2 + (\|w_0\|_1 + \nu\|g\|_1)(M_\varepsilon + 2L_\varepsilon))t).$$

Integrating (4.8) w.r.t. time, dealing with the stochastic integral thanks to Doob's inequality and using that the r.h.s. of (4.9) is smaller than $f(t)$, we get

$$\begin{aligned} E(\sup_{s \leq t} |Z_s^{in,\varepsilon} - \bar{Z}_s^{i,\varepsilon}|^2) \quad &\leq \quad \left(K_H \int_0^t E(|Z_s^{in,\varepsilon} - \bar{Z}_s^{i,\varepsilon}|^4) ds \right)^{1/2} + f(t) \\ &\leq \quad d(\Theta) \left(K_H \int_0^t E(|Z_s^{in,\varepsilon} - \bar{Z}_s^{i,\varepsilon}|^2) ds \right)^{1/2} + f(t) \\ &\leq \quad d(\Theta)\sqrt{f(t)} + f(t) \ \text{ since the r.h.s. of (4.10) is smaller than } f(t) \end{aligned}$$

The l.h.s. being smaller than $d(\Theta)^2$, it is smaller than $2d(\Theta)\sqrt{f(t)}$ when $f(t) \geq d(\Theta)^2$ and the r.h.s. is smaller than $2d(\Theta)\sqrt{f(t)}$ otherwise. We deduce the desired estimate for $E(\sup_{s\leq t}|Z_s^{in,\varepsilon} - \bar{Z}_s^{i,\varepsilon}|^2)$.
Now remarking that

$$\sup_{s\leq t}|k_s^{in,\varepsilon} - \bar{k}_s^{i,\varepsilon}| \leq \int_0^t |K_\varepsilon \tilde{\mu}_s^{n,\varepsilon}(Z_s^{in,\varepsilon}) - K_\varepsilon \tilde{Q}_s^\varepsilon(\bar{Z}_s^{i,\varepsilon})|ds + \sup_{s\leq t}|Z_s^{in,\varepsilon} - \bar{Z}_s^{i,\varepsilon}|$$

and using arguments developped above we obtain the other estimate. □

Remark 4.6 *Let us remark that if $\bar{\Theta}$ is a convex region then the rate of convergence is easier to obtain. Indeed the constant C_{sp} defined in (4.7) can be chosen equal to 0 :*

$$\forall x \in \partial\Theta \ , \ \forall x' \in \bar{\Theta} \ , \ n(x).(x - x') \geq 0. \tag{4.11}$$

In the expression of $|Z_t^{in,\varepsilon} - \bar{Z}_t^{i,\varepsilon}|^2$ given by Itô's formula, the local times terms are non-positive and therefore

$$\begin{aligned} E(\sup_{s\leq t}|Z_s^{in,\varepsilon} - \bar{Z}_s^{i,\varepsilon}|^2) \quad &\leq \quad (1 + 2(\|w_0\|_1 + \nu\|g\|_1)L_\varepsilon)) \int_0^t E(\sup_{u\leq s}|Z_u^{in,\varepsilon} - \bar{Z}_u^{i,\varepsilon}|^2)ds \\ &\quad + \frac{4(\|w_0\|_1 + \nu\|g\|_1)^2 M_\varepsilon^2 t}{n} \end{aligned}$$

and we conclude by Gronwall's Lemma.

4.2 Convergence of the limit laws

We want to prove that the law Q^ε of $(\tau^1, \bar{Z}^{1,\varepsilon}, \bar{k}^{1,\varepsilon})$ converges to the unique solution P of problem $\mathcal{M}_T$ as ε tends to 0. We are first going to check that the drift coefficient $K_\varepsilon \tilde{Q}_s^\varepsilon$ converges to $K\tilde{p}_s$.
By Girsanov's theorem, it turns out that $\forall s > 0$, the measure $\tilde{Q}_s^\varepsilon$ admits a density function q_s^ε. Moreover, reasoning like in the proof of Theorem 3.4 and using the boundedness of the kernel K_ε, we show that q^ε is the unique solution in $L_T^1 = \{p_t, |||p|||_T = \sup_{t\leq T}\|p_t\|_{L^1} < +\infty\}$ of the evolution equation

$$q_t^\varepsilon(x) = P_t^\nu w_0(x) + \int_0^t \nabla_x P_{t-s}^\nu.(q_s^\varepsilon K_\varepsilon q_s^\varepsilon)(x)ds + \nu\int_0^t\int_{\partial\Theta} P_{t-s}^\nu g(s,y)d\sigma(y)ds. \tag{4.12}$$

On the other hand, thanks to Lemma 4.3 1), we can apply to the equation

$$\begin{aligned} &\partial_t w(t,x) + \nabla.(wK_\varepsilon w)(t,x) = \nu\Delta w(t,x) \ \text{ in } \Theta; \\ &w(x,0) = w_0 \text{ in } \Theta; \\ &\partial_n w = \nabla w.n = g \text{ on } \partial\Theta \end{aligned} \tag{4.13}$$

all what we have done for the equation (2.1). We can then prove the existence of a unique weak solution w^ε belonging to $L_t^\infty(L_x^2) \cap L_t^2(H_x^1)$. Now, like in Proposition 2.6, we obtain that w^ε is also solution of (4.12). Since it belongs to L_T^1 (Θ is bounded), $w^\varepsilon = q^\varepsilon$. Thanks to (4.3), one can check that the *a priori* estimate (2.17) holds for $w^\varepsilon = q^\varepsilon$ with a constant C_T independent of ε. Following Remark 2.3, we deduce that

$$\sup_{\varepsilon \in (0,1]} \left(\|q^\varepsilon\|_{L_t^\infty(L_x^2)} + \|q^\varepsilon\|_{L_t^2(H_x^1)} + \|\partial_t q^\varepsilon\|_{L_t^2(H_x^{1'})} + \|q^\varepsilon\|_{L_t^4(L_x^4)} \right) < +\infty. \tag{4.14}$$

Remark 4.7 *Similarly the non-negative measures $B \in \mathcal{B}(\bar{\Theta}) \to E^{Q^\varepsilon}(\mathbf{1}_{\{\tau \le t\}} 1_B(X_t))$ have densities p_t^ε w.r.t. the Lebesgue measure which are the unique solution in L_T^1 of the mild equation obtained by replacing respectively w_0 and g by $|w_0|/(\|w_0\|_1 + \nu\|g\|_1)$ and $|g|/(\|w_0\|_1 + \nu\|g\|_1)$ in (4.12).*
Identifying p^ε with the unique weak solution of the problem obtained from (4.13) by replacing w_0 and g in the same way, we check that (4.14) holds for p^ε.

We can now prove the convergence of q^ε to w.

Proposition 4.8

$$\lim_{\varepsilon \to 0} \|q^\varepsilon - w\|_{L_t^2(L_x^2)} = 0 \ ; \ \lim_{\varepsilon \to 0} \|K_\varepsilon q^\varepsilon - Kw\|_{L_t^2(L_x^2)} = 0.$$

Proof. Thanks to (4.14), one can extract from each sequence q^{ε_n} with ε_n tending to 0, a sub-sequence (still denoted q^{ε_n} for simplicity), which converges strongly in $L_t^2(L_x^2)$ and in $L_t^2(H_x^1)$ and weakly* in $L_t^\infty(L_x^2)$ to $\tilde{w}$. By adapting the proof of Theorem 2.5, we get that $\tilde{w}$ is a weak solution of (2.1) and conclude that $\tilde{w} = w$ by uniqueness for this equation.

The only difference comes from the term $(K_{\varepsilon_n} - K)q_s^{\varepsilon_n}$. Let $1 < p < 2$. Combining the Sobolev inequality $\|q_s^{\varepsilon_n}\|_{L^{\frac{p}{p-1}}} \le C\|q_s^{\varepsilon_n}\|_{H^1}$, Lemma 4.1 2) and (4.14), we deduce that this term converges to 0 in $L_t^2(L_x^\infty)$.

Now, by writing

$$\|K_\varepsilon q^\varepsilon - Kw\|_{L_t^2(L_x^2)} \le \|K(q^\varepsilon - w)\|_{L_t^2(L_x^2)} + \|(K_\varepsilon - K)q^\varepsilon\|_{L_t^2(L_x^2)},$$

and using (2.11), one easily deduces the second assertion. $\square$

Theorem 4.9 *The probability measures Q^ε on $[0,T] \times \mathcal{C}([0,T], \bar{\Theta}) \times \mathcal{C}([0,T], I\!R^2)$ converge weakly to the unique solution P of the nonlinear martingale problem $(\mathcal{M}_T)$, as ε tends to 0.*

Proof. As the weak convergence topology is metrizable, we are going to check that $(Q^n = Q^{\varepsilon_n})_{n \in I\!N}$ converges weakly to P when ε_n is a sequence which tends to 0 as n tends to $+\infty$. Let us prove the uniform tightness of the sequence $(Q^n)_n$ before identifying the limit of any weakly convergent subsequence.

1) By (4.3) and (4.14), we easily obtain that

$$\sup_n \|K_{\varepsilon_n} q_s^{\varepsilon_n}\|_{L_t^4(L_x^\infty)} < +\infty. \tag{4.15}$$

Then the Kolmogorov tightness criterion is satisfied for the laws of

$$\bar{Y}_t^{1,\varepsilon_n} = Z_0^1 + \sqrt{2\nu}\int_0^t \mathbf{1}_{\{\tau^1\le s\}} dB_s^1 + \int_0^t \mathbf{1}_{\{\tau^1\le s\}} K_{\varepsilon_n} q_s^n(\bar{Z}_s^{1,\varepsilon_n}) ds.$$

Now the uniform tightness of the laws Q^n of the processes $(\tau^1, \bar{Z}^{1,\varepsilon_n}, \bar{k}^{1,\varepsilon_n})$ is a simple consequence of the fact that the application sending $y \in \mathcal{C}([0,T],\mathbb{R}^2)$ on the solution $(x,k) \in \mathcal{C}([0,T],\bar{\Theta}) \times \mathcal{C}([0,T],\mathbb{R}^2)$ of the Skorohod problem is continuous (See [13]).

2) Let us now denote by Q^∞ a limit value of a convergent subsequence still denoted by (Q^n) for simplicity and prove by arguments inspired from Sznitman ([21]) that $Q^\infty = P$.

If as usual (τ, X, k) denotes the canonical process on $[0,T]\times\mathcal{C}([0,T],\bar{\Theta})\times\mathcal{C}([0,T],\mathbb{R}^2)$, let us define, for $p \in \mathbb{N}^*$, $0 \le s_1 \le ... \le s_p \le s < t \le T$, $\phi \in \mathcal{C}_b^2(\mathbb{R}^2)$, $g \in \mathcal{C}_b([0,T],(\bar{\Theta}\times\mathbb{R}^2)^p)$ the function

$$\begin{aligned} G_n(\tau,X,k) \quad = \quad & g(\tau, X_{s_1}, k_{s_1}, ..., X_{s_p}, k_{s_p})\Big(\phi(X_t+k_t) - \phi(X_s+k_s) \\ & - \int_s^t \mathbf{1}_{\{\tau\le u\}}\Big(\nu\Delta\phi(X_u+k_u) + K_{\varepsilon_n} q_u^{\varepsilon_n}(X_u).\nabla\phi(X_u+k_u)\Big)du\Big) \end{aligned}$$

Then $E^{Q^n}(G_n(\tau,X,k)) = 0$. Now if we define the function G by replacing $K_{\varepsilon_n} q_s^{\varepsilon_n}$ by Kw_s in (4.16), we want to prove that $E^{Q^\infty}(G(\tau,X,k)) = 0$.

$$E^{Q^\infty}(G(\tau,X,k)) = E^{Q^\infty}(G(\tau,X,k)) - E^{Q^n}(G(\tau,X,k)) + E^{Q^n}(G(\tau,X,k) - G^n(\tau,X,k)).$$

Since $w \in L_t^4(L_x^4)$, by (2.10), ds a.e. in $[0,T]$ $x \in \bar{\Theta} \to Kw_s(x)$ is continuous and $Kw_s \in L_t^4(L_x^\infty)$. We deduce that $G(\tau,X,k)$ is a continuous function on the path space, and the first term of the r.h.s. tends to 0 as n tends to infinity. On the other hand, using Remark 4.7 and Proposition 4.8, we obtain

$$\begin{aligned} E^{Q^n}|G^n(\tau,X,k) - G(\tau,X,k)| \quad & \le \quad CE\Big(\int_0^t \mathbf{1}_{\{\tau^1\le s\}}|K_{\varepsilon_n} q_s^{\varepsilon_n}(\bar{Z}_s^{1,\varepsilon_n}) - Kw_s(\bar{Z}_s^{1,\varepsilon_n})|ds\Big) \\ & \le \quad C\|p^{\varepsilon_n}\|_{L_t^2(L_x^2)}\|K_{\varepsilon_n} q^{\varepsilon_n} - Kw\|_{L_t^2(L_x^2)} \to 0 \text{ as } n \to +\infty. \end{aligned}$$

Hence $E^{Q^\infty}(G(\tau,X,k)) = 0$. Since $\forall n$, $Q^n\circ(\tau,X_0,k_0)^{-1} = P_0\otimes\delta_{(0,0)}$, $Q^\infty\circ(\tau,X_0,k_0)^{-1} = P_0\otimes\delta_{(0,0)}$. We are now going to prove that Q^∞-almost surely,

$$|k|_T < \infty \text{ and } \forall t \in [0,T],\ |k|_t = \int_0^t \mathbf{1}_{\{X_s\in\partial\Theta\}}\mathbf{1}_{\{\tau\le s\}} d|k|_s \ ;\ k_t = \int_0^t n(X_s)d|k|_s.$$

As according to the proof of Theorem 3.4, P is the unique solution of the linear martingale problem defined like $\mathcal{M}_T$ but with known drift coefficient Kw_s, we will conclude that $Q^\infty = P$. According to the following Lemma the proof of which is postponed,

Lemma 4.10 *For any $A \geq 0$, the following subset of* $[0,T] \times \mathcal{C}([0,T],\bar{\Theta}) \times \mathcal{C}([0,T],\mathbb{R}^2)$

$$F_A = \left\{(u,x,k) : \ |k|_T = \int_0^T \mathbf{1}_{\{u\leq s\}}\mathbf{1}_{\{x_s\in\partial\Theta\}} d|k|_s \leq A \text{ and } \forall t\in[0,T], \ k_t = \int_0^t n(x_s)d|k|_s\right\}$$

is closed.

$$Q^\infty\left(\bigcup_{A>0} F_A\right) \geq 1 - \lim_{A\to+\infty}\liminf_{n\to+\infty} Q^n(F_A^c) \geq 1 - \lim_{A\to+\infty}\frac{\sup_{n\in\mathbb{N}} E^{Q^n}|k|_T}{A}.$$

Therefore it is enough to check that $\sup_{n\in\mathbb{N}} E|\bar{k}^{1,\varepsilon_n}|_T < +\infty$ to conclude the proof. Since $\nabla H = -n$ on $\partial\Theta$, applying Itô's formula to compute $H(\bar{Z}_T^{1,\varepsilon_n})$, we get that $|\bar{k}^{1,\varepsilon_n}|_T$ is equal to

$$H(\bar{Z}_T^{1,\varepsilon_n}) - H(Z_0^1) - \int_0^T \mathbf{1}_{\{\tau^1\leq s\}}\left((\nu\Delta H + K_{\varepsilon_n} q_s^{\varepsilon_n}.\nabla H)(\bar{Z}_s^{1,\varepsilon_n})ds + \sqrt{2\nu}\nabla H(\bar{Z}_s^{1,\varepsilon_n}).dB_s^1\right).$$

Taking expectations and using (4.15), we obtain the desired result. □

Proof.of Lemma 4.10 Let $(u^n,x^n,k^n) \in F_A$ converge to (u,x,k) as $n\to+\infty$. Since $\sup_n |k^n|_T \leq A$, by extraction of a subsequence, we can suppose that the measure $d|k^n|$ (resp. dk^n) converges weakly to a positive measure da with mass smaller than A (resp. to db_s). Of course $db_s = \lambda(s)da_s$ for some measurable function $\lambda : [0,T] \to \mathbb{R}^2$ and since k^n converges uniformly on $[0,T]$ to k, $db_s = dk_s$. Since $d(x_s^n,\partial\Theta)$, where $d(.,\partial\Theta)$ denotes the (continuous) distance from the boundary function, converges uniformly on $[0,T]$ to $d(x_s,\partial\Theta)$,

$$\int_0^T d(x_s,\partial\Theta)da_s = \lim_n \int_0^T d(x_s^n,\partial\Theta)d|k|_s^n = 0.$$

We deduce that da_s a.e. and therefore $d|k|_s$ a.e., $x_s\in\partial\Theta$. Since the functions k^n which are equal to $(0,0)$ on $[0,u^n]$ converge uniformly to k, this function is equal to $(0,0)$ on $[0,u]$ and $|k|_u = 0$. To check the only lacking property : $dk_s = n(x_s)d|k|_s$, we remark that

$$\forall f\in\mathcal{C}([0,T],\mathbb{R}_+), \ \forall g\in\mathcal{C}([0,T],\bar{\Theta}), \ \int_0^T f(s)\left((x_s - g(s)).dk_s + C_{sp}|x_s-g(s)|^2 da_s\right) \geq 0$$

by taking the limit $n\to+\infty$ in the similar inequalities satisfied with (x,dk,da) replaced by $(x^n,dk^n,d|k^n|)$ according to the uniform "exterior sphere" condition (4.7). We deduce that $dk_s = |\lambda(s)|n(x_s)da_s$ which implies the desired property. □

4.3 The convergence theorem

We now consider a sequence (ε_n) tending to 0 as n tends to infinity, in such a way that

$$\lim_{n\to+\infty} L^2_{\varepsilon_n}\sqrt{\frac{A_{\varepsilon_n}}{n}}\exp(K_H(1+(\|w_0\|_1+\nu\|g\|_1)(M_{\varepsilon_n}/2+L_{\varepsilon_n})T))+\frac{M_{\varepsilon_n}}{\sqrt{n}}=0. \quad (4.16)$$

This is possible, even if the convergence of ε_n to 0 is then very slow. Let us now consider for each n the system of processes (τ^i, Z^{in}, k^{in}) where $Z^{in} = Z^{in,\varepsilon_n}$ and $k^{in} = k^{in,\varepsilon_n}$ are defined as in (4.4) but with K_{ε_n} replacing K_ε. We are now able to obtain our main theorem.

Theorem 4.11 *1) The laws of the n-particle system $(\tau^i, Z^{in}, k^{in})_{1\le i\le n}$, are P-chaotic (where P is the solution of the problem $(\mathcal{M}_T)$):*

$$\forall p\in I\!N^*\ ,\ \mathcal{L}((\tau^1,Z^{1n},k^{1n}),...,(\tau^p,Z^{pn},k^{pn}))\ \overset{weakly}{\Longrightarrow}\ P^{\otimes p}\ \textit{as}\ n\to+\infty. \quad (4.17)$$

2) The approximate velocity field converges to Kw:

$$\lim_{n\to+\infty} E(\|K_{\varepsilon_n}\tilde\mu^{n,\varepsilon_n}_t(x)-Kw_t(x)\|^2_{L^2_t(L^2_x)})=0. \quad (4.18)$$

Proof.

1) Since the processes $(\tau^i, \bar Z^{i,\varepsilon_n}, \bar k^{i,\varepsilon_n})_i$ are independent, Theorem 4.9 implies that for every fixed $p\in I\!N^*$, the law of $((\tau^1,\bar Z^{1,\varepsilon_n},\bar k^{1,\varepsilon_n}),...,(\tau^p,\bar Z^{p,\varepsilon_n},\bar k^{p,\varepsilon_n}))$ converges weakly to $P^{\otimes p}$. Let $\mathcal{C}_T=[0,T]\times\mathcal{C}([0,T],\bar\Theta)\times\mathcal{C}([0,T],I\!R^2)$. We endow $\mathcal{C}^p_T$ with the metric

$$\begin{aligned} d\Big(\ \big((u^1,x^1,k^1),...,(u^p,x^p,k^p)\big)\quad &,\quad \big((\bar u^1,\bar x^1,\bar k^1),...,(\bar u^p,\bar x^p,\bar k^p)\big)\Big)\\ &= \sum_{i=1}^p\Big(|u^i-\bar u^i|+\sup_{[0,T]}|x^i_t-\bar x^i_t|+\sup_{[0,T]}|k^i_t-\bar k^i_t|\Big). \end{aligned}$$

and $\mathcal{P}(\mathcal{C}^p_T)$ with the metric

$$\rho(\mu,\nu)=\inf\Big\{\int_{\mathcal{C}^p_T\times\mathcal{C}^p_T} d(x,y)\wedge 1R(dx,dy); R \text{ has marginals } \mu \text{ and } \nu\Big\}$$

which is compatible with the topology of the weak convergence. Hence

$$\rho(\mathcal{L}((\tau^1,\bar Z^{1,\varepsilon_n},\bar k^{1,\varepsilon_n}),...,(\tau_p,\bar Z^{p,\varepsilon_n},\bar k^{p,\varepsilon_n})),P^{\otimes p})\to 0 \text{ as } n\to+\infty.$$

By Proposition 4.4, and (4.16)

$$\lim_{n\to+\infty} E\left(d\left(((\tau^1,Z^{1n},k^{1n}),...,(\tau^p,Z^{pn},k^{pn})),((\tau^1,\bar Z^{1,\varepsilon_n},\bar k^{1,\varepsilon_n}),...,(\tau_p,\bar Z^{p,\varepsilon_n},\bar k^{p,\varepsilon_n}))\right)\right)=0$$

which ensures that

$$\lim_{n\to+\infty}\rho\left(\mathcal{L}\left((\tau^1,Z^{1n},k^{1n}),...,(\tau^p,Z^{pn},k^{pn})\right),\mathcal{L}\left((\tau^1,\bar{Z}^{1,\varepsilon_n},\bar{k}^{1,\varepsilon_n}),...,(\tau^p,\bar{Z}^{p,\varepsilon_n},\bar{k}^{p,\varepsilon_n})\right)\right)=0.$$

We conclude that $\rho(\mathcal{L}((\tau^1,Z^{1n},k^{1n}),...,(\tau^p,Z^{pn},k^{pn})),P^{\otimes p})$ converges to 0.

2) On the other hand,

$$E(|K_{\varepsilon_n}\tilde{\mu}_t^{n,\varepsilon_n}(x)-Kw_t(x)|^2)\leq 3E\Bigg(\left|K_{\varepsilon_n}\tilde{\mu}_t^{n,\varepsilon_n}(x)-\frac{1}{n}\sum_{i=1}^n \mathbf{1}_{\{\tau^i\leq t\}}h(\tau^i,Z_0^i)K_{\varepsilon_n}(x,\bar{Z}_t^{i,\varepsilon_n})\right|^2$$

$$+\left|\frac{1}{n}\sum_{i=1}^n \mathbf{1}_{\{\tau^i\leq t\}}h(\tau^i,Z_0^i)K_{\varepsilon_n}(x,\bar{Z}_t^{i,\varepsilon_n})-K_{\varepsilon_n}\tilde{Q}_t^{\varepsilon_n}(x)\right|^2+|K_{\varepsilon_n}\tilde{Q}_t^{\varepsilon_n}(x)-Kw_t(x)|^2\Bigg)$$

$$\leq 3\Bigg((\|w_0\|_1+\nu\|g\|_1)^2(L_{\varepsilon_n}^2E(\sup_{s\leq t}|Z_s^{in}-\bar{Z}_s^{i,\varepsilon_n}|^2)+\frac{4M_{\varepsilon_n}^2}{n})+|K_{\varepsilon_n}q_t^{\varepsilon_n}(x)-Kw_t(x)|^2\Bigg).$$

We conclude using (4.16), Proposition 4.4 and Proposition 4.8. □

Remark 4.12 *Since the laws $\mathcal{L}((\tau^1,Z^{1n},k^{1n}),...,(\tau^n,Z^{nn},k^{nn}))$ are exchangeable, the propagation of chaos result is equivalent to the convergence in probability of the empirical measures to P, as probability measures on the path space (cf. [22]). As a consequence, if the space of finite measures on $\bar{\Theta}$ is endowed with the weak convergence topology, then for $t\in[0,T]$, the random finite measures $\tilde{\mu}_t^{n,\varepsilon_n}=\frac{1}{n}\sum_{i=1}^n \mathbf{1}_{\{\tau^i\leq t\}}h(\tau^i,Z_0^i)\delta_{Z_t^{in}}$ converge in probability to $w_t(x)dx$, w being the unique solution of the vortex equation.*

We finally deduce from this study an algorithm for the simulation of the solution of the vortex equation. To approximate numerically this solution, it is necessary to discretize in time the particle system. This can be achieved thanks to the Euler scheme for reflected diffusions proposed by Gobet [9]. In our situation, with identity diffusion matrix and normal reflection, the weak rate of convergence of this scheme is $\mathcal{O}(\Delta t)$, where Δt denotes the time-step. Like in Bossy-Jourdain [2], one could try to prove that if $\bar{\mu}_{l\Delta t}^n$ denotes the weighted empirical measure of the discretized system, $K_{\varepsilon_n}\bar{\mu}_{l\Delta t}^n$ converges to $Kw_{l\Delta t}$ with rate $\mathcal{O}(\Delta t+\frac{1}{\sqrt{n}})$.

References

[1] Benachour, S.; Roynette, B.; Vallois, P.: Branching process associated with 2d-Navier-Stokes equation, Revista Mathematica Iberoamericana, Vol. 17(2), 331-373, (2001).

[2] Bossy, M.; Jourdain, B.: Rate of convergence of a particle method for the solution of a 1d viscous scalar conservation law in a bounded interval, Preprint CERMICS 2000-197, to appear in Annals of Probability.

[3] Brezis, H. :*Analyse fonctionnelle*, Masson (1993).

[4] Chorin, A.J.: *Vorticity and turbulence*, Applied Mathematical Sciences 103, Springer-Verlag (1994).

[5] Cottet, P.H.: Boundary conditions and deterministic vortex methods for the Navier-Stokes equations, in *Mathematical aspects of vortex dynamics*, SIAM, Philadelphia PA, (1989).

[6] Fernandez, B.; Méléard, S.: Asymptotic behaviour for interacting diffusion processes with space-time random birth, Bernoulli 6 (1), 91-111 (2000).

[7] Giet, J.S.: Processus stochastiques: application à l'équation de Navier-Stokes; simulation de la loi de diffusions irrégulières; vitesse du schéma d'Euler pour des fonctionnelles, Thèse de doctorat de l'Université Henri Poincaré, Nancy 1, (2000).

[8] Gilbarg, D.; Trudinger, N.S.: *Elliptic partial differential equations of second order*, 2nd ed., Berlin-Heidelberg-New-York, Springer-Verlag (1983).

[9] Gobet, E.: Euler schemes and half-space approximations for the simulation of diffusion in a domain, ESAIM P&S, vol. 5, 261-293 (2001).

[10] Jourdain, B.; Méléard, S.: Propagation of chaos and fluctuations for a moderate model with smooth initial data, Annales de l'IHP, Vol. 34, 6, 727-767 (1998).

[11] Jourdain, B.; Diffusion processes associated with nonlinear evolution equations for signed initial measures,Methodol. Comput. Appl. Probab. 2 (1), 69-91 (2000).

[12] Ladyzenskaja, O.A., Solonnikov, V.A. and Ural'ceva, N.N.; *Linear and quasilinear equations of parabolic type*, AMS (1968).

[13] Lions, P.L.; Sznitman, A.S.: Stochastic differential equations with reflecting boundary conditions, Communications on pure and applied mathematics, 37, 511-537, (1984).

[14] Marchioro, C. ; Pulvirenti, M.: Hydrodynamics in two dimensions and vortex theory, Commun. Math. Phys. 84, 483-503 (1982).

[15] Méléard, S.: Asymptotic behaviour of some interacting particle systems, McKean-Vlasov and Boltzmann models, CIME lectures, L.N. in Math. 1627, 42-95, Springer (1996).

[16] Méléard, S.: A trajectorial proof of the vortex method for the two-dimensional Navier-Stokes equation, Ann. Appl. Prob. 10 (4), 1197-1211 (2000).

[17] Méléard, S.: Monte-Carlo approximations of the solution of two-dimensional Navier-Stokes equations with finite measure initial data, P.T.R.F. 121, no 3, 367-388 (2001).

[18] Meyer, P.A.: *Probabilités et Potentiel*, Hermann (1966).

[19] Pommerenke, Ch.: *Boundary behaviour of conformal maps*, Springer-Verlag (1992).

[20] Sato, K. and Ueno, T.: Multi-dimensional diffusion and the Markov process on the boundary, J. Math. Kyoto Univ. 4(3) 529-605 (1965)

[21] Sznitman, A.S.: Nonlinear reflecting diffusion process, and the propagation of chaos and fluctuations associated, Journal of functional analysis 56, 311-336 (1984).

[22] Sznitman, A.S.: Topics in propagation of chaos, Ecole d'été de Probabilités de Saint-Flour XIX - 1989, L.N. in Math. 1464, Springer-Verlag (1991).

[23] Temam, R.: *Navier-Stokes equations*, North Holland, 1979.