MATHEMATICAL ANALYSIS OF A STOCHASTIC DIFFERENTIAL EQUATION ARISING IN THE MICRO-MACRO MODELLING OF POLYMERIC FLUIDS.

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We analyze the properties of a stochastic differential equation (SDE) arising in the modeling of polymeric fluids. More precisely, we focus on the so-called FENE (Finite Extensible Nonlinear Elastic) model, for which the drift term in the SDE is singular.

1. Introduction

The rheology of non-newtonian fluids is a very lively field of modern fluid mechanics. The challenge is to find a good relation linking within the fluid the stress tensor to the velocity field in order to reproduce the behavior of the fluid in some classical situations (shear flow, elongational flow) and to simulate it in some more complex cases. This relation may be complicated since the stress generally depends on the whole history of the velocity field. Many approaches consist in deriving this relation from the microscopic structure of the fluid. In some cases, it is possible to directly attack the full system coupling the evolution of these microscopic structures to the macroscopic quantities (such as velocity or pressure) : this is the so-called micro-macro approach.

We are here interested in the modeling of polymeric fluids. More precisely, we consider dilute solutions of polymers, so that the chains of polymers (the "microscopic structures") do not interact with each other. In order to describe the microscopic structure of this fluid, one can model a polymer by a chain of beads and rods (this is the Kramers model) or more simply by some beads linked by springs (see Figure 1). We consider here the simplest model consisting in two beads linked by one spring : this is the dumbbell model. In this model, the evolution of the end-to-end vector (which joins the two beads) is described by a SDE. We refer the interested reader to Refs [10,1,2,6] for the general physical background of these models. This SDE is actually coupled to the Navier-Stokes equation through the expression of the stress tensor as an expectation value built from the end-to-end vector.

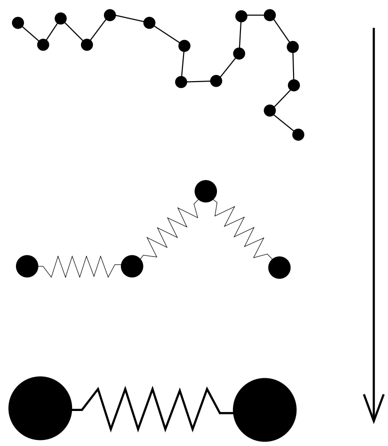

Figure 1.: A hierarchy of models : from Kramers chain (top) to dumbbell (bottom).

The spring force can be linear (Hookean dumbbell model) or explosive (Finite Extensible Nonlinear Elastic dumbbell model).

In the following, we consider the start-up of a Couette flow of a polymeric fluid (see Figure 2) : the fluid is initially at rest, and for $t > 0$, the upper plate moves with a constant velocity. For a complete analysis (existence, uniqueness, convergence of a finite element method coupled with a Monte Carlo method) of this model in the Hookean dumbbell case, we refer to Ref. [8]. This reference also contains a more detailed introduction to these types of models and the way to discretize the corresponding system of coupled PDE-SDE.

We here complement the mathematical analysis of the FENE model presented in Ref. [3] by focusing on the SDE modeling the evolution of the conformation of the polymers in the FENE case. It is proven in Ref. [3] that a solution to the coupled micro-macro system uniquely exists under natural assumptions. Our concern in the present paper is in particular to investigate the role played by the finite extensibility coefficient b (see formulas (2) and (3) below) in the existence and uniqueness of solution of the SDE itself, the fluid velocity being considered known.

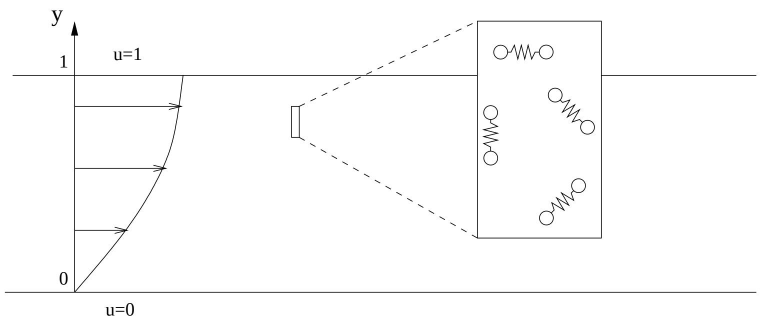

Figure 2.: Velocity profile in a shear flow of a dilute solution of polymers.

Let us now introduce the equations we deal with. They read, in a non-dimensional form :

$$\partial_t u - \partial_{yy} u = \partial_y \tau + f_{ext}, \tag{1}$$

$$\tau = \mathbb{E}\left(\frac{X_t^y Y_t^y}{1 - \frac{(X_t^y)^2 + (Y_t^y)^2}{b}} \right), \tag{2}$$

$$\begin{cases} dX_t^y = \left(-\frac{1}{2} \frac{X_t^y}{1 - \frac{(X_t^y)^2 + (Y_t^y)^2}{b}} + \partial_y u \, Y_t^y \right) dt + dV_t, \\ dY_t^y = \left(-\frac{1}{2} \frac{Y_t^y}{1 - \frac{(X_t^y)^2 + (Y_t^y)^2}{b}} \right) dt + dW_t. \end{cases} \tag{3}$$

where the parameter $b > 0$ measures the finite extensibility of the polymer. The space variable y varies in $\mathcal{O} = (0,1)$ and t varies in the whole of $\mathbb{R}_+$. The random variables are defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. The random process (V_t, W_t) is a $(\mathcal{F}_t)$-two-dimensional Brownian motion. We take Dirichlet boundary conditions on the velocity. The initial velocity is $u(t=0,.) = u_0$, and (X_0, Y_0) is a $\mathcal{F}_0$-measurable random variable. We will suppose that (X_0, Y_0) is either such that $\mathbb{P}\left(X_0^2 + Y_0^2 > b\right) = 0$ (Section 2) or such that $\mathbb{P}\left(X_0^2 + Y_0^2 \geq b\right) = 0$ (Sections 3 and 4).

We fix y in $\mathcal{O}$, set $g(t) = \partial_y u(y,t)$ and suppose throughout this paper that we have at least the following regularity on g :

$$g \in L^1_{\text{loc}}(\mathbb{R}_+) \tag{4}$$

where $\mathbb{R}_+ = [0, +\infty)$. We are then interested in solving for $t \geq 0$ the following SDE, which is a rewriting of the SDE (3) of the initial coupled system :

$$\begin{cases} dX_t^g = \left(-\frac{1}{2} \frac{X_t^g}{1 - \frac{(X_t^g)^2 + (Y_t^g)^2}{b}} + g(t)\, Y_t^g \right) dt + dV_t, \\ dY_t^g = \left(-\frac{1}{2} \frac{Y_t^g}{1 - \frac{(X_t^g)^2 + (Y_t^g)^2}{b}} \right) dt + dW_t, \end{cases} \tag{5}$$

with initial condition (X_0, Y_0).

Let us begin by recalling from Ref. [3] the precise mathematical meaning we give to (5).

Definition 1.1. Let $\boldsymbol{X}_0 = (X_0, Y_0)$ and $\boldsymbol{W}_t = (V_t, W_t)$. We shall say that a $(\mathcal{F}_t)$-adapted process $\boldsymbol{X}_t^g = (X_t^g, Y_t^g)$ is a solution to (5) when : for $\mathbb{P}$-a.e. ω, $\forall t \geq 0$,

$$\begin{cases} \displaystyle\int_0^t \left| \frac{1}{1 - \frac{|\boldsymbol{X}_s^g|^2}{b}} \right| ds < \infty, \\ \text{with the convention } , \forall \boldsymbol{x} = (x, y) \in \mathbb{R}^2, \frac{1}{1 - \frac{|\boldsymbol{x}|^2}{b}} = +\infty \text{ if } |\boldsymbol{x}|^2 = b, \\ \displaystyle\boldsymbol{X}_t^g = \boldsymbol{X}_0 - \frac{1}{2} \int_0^t \frac{\boldsymbol{X}_s^g}{1 - \frac{|\boldsymbol{X}_s^g|^2}{b}} ds + \int_0^t (g(s) Y_s^g, 0)\, ds + \boldsymbol{W}_t \end{cases} \tag{6}$$

Remark 1.1. Because of the convention $\frac{1}{1 - \frac{|\boldsymbol{x}|^2}{b}} = +\infty$ if $|\boldsymbol{x}|^2 = b$, we deduce that a solution to (6) is such that the subset of $\mathbb{R}_+$ $\{0 \leq u < \infty, |\boldsymbol{X}_u^g|^2 = b\}$ has $\mathbb{P}$-a.s. zero Lebesgue measure.

The paper is organized as follows : in Section 2, we prove the existence and uniqueness of the solution to (6) with values in $\overline{B}$, where

$$B = B(0, \sqrt{b}) = \left\{ (x, y), x^2 + y^2 < b \right\}.$$

The existence of such a solution is derived from results concerning multivalued SDEs (see Refs [4,5]). We then focus on the probability for this solution to reach the boundary of $\overline{B}$ (see Section 3). When $b < 2$ and $\mathbb{P}(|\boldsymbol{X}_0|^2 < b) = 1$, this probability is equal to one. This enables us to construct (for $g = 0$) a solution to (6) that leaves a.s. $\overline{B}$. Hence, if $b < 2$, uniqueness of solutions does not hold for solutions to (6) without the additional requirement to take values in $\overline{B}$. When $b \geq 2$ and again $\mathbb{P}(|\boldsymbol{X}_0|^2 < b) = 1$, the probability to reach the boundary is equal to zero and trajectorial uniqueness holds. We exhibit the unique invariant probability measure of the SDE (6) with $g = 0$ (see Section 4). All these results on the SDE have an impact on the analysis and the understanding of the coupled SDE-PDE system (for which we refer to Ref. [3]). They show that the assumption $b \geq 2$ adopted in Ref. [3] to prove existence and uniqueness of solution to the coupled system is in some sense “optimal”.

2. Existence and uniqueness

In this section, we suppose that (X_0, Y_0) is such that $\mathbb{P}\left(X_0^2 + Y_0^2 > b\right) = 0$. Our aim is to prove the following :

Proposition 2.1. *Under assumption (4), for any $b > 0$ and for any initial condition (X_0, Y_0) such that $\mathbb{P}\left(X_0^2 + Y_0^2 > b\right) = 0$, there exists a unique solution to (6) with values in $\overline{B}$.*

We first prove the uniqueness statement (Section 2.1), then turn to the existence first when $g \in L_t^\infty$ (Section 2.2) and finally when $g \in L^1_{\text{loc}}$ (Section 2.3). In the following, the point is to notice that the singular term in the drift derives from a convex potential $\Pi : \mathbb{R}^2 \to]-\infty, +\infty]$:

$$\Pi(x,y) = \begin{cases} -\frac{b}{4}\ln\left(1 - \frac{x^2+y^2}{b}\right) & \text{if } x^2 + y^2 < b, \\ +\infty & \text{otherwise.} \end{cases} . \tag{7}$$

We have : $\forall x \in B$, $\nabla\Pi(\boldsymbol{x}) = \frac{1}{2}\frac{\boldsymbol{x}}{1 - \frac{|\boldsymbol{x}|^2}{b}}$. Moreover, the function Π is a continuous convex function with domain B.

2.1. *Trajectorial uniqueness for solutions with values in $\overline{B}$*

Let us begin with the uniqueness.

Proposition 2.2. *Let us suppose we have two solutions $\boldsymbol{X}_t^g$ and $\tilde{\boldsymbol{X}}_t^g$ to (6) and such that $\mathbb{P}$-a.s., $\boldsymbol{X}_0^g = \tilde{\boldsymbol{X}}_0^g$. Then these two solutions are indistiguishable until one of the processes leaves $\overline{B}$. In addition, if $\mathbb{P}(\exists t \geq 0 |\boldsymbol{X}_t^g|^2 = b) = 0$, then $\boldsymbol{X}_t^g$ and $\tilde{\boldsymbol{X}}_t^g$ are indistiguishable.*

Proof : Let us consider $\tau = \inf\{t \geq 0, (|\tilde{\boldsymbol{X}}_t^g|^2 \vee |\boldsymbol{X}_t^g|^2) > b\}$ and $\boldsymbol{Z}_t = \left(\boldsymbol{X}_t^g - \tilde{\boldsymbol{X}}_t^g\right)$. By Itô's formula, we have :

$$\begin{aligned} d|\boldsymbol{Z}|_t^2 &= 2\boldsymbol{Z}_t.d\boldsymbol{Z}_t, \\ &= -2(\nabla\Pi(\boldsymbol{X}_t^g) - \nabla\Pi(\tilde{\boldsymbol{X}}_t^g)).\boldsymbol{Z}_t\,dt + 2g(t)(X_t^g - \tilde{X}_t^g)(Y_t^g - \tilde{Y}_t^g)dt, \end{aligned}$$

where $\boldsymbol{x}.\boldsymbol{y}$ denotes the scalar product of $\boldsymbol{x}$ and $\boldsymbol{y} \in \mathbb{R}^2$.

Using the fact that, since Π is convex, for any $\boldsymbol{x}$ and $\tilde{\boldsymbol{x}} \in B$, $(\nabla\Pi(\boldsymbol{x}) - \nabla\Pi(\tilde{\boldsymbol{x}})).(\boldsymbol{x} - \tilde{\boldsymbol{x}}) \geq 0$, we obtain, for any $t \geq 0$:

$$\begin{aligned} |\boldsymbol{Z}_{t\wedge\tau}|^2 &\leq 2\int_0^{t\wedge\tau} |g(s)| \left|X_s^g - \tilde{X}_s^g\right| \left|Y_s^g - \tilde{Y}_s^g\right| ds, \\ &\leq \int_0^t |g(s)||\boldsymbol{Z}_{s\wedge\tau}|^2\,ds. \end{aligned}$$

Using Gronwall Lemma and the fact that $g \in L^1_{\text{loc}}(\mathbb{R}_+)$, we have thus shown that $\mathbb{P}$-a.s, $\forall t \geq 0$, $\boldsymbol{X}^g_{t\wedge\tau} = \tilde{\boldsymbol{X}}^g_{t\wedge\tau}$. Therefore, on $\{\tau < \infty\}$, $|\boldsymbol{X}^g_\tau|^2 = b$. We deduce that in case $\mathbb{P}(\exists t \geq 0,\, |\boldsymbol{X}^g_t|^2 = b) = 0$, $\tau = \infty$ $\mathbb{P}$-a.s. . $\diamondsuit$

2.2. *Existence in the case $g \in L^\infty_t$*

In this section, we suppose :

$$g \in L^\infty(\mathbb{R}_+)\,. \tag{8}$$

In order to prove an existence result, we will use a multivalued stochastic differential equation. In this section, we use the results of E. Cépa[4] and E. Cépa and D. Lépingle[5].
Since the function Π is convex on the open set B, its subdifferential $\partial\Pi$ is a simple-valued maximal monotone operator on $\mathbb{R}^2$ with domain B :

$$\partial\Pi(\boldsymbol{x}) = \begin{cases} \{\nabla\Pi(\boldsymbol{x})\} & \text{if } \boldsymbol{x} \in B, \\ \qquad\emptyset & \text{if } \boldsymbol{x} \notin B. \end{cases}$$

Let us now consider the two-dimensional process $\boldsymbol{X}_t$ solution of the following multivalued SDE :

$$\begin{cases} d\boldsymbol{X}^g_t + \partial\Pi(\boldsymbol{X}^g_t)\,dt \ni (g(t)Y^g_t, 0)\,dt + d\boldsymbol{W}_t, \\ \boldsymbol{X}^g_0 = \boldsymbol{X}_0 = (X_0, Y_0), \end{cases} \tag{9}$$

We first recall the precise meaning of a solution to (9).

Definition 2.1. We shall say that a continuous $(\mathcal{F}_t)$-adapted process $\boldsymbol{X}^g_t = (X^g_t, Y^g_t)$ with values in $\overline{B}$ is a solution to (9) if and only if $\boldsymbol{X}^g_0 = \boldsymbol{X}_0$ and the process $\boldsymbol{K}^g_t = \boldsymbol{W}_t + \int_0^t (g(s)Y^g_s, 0)\,ds - (\boldsymbol{X}^g_t - \boldsymbol{X}^g_0)$ is a continuous process with finite variation such that : for any continuous $(\mathcal{F}_t)$-adapted process $\boldsymbol{\alpha}_t$ with values in $\mathbb{R}^2$, for $\mathbb{P}$-a.e. ω, $\forall\, 0 \leq s \leq t < \infty$,

$$\int_s^t \Pi(\boldsymbol{X}^g_u)\,du \leq \int_s^t \Pi(\boldsymbol{\alpha}_u)\,du + \int_s^t (\boldsymbol{X}^g_u - \boldsymbol{\alpha}_u).d\boldsymbol{K}^g_u. \tag{10}$$

Remark 2.1. A condition equivalent to (10) is the following : for any continuous $(\mathcal{F}_t)$-adapted process $\boldsymbol{\alpha}_t$ with values in B, the measure on $\mathbb{R}_+$:

$$(\boldsymbol{X}^g_u - \alpha_u)\,.\,(d\boldsymbol{K}^g_u - \nabla\Pi(\alpha_u)\,du)$$

is $\mathbb{P}$-a.s. nonnegative.

Since (8) ensures that $\boldsymbol{x} = (x, y) \mapsto (g(t)y, 0)$ is (uniformly in time) Lipschitz and with linear growth, according to E. Cépa[4], we have :

Proposition 2.3. *Under the assumption (8), for any $b > 0$, the multivalued SDE (9) has a unique strong solution.*

We are now going to recover a solution to (6) from the solution of (9). More precisely, we follow the method of E. Cépa and D. Lépingle[5] (see Lemmas 3.3, 3.4 and 3.6) in order to identify the process $\boldsymbol{K}_t^g$.

We can thus show that for all $0 < t < \infty$, we have :

$$\mathbb{E}\left(\int_0^t |\partial\Pi(\boldsymbol{X}_u^g)|\, du\right) < \infty, \text{ with convention } |\partial\Pi(\boldsymbol{x})| = +\infty \text{ if } \boldsymbol{x} \notin B.$$

As a consequence, for any $0 < t < \infty$, $\mathbb{P}$-a.s.,

$$\int_0^t \frac{1}{1 - \frac{|\boldsymbol{X}_u^g|^2}{b}}\, du < \infty \text{ with convention } \frac{1}{1-\frac{b}{b}} = +\infty. \tag{11}$$

Moreover, the process $\boldsymbol{K}_t^g$ is $\mathbb{P}$-a.s. absolutely continuous on $\{0 \le u < \infty,\ \boldsymbol{X}_u^g \in B\}$, with density $\nabla\Pi(\boldsymbol{X}_u^g)$ so that $d\boldsymbol{K}_u^g$ has the following form :

$$d\boldsymbol{K}_u^g = \nabla\Pi(\boldsymbol{X}_u^g)\, du + d\boldsymbol{G}_u^g, \tag{12}$$

where $\boldsymbol{G}^g$ is a continuous boundary process with finite variation $|\boldsymbol{G}^g|$:

$$\boldsymbol{G}_t^g = \int_0^t 1_{\{\boldsymbol{X}_u^g \in \partial B\}} d\boldsymbol{K}_u^g = \int_0^t 1_{\{\boldsymbol{X}_u^g \in \partial B\}} d\boldsymbol{G}_u^g. \tag{13}$$

Finally, one can identify this process $\boldsymbol{G}_t^g$: for all $t \ge 0$,

$$\boldsymbol{G}_t^g = \int_0^t 1_{\{\boldsymbol{X}_u^g \in \partial B\}} \boldsymbol{n}(\boldsymbol{X}_u^g) d|\boldsymbol{G}^g|_u = \int_0^t \frac{\boldsymbol{X}_u^g}{\sqrt{b}} d|\boldsymbol{G}^g|_u,$$

where, for any $\boldsymbol{x} \in \partial B$, $\boldsymbol{n}(\boldsymbol{x}) = \frac{\boldsymbol{x}}{\sqrt{b}}$ is the unitary outward normal to B at the point $\boldsymbol{x}$.

Hence the process $\boldsymbol{X}_t^g$ is solution of the following SDE with normal reflexion at the boundary of B :

$$d\boldsymbol{X}_t^g = -\nabla\Pi(\boldsymbol{X}_t^g)\, dt + (g(t)Y_t^g, 0)\, dt + d\boldsymbol{W}_t - 1_{\{\boldsymbol{X}_t^g \in \partial B\}} \boldsymbol{n}(\boldsymbol{X}_t^g) d|\boldsymbol{G}^g|_t.$$

It just remains to show that $|\boldsymbol{G}^g|_u = 0$, for $u \ge 0$, in order to recover (6). Notice in particular that by (11), the property of integrability of the drift term in (6) holds for the solution $\boldsymbol{X}_t^g$ of the multivalued SDE (9).

Lemma 2.1. $|\boldsymbol{G}^g| = 0$.

Proof : We follow here again the ideas of E. Cépa and D. Lépingle[5] (see Lemma 3.8 p. 438) to prove that $|\boldsymbol{G}^g| = 0$. Let us consider $R_t^g = b - |\boldsymbol{X}_t^g|^2$. By Itô's formula,

$$\begin{aligned} dR_t^g &= -2\boldsymbol{X}_t^g . d\boldsymbol{X}_t^g - 2\, dt, \\ &= -2\nabla\Pi(\boldsymbol{X}_t^g).\boldsymbol{X}_t^g\, dt - 2g(t)X_t^g Y_t^g\, dt - 2\, dt - 2\boldsymbol{X}_t^g.d\boldsymbol{W}_t + 2\frac{|\boldsymbol{X}_t^g|^2}{\sqrt{b}}\, d|\boldsymbol{G}^g|_t, \\ &= \frac{b^2}{R_t^g}\, dt - 2g(t)X_t^g Y_t^g\, dt - (2+b)\, dt - 2\boldsymbol{X}_t^g.d\boldsymbol{W}_t + 2\sqrt{b}\, d|\boldsymbol{G}^g|_t, \end{aligned} \tag{14}$$

the last equality using the fact that $d|\boldsymbol{G}^g|_t = 1_{\{\boldsymbol{X}_t^g \in \partial B\}} d|\boldsymbol{G}^g|_t$.
We know that R_t^g is a continuous semimartingale with values in $[0,b]$. We want to prove that $dR_t^g = 1_{R_t^g>0} dR_t^g$. Using Tanaka's formula (see[11] p. 213),

$$R_t^g = (R_t^g)^+ = (R_0^g)^+ + \int_0^t 1_{R_s^g>0}\, dR_s^g + \frac{1}{2} L_t^0, \tag{15}$$

where, for any $a \in [0,b]$, L_t^a denotes the local time in a of R^g. Using now the occupation times formula (see Ref. [11] p. 215), we know (using (11)) that, for any fixed $t > 0$:

$$\int_0^b \frac{1}{a} L_t^a \, da = \int_0^t \frac{1}{R_s^g}\, d < R^g >_s \leq 4 \int_0^t \frac{1}{1 - \frac{|\boldsymbol{X}_s^g|^2}{b}}\, ds < \infty.$$

Since $a \mapsto L_t^a$ is a.s. cadlag (see[11] p. 216), we deduce that for any $t > 0$, $\mathbb{P}$-a.s., $L_t^0 = 0$. Using this in (15), we obtain

$$dR_t^g = 1_{R_t^g>0}\, dR_t^g.$$

Using this equality in (14), we have : $\forall t \geq 0$,

$$\int_0^t 1_{R_s^g=0} d|\boldsymbol{G}^g|_s =$$
$$\frac{1}{2\sqrt{b}} \int_0^t 1_{R_s^g=0} \left(-\frac{b^2}{R_s^g}\, ds + 2g(s) X_s^g Y_s^g\, ds + (2+b)\, ds + 2\boldsymbol{X}_s^g . d\boldsymbol{W}_s \right).$$

Since, according to (11), $\mathbb{P}$-a.s., $\{0 \leq t < \infty, R_t^g = 0\}$ has zero Lebesgue measure, the right hand side is nul. We conclude by using $d|\boldsymbol{G}^g|_t = 1_{R_t^g=0} d|\boldsymbol{G}^g|_t$. $\diamondsuit$

We have thus shown the following properties on the process $\boldsymbol{X}_t^g$:

- for any $0 < t < \infty$, $\mathbb{P}$-a.s., $\displaystyle\int_0^t \frac{1}{1 - \frac{|\boldsymbol{X}_u^g|^2}{b}}\, du < \infty$,
- $d\boldsymbol{X}_t^g = -\nabla \Pi(\boldsymbol{X}_t^g)\, dt + (g(t) Y_t^g, 0)\, dt + d\boldsymbol{W}_t$.

We have thus built a solution $\boldsymbol{X}_t^g = (X_t^g, Y_t^g)$ to our initial problem (6) in case $g \in L^\infty(\mathbb{R}_+)$. This result is not sufficient in our context since the energy estimates on the coupled system (1-3) yields less regularity on g (see Ref. [8]).

2.3. *Existence in the case* $g \in L^1_{loc}(\mathbb{R}_+)$

We now want to build a solution to (6) using the multivalued SDE (9), but with a weaker assumption on g, namely (4). In this case, the general results of existence on multivalued SDE do not apply immediatly.

Therefore, we consider the following sequence of approximations of this problem :

$$\begin{cases} d\boldsymbol{X}_t^{g^n} + \partial\Pi(\boldsymbol{X}_t^{g^n})\,dt \ni (g^n(t)Y_t^{g^n},0)\,dt + d\boldsymbol{W}_t, \\ \boldsymbol{X}_0^{g^n} = \boldsymbol{X}_0, \end{cases} \tag{16}$$

where $n \in \mathbb{N}^*$ and $g^n(t) = -n \vee (n \wedge g(t))$. Since g^n is bounded, the results of the previous section apply and we obtain a unique solution $\boldsymbol{X}_t^{g^n}$ of the multivalued SDE (16). Moreover, these processes $\boldsymbol{X}_t^{g^n}$ are such that :

- for any $0 < t < \infty$, $\mathbb{P}$-a.s., $\displaystyle\int_0^t \frac{1}{1 - \frac{|\boldsymbol{X}_u^{g^n}|^2}{b}}\,du < \infty$,
- $$\boldsymbol{X}_t^{g^n} = \boldsymbol{X}_0 - \int_0^t \nabla\Pi(\boldsymbol{X}_s^{g^n})\,ds + \int_0^t (g^n(s)Y_s^{g^n},0)\,ds + \boldsymbol{W}_t. \tag{17}$$

We now want to let n go to ∞ in Definition 2.1 (notice that by (17), $d\boldsymbol{K}_t^{g^n} = \nabla\Pi(\boldsymbol{X}_t^{g^n})\,dt$). In the following, we choose $T > 0$ and we work on the time interval $[0,T]$. We know that for all n, $\sup_{t\geq 0}\left|\boldsymbol{X}_t^{g^n}\right|^2 \leq b$. For any $n \geq m$, we have, by Itô's formula,

$$\begin{aligned} d\left|\boldsymbol{X}_t^{g^n} - \boldsymbol{X}_t^{g^m}\right|^2 &= -\left(\nabla\Pi(\boldsymbol{X}_t^{g^n}) - \nabla\Pi(\boldsymbol{X}_t^{g^m})\right).\left(\boldsymbol{X}_t^{g^n} - \boldsymbol{X}_t^{g^m}\right)\,dt \\ &\quad + \left(g^n(t)Y_t^{g^n} - g^m(t)Y^{g^m}(t)\right)\left(X_t^{g^n} - X_t^{g^m}\right)\,dt. \end{aligned}$$

Using the fact that, since Π is convex, for any $\boldsymbol{x}$ and $\boldsymbol{y} \in B$, $(\nabla\Pi(\boldsymbol{x}) - \nabla\Pi(\boldsymbol{y})).(\boldsymbol{x}-\boldsymbol{y}) \geq 0$, we obtain : $\forall t \in [0,T]$,

$$\left|\boldsymbol{X}_t^{g^n} - \boldsymbol{X}_t^{g^m}\right|^2 \leq \int_0^t \left(g^n(s)Y^{g^n}(s) - g^m(s)Y^{g^m}(s)\right)\left(X_s^{g^n} - X_s^{g^m}\right)\,ds$$

so that : $\forall t \in [0,T]$,

$$\begin{aligned} \left|\boldsymbol{X}_t^{g^n} - \boldsymbol{X}_t^{g^m}\right|^2 &\leq \int_0^t \left|g^n(s)Y_s^{g^n} - g^m(s)Y_s^{g^m}\right|\left|X_s^{g^n} - X_s^{g^m}\right|\,ds \\ &\leq \int_0^t \left(|g^n(s)|\left|Y_s^{g^n} - Y_s^{g^m}\right| + \left|Y_s^{g^m}\right||g^n(s) - g^m(s)|\right)\left|X_s^{g^n} - X_s^{g^m}\right|\,ds \\ &\leq \frac{1}{2}\int_0^t |g|(s)\left|\boldsymbol{X}_s^{g^n} - \boldsymbol{X}_s^{g^m}\right|^2\,ds + 2b\int_0^t |g^n(s) - g^m(s)|\,ds. \end{aligned}$$

Using Gronwall Lemma, we then obtain :

$$\begin{aligned} \sup_{t\in[0,T]}\left|\boldsymbol{X}_t^{g^n} - \boldsymbol{X}_t^{g^m}\right|^2 &\leq 2b\int_0^T \exp\left(\frac{1}{2}\int_s^T |g(u)|\,du\right)|g^n(s) - g^m(s)|\,ds \\ &\leq 2b\exp\left(\frac{1}{2}||g||_{L^1_t([0,T])}\right)\int_0^T |g^n(s) - g^m(s)|\,ds. \end{aligned}$$

From this inequality and the fact that $g \in L^1_t([0,T])$, we deduce that there exists a continuous adapted process $\boldsymbol{X}^g_t$ with values in $\overline{B}$ such that $\boldsymbol{X}^{g^n}_t \to \boldsymbol{X}^g_t$ in $L^\infty_\omega(L^\infty_t([0,T]))$.

One has the following estimate on the total variation of $\nabla\Pi(\boldsymbol{X}^{g^n}_u)\,du$ on $[0,T]$:

$$\int_0^T \left|\nabla\Pi(\boldsymbol{X}^{g^n}_t)\right|\,dt \le \frac{\sqrt{b}}{2}\int_0^T \frac{1}{1-\frac{|\boldsymbol{X}^{g^n}_s|^2}{b}}\,ds. \tag{18}$$

By Itô's formula, we know that : $\forall t\in[0,T]$,

$$|\boldsymbol{X}^{g^n}_t|^2 = |\boldsymbol{X}_0|^2 - \int_0^t \frac{|\boldsymbol{X}^{g^n}_s|^2}{1-\frac{|\boldsymbol{X}^{g^n}_s|^2}{b}}\,ds + 2\int_0^t g^n(s)\,X^{g^n}_s\,Y^{g^n}_s\,ds + 2t + 2\int_0^t \boldsymbol{X}^{g^n}_s.d\boldsymbol{W}_s,$$

which yields : $\forall t\in[0,T]$,

$$\int_0^t \frac{1}{1-\frac{|\boldsymbol{X}^{g^n}_s|^2}{b}}\,ds = \frac{1}{b}\left(-|\boldsymbol{X}^{g^n}_t|^2 + |\boldsymbol{X}_0|^2 + (2+b)t + 2\int_0^t g^n(s)\,X^{g^n}_s\,Y^{g^n}_s\,ds \right.$$
$$\left. +2\int_0^t \boldsymbol{X}^{g^n}_s.d\boldsymbol{W}_s\right). \tag{19}$$

It is obvious that $\int_0^t \boldsymbol{X}^{g^n}_s.d\boldsymbol{W}_s \to \int_0^t \boldsymbol{X}_s.d\boldsymbol{W}_s$ in $L^2_\omega(L^\infty_t([0,T]))$-norm. Up to the extraction of a subsequence, we can suppose that this convergence holds for almost every ω. Using this property together with (18) and (19), we deduce that for a.e. ω, the measure $\nabla\Pi(\boldsymbol{X}^{g^n}_t)\,dt$ on $[0,T]$ is such that $\int_0^T \left|\nabla\Pi(\boldsymbol{X}^{g^n}_t)\right|\,dt < C(T,\omega)$ where $C(T,\omega)$ is a constant only depending on T and ω. One can thus extract a weakly converging subsequence of $\left(\nabla\Pi(\boldsymbol{X}^{g^n}_t)\,dt\right)_{n\ge1}$. On the other hand, taking the limit $n\to\infty$ in (17), we see that $\int_0^t \nabla\Pi(\boldsymbol{X}^{g^n}_u)\,du$ uniformly converges on $[0,T]$ to $\boldsymbol{K}^g_t$ satisfying : $\forall t\in[0,T]$,

$$\boldsymbol{K}^g_t = \int_0^t (g(u)Y^g_u, 0)\,du + \boldsymbol{W}_t - (\boldsymbol{X}^g_t - \boldsymbol{X}_0).$$

By identification of the limit, we have $\nabla\Pi(\boldsymbol{X}^{g^n}_t)\,dt \rightharpoonup d\boldsymbol{K}^g_t$ weakly.

By Definition 2.1, the processes $\boldsymbol{X}^{g^n}_t$ are such that for any continuous $(\mathcal{F}_t)$-adapted process $\boldsymbol{\alpha}_t$ with values in $\mathbb{R}^2$, for $\mathbb{P}$-a.e. ω, $\forall\, 0\le s\le t<\infty$,

$$\int_s^t \Pi(\boldsymbol{X}^{g^n}_u)\,du \le \int_s^t \Pi(\boldsymbol{\alpha}_u)\,du + \int_s^t (\boldsymbol{X}^{g^n}_u - \boldsymbol{\alpha}_u).\nabla\Pi(\boldsymbol{X}^{g^n}_u)\,du. \tag{20}$$

One can pass to the limit $n\to\infty$ in (20), using the fact that $\Pi(\boldsymbol{X}^{g^n}_u)\to\Pi(\boldsymbol{X}^g_u)$ pointwise in u and that $\Pi(\boldsymbol{X}^{g^n}_u)$ is uniformly integrable. Indeed, for any $A \geq \frac{b}{4}$, if we set $M_u = \left(1 - \frac{|\boldsymbol{X}_u^{g^n}|^2}{b}\right)^{-1}$, we have (since $x \mapsto \frac{\ln(x)}{x}$ is decreasing on $[e, +\infty)$) :

$$\begin{aligned} \int_0^T 1_{|\Pi(\boldsymbol{X}_u^{g^n})| \geq A} \, \Pi(\boldsymbol{X}_u^{g^n}) \, du &= \frac{b}{4} \int_0^T 1_{M_u \geq \exp(4A/b)} \frac{\ln(M_u)}{M_u} M_u \, du \\ &\leq \frac{A}{\exp(4A/b)} C(T, \omega), \end{aligned}$$

so that $\displaystyle\int_0^T 1_{|\Pi(\boldsymbol{X}_u^{g^n})| \geq A} \Pi(\boldsymbol{X}_u^{g^n}) \, du \to 0$ uniformly in n when $A \to \infty$. We have thus obtained a continuous process $\boldsymbol{X}_t^g$ on $[0, T]$ and a continuous process with finite variation $\boldsymbol{K}_t^g = \int_0^t (g(u) Y_u^g, 0) \, du + \boldsymbol{W}_t - (\boldsymbol{X}_t^g - \boldsymbol{X}_0)$ on $[0, T]$ such that for any continuous $(\mathcal{F}_t)$-adapted process $\boldsymbol{\alpha}_t$ with values in $\mathbb{R}^2$, for $\mathbb{P}$-a.e. ω, $\forall\, 0 \leq s \leq t < T$,

$$\int_s^t \Pi(\boldsymbol{X}_u^g) \, du \leq \int_s^t \Pi(\boldsymbol{\alpha}_u) \, du + \int_s^t (\boldsymbol{X}_u^g - \boldsymbol{\alpha}_u) . d\boldsymbol{K}_u^g .$$

This shows that we have built a solution to the multivalued SDE (9) on the time interval $[0, T]$. Since T is arbitrary, using Proposition 2.2, we have built a solution on $\mathbb{R}_+$. Following again the arguments of the last section it is easy to show that :

- for any $0 < t < \infty$, $\mathbb{P}$-a.s., $\displaystyle\int_0^t \frac{1}{1 - \frac{|\boldsymbol{X}_u^g|^2}{b}} \, du < \infty$,
- $d\boldsymbol{X}_t^g = -\nabla \Pi(\boldsymbol{X}_t^g) \, dt + (g(t) Y_t^g, 0) \, dt + d\boldsymbol{W}_t$.

This shows that $\boldsymbol{X}_t^g$ is a solution to (6) and completes the proof of Proposition 2.1.

3. Does the solution reach the boundary ?

In this section, we want to determine whether or not the process $\boldsymbol{X}_t^g$ we have built in the previous section reaches the boundary of B. Should the occasion arise, we deduce that uniqueness does not hold for (6), at least in the case $g = 0$. Throughout this section, we suppose that the initial condition is such that $\mathbb{P}(|\boldsymbol{X}_0|^2 < b) = 1$.

3.1. *Necessary and sufficient conditions*

In this section, we want to analyze the event $\{\exists t > 0, |\boldsymbol{X}_t^g|^2 = b\}$. We are going to prove :

Proposition 3.1. *Assume*

$$g \in L^2(\mathbb{R}_+), \tag{21}$$

and that $\mathbb{P}(|\boldsymbol{X}_0|^2 < b) = 1$. *Let us consider the process* $\boldsymbol{X}_t^g$ *solution to (6) built above. We have :*

- *if* $b \geq 2$, *then* $\mathbb{P}\left(\exists t > 0, |\boldsymbol{X}_t^g|^2 = b\right) = 0$,
- *if* $b < 2$, *then* $\mathbb{P}\left(\exists t > 0, |\boldsymbol{X}_t^g|^2 = b\right) = 1$.

In view of Proposition 2.2, we deduce immediatly :

Corollary 3.1. *If* $b \geq 2$ *and* $\mathbb{P}(|\boldsymbol{X}_0|^2 < b) = 1$, *then trajectorial uniqueness holds for (6).*

Proof. First, by Girsanov Lemma, one can suppose $g = 0$. Indeed, let us consider the process $\boldsymbol{X}_t^g$ we have built in last section. Under the probability $\mathbb{P}^g$ defined by

$$\frac{d\mathbb{P}^g}{d\mathbb{P}} = \exp\left(-\int_0^\infty g(s) Y_s^g \, dV_s - \frac{1}{2}\int_0^\infty \left(g(s)Y_s^g\right)^2 \, ds\right),$$

the process $(V_t^g, W_t^g) = (V_t + \int_0^t g(s)Y_s^g \, ds, W_t)$ is a Brownian motion and therefore $(X_t^g, Y_t^g, V_t^g, W_t^g, \mathbb{P}^g)_{t \in \mathbf{R}_+}$ is a weak solution of the SDE (5) with $g = 0$. Since this solution is with values in $\overline{B}$, it is also a weak solution of the multivalued SDE (9), with $g = 0$, for which uniqueness in law holds. Since $\mathbb{P}^g$ and $\mathbb{P}$ are equivalent on $\mathcal{F}$, we can then deduce the properties of Proposition 3.1 in case $g \in L^2(\mathbb{R}_+)$ from the properties of Proposition 3.1 in case $g = 0$.

In the following, we focus on the solution to (9) with $g = 0$, which we denote by $\boldsymbol{X}_t = (X_t, Y_t)$. We fix $\boldsymbol{x} \in B$ and the superscript $\boldsymbol{x}$ means that we consider the solution to (9) with $g = 0$ such that $\boldsymbol{X}_0 = \boldsymbol{x}$.

Let us first suppose that $|\boldsymbol{x}| > 0$. Let us consider the process $R_t^{\boldsymbol{x}} = b - |\boldsymbol{X}_t^{\boldsymbol{x}}|^2$. We know that :

$$dR_t^{\boldsymbol{x}} = \frac{b^2}{R_t^{\boldsymbol{x}}} \, dt - (2 + b) \, dt - 2\boldsymbol{X}_t^{\boldsymbol{x}}.d\boldsymbol{W}_t. \tag{22}$$

Let us introduce the stopping time

$$\tau_n^{\boldsymbol{x}} = \inf\left\{t \geq 0, |\boldsymbol{X}_t^{\boldsymbol{x}}|^2 \geq b\left(1 - \frac{1}{n}\right)\right\}.$$

Let fix $t > 0$. By Girsanov Lemma, one shows that $\mathbb{P}$-a.s., $\left|\boldsymbol{X}_{t \wedge \tau_n^{\boldsymbol{x}}}^{\boldsymbol{x}}\right| \neq 0$. Indeed, by definition of $\tau_n^{\boldsymbol{x}}$,

$$\mathbb{P}(|\boldsymbol{X}_{t \wedge \tau_n^{\boldsymbol{x}}}^{\boldsymbol{x}}| = 0) = \mathbb{P}(|\boldsymbol{X}_t^{\boldsymbol{x}}| = 0 \text{ and } t < \tau_n^{\boldsymbol{x}}).$$

Let $\mathbb{P}_n^{\boldsymbol{x}}$ be defined by :

$$\frac{d\mathbb{P}_n^{\boldsymbol{x}}}{d\mathbb{P}} = \exp\left(\int_0^t \nabla\Pi(\boldsymbol{X}^{\boldsymbol{x}}_{s\wedge\tau_n^{\boldsymbol{x}}})\,.d\boldsymbol{W}_s - \frac{1}{2}\int_0^t \left|\nabla\Pi(\boldsymbol{X}^{\boldsymbol{x}}_{s\wedge\tau_n^{\boldsymbol{x}}})\right|^2\,ds\right)$$

and $\mathbb{E}_n^{\boldsymbol{x}}$ denote the corresponding expectation. By Girsanov Theorem, $\left(\boldsymbol{B}_s^{\boldsymbol{x}} = \boldsymbol{x} + \boldsymbol{W}_s - \int_0^s \nabla\Pi(\boldsymbol{X}^{\boldsymbol{x}}_{u\wedge\tau_n^{\boldsymbol{x}}})\,du\right)_{s\le t}$ is a $\mathbb{P}_n^{\boldsymbol{x}}$-Brownian motion starting from $\boldsymbol{x}$. Since on $t \le \tau_n^{\boldsymbol{x}}$, $\boldsymbol{X}_t^{\boldsymbol{x}} = \boldsymbol{B}_t^{\boldsymbol{x}}$,

$$\begin{aligned}\mathbb{P}(|\boldsymbol{X}_t^{\boldsymbol{x}}| = 0 \text{ and } t < \tau_n^{\boldsymbol{x}}) &\le \mathbb{P}(|\boldsymbol{B}_t^{\boldsymbol{x}}| = 0)\\ &= \mathbb{E}_n^{\boldsymbol{x}}\left(1_{|\boldsymbol{B}_t^{\boldsymbol{x}}|=0}\frac{d\mathbb{P}}{d\mathbb{P}_n^{\boldsymbol{x}}}\right)\\ &= 0.\end{aligned} \tag{23}$$

One can therefore show that $|\boldsymbol{X}_t^{\boldsymbol{x}}| > 0$ on $[0, T^{\boldsymbol{x}})$, where

$$T^{\boldsymbol{x}} = \lim_{n\to\infty}\tau_n^{\boldsymbol{x}} = \inf\left\{t \ge 0, |\boldsymbol{X}_t^{\boldsymbol{x}}|^2 = b\right\} = \inf\left\{t \ge 0, R_t^{\boldsymbol{x}} = 0\right\}.$$

Thus, one can write, for $t \in [0, T^{\boldsymbol{x}})$:

$$dR_t^{\boldsymbol{x}} = \frac{b^2}{R_t^{\boldsymbol{x}}}\,dt - (2+b)\,dt + 2\sqrt{b - R_t^{\boldsymbol{x}}}\,d\beta_t, \tag{24}$$

where β_t is a $\mathcal{F}_t$-adapted 1-dimensional Brownian motion.

Let us now introduce the stopping time

$$S^{\boldsymbol{x}} = \inf\left\{t \ge 0, R_t^{\boldsymbol{x}} \notin (0, b)\right\}.$$

We have, $\mathbb{P}$-a.s., $S^{\boldsymbol{x}} \le T^{\boldsymbol{x}}$. We refer here to I. Karatzas and S.E. Shreve[9] (see Section 5.5 p. 342-351).

We introduce a scale function p such that :

$$\left(\frac{b^2}{r} - (2+b)\right)p'(r) + 2(b-r)p''(r) = 0,$$

which leads to :

$$p'(r) = C(b-r)^{-1}r^{-b/2},$$

where $C > 0$. We have therefore $p(b-) = +\infty$ and $(b < 2 \iff p(0+) > -\infty)$. Using this property of the scale function and the results of I. Karatzas and S.E. Shreve, one can conclude that :

- if $b \ge 2$, then $\mathbb{P}\left(S^{\boldsymbol{x}} = +\infty\right) = \mathbb{P}\left(T^{\boldsymbol{x}} = +\infty\right) = 1,$ (25)
- if $b < 2$, then $\mathbb{P}\left(\lim_{t\to S^{\boldsymbol{x}}}|\boldsymbol{X}_t^{\boldsymbol{x}}|^2 = b\right) = 1.$

In case $b < 2$, we can deduce from the second item that $S^{\boldsymbol{x}} = T^{\boldsymbol{x}}$. We now want to know whether $S^{\boldsymbol{x}} = +\infty$ or not in this case. Let us introduce the speed measure m on $(0, b)$ defined by

$$m(dr) = \frac{2\,dr}{4(b-r)p'(r)} = \frac{r^{b/2}dr}{2C},$$

and the function v such that, for any $r \in (0, b)$,

$$v(r) = \int_a^r (p(r) - p(s))m(ds) = \int_a^r (p(r) - p(s))\frac{s^{b/2}}{2C}\,ds.$$

We have $p(b-) = +\infty$ and therefore $v(b-) = +\infty$. In case $b < 2$, it is easy to check that $v(0+) < \infty$. Using again the results of I. Karatzas and S.E. Shreve, we can deduce from this that in case $b < 2$, we have

$$\mathbb{P}(S^{\boldsymbol{x}} < \infty) = \mathbb{P}(T^{\boldsymbol{x}} < \infty) = 1. \tag{26}$$

In case $|\boldsymbol{x}| = 0$, the former results (25) and (26) still hold. Indeed, let us suppose that $\boldsymbol{x} = \mathbf{0}$ and let us introduce the stopping time $\tau = \inf\left\{t \geq 0, |\boldsymbol{X}_t^{\mathbf{0}}|^2 \geq \frac{b}{2}\right\}$. Obvisouly, one has :

$$\mathbb{P}\left(\exists t > 0, |\boldsymbol{X}_t^{\mathbf{0}}|^2 = b\right) = \mathbb{P}\left(\exists t > 0, |\boldsymbol{X}_t^{\mathbf{0}}|^2 = b \text{ and } \tau < \infty\right).$$

In case $b \geq 2$, using the strong Markov property of $\boldsymbol{X}_t^{\mathbf{0}}$ (see E. Cépa[4] p. 86), one has :

$$\begin{aligned}\mathbb{P}\left(\exists t > 0, |\boldsymbol{X}_t^{\mathbf{0}}|^2 = b\right) &= \mathbb{P}\left(\exists t > 0, |\boldsymbol{X}_t^{\mathbf{0}}|^2 = b \text{ and } \tau < \infty\right), \\ &= \mathbb{E}\left(1_{\tau<\infty}\mathbb{P}\left(\exists t > 0, |\boldsymbol{X}_t^{\boldsymbol{x}}|^2 = b\right)|_{\boldsymbol{x}=\boldsymbol{X}_\tau}\right), \\ &= 0.\end{aligned}$$

In case $b < 2$, we use the fact that, due to the proof of (23), $\mathbb{P}(|\boldsymbol{X}_{1\wedge\tau}^{\mathbf{0}}| = 0) = 0$. By the strong Markov property and since $\mathbb{P}$-a.s., $\sup_{t\in[0,1\wedge\tau]}|\boldsymbol{X}_t^{\mathbf{0}}|^2 < b$, we have $\mathbb{P}\left(\exists t > 0, |\boldsymbol{X}_t^{\mathbf{0}}|^2 = b\right) = \mathbb{E}\left(\mathbb{P}\left(\exists t > 0, |\boldsymbol{X}_t^{\boldsymbol{x}}|^2 = b\right)|_{\boldsymbol{x}=\boldsymbol{X}_{1\wedge\tau}}\right) = 1$.

In case of a non-deterministic initial condition $\boldsymbol{X}_0$ with law μ_0, we can deduce the properties of Proposition 3.1 from the fact that (by uniqueness of the solution) :

$$\mathbb{P}\left(\exists t > 0, |\boldsymbol{X}_t|^2 = b\right) = \int \mathbb{P}\left(\exists t > 0, |\boldsymbol{X}_t^{\boldsymbol{x}}|^2 = b\right)\,d\mu_0(\boldsymbol{x}). \qquad \square$$

Remark 3.1. In case $g \in L^2_{\text{loc}}(\mathbb{R}_+)$, what we can conclude is the following :

- if $b \geq 2$, then $\mathbb{P}\left(\exists t > 0, |\boldsymbol{X}_t^g|^2 = b\right) = 0$,
- if $b < 2$, then $\mathbb{P}\left(\exists t > 0, |\boldsymbol{X}_t^g|^2 = b\right) > 0$.

3.2. *Non-uniqueness in case $b < 2$*

In this section, we suppose $b < 2$ and $\mathbb{P}(|\boldsymbol{X}_0|^2 < b) = 1$. We restrict our attention to the case $g = 0$. We are going to construct another process $\tilde{\boldsymbol{X}}_t$ weak solution to (6) and such that $\mathbb{P}(\exists t > 0, \tilde{\boldsymbol{X}}_t \notin \overline{B}) = 1$. In other words, we will build a solution to (6) which, unlike $\boldsymbol{X}_t$, goes out of the ball $\overline{B}$. This will show that (6) admits at least two different solutions.

Let us consider the solution $\boldsymbol{X}_t$ to (6) we have built in Section 2. We know that $\mathbb{P}$-a.s., the process $\boldsymbol{X}_t$ reaches the boundary of B in finite time (see Proposition 3.1). Let us introduce the stopping time $T = \inf\{t \geq 0, |\boldsymbol{X}_t|^2 \geq b\}$. In polar coordinate, we write $\boldsymbol{X}_T = (\sqrt{b}, \theta_0) : (X_T, Y_T) = (\sqrt{b}\cos(\theta_0), \sqrt{b}\sin(\theta_0))$, where $\theta_0 \in [0, 2\pi)$ denotes the polar angle. We now want to construct a solution to (6), which takes (X_T, Y_T) as initial value, and lives outside of the ball $\overline{B}$. Let us introduce a two-dimensional standard Brownian motion (β_t, γ_t) independent of $\boldsymbol{W}_t$. We use a polar representation $(\sqrt{r_t}, \theta_t)$ of the process we want to build. We consider the solution r_t to the following multivalued SDE :

$$\begin{cases} dr_t + \partial f(r_t)\, dt \ni (2+b)\, dt + 2\sqrt{r_t} d\beta_t, \\ r_0 = b, \end{cases} \tag{27}$$

where $f : \mathbb{R} \to]-\infty, +\infty]$ is the convex function defined by :

$$f(r) = \begin{cases} -b^2 \ln(r - b) & \text{if } r > b, \\ +\infty & \text{otherwise.} \end{cases} \tag{28}$$

so that ∂f is a simple-valued maximal monotone operator with domain $I = (b, \infty)$ (for all $r > b$, $\partial f(r) = \{\nabla f(r)\} = \{\frac{b^2}{b-r}\}$). By E. Cépa[4], there exists a unique process r_t solution to (27). Following exactly the arguments of Lemma 2.1, one can show that this process r_t is such that :

- for any $0 < t < \infty$, $\mathbb{P}$-a.s., $\displaystyle\int_0^t \left| \frac{1}{r_u - b} \right| du < \infty$, with convention $\frac{1}{b-b} = +\infty$,
- $dr_t = -\frac{b^2}{b - r_t}\, dt + (2+b)\, dt + 2\sqrt{r_t} d\beta_t$.

Let us now consider the process θ_t defined by :

$$\theta_t = \theta_0 + \int_0^t \frac{1}{\sqrt{r_s}} d\gamma_s, \tag{29}$$

and the random process $\bar{\boldsymbol{X}}_t$ in $\mathbb{R}^2$ defined by :

$$\bar{\boldsymbol{X}}_t = (\sqrt{r_t}\, \cos(\theta_t), \sqrt{r_t}\, \sin(\theta_t)) \,.$$

By Itô's formula, we have :

$$d\bar{\boldsymbol{X}}_t = -\frac{1}{2}\frac{\bar{\boldsymbol{X}}_t}{1-\frac{|\bar{\boldsymbol{X}}_t|^2}{b}}\,dt + (-\sin(\theta_t),\cos(\theta_t))d\gamma_t + (\cos(\theta_t),\sin(\theta_t))d\beta_t.$$

Using Paul Lévy characterisation, one can show that $(-\sin(\theta_t),\cos(\theta_t))d\gamma_t + (\cos(\theta_t),\sin(\theta_t))d\beta_t = d\boldsymbol{B}_t$ where $\boldsymbol{B}_t$ is a two-dimensional Brownian motion, independent of $\boldsymbol{W}_t$.

Let us now consider $\tilde{\boldsymbol{X}}_t$ defined by $\tilde{\boldsymbol{X}}_t = 1_{0\le t\le T}\boldsymbol{X}_t + 1_{t>T}\bar{\boldsymbol{X}}_{t-T}$ and the process $\tilde{\boldsymbol{W}}_t$ defined by $\tilde{\boldsymbol{W}}_t = \boldsymbol{W}_{t\wedge T} + 1_{t>T}\boldsymbol{B}_{t-T}$. It is obvious (for example by Paul Lévy characterisation) that $\tilde{\boldsymbol{W}}_t$ is a Brownian motion. In addition, the process $\tilde{\boldsymbol{X}}_t$ is a solution to (6) with $g=0$, such that $\mathbb{P}(\exists t>0, \tilde{\boldsymbol{X}}_t \notin \overline{B}) = 1$. This shows that the problem (6) with $g=0$ does not admit a unique solution.

Remark 3.2. In case $g \in L^{\infty}_{\text{loc}}(\mathbb{R}_+)$, using the solution (r_t,θ_t) of the multivalued SDE : $(r_0,\theta_0) = (b,\theta_0)$ and

$$\begin{aligned} &d(r_t,\theta_t) + \partial h(r_t,\theta_t)\,dt \ni \\ &\qquad \big((2+b) + r_t\sin(\theta_t)g(t), -\sin^2(\theta_t)g(t)\big)\;dt + (2\sqrt{r_t}, \tfrac{1}{\sqrt{r_t}})d(\beta_t,\gamma_t), \end{aligned}$$

where $h:\mathbb{R}^2 \to]-\infty,+\infty]$ is the convex function defined by $h(r,\theta) = f(r)$ (see formula (28)), one can by the same arguments prove that there is non-uniqueness in law for the solutions to (6).

We have summarized in Table 1 some of the results we have obtained in the last two sections.

	$b<2.$	$b\ge 2.$
$\mathbb{P}(\lvert\boldsymbol{X}_0\rvert^2 = b) = 0.$	Existence. $\mathbb{P}\left(\exists t\ge 0, \lvert\boldsymbol{X}_t\rvert^2 = b\right) = 1.$ Non-uniqueness.	Existence. $\mathbb{P}\left(\exists t\ge 0, \lvert\boldsymbol{X}_t\rvert^2 = b\right) = 0.$ Uniqueness.
$\mathbb{P}(\lvert\boldsymbol{X}_0\rvert^2 = b) > 0.$	Existence. Non-uniqueness.	Existence. Non-uniqueness

Table 1.: Properties of solutions to (6) when $g=0$. We suppose $\mathbb{P}(|\boldsymbol{X}_0|^2 \le b) = 1$. In any case, uniqueness holds for solutions with values in $\overline{B}$ according to Proposition 2.2. The terminology uniqueness and non uniqueness relates to a solution that is not enforced to take values in $\overline{B}$.

4. Invariant probability measure in case $g = 0$ and $b \geq 2$

In this section we are interested in invariant probability measures for the SDE (6) with $g = 0$ in case $b \geq 2$.

The motivation for this study is twofold. First, since we consider a fluid which is initially at rest, it is natural from a physical point of view to choose an invariant probability for the SDE (6) with $g = 0$ as law for $\boldsymbol{X}_0$. Second, in the analysis of the coupled system (1-3), we are interested in the regularity of the stress $\tau(t, y) = \mathbb{E}\left(\frac{X_t^y Y_t^y}{1 - \frac{(X_t^y)^2 + (Y_t^y)^2}{b}}\right)$ which, by Girsanov, can also be written in the following form :

$$\mathbb{E}\left(\left(\frac{X_t Y_t}{1 - \frac{X_t^2 + Y_t^2}{b}}\right) \exp\left(\int_0^t \partial_y u(s, y) Y_s \, dV_s - \frac{1}{2}\int_0^t (\partial_y u(s, y) Y_s)^2 \, ds\right)\right),$$

where $\boldsymbol{X}_t = (X_t, Y_t)$ denotes (as in last section) the solution with values in $\overline{B}$ to (6) with $g = 0$ (see Ref. [3]). This expression of the stress yields the following estimate (using Hölder inequality) : for almost all y and t,

$$|\tau(y, t)| \leq \mathbb{E}\left(\left(\frac{X_t Y_t}{1 - \frac{X_t^2 + Y_t^2}{b}}\right)^p\right)^{1/p} \exp\left(\frac{q-1}{2} b \int_0^t (\partial_y u(s, y))^2 \, ds\right),$$

where $p = \frac{q}{q-1}$.

It is thus important to estimate the quantities $\mathbb{E}\left(\left(\frac{X_t Y_t}{1 - \frac{X_t^2 + Y_t^2}{b}}\right)^p\right)$, which is simple if we identify and start under an invariant probability measure (see formula (31)).

The density p_0 defined by :

$$p_0(\boldsymbol{x}) = \frac{\exp(-2\Pi(\boldsymbol{x}))}{\int \exp(-2\Pi(\boldsymbol{x}))\, d\boldsymbol{x}} = \frac{b+2}{2\pi b}\left(1 - \frac{|\boldsymbol{x}|^2}{b}\right)^{b/2} 1_{|\boldsymbol{x}|^2 < b} \tag{30}$$

obviously solves $\operatorname{div}_{\boldsymbol{x}}\left(-(\nabla_{\boldsymbol{x}}\Pi) p_0 + \frac{1}{2}(\nabla_{\boldsymbol{x}} p_0)\right) = 0$ and is therefore a natural candidate to be invariant. This is indeed the case as shown by :

Proposition 4.1. *For $b \geq 2$, $p_0(\boldsymbol{x})\, d\boldsymbol{x}$ is the unique invariant probability measure on B for the SDE (6) with $g = 0$.*

This proposition is a consequence of the following lemma :

Lemma 4.1. *Let $b \geq 2$. For any $\boldsymbol{x} \in B$, $t > 0$, the solution $\boldsymbol{X}_t^{\boldsymbol{x}}$ of the SDE (6) with $g = 0$ and $\boldsymbol{X}_0 = \boldsymbol{x}$ has a density $p(t, \boldsymbol{x}, \boldsymbol{y})$ with respect to the Lebesgue measure on B. In addition, $\forall t \geq 0$,*

(i) $d\boldsymbol{x}\, d\boldsymbol{y}$*-a.e.,* $\exp(-2\Pi(\boldsymbol{x})) p(t, \boldsymbol{x}, \boldsymbol{y}) = \exp(-2\Pi(\boldsymbol{y})) p(t, \boldsymbol{y}, \boldsymbol{x})$,

(ii) $\forall \boldsymbol{x} \in B$, $d\boldsymbol{y}$-a.e., $p(t, \boldsymbol{x}, \boldsymbol{y}) > 0$.

Indeed, by (i), one easily checks that $p_0(\boldsymbol{x})\, d\boldsymbol{x}$ is invariant. By (ii), any invariant probability measure is equivalent to the Lebesgue measure on B which implies uniqueness (see Proposition 6.1.9 p. 188 of M. Duflo [7]).

With Proposition 4.1, it is then straightforward to prove that, if $\boldsymbol{X}_0$ has the density $p_0(\boldsymbol{x})$, then we have :

$$\mathbb{E}\left(\left(\frac{X_t Y_t}{1 - \frac{X_t^2 + Y_t^2}{b}}\right)^p\right) < \infty \iff b > 2(p-1). \tag{31}$$

Let us now prove Lemma 4.1.

Proof. In order to prove (i), we regularize the potential Π so that the results of L.C.G. Rogers [12] (see p. 161) apply. Let Π_n be defined by :

$$\Pi_n(\boldsymbol{x}) = \pi_n(|\boldsymbol{x}|^2), \tag{32}$$

$$\pi_n(r) = \begin{cases} -\dfrac{b}{4} \ln\left(1 - \dfrac{r}{b}\right) & \text{if } r \leq b\left(1 - \frac{1}{n}\right), \\ \sqrt{r} + \frac{b}{4}\ln(n) & \text{if } r \geq b\ , \end{cases} \tag{33}$$

and π_n is increasing and $\mathcal{C}^2(\mathbb{R}_+, \mathbb{R}_+)$, so that $\nabla \Pi_n$ is bounded with continuous derivatives of first order. Let $t > 0$ and $\boldsymbol{x} \in \mathbb{R}^2$. According to L.C.G. Rogers, the solution $\boldsymbol{X}^{n,\boldsymbol{x}}$ of the SDE :

$$\boldsymbol{X}_t^{n,\boldsymbol{x}} = \boldsymbol{x} - \int_0^t \nabla \Pi_n(\boldsymbol{X}_s^{n,\boldsymbol{x}})\, ds + \boldsymbol{W}_t, \tag{34}$$

has a density $p_n(t, \boldsymbol{x}, \boldsymbol{y})$ with respect to the Lebesgue measure on $\mathbb{R}^2$ which satisfies $d\boldsymbol{x}\, d\boldsymbol{y}$-a.e., $\exp(-2\Pi_n(\boldsymbol{x})) p_n(t, \boldsymbol{x}, \boldsymbol{y}) = \exp(-2\Pi_n(\boldsymbol{y})) p_n(t, \boldsymbol{y}, \boldsymbol{x})$. For $\boldsymbol{x} \in B$, let $\tau_n^{\boldsymbol{x}} = \inf\left\{t \geq 0, |\boldsymbol{X}_t^{\boldsymbol{x}}|^2 \geq b\left(1 - \frac{1}{n}\right)\right\}$. Since $\mathbb{P}\left(\boldsymbol{X}_t^{n,\boldsymbol{x}} \neq \boldsymbol{X}_t^{\boldsymbol{x}}\right) \leq \mathbb{P}(\tau_n^{\boldsymbol{x}} < t)$, according to Proposition 3.1,

$$\lim_{n \to \infty} \mathbb{P}\left(\boldsymbol{X}_t^{n,\boldsymbol{x}} \neq \boldsymbol{X}_t^{\boldsymbol{x}}\right) = 0. \tag{35}$$

We deduce that for a fixed $\boldsymbol{x} \in B$, $p_n(t, \boldsymbol{x}, \boldsymbol{y})$ converges in $L^1_{\boldsymbol{y}}(\mathbb{R}^2)$ to $p(t, \boldsymbol{x}, \boldsymbol{y})$, which is the density of $\boldsymbol{X}_t^{\boldsymbol{x}}$.

As the non-negative potential Π_n converges pointwise to Π in B, we deduce that $\exp(-2\Pi_n(\boldsymbol{x})) p_n(t, \boldsymbol{x}, \boldsymbol{y})$ converges to $\exp(-2\Pi(\boldsymbol{x})) p(t, \boldsymbol{x}, \boldsymbol{y})$ in $L^1_{\boldsymbol{x},\boldsymbol{y}}(B \times B)$ and conclude that (i) holds.

We are now going to check (ii) for a fixed $\boldsymbol{x} \in B$ and $t > 0$. Let A be a Borel subset of B such that $\int 1_A\, d\boldsymbol{x} > 0$. We choose $n \in \mathbb{N}^*$ such that

$|\boldsymbol{x}|^2 < b\left(1-\frac{1}{n}\right)$ and $\int 1_{A_n}\,d\boldsymbol{x} > 0$ where $A_n = A \cap B\left(0, \sqrt{b\left(1-\frac{1}{n}\right)}\right)$. By Girsanov Theorem, under $\mathbb{P}_n^{\boldsymbol{x}}$ defined by :

$$\frac{d\mathbb{P}_n^{\boldsymbol{x}}}{d\mathbb{P}} = \exp\left(\int_0^t \nabla\Pi(\boldsymbol{X}_{s\wedge\tau_n^{\boldsymbol{x}}}^{\boldsymbol{x}})\,.d\boldsymbol{W}_s - \frac{1}{2}\int_0^t \left|\nabla\Pi(\boldsymbol{X}_{s\wedge\tau_n^{\boldsymbol{x}}}^{\boldsymbol{x}})\right|^2 ds\right),$$

where $\tau_n^{\boldsymbol{x}}$ is as above, $(\boldsymbol{X}_{s\wedge\tau_n^{\boldsymbol{x}}}^{\boldsymbol{x}})_{s\le t}$ is a Brownian motion starting from $\boldsymbol{x}$ and stopped at the boundary of $B\left(0, \sqrt{b\left(1-\frac{1}{n}\right)}\right)$ so that $\mathbb{P}_n^{\boldsymbol{x}}\left(\boldsymbol{X}_{t\wedge\tau_n^{\boldsymbol{x}}}^{\boldsymbol{x}} \in A_n\right) > 0$. Therefore, $\mathbb{P}\left(\boldsymbol{X}_t^{\boldsymbol{x}} \in A\right) \ge \mathbb{P}\left(\boldsymbol{X}_{t\wedge\tau_n^{\boldsymbol{x}}}^{\boldsymbol{x}} \in A_n\right) = \mathbb{E}_n^{\boldsymbol{x}}\left(1_{A_n}\left(\boldsymbol{X}_{t\wedge\tau_n^{\boldsymbol{x}}}^{\boldsymbol{x}}\right)\frac{d\mathbb{P}}{d\mathbb{P}_n^{\boldsymbol{x}}}\right) > 0$, which concludes the proof. $\square$

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