SUPER BROWNIAN MOTION WITH INTERACTION

JEAN-FRANÇOIS DELMAS AND JEAN-STÉPHANE DHERSIN

Abstract. Using an approximating scheme with the Brownian snake, we prove the existence of solution to a martingale problem for super Brownian motion with interaction.

1. Introduction

Let $\mathcal{B}(\mathbb{R}^d)$ be the set of real valued measurable functions defined on $\mathbb{R}^d$. Let $\mathcal{M}_f$ be the set of finite measures on $\mathbb{R}^d$, endowed with the topology of the weak convergence. For $\mu \in \mathcal{M}_f$ and $\varphi \in \mathcal{B}(\mathbb{R}^d)$, bounded, we denote $(\mu, \varphi) = \int \varphi(x)\, \mu(dx)$.

Consider a super Brownian motion $X = (X_t, t \geq 0)$ started at $X_0 = \mu_0 \in \mathcal{M}_f$. It is the unique solution of the martingale problem on $\mathcal{M}_f$: for any bounded C^2 function, φ, with bounded derivatives,

$$X_0 = \mu_0$$

$$(X_t, \varphi) = (X_0, \varphi) + \int_0^t (X_s, \frac{\Delta}{2}\varphi)\, ds + M(\varphi)_t,$$

where $M(\varphi)$ is a continuous martingale (with respect to the filtration generated by X) with quadratic variation

$$\langle M(\varphi) \rangle_t = 4 \int_0^t (X_s, \varphi^2)\, ds.$$

Let ϕ be a C^1 function defined on $\mathbb{R}^+$, s.t. $\phi(0) = 0$ and $\phi'(t) > 0$ for all $t \geq 0$. If we consider the changed time process $Y_t = X_{\phi(t)}$, for $t \geq 0$, then it is easy to check that $Y = (Y_t, t \geq 0)$ is a solution to the martingale problem on $\mathcal{M}_f$: for any bounded C^2 function, φ, with bounded derivatives,

$$Y_0 = \mu_0$$

$$(Y_t, \varphi) = (Y_0, \varphi) + \int_0^t (Y_s, \phi'(s)\frac{\Delta}{2}\varphi)\, ds + M(\varphi)_t,$$

where $M(\varphi)$ is a continuous martingale with quadratic variation

$$\langle M(\varphi) \rangle_t = 4 \int_0^t (Y_s, \phi'(s)\varphi^2)\, ds.$$

By using the inverse of the time change ϕ, in order to recover X from Y, it is clear that the solution of this martingale problem is unique.

The aim of this paper is to use a random time change procedure to transform the martingale problem (see [3] chapter 6). We prove in Theorem 1 the existence of solutions

Date: September 24, 2002.

to the following martingale problem (MP) on $\mathcal{M}_f$: for any bounded C^2 function, φ, with bounded derivatives,

$$Y_0 = \mu_0$$
$$(Y_t, \varphi) = (Y_0, \varphi) + \int_0^t (Y_s, \theta(Y_s) A(Y_s)\varphi)\, ds + M(\varphi)_t,$$

where $A(\mu)$ is the infinitesimal generator of a diffusion, with diffusion coefficient $\sigma(\mu, x)$ and drift $b(\mu, x)$, and $M(\varphi)$ is a continuous martingale with quadratic variation

$$\langle M(\varphi)\rangle_t = 4 \int_0^t (Y_s, \theta(Y_s)\varphi^2)\, ds.$$

The functions θ, b and σ are bounded continuous functions defined on $\mathcal{M}_f \times \mathbb{R}^d$ taking values respectively in $\mathbb{R}, \mathbb{R}^d$ and $\mathbb{R}^{d\times d}$. And the functions θ and σ are positive.

The existence of solutions will be proved by using approximating schemes of the martingale problem. In fact we prove the tightness of the approximating scheme and that the limit points are solution to the above martingale problem. Except in very particular case, we were unable to prove the uniqueness of the limit, as well as the uniqueness of the solutions to the martingale problem (see [8] p.7 on this last question).

Our approach relies on the Brownian snake representation of the super Brownian motion.

Roughly speaking, the super Brownian motion, X_t, can be described as the integral with respect to the local time of the snake lifetime process at level t of terminal points of underlying paths. Now, using a random time change for each path, we integrate the terminal points of diffusions with respect to the local time of the snake lifetime process along a random curve instead of a deterministic line. This procedure modifies the underlying branching tree of the life time process.

The approach differs from Perkins [8, 7], where the interactions was introduced through a stochastic integral along the paths as well as a different weighting for each path. In particular, the structure of the underlying branching tree (that is the lifetime process for the Brownian snake representation) was the same for the superprocess and the interacting superprocess.

On the other side, Dhersin and Serlet [1], see also Watanabe [11], introduced a change in the underlying branching tree of the Brownian snake through a killing rate which depends on the path of the particle. This approach was a first step to introduce interaction in the underlying branching tree.

2. The approximating scheme

2.1. **The Brownian snake.** We first recall the Brownian snake representation of the super Brownian motion.

Let $C = C(\mathbb{R}^+, \mathbb{R}^d)$ be the set of continuous function defined on $\mathbb{R}^+$ with values in $\mathbb{R}^d$. We shall denote by $\mathbb{N}_{x,A}[dW]$ the excursion measure on C of the Brownian snake $W = (W_s, s \geq 0)$ started at $x \in \mathbb{R}^d$ with underlying process a diffusion with infinitesimal generator A. We refer to [5] for the definition and properties of the Brownian snake. We recall that under $\mathbb{N}_{x,A}$, the law of the lifetime process $\zeta = (\zeta_s, s \geq 0)$ is the law of a positive excursion of linear Brownian motion. We take the normalization $\mathbb{N}_{x,A}[\sup_{s\geq 0} \zeta_s > \varepsilon] = \frac{1}{2\varepsilon}$.

Under $\mathbb{N}_{x,A}$, conditionally on the lifetime process, W is a continuous C-valued Markov process started at the constant path equal to $x \in \mathbb{R}^d$. Conditionally, on the lifetime process and on $(W_u, u \in [0,s])$, the law of $W_{s'}$, with $s' \geq s$ is as follow: the two paths W_s and $W_{s'}$ coincide up to time $m = \inf_{u\in[s,s']} \zeta_u$, and $(W_{s'}(t+m), t \geq 0)$ is a diffusion with infinitesimal generator A, constant after $\zeta_{s'} - m$, which depends of $(W_u, u \in [0,s])$ only through its starting point $W_{s'}(m) = W_s(m)$.

Starting from the Brownian snake W, we can construct a continuous measure valued process $(X_s(W), s \geq 0)$ defined by $X_t(W) = \displaystyle\int_{s\geq 0} \delta_{W_s(\zeta_s)} dL_s^t$, where δ_y is the Dirac mass at point y, and L_s^t is the local time of the lifetime process at level t up to time s. Notice the integration over s is up to time σ, which is the duration of the excursion ζ under $\mathbb{N}_{x,A}$.

Let $x \in \mathbb{R}^d$ and $\mu_0 \in \mathcal{M}_f$. And consider the Poisson point measure on C, $\sum_{i\in I} \delta_{W^i}$, with intensity measure $\displaystyle\int \mu_0(dx)\mathbb{N}_{x,A}[dW]$. It is well known that the process $X = (X_t, t \geq 0)$ defined by $X_0 = \mu_0$, and

$$X_t = \sum_{i\in I} X_t(W^i) \tag{1}$$

is the usual superdiffusion started at μ_0 with underlying process a diffusion with infinitesimal generator A and branching mechanism $\psi(z) = 2z^2$.

We intend to replace the local time of the lifetime process at level t, by the local time along a random curve $\phi = (\phi_s^i, s \geq 0, i \in I)$, where $\phi_s^i \in [0,\infty]$. This curve ϕ needs to have particular properties (see also [10] for the definition of the local time along a random curve). This was already done for $\phi_s^i = \phi(W_s^i)$ defined as the first exit time of a domain D. The random measure associated to this curve is the so called exit measure of D (see [5]). From this example we expect the following "tree property" to be in force:

If $\zeta_s^i > \phi_s^i$, where ζ_s^i is the lifetime of W_s^i, then for all s' such that $\inf_{u\in[s,s']} \zeta_u^i > \phi_s^i$, we have $\phi_{s'}^i = \phi_s^i$.

It is a natural condition when one deals with the excursion filtration (see also the definition of identifiable curve in [10]). Furthermore, in order to get the so called special Markov property, we need that conditionally on what is "below" the curve ϕ, the excursions of the snake above the curve ϕ are distributed according to $\int X^\phi(dx)\mathbb{N}_{x,A}[dW]$, where X^ϕ is the exit measure of the superdiffusion above level ϕ. This will be stated precisely in property (B).

Eventually we will define for each t a random curve $\phi(t)$ and the corresponding exit measure X_t^ϕ in such a way that X_t^ϕ solves the martingale problem (MP).

The random time change will formally be given by the following equations:

- Stochastic differential equation and time change for the path W_s^i of the Brownian snake W^i:

$$d_t V_s^i(t) = \sigma(Y_t, V_s^i(t)) d_t W_s^i(\phi_s^i(t)) + b(Y_t, V_s^i(t)) d_t \phi_s^i(t). \tag{2}$$

- Differential equation for the time change at time t:

$$d_t \phi_s^i(t) = \theta(Y_t, V_s^i(t)) dt \quad \text{for} \quad \phi_s^i(t) \leq \zeta_s^i. \tag{3}$$

- Definition of the random measure Y_t:

$$Y_t = \sum_{i\in I} \int_0^\infty \delta_{V_s^i(t)} \; d_s L_s^{\phi_s^i(t),i}, \tag{4}$$

where $L^{\phi_\cdot,i}_\cdot$ is the "local time" of the lifetime of process of W^i on the random curve ϕ.

Notice that in general, the function $s \mapsto \phi_s^i(t)$ is not adapted to the filtration $(\mathcal{F}_s^i = \sigma(W_u^i, u \le s), s \ge 0)$ generated by the snake, because the measure Y_t take into account path $W_{s'}^i$ for $s' \ge s$.

We will present a discrete version of those equations and prove that X^ε, the discrete versions of Y, are tight and that any limit is solution to the martingale problem (MP).

We are now ready to present our approximating scheme.

2.2. **The approximating scheme.** Let θ, b and σ be bounded continuous functions defined on $\mathcal{M}_f \times \mathbb{R}^d$ taking values respectively in $\mathbb{R}, \mathbb{R}^d$ and $\mathbb{R}^{d\times d}$. We also assume the functions θ and σ are positive. Let $\mu' \in \mathcal{M}_f$ and $x' \in \mathbb{R}^d$. We will denote by $A(\mu', x')$ the infinitesimal generator of the d-dimensional Brownian motion with constant drift $b(\mu', x')$ and constant diffusion coefficient $\sigma(\mu', x')$.

Let $\mu_0 \in \mathcal{M}_f$. We consider the Poisson point measure on $\mathbb{R}^d \times C$, $\sum_{i\in I} \delta_{(x^i, W^i)}$, with intensity measure $\mu_0(dx)\mathbb{N}_{0,\frac{\Delta}{2}}[dW]$. For $i \in I$, let σ_i be the duration of the lifetime process of the snake W^i. Recall W_s^i is the path at time s of the snake W^i.

Let $\varepsilon > 0$. We define by induction at time $k\varepsilon$ with $k \in \mathbb{N}$, the random time change $\phi^\varepsilon = ((\phi_s^{i,\varepsilon}(k\varepsilon), s \ge 0, i \in I), k \in \mathbb{N})$, the starting point $V = ((V_s^i(k\varepsilon), s \ge 0, i \in I), k \in \mathbb{N})$, the random measure $X^\varepsilon = (X_{k\varepsilon}^\varepsilon, k \in \mathbb{N})$ and the filtration $\mathcal{G}^\varepsilon = (\mathcal{G}_{k\varepsilon}^\varepsilon, k \in \mathbb{N})$ such that the following hypothesis (A) and (B) are in force.

(A) *$\phi^{i,\varepsilon}(k\varepsilon)$ enjoys the "tree property". And for all $i \in I$, the sets $\{s \in [0, \sigma^i]\ ;\ \phi_s^{i,\varepsilon}(k\varepsilon) < \zeta_s^i\}$ are open.*

The sets $\{s \in [0, \sigma^i]\ ;\ \phi_s^{i,\varepsilon}(k\varepsilon) < \zeta_s^i\}$ can be described as the union of the open non overlapping intervals (a^{i,j_k}, b^{i,j_k}) for $j_k \in J_k^i$, where the set J_k^i is possibly empty. We assume the family of indices J_l^i are non overlapping for $i \in I$, $0 \le l \le k$. Notice that from property (A), $\phi^{i,\varepsilon}(k\varepsilon)$ is constant over each interval (a^{i,j_k}, b^{i,j_k}). For $i \in I$, $j_k \in J_k$, $s \in [a^{i,j_k}, b^{i,j_k}]$, we consider the increments of the paths of the Brownian snake after time $\phi^{i,\varepsilon}(k\varepsilon)$:

$$\bar{W}_s^{i,j_k}(u) = W_s^i(u + \phi_s^{i,\varepsilon}(k\varepsilon)) - W_s^i(\phi_s^{i,\varepsilon}(k\varepsilon)). \tag{5}$$

And for $i \in I$, $j_k \in J_k^i$, we define the snake excursions $\bar{W}^{i,j_k} = (\bar{W}_s^{i,j_k}, s \in [a^{i,j_k}, b^{i,j_k}])$.

Let $\kappa_k^i(s) = \inf\{r \ge 0; \int_0^r du\ \mathbf{1}_{\{\phi_u^{i,\varepsilon}(k\varepsilon) \ge \zeta_u^i\}} > s\}$, the inverse of the time spent under $\phi^{i,\varepsilon}(k\varepsilon)$ by the life time of the snake W^i. We define the snake $W^{i;\kappa_k^i} = (W_{\kappa_k^i(s)}^i, s \ge 0)$ and the σ-field $\mathcal{G}_{k\varepsilon}^\varepsilon$ generated by $(W^{i;\kappa_k^i}, i \in I)$. Roughly speaking, $\mathcal{G}_{k\varepsilon}^\varepsilon$ represent all the information available on the Brownian snake up to level $\phi^{i,\varepsilon}(k\varepsilon)$.

(B) *The random measure $X_{k\varepsilon}^\varepsilon$ is $\mathcal{G}_{k\varepsilon}^\varepsilon$-measurable. The function $V_\cdot^i(k\varepsilon)$ is constant over each excursion interval (a^{i,j_k}, b^{i,j_k}), and let $V^{i,j_k}(k\varepsilon)$ be its value. Conditionally* *on* $\mathcal{G}^\varepsilon_{k\varepsilon}$*, the measure* $\displaystyle\sum_{i\in I, j_k\in J^i_k} \delta_{(V^{i,j_k},\bar W^{i,j_k})}$ *is a Poisson point measure with intensity* $X^\varepsilon_{k\varepsilon}(dx)\,\mathbb{N}_{0,\frac{\Delta}{2}}[dW]$.

For $k=0$, we set for $i\in I$, $s\in[0,\sigma_i]$:

- the time change: $\phi^{i,\varepsilon}_s(0)=0$,
- the measure: $X^\varepsilon_0=\mu_0$,
- the starting point: $V^i_s(0)=x^i$,
- the tribe: $\mathcal{G}^\varepsilon_0=\sigma(X^\varepsilon_0, V^i_s(0); s\ge 0, i\in I)$,
- the excursion intervals: $(a^{i,j_0}, b^{i,j_0})=(0,\sigma^i)$, where $j_0\in J^i_0=\{i\}$,
- the transformed snake above level $\phi^{i,\varepsilon}_s(0)$: for $i\in I$, $j_0\in J^i_0$, we set $U^{i,j_0}=(U^{i,j_0}_s, s\in [a^{i,j_0}, b^{i,j_0}])$, where for $s\in[a^{i,j_0}, b^{i,j_0}]$, $u\ge 0$,

$$U^{i,j_0}_s(u)=V^i_s(0)+\sigma(X^\varepsilon_0, V^i_s(0))W^i_s(u)+b(X^\varepsilon_0, V^i_s(0))u.$$

Notice that conditionally on $\mathcal{G}^\varepsilon_0$, $\sum_{i\in I, j_0\in J^i_0}\delta_{U^{i,j_0}}$ is a Poisson point measure with intensity $\int \mu_0(dx)\,\mathbb{N}_{x,A(\mu_0,x)}[dW]$, with $\mu_0=X^\varepsilon_0$. Notice also that properties (A) and (B) are in force for $k=0$.

Let $k\ge 0$. Assume $\phi^{i,\varepsilon}(k\varepsilon)$, $X^\phi_{k\varepsilon}$, $V^i(k\varepsilon)$ are built in such a way that properties (A) and (B) are satisfied. Let us now built $\phi^{i,\varepsilon}((k+1)\varepsilon)$, $X^\phi_{(k+1)\varepsilon}$, $V^i((k+1)\varepsilon)$ and check the properties (A) and (B) are in force if k is replaced by $k+1$. We set for $s\in[a^{i,j_k}, b^{i,j_k}], i\in I, j_k\in J^i_k$,

$$\phi^{i,\varepsilon}_s((k+1)\varepsilon)=\phi^{i,\varepsilon}_s(k\varepsilon)+\theta(X^\varepsilon_{k\varepsilon}, V^i_s(k\varepsilon))\,\varepsilon,$$

and $\phi^{i,\varepsilon}_s((k+1)\varepsilon)=+\infty$ if $s\notin[a^{i,j_k}, b^{i,j_k}]$ for any $i\in I, j_k\in J^i_k$. This equation is the discrete version of (3).

Notice that on each interval $[a^{i,j_k}, b^{i,j_k}]$, $\phi^{i,\varepsilon}((k+1)\varepsilon)$ is constant, and that outside those intervals $\phi^{i,\varepsilon}_s((k+1)\varepsilon)=+\infty>\zeta^i_s$. Therefore property (A) is true for k replaced by $k+1$.

We then describe the transformed snake U^{i,j_k} and built the random measure $X_{(k+1)\varepsilon}$. We set for $i\in I$, $j_k\in J_k$, $s\in[a^{i,j_k}, b^{i,j_k}]$, $u\ge 0$,

$$U^{i,j_k}_s(u)=V^i_s(k\varepsilon)+\sigma(X^\varepsilon_{k\varepsilon}, V^i_s(k\varepsilon))\bar W^{i,j_k}_s(u)+b(X^\varepsilon_{k\varepsilon}, V^i_s(k\varepsilon))u. \tag{6}$$

And for $i\in I$, $j\in J^i_k$, we define the snake excursions $U^{i,j_k}=(U^{i,j_k}_s, s\in[a^{i,j_k}, b^{i,j_k}])$. From property (B), we deduce that the measure $\sum_{i\in I, j_k\in J^i_k}\delta_{U^{i,j_k}}$ is a Poisson point measure with intensity $\int \mathbb{N}_{x,A(\mu,x)}[dW]\,\mu(dx)$, where $\mu=X^\varepsilon_{k\varepsilon}$.

Considering the snake excursion U^{i,j_k}, we define using (1) for $u>0$, the random measures

$$\tilde X^{i,j_k}_u=X_u(U^{i,j_k}). \tag{7}$$

For $j_k\in J^i_k$, $(\tilde X^{i,j_k}_u, u\ge 0)$ is distributed, conditionally on $\mathcal{G}^\varepsilon_{k\varepsilon}$, according to $(X_s(W), s\ge 0)$ under $\mathbb{N}_{x,A(\mu,x)}[dW]$, with $x=V^i_{a^{i,j_k}}(k\varepsilon)$ and $\mu=X^\varepsilon_{k\varepsilon}$.

Recall that on each interval $[a^{i,j_k}, b^{i,j_k}]$, $\phi_\cdot^{i,\varepsilon}((k+1)\varepsilon)$, $\phi_\cdot^{i,\varepsilon}(k\varepsilon)$ and $V_\cdot^i(k\varepsilon)$ are constant and $\mathcal{G}_{k\varepsilon}^\varepsilon$-measurable. In particular the random measure

$$X_{(k+1)\varepsilon}^{i,j_k} = \tilde{X}_{u_k}^{i,j_k} = X_{u_k}(U^{i,j_k}), \tag{8}$$

where

$$u_k = \phi_\cdot^{i,\varepsilon}((k+1)\varepsilon) - \phi_\cdot^{i,\varepsilon}(k\varepsilon) = \varepsilon\theta(X_{k\varepsilon}^\varepsilon, V_\cdot^i(k\varepsilon)), \tag{9}$$

is well define.

From the special Markov property of the Brownian snake (see for example [6]), we get that $X_{(k+1)\varepsilon}^{i,j_k}$ is measurable with respect to the σ-field generated by $\mathcal{G}_{k\varepsilon}^\varepsilon$ and by $(\bar{W}_{\kappa(s)}^{i,j_k}, s \geq 0)$, where $\kappa(s) = \inf\{r \in [a^{i,j_k}, b^{i,j_k}]; \int_{a^{i,j_k}}^r du\ \mathbf{1}_{\{\phi_u^{i,\varepsilon}((k+1)\varepsilon)\geq\zeta_u^i\}} > s\}$ (the inverse of the time spent under $\phi^{i,\varepsilon}((k+1)\varepsilon)$ by the snake $\bar{W}^{i,j_k}$). In particular it is measurable with respect to $\mathcal{G}_{(k+1)\varepsilon}^\varepsilon$ which is defined as $\mathcal{G}_{k\varepsilon}^\varepsilon$ with k replaced by $(k+1)$. If we define

$$X_{(k+1)\varepsilon}^\varepsilon = \sum_{i\in I, j_k\in J_k} X_{(k+1)\varepsilon}^{i,j_k}, \tag{10}$$

it is clear that the first sentence of property (B) is true for k replaced by $k+1$. Notice the above definition is the discrete version of (4).

To prove the second part, we have to define the functions $V^i((k+1)\varepsilon)$. Let us consider the excursions of the Brownian snake above level $\phi_\cdot^{\cdot,\varepsilon}((k+1)\varepsilon)$. We focus on the snake $\bar{W}^{i,j_k}$. The set $\{s \in (a^{i,j_k}, b^{i,j_k})\ ;\ \phi_s^{i,\varepsilon}((k+1)\varepsilon) < \zeta_s^i\}$ is open. It is the union of the open non overlapping intervals $(a^{i,j_{k+1}}, b^{i,j_{k+1}})$ for $j_{k+1} \in J_{k+1}^{i,j_k}$, where the set of indices J_{k+1}^{i,j_k} is possibly empty. Recall $\phi_s^{i,\varepsilon}((k+1)\varepsilon)$ and $\phi_s^{i,\varepsilon}(k\varepsilon)$ are constant functions over $(a^{i,j_{k+1}}, b^{i,j_{k+1}})$. Using u_k defined in (9), we define for $s \in [a^{i,j_{k+1}}, b^{i,j_{k+1}}]$, $u \geq 0$,

$$\begin{aligned}\bar{W}_s^{i,j_{k+1}}(u) &= \bar{W}_s^{i,j_k}(u+u_k) - \bar{W}_s^{i,j_k}(u_k)\\ &= W_s^i(u+\phi_s^{i,\varepsilon}((k+1)\varepsilon)) - W_s^i(\phi_s^{i,\varepsilon}((k+1)\varepsilon)).\end{aligned}$$

Define for $j_{k+1} \in J_{k+1}^{i,j_k}$ the snakes $\bar{W}^{i,j_{k+1}} = (\bar{W}_s^{i,j_{k+1}}, s \in [a^{i,j_{k+1}}, b^{i,j_{k+1}}])$. This last formula coincides with definition (5), with k replaced by $k+1$. And we set using (6)

$$\begin{aligned}V_s^i((k+1)\varepsilon) &= U_s^{i,j_k}(u_k)\\ &= V_s^i(k\varepsilon) + \sigma(X_{k\varepsilon}^\varepsilon, V_s^i(k\varepsilon))[W_s^i(\phi_s^{i,\varepsilon}((k+1)\varepsilon)) - W_s^i(\phi_s^{i,\varepsilon}(k\varepsilon))]\\ &\qquad + b(X_{k\varepsilon}^\varepsilon, V_s^i(k\varepsilon))[\phi_s^{i,\varepsilon}((k+1)\varepsilon) - \phi_s^{i,\varepsilon}(k\varepsilon)].\end{aligned}$$

The above definition is the discrete version of (2).

Notice the function $V_\cdot^i((k+1)\varepsilon)$ is constant over each excursion interval $[a^{i,j_{k+1}}, b^{i,j_{k+1}}]$, and let $V^{i,j_{k+1}}((k+1)\varepsilon)$ be its value. This proves the second sentence of property (B), with k replaced by $k+1$.

Again from the special Markov property of the Brownian snake, the random measure $\displaystyle\sum_{j_{k+1}\in J_{k+1}^{i,j_k}} \delta_{(V^{i,j_{k+1}}((k+1)\varepsilon), \bar{W}^{i,j_{k+1}})}$ is, conditionally on the σ-field $\mathcal{G}_{(k+1)\varepsilon}^\varepsilon$, distributed according to a Poisson point measure with intensity $X_{(k+1)\varepsilon}^{i,j_k}(dx)\ \mathbb{N}_{0,\frac{\Delta}{2}}[dW]$.

Let $J^i_{k+1} = \bigcup_{j_k \in J^i_k} J^{i,j_k}_{k+1}$. Since $X^\varepsilon_{(k+1)\varepsilon} = \sum_{i \in I, j_k \in J_k} X^{i,j_k}_{(k+1)\varepsilon}$, we then deduce that the random measure

$$\sum_{i \in I} \sum_{j_{k+1} \in J^i_{k+1}} \delta_{(V^{i,j_{k+1}}((k+1)\varepsilon), \bar{W}^{i,j_{k+1}})}$$

is, conditionally on the σ-field $\mathcal{G}^\varepsilon_{(k+1)\varepsilon}$, distributed according to a Poisson point measure with intensity $X^\varepsilon_{(k+1)\varepsilon}(dx)\ \mathbb{N}_{0,\frac{\Delta}{2}}[dW]$. Hence property (B) is fulfilled for k replaced by $k+1$.

2.3. **Results.** Let $X^\varepsilon = (X^\varepsilon_t, t \geq 0)$ be the right continuous step function which is the extension of $(X^\varepsilon_{k\varepsilon}, k \in \mathbb{N})$. Let $D = D(\mathbb{R}^+, \mathcal{M}_f)$ be the Polish space of càdlàg paths from $\mathbb{R}^+$ to $\mathcal{M}_f$, with the Skorokhod topology. Let θ, b and σ be bounded continuous functions defined on $\mathcal{M}_f \times \mathbb{R}^d$ taking values respectively in $\mathbb{R}, \mathbb{R}^d$ and $\mathbb{R}^{d \times d}$. We also assume the functions θ and σ are positive. We write $A(\mu)$ for the infinitesimal generator of the d-dimensional diffusion with drift $b(\mu, \cdot)$ and diffusion coefficient $\sigma(\mu, \cdot)$ $(\mu \in \mathcal{M}_f)$.

Theorem 1. *The family of law of X^ε, for $\varepsilon \in (0, 1]$ is C-tight in D as ε decreases to 0. Any limiting measure valued process $Y = (Y_t, t \geq 0)$ satisfies the martingale problem (MP): for any bounded C^2 function, φ, with bounded derivatives,*

$$Y_0 = \mu_0$$

$$(Y_t, \varphi) = (Y_0, \varphi) + \int_0^t (Y_s, \theta(Y_s) A(Y_s)\varphi)\, ds + M(\varphi)_t, \tag{11}$$

where $M(\varphi)$ is a continuous martingale with quadratic variation

$$\langle M(\varphi) \rangle_t = \int_0^t (Y_s, \theta(Y_s)\varphi^2)\, ds.$$

Furthermore any limiting measure valued process Y has a continuous version.

We will follow an idea due to Perkins [9] to prove this theorem.

Unfortunately, we were unable to prove the uniqueness of the martingale problem, even for the historical process (see in [8] why it is more convenient to look at the historical processes for uniqueness to martingale problem). Uniqueness is trivially proved in the very particular cases of the next two remarks, where in fact the interaction disappears.

Remark. The particular case $\sigma = \sigma(x)$, $b = b(x)$ and $\theta = \theta(x)$ with the additional condition $\theta(x) \geq \theta_0 > 0$ correspond to the usual super process with underlying process a diffusion with diffusion coefficient $\sigma(x)$ and drift $b(x)$, and branching mechanism $2\theta(x)z^2$ (see [2]). In this case the martingale problem (MP) has a unique solution.

Remark. One can also consider the other particular case $\sigma = \sigma(x)$, $b = b(x)$ and $\theta = \theta(\mu)$ with the additional condition $\theta(\mu) \geq \theta_0 > 0$. Then we consider the super process X with underlying process a diffusion coefficient $\sigma(x)$ and drift $b(x)$ and branching mechanism $2z^2$. We define the continuous additive functional of X by : $Q_t = \displaystyle\int_0^t \frac{du}{\theta(X_u)}$ and its continuous inverse $R_t = Q_t^{-1}$. It is the easy to check that the process $Y = (Y_t = X_{R_t}, t \geq 0)$ is solution to the martingale problem (MP). To prove the solution of (MP) is unique, consider $\tilde{Y}$ an other solution to (MP). Set $R_t = \int_0^t \theta(\tilde{Y}_u)\, du$ and consider its continuous inverse $\tilde{Q}_t = \tilde{R}_t^{-1}$. It is then easy to check that the process $\tilde{X} = (\tilde{X}_t = \tilde{Y}_{\tilde{Q}_t}, t \geq 0)$ is solution to the martingale problem (MP) with $\theta = 1$. Since this martingale problem has a unique solution, we get that X and $\tilde{X}$ are equally distributed. And so Y and $\tilde{Y}$ have the same law. In this case the martingale problem has also a unique solution in law.

3. Intermediate results

Before giving the proof of Theorem 1, we set 5 lemmas. Let c denote a constant which may vary from line to line. For $t \in [0, +\infty)$ we set $[t]$ the unique integer such that $[t] \leq t < [t] + 1$. For $f \in \mathcal{B}(\mathbb{R}^d)$ we will consider the following norms: $\| f \|_\infty = \sup_{x \in \mathbb{R}^d} |f(x)|$,

$$\| f \|_{\mathrm{Lip}} = \sup_{x,y \in \mathbb{R}^d; x \neq y} \frac{|f(x) - f(y)|}{|x - y|},$$

as well as for $\varphi \in C^2$,

$$\| \varphi \|_* = \| \varphi \|_{\mathrm{Lip}} + \sum_{l=1}^d \left(\| \frac{\partial \varphi}{\partial x_l} \|_\infty + \| \frac{\partial \varphi}{\partial x_l} \|_{\mathrm{Lip}} + \sum_{k=1}^d \left(\| \frac{\partial^2 \varphi}{\partial x_l \partial x_k} \|_\infty + \| \frac{\partial^2 \varphi}{\partial x_l \partial x_k} \|_{\mathrm{Lip}} \right) \right).$$

Let φ be a C^2 real function defined on $\mathbb{R}^d$, bounded with bounded Lipschitz derivatives. Let $T > 0$ be fixed.

We want to use the structure of the snake excursion U^{i,j_k} (see (6)) to express $(X^{i,j_k}_{k\varepsilon}, \varphi)$ as a sum of martingales and a process of finite variation. Recall the functions $\phi^{i,\varepsilon}_{\cdot}(k\varepsilon)$, $\phi^{i,\varepsilon}_{\cdot}((k+1)\varepsilon)$ as well as $V^i_{\cdot}(k\varepsilon)$ are constant on the time indices where the excursion U^{i,j_k} is defined. We set for $i \in I$, $j_k \in J^i_k$,

$$\Delta M^{i,j_k}(\varphi) = (X^{i,j_k}_{(k+1)\varepsilon}, \varphi) - \int_{\phi^{i,\varepsilon}_{\cdot}(k\varepsilon)}^{\phi^{i,\varepsilon}_{\cdot}((k+1)\varepsilon)} (\tilde{X}^{i,j_k}_u, A(X^\varepsilon_{k\varepsilon}, V^i_{\cdot}(k\varepsilon))\varphi) \, du,$$

where $\tilde{X}^{i,j_k}$ has been defined in (7). And now we define what will be a $\mathcal{G}^\varepsilon$-martingale: $M(\varphi)_0 = 0$ and for $k \geq 0$,

$$M(\varphi)_{(k+1)\varepsilon} = M(\varphi)_{k\varepsilon} - (X^\varepsilon_{k\varepsilon}, \varphi) + \sum_{i \in I, j_k \in J^i_k} \Delta M^{i,j_k}(\varphi).$$

In particular we have

$$(X^\varepsilon_{(k+1)\varepsilon}, \varphi) - M(\varphi)_{(k+1)\varepsilon} = (X^\varepsilon_{k\varepsilon}, \varphi) - M(\varphi)_{k\varepsilon} + \sum_{i \in I, j_k \in J^i_k} \int_{\phi^{i,\varepsilon}_{\cdot}(k\varepsilon)}^{\phi^{i,\varepsilon}_{\cdot}((k+1)\varepsilon)} (\tilde{X}^{i,j_k}_u, A(X^\varepsilon_{k\varepsilon}, V^i_{\cdot}(k\varepsilon))\varphi) \, du. \tag{12}$$

We rewrite this as

$$(X^\varepsilon_{(k+1)\varepsilon}, \varphi) = (X^\varepsilon_{k\varepsilon}, \varphi) + M(\varphi)_{(k+1)\varepsilon} - M(\varphi)_{k\varepsilon} + \varepsilon(X^\varepsilon_{k\varepsilon}, \theta(X^\varepsilon_{k\varepsilon}) A(X^\varepsilon_{k\varepsilon})\varphi) + \eta^\varepsilon_{k+1}, \tag{13}$$

where

$$\eta^\varepsilon_{k+1} = \sum_{i \in I, j_k \in J^i_k} \int_{\phi^{i,\varepsilon}_{\cdot}(k\varepsilon)}^{\phi^{i,\varepsilon}_{\cdot}((k+1)\varepsilon)} (\tilde{X}^{i,j_k}_u, A(X^\varepsilon_{k\varepsilon}, V^i_{\cdot}(k\varepsilon))\varphi) \, du - \varepsilon(X^\varepsilon_{k\varepsilon}, \theta(X^\varepsilon_{k\varepsilon}) A(X^\varepsilon_{k\varepsilon})\varphi). \tag{14}$$

From (9), and property (B), we get that u_k is $\mathcal{G}^\varepsilon_{k\varepsilon}$-measurable. Since $\sum_{i\in I, j_k\in J^i_k}\delta_{U^{i,j_k}}$ is conditionally on $\mathcal{G}^\varepsilon_{k\varepsilon}$ distributed according to a Poisson point measure with intensity $\int X^\varepsilon_{k\varepsilon}(dx)\mathbb{N}_{x,A(X^\varepsilon_{k\varepsilon},x)}[dW]$, we get from the definition of $X^\varepsilon_{(k+1)\varepsilon}$, formula (8) and (10), that

$$\mathbb{E}[(X^\varepsilon_{(k+1)\varepsilon},\varphi)|\mathcal{G}^\varepsilon_{k\varepsilon}] = \int \mu(dx)\mathbb{N}_{x,A(\mu,x)}[(X_{\varepsilon\theta(\mu,x)},\varphi)], \tag{15}$$

with $\mu = X^\varepsilon_{k\varepsilon}$.

Lemma 2. *The process $((X^\varepsilon_{k\varepsilon},\mathbf{1}), k\in\mathbb{N})$ is an L^2 $\mathcal{G}^\varepsilon_{k\varepsilon}$-martingale. Moreover, we have :*

$$\mathbb{E}[\sup_{k\leq T/\varepsilon}(X^\varepsilon_{k\varepsilon},\mathbf{1})^2] \leq 4\,T\,\|\,\theta\,\|_\infty(\mu_0,\mathbf{1}) + 4\,(\mu_0,\mathbf{1})^2. \tag{16}$$

Proof. We use the notation $\mu = X^\varepsilon_{k\varepsilon}$. From (15), we get

$$\mathbb{E}[(X^\varepsilon_{(k+1)\varepsilon},\mathbf{1})|\mathcal{G}^\varepsilon_{k\varepsilon}] = \int \mu(dx)\;\mathbb{N}_{x,A(\mu,x)}[(X_{\varepsilon\theta(\mu,x)},\mathbf{1})] = (\mu,\mathbf{1}).$$

Hence the process $((X^\varepsilon_{k\varepsilon},\mathbf{1}), k\in\mathbb{N})$ is a nonnegative $\mathcal{G}^\varepsilon_{k\varepsilon}$-martingale. Using the second moment formula for Poisson point measure and (35), we get

$$\begin{aligned}\mathbb{E}[(X^\varepsilon_{(k+1)\varepsilon},\mathbf{1})^2|\mathcal{G}^\varepsilon_{k\varepsilon}] &= (\mu,\mathbf{1})^2 + \int \mu(dx)\;\mathbb{N}_{x,A(\mu,x)}[(X_{\varepsilon\theta(\mu,x)},\mathbf{1})^2]\\ &= (\mu,\mathbf{1})^2 + 4\int \mu(dx)\;\varepsilon\theta(\mu,x).\end{aligned}$$

We set $M_k = (X^\varepsilon_{k\varepsilon},\mathbf{1})$. We deduce from the previous equality that

$$\begin{aligned}\langle M\rangle_{k+1} - \langle M\rangle_k &= \mathbb{E}[(X^\varepsilon_{(k+1)\varepsilon},\mathbf{1})^2 - (X^\varepsilon_{k\varepsilon},\mathbf{1})^2|\mathcal{G}^\varepsilon_{k\varepsilon}]\\ &= 4\int \mu(dx)\;\varepsilon\theta(\mu,x)\\ &\leq 4\,\varepsilon\,\|\,\theta\,\|_\infty(X^\varepsilon_{k\varepsilon},\mathbf{1}).\end{aligned}$$

Hence, we have

$$\mathbb{E}[\langle M\rangle_k] \leq 4\,\varepsilon\,\|\,\theta\,\|_\infty\sum_{l=0}^{k-1}\mathbb{E}[(X^\varepsilon_{l\varepsilon},\mathbf{1})] \leq 4\,k\varepsilon\,\|\,\theta\,\|_\infty(\mu_0,\mathbf{1}).$$

Using Doob's inequality for $N\in\mathbb{N}$, we get

$$\begin{aligned}\mathbb{E}[\sup_{k\leq T/\varepsilon}(X^\varepsilon_{k\varepsilon},\mathbf{1})^2] &\leq 4\,\mathbb{E}[(X^\varepsilon_{[T/\varepsilon]\varepsilon},\mathbf{1})^2]\\ &= 4\,\mathbb{E}[\langle M\rangle_{[T/\varepsilon]\varepsilon}] + 4(\mu_0,1)^2\\ &\leq 4T\,\|\,\theta\,\|_\infty(\mu_0,\mathbf{1}) + 4\,(\mu_0,\mathbf{1})^2.\end{aligned}$$

□

Lemma 3. *The process $(M(\varphi)_{k\varepsilon}, k\in\mathbb{N})$ is an L^2 $\mathcal{G}^\varepsilon_{k\varepsilon}$-martingale. We also have:*

1. *For $k\leq T/\varepsilon$,*

$$\mathbb{E}[(M(\varphi)_{(k+1)\varepsilon} - M(\varphi)_{k\varepsilon})^2|\mathcal{G}^\varepsilon_{k\varepsilon}] \leq c\,\varepsilon\,\|\,\varphi\,\|^2_\infty\sup_{l\leq T/\varepsilon}(X^\varepsilon_{l\varepsilon},\mathbf{1}), \tag{17}$$

and

$$\mathbb{E}[(M(\varphi)_{(k+1)\varepsilon} - M(\varphi)_{k\varepsilon})^4 | \mathcal{G}^\varepsilon_{k\varepsilon}] \leq c\,\varepsilon^2 \,\|\,\varphi\,\|^4_\infty (1 + \sup_{l\leq T/\varepsilon} (X^\varepsilon_{l\varepsilon}, \mathbf{1})^2), \tag{18}$$

where the constant c is independent of φ, k and ε.

2. *For $t = k\varepsilon$, $s = l\varepsilon$, where $l, k \in \mathbb{N}$, and $0 \leq s \leq t \leq T$,*

$$\langle M(\varphi)\rangle_t - \langle M(\varphi)\rangle_s \leq c(t-s)\,\|\,\varphi\,\|^2_\infty \sup_{k'\leq T/\varepsilon} (X^\varepsilon_{k'\varepsilon}, \mathbf{1}), \tag{19}$$

where the constant c is independent of φ, t, s and ε.

3. *We have*

$$\mathbb{E}[\langle M(\varphi)\rangle^2_{[T/\varepsilon]\varepsilon}] \leq c\,T^2\,\|\,\varphi\,\|^4_\infty, \tag{20}$$

where the constant c is independent of φ and ε.

Proof. We still use the notation $\mu = X^\varepsilon_{k\varepsilon}$.

From (13), it is easy to prove by induction that $M(\varphi)_{k\varepsilon}$ is integrable. Let us now prove that $(M(\varphi)_k, k \in \mathbb{N})$ is a martingale. From (12), using (7) the definition of $\tilde{X}^{i,j_k}$, the fact that $\sum_{i\in I, j_k\in J^i_k} \delta_{U^{i,j_k}}$ is conditionally on $\mathcal{G}^\varepsilon_{k\varepsilon}$ distributed according to a Poisson point measure with intensity $\int \mu(dx)\mathbb{N}_{x,A(\mu,x)}[dW]$ and eventually (15) we get

$$\begin{aligned}
&\mathbb{E}[M(\varphi)_{(k+1)\varepsilon} - M(\varphi)_{k\varepsilon} | \mathcal{G}^\varepsilon_{k\varepsilon}] \\
&= \mathbb{E}[(X^\varepsilon_{(k+1)\varepsilon}, \varphi)|\mathcal{G}^\varepsilon_{k\varepsilon}] - (\mu,\varphi) - \mathbb{E}[\sum_{i\in I, j_k\in J^i_k} \int_{\phi^{i,\varepsilon}_\cdot(k\varepsilon)}^{\phi^{i,\varepsilon}_\cdot((k+1)\varepsilon)} (\tilde{X}^{i,j_k}_u, A(X^\varepsilon_{k\varepsilon}, V^i_\cdot(k\varepsilon))\varphi)\; du | \mathcal{G}^\varepsilon_{k\varepsilon}] \\
&= \int \mu(dx)\; \mathbb{N}_{x,A(\mu,x)}[(X_{\varepsilon\theta(\mu,x)}, \varphi)] - (\mu,\varphi) - \int \mu(dx)\; \mathbb{N}_{x,A(\mu,x)}[\int_0^{\varepsilon\theta(x,\mu)} du\; (X_u, A(\mu,x)\varphi)] \\
&= \int \mu(dx) \left(\mathbb{E}_x[\varphi(Z_{\varepsilon\theta(x,\mu)}) - \varphi(x) - \int_0^{\varepsilon\theta(x,\mu)} du\; A(\mu,x)\varphi(Z_u)] \right) \\
&= 0.
\end{aligned}$$

For the third equality, we introduced the process $(Z_s, s \geq 0)$ which is under $\mathbb{E}_x$ a diffusion with infinitesimal generator $A(\mu, x)$ started at point x, and we used the first moment formula (33) for the Brownian snake.

Hence $(M(\varphi)_{k\varepsilon}, k \in \mathbb{N})$ is a martingale.

1. Let us now compute $\mathbb{E}[(M(\varphi)_{(k+1)\varepsilon} - M(\varphi)_{k\varepsilon})^p|\mathcal{G}^\varepsilon_{k\varepsilon}]$ for $p = 2, 4$. We have

$$\begin{aligned}
\mathbb{E}[(M(\varphi)_{(k+1)\varepsilon} &- M(\varphi)_{k\varepsilon})^p|\mathcal{G}^\varepsilon_{k\varepsilon}] \\
&= \mathbb{E}[((\mu,\varphi) - \sum_{i\in I, j_k\in J^i_k} \Delta M^{i,j_k}(\varphi))^p|\mathcal{G}^\varepsilon_{k\varepsilon}] \\
&= \sum_{m=0}^p (-1)^m C^m_p (\mu,\varphi)^{p-m} \mathbb{E}[(\sum_{i\in I, j_k\in J^i_k} \Delta M^{i,j_k}(\varphi))^m|\mathcal{G}^\varepsilon_{k\varepsilon}],
\end{aligned} \tag{21}$$

with

$$\Delta M^{i,j_k}(\varphi) = (X^{i,j_k}_{(k+1)\varepsilon}, \varphi) - \int_{\phi^{i,\varepsilon}_\cdot(k\varepsilon)}^{\phi^{i,\varepsilon}_\cdot((k+1)\varepsilon)} (\tilde{X}^{i,j_k}_u, A(X^\varepsilon_{k\varepsilon}, V^i_\cdot(k\varepsilon))\varphi)\; du.$$

First of all, let us compute the Laplace transform

(22)

$$\mathcal{A} = \mathbb{E}\left[\exp - \left\{ \lambda \sum_{i\in I, j_k \in J_k^i} \left((X^{i,j_k}_{(k+1)\varepsilon}, \varphi) + \int_{\phi_\cdot^{i,\varepsilon}(k\varepsilon)}^{\phi_\cdot^{i,\varepsilon}((k+1)\varepsilon)} du\ (\tilde{X}^{i,j_k}_u, \psi(V_\cdot^i(k\varepsilon), \cdot)) \right) \right\} \middle| \mathcal{G}^\varepsilon_{k\varepsilon} \right]$$

where φ and ψ are non negative bounded measurable functions defined respectively on $\mathbb{R}^d$ and $\mathbb{R}^d \times \mathbb{R}^d$, and $\lambda \geq 0$. Using the Laplace transform for Poisson point measure, we have

$$\mathcal{A} = \exp - \int \mu(dx)\ \mathbb{N}_{x,A(\mu,x)}[1 - \exp -\lambda((X_{\varepsilon\theta(\mu,x)}, \varphi) + \int_0^{\varepsilon\theta(\mu,x)} du\ (X_u, \psi(x, \cdot)))].$$

Let us introduce $(P_t, t \geq 0)$ the transition kernel of the diffusion with infinitesimal generator $A(\mu, x_0)$ for $x_0 \in \mathbb{R}^d$ fixed. If we define

$$v_{\lambda,x_0}(t,x) = \mathbb{N}_{x,A(\mu,x_0)}[1 - \exp -\lambda((X_t, \varphi) + \int_0^t du\ (X_u, \psi(x_0, \cdot)))],$$

then v_{λ,x_0} solves the equation

$$v_{\lambda,x_0}(t,x) + 2\int_0^t P_{t-s}(v_{\lambda,x_0}(s)^2)(x)\,ds = \lambda \mathbb{N}_{x,A(\mu,x_0)}[(X_t, \varphi) + \int_0^t du\ (X_u, \psi(x_0, \cdot))].$$

For λ small enough, the function v_{λ,x_0} can be developed as a power series in λ. In particular

$$v_{\lambda,x_0}(t,x) = \lambda\alpha_{1,x_0}(t,x) + \lambda^2\alpha_{2,x_0}(t,x) + \lambda^3\alpha_{3,x_0}(t,x) + \lambda^4\alpha_{4,x_0}(t,x) + \lambda^5 g_{\lambda,x_0}(t,x),$$

where g is uniformly bounded in $(t, x, x_0, \lambda) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times [0,1]$. Using the previous integral equation, we have

$$\begin{aligned}
\alpha_{1,x_0}(t,x) &= \mathbb{N}_{x,A(\mu,x_0)}[(X_t, \varphi) + \int_0^t du (X_u, \psi(x_0, \cdot))],\\
\alpha_{2,x_0}(t,x) &= -2\int_0^t P_{t-s}(\alpha_{1,x_0}(s)^2)(x)\,ds,\\
\alpha_{3,x_0}(t,x) &= -4\int_0^t P_{t-s}(\alpha_{1,x_0}(s)\alpha_{2,x_0}(s))(x)\,ds,\\
\alpha_{4,x_0}(t,x) &= -2\int_0^t P_{t-s}(\alpha_{2,x_0}(s)^2 + 2\alpha_{1,x_0}(s)\alpha_{3,x_0}(s))(x)\,ds.
\end{aligned}$$

So we have, with the notation $\alpha_i = \alpha_{i,\cdot}(\varepsilon\theta(\mu,\cdot),\cdot)$,

$$\begin{aligned}
\mathcal{A} =& \exp - \int \mu(dx)\ \mathbb{N}_{x,A(\mu,x)}\left[1 - \exp[-\lambda((X_t, \varphi) + \int_0^t du (X_u, \psi(x, \cdot)))]\right]\\
=& 1 - \lambda(\mu, \alpha_1) + \lambda^2\left[-(\mu,\alpha_2) + \frac{1}{2}(\mu,\alpha_1)^2\right] + \lambda^3\left[-(\mu,\alpha_3) + (\mu,\alpha_2)(\mu,\alpha_1) - \frac{1}{6}(\mu,\alpha_1)^3\right]\\
&+ \lambda^4\left[-(\mu,\alpha_4) + (\mu,\alpha_3)(\mu,\alpha_1) + \frac{1}{2}(\mu,\alpha_2)^2 - \frac{1}{2}(\mu,\alpha_2)(\mu,\alpha_1)^2 + \frac{1}{24}(\mu,\alpha_1)^4\right] + o(\lambda^4).
\end{aligned}$$

We deduce from (22) that

$$
\begin{aligned}
&\mathbb{E}[(\sum_{i\in I, j_k\in J_k^i} X^{i,j_k}_{(k+1)\varepsilon},\varphi) + \int_{\phi^{i,\varepsilon}_\cdot(k\varepsilon)}^{\phi^{i,\varepsilon}_\cdot((k+1)\varepsilon)} (\tilde X^{i,j_k}_u, \psi(V^i_\cdot(k\varepsilon),\cdot))\; du | \mathcal{G}^\varepsilon_{k\varepsilon}] = (\mu,\alpha_1),\\
&\mathbb{E}[(\sum_{i\in I, j_k\in J_k^i} X^{i,j_k}_{(k+1)\varepsilon},\varphi) + \int_{\phi^{i,\varepsilon}_\cdot(k\varepsilon)}^{\phi^{i,\varepsilon}_\cdot((k+1)\varepsilon)} (\tilde X^{i,j_k}_u, \psi(V^i_\cdot(k\varepsilon),\cdot))\; du)^2 | \mathcal{G}^\varepsilon_{k\varepsilon}]\\
&\qquad\qquad = (\mu,\alpha_1)^2 - 2(\mu,\alpha_2),\\
&\mathbb{E}[(\sum_{i\in I, j_k\in J_k^i} X^{i,j_k}_{(k+1)\varepsilon},\varphi) + \int_{\phi^{i,\varepsilon}_\cdot(k\varepsilon)}^{\phi^{i,\varepsilon}_\cdot((k+1)\varepsilon)} (\tilde X^{i,j_k}_u, \psi(V^i_\cdot(k\varepsilon),\cdot))\; du)^3 | \mathcal{G}^\varepsilon_{k\varepsilon}]\\
&\qquad\qquad = (\mu,\alpha_1)^3 - 6(\mu,\alpha_1)(\mu,\alpha_2) + 6(\mu,\alpha_3),\\
&\mathbb{E}[(\sum_{i\in I, j_k\in J_k^i} X^{i,j_k}_{(k+1)\varepsilon},\varphi) + \int_{\phi^{i,\varepsilon}_\cdot(k\varepsilon)}^{\phi^{i,\varepsilon}_\cdot((k+1)\varepsilon)} (\tilde X^{i,j_k}_u, \psi(V^i_\cdot(k\varepsilon),\cdot))\; du)^4 | \mathcal{G}^\varepsilon_{k\varepsilon}]\\
&\qquad = (\mu,\alpha_1)^4 - 12(\mu,\alpha_1)^2(\mu,\alpha_2) + 12(\mu,\alpha_2)^2 + 24(\mu,\alpha_1)(\mu,\alpha_3) - 24(\mu,\alpha_4).
\end{aligned}
\tag{23}
$$

Using a polarization argument, we have the same result for any bounded measurable function φ and ψ. In particular, we can take $\psi(x_0,\cdot) = A(\mu,x_0)\varphi$. Moreover, in that case,

$$
\begin{aligned}
\alpha_{1,x_0}(t,x) &= \mathbb{N}_{x,A(\mu,x_0)}[(X_t,\varphi) + \int_0^t du\; (X_u, \psi(x_0,\cdot))]\\
&= \mathbb{E}_x[\varphi(Z_t) - \int_0^t A(\mu,x_0)\varphi(Z_s)\; ds]\\
&= \varphi(x),
\end{aligned}
$$

where $(Z_t, t\geq 0)$ is a diffusion with infinitesimal generator $A(\mu,x_0)$ started at x.

We also have upper bounds for the others α_i :

$$
\begin{aligned}
|\alpha_{2,x_0}(t,x)| &= \left| -2\int_0^t P_{t-s}(\alpha_{1,x_0}(s)^2)(x)\, ds \right| \leq 2t \,\|\varphi\|_\infty^2\\
|\alpha_{3,x_0}(t,x)| &= \left| -4\int_0^t P_{t-s}(\alpha_{1,x_0}(s)\alpha_{2,x_0}(s))(x)\, ds \right| \leq 4t^2 \,\|\varphi\|_\infty^3\\
|\alpha_{4,x_0}(t,x)| &= \left| -2\int_0^t P_{t-s}(\alpha_{2,x_0}(s)^2 + 2\alpha_{1,x_0}(s)\alpha_{3,x_0}(s))(x)\, ds \right| \leq 8t^3 \,\|\varphi\|_\infty^4 .
\end{aligned}
$$

Since θ is bounded from above, we get using the previous computations and formula (21)

$$
\begin{aligned}
\mathbb{E}[(M(\varphi)_{(k+1)\varepsilon} - M(\varphi)_{k\varepsilon})^2 | \mathcal{G}^\varepsilon_{k\varepsilon}] &= -2(\mu,\alpha_2)\\
&\leq c\varepsilon(\mu,1)\,\|\varphi\|_\infty^2\\
&\leq c\,\varepsilon\,\|\varphi\|_\infty^2 \sup_{l\leq T/\varepsilon} (X^\varepsilon_{l\varepsilon},\mathbf{1}),
\end{aligned}
\tag{24}
$$

and

$$\begin{aligned}\mathbb{E}[(M(\varphi)_{(k+1)\varepsilon}-M(\varphi)_{k\varepsilon})^4|\mathcal{G}^{\varepsilon}_{k\varepsilon}] &= 12(\mu,\alpha_2)^2-24(\mu,\alpha_4)\\ &\le c\,\varepsilon^2\,\|\varphi\|^4_\infty(\sup_{l\le T/\varepsilon}(X^{\varepsilon}_{l\varepsilon},\mathbf{1})^2+\sup_{l\le T/\varepsilon}(X^{\varepsilon}_{l\varepsilon},\mathbf{1}))\\ &\le c\,\varepsilon^2\,\|\varphi\|^4_\infty(1+\sup_{l\le T/\varepsilon}(X^{\varepsilon}_{l\varepsilon},\mathbf{1})^2).\end{aligned}$$

2. It is a direct consequence of (17).
3. Recall c denotes a constant which value may vary from line to line. From (19), we deduce that

$$\begin{aligned}\mathbb{E}[\langle M(\varphi)\rangle^2_{k\varepsilon}] &\le c\varepsilon^2k^2\,\|\varphi\|^4_\infty\,\mathbb{E}[\sup_{k\le T/\varepsilon}(X^{\varepsilon}_{k\varepsilon},\mathbf{1})^2]\\ &\le c\varepsilon^2k^2\,\|\varphi\|^4_\infty(T(\mu_0,\mathbf{1})+(\mu_0,\mathbf{1})^2),\end{aligned}$$

where we used (16). We deduce (20). □

Recall the definition of η^{ε}_k:

$$\eta^{\varepsilon}_{k+1}=\sum_{i\in I,j_k\in J^i_k}\int_{\phi^{i,\varepsilon}_{\cdot}(k\varepsilon)}^{\phi^{i,\varepsilon}_{\cdot}((k+1)\varepsilon)}(\tilde{X}^{i,j_k}_u,A(X^{\varepsilon}_{k\varepsilon},V^i_{\cdot}(k\varepsilon))\varphi)\,du-\varepsilon(X^{\varepsilon}_{k\varepsilon},\theta(X^{\varepsilon}_{k\varepsilon})A(X^{\varepsilon}_{k\varepsilon})\varphi).$$

Lemma 4. *We have the convergence of* $\sup_{0\le l\le[T/\varepsilon]}\left|\sum_{k=1}^{l}\eta^{\varepsilon}_k\right|$ *to* 0 *in* L^1 *as* ε *decreases to* 0.

Proof. We still use the notation $\mu=X^{\varepsilon}_{k\varepsilon}$. We first prove that $\mathbb{E}[(\eta^{\varepsilon}_{k+1})^2|\mathcal{G}^{\varepsilon}_{k\varepsilon}]$ can be bounded from above by $c\varepsilon^3(1+(\mu,\mathbf{1})^2)$. Using (23) with $\varphi=0$ and $\psi(x,\cdot)=A(\mu,x)\varphi(\cdot)$, we have

$$
\begin{aligned}
&\mathbb{E}[(\eta^\varepsilon_{k+1})^2|\mathcal{G}^\varepsilon_{k\varepsilon}] \\
&\qquad =(\mu,\alpha_1)^2-2(\mu,\alpha_2)+\varepsilon^2\left(\int \mu(dx)\; \theta(\mu,x)A(\mu,x)\varphi(x)\right)^2 \\
&\qquad\qquad -2\varepsilon(\mu,\alpha_1)\int \mu(dx)\; \theta(\mu,x)A(\mu,x)\varphi(x) \\
&\qquad =\int \mu(dx)\; \mathbb{N}_{x,A(\mu,x)}[(\int_0^{\varepsilon\theta(\mu,x)} du\; (X_u, A(\mu,x)\varphi))^2] \\
&\qquad\qquad +\left(\int \mu(dx)\; \mathbb{N}_{x,A(\mu,x)}[\int_0^{\varepsilon\theta(\mu,x)} du\; (X_u, A(\mu,x)\varphi)]\right)^2 \\
&\qquad\qquad +\varepsilon^2\left(\int \mu(dx)\; \theta(\mu,x)A(\mu,x)\varphi(x)\right)^2 \\
&\qquad\qquad -2\varepsilon\int \mu(dx)\; \theta(\mu,x)A(\mu,x)\varphi(x) \\
&\qquad\qquad\qquad\qquad \int \mu(dx')\; \mathbb{N}_{x',A(\mu,x')}[\int_0^{\varepsilon\theta(\mu,x')} du\; (X_u, A(\mu,x')\varphi)] \\
&\qquad =2\int \mu(dx)\; \int_0^{\varepsilon\theta(\mu,x)} du\int_0^u dv\; \mathbb{N}_{x,A(\mu,x)}[(X_v,A(\mu,x)\varphi)(X_v,\mathrm{P}_{u-v}(A(\mu,x)\varphi))] \\
&\qquad\qquad +\left(\int \mu(dx)\; \int_0^{\varepsilon\theta(\mu,x)} du\; [\mathrm{P}_u(A(\mu,x)\varphi)-A(\mu,x)\varphi]\right)^2,
\end{aligned}
$$

where we used (33) and (34) to compute the first and second moment of the super diffusion under $\mathbb{N}$, with the notation $(\mathrm{P}_u, u\geq 0)$ for the transition semi-group with infinitesimal generator $A(\mu,x)$. We deduce from (35) that

$$\mathbb{N}_{x,A(\mu,x)}[(X_v,A(\mu,x)\varphi)(X_v,\mathrm{P}_{u-v}(A(\mu,x)\varphi))]\leq c\,\|\varphi\|_*^2\, v,$$

where c depends on θ, b and σ. Since the coefficients of $A(\mu,x)$ are uniformly bounded, we deduce that for $0\leq u\leq \|\theta\|_\infty$,

$$\|\mathrm{P}_u(A(\mu,x)\varphi)-A(\mu,x)\varphi\|_\infty\leq c\,\|\varphi\|_*\sqrt{u}.$$

We get that for $\varepsilon\in(0,1]$,

$$\mathbb{E}[(\eta^\varepsilon_{k+1})^2|\mathcal{G}^\varepsilon_{k\varepsilon}]\leq c\varepsilon^3(\mu,\mathbf{1})\,\|\varphi\|_*^2+c\varepsilon^3(\mu,\mathbf{1})^2\,\|\varphi\|_*^2.$$

In particular we have for $\varepsilon\in(0,1]$,

$$\mathbb{E}[\left|\eta^\varepsilon_{k+1}\right||\mathcal{G}^\varepsilon_{k\varepsilon}]\leq \mathbb{E}[(\eta^\varepsilon_{k+1})^2|\mathcal{G}^\varepsilon_{k\varepsilon}]^{1/2}\leq c\varepsilon^{3/2}\,\|\varphi\|_*((X^\varepsilon_{k\varepsilon},\mathbf{1})+1),$$

where the constant c depends only on the bounds of θ, b and σ. Therefore we deduce that for $T>0$,

$$\mathbb{E}[\sum_{k=1}^{[T/\varepsilon]}|\eta_k|]\leq c\sqrt{\varepsilon}T\,\|\varphi\|_*(1+\mathbb{E}[\sup_{0\leq k\leq [T/\varepsilon]}(X^\varepsilon_{k\varepsilon},\mathbf{1})]).$$

From Lemma 2, we deduce that

$$\mathbb{E}[\sum_{k=1}^{[T/\varepsilon]} |\eta_k|] \leq c\sqrt{\varepsilon}T(1+T)\,\|\varphi\|_*, \tag{25}$$

where c depends only on the bounds of θ, b and σ. Therefore we have the convergence of $\sup_{0\leq l\leq [T/\varepsilon]}\left|\sum_{k=1}^{l}\eta_k\right|$ to 0 in L^1 as ε decreases to 0.

□

Lemma 5. *We have*

$$\langle M(\varphi)\rangle_{(k+1)\varepsilon} = \langle M(\varphi)\rangle_{k\varepsilon} + 4\varepsilon(X^\varepsilon_{k\varepsilon}, \theta(X^\varepsilon_{k\varepsilon})\varphi^2) + \kappa_k,$$

where $\sum_{k=0}^{[T/\varepsilon]}\kappa_k$ converge in L^2 to 0 as ε decreases to 0.

Proof. We still write μ for $X^\varepsilon_{k\varepsilon}$. Recall from (24) that

$$\langle M(\varphi)\rangle_{(k+1)\varepsilon} - \langle M(\varphi)\rangle_{k\varepsilon} = 4\int \mu(dx)\int_0^{\varepsilon\theta(\mu,x)} ds\; \mathbb{E}_x[\varphi(Z_s)^2],$$

where $(Z_s, s\geq 0)$ is under $\mathbb{E}_x$ a diffusion with infinitesimal generator $A(\mu,x)$ started at point x. In particular, for $s\in[0,\|\theta\|_\infty]$,

$$\begin{aligned}\mathbb{E}_x[\left|\varphi(Z_s)^2-\varphi(Z_0)^2\right|] &\leq 2\,\|\varphi\|_\infty\,\|\varphi\|_{\mathrm{Lip}}\,\mathbb{E}_x[|Z_s-Z_0|]\\ &\leq c\,\|\varphi\|_\infty\,\|\varphi\|_*\sqrt{s},\end{aligned}$$

where the constant c depends only on θ, b, σ and T. Therefore, we have for $\varepsilon\in(0,1]$,

$$\begin{aligned}|\kappa_k| &\leq 4\int\mu(dx)\;\int_0^{\varepsilon\theta(\mu,x)} ds\;\mathbb{E}_x[\left|\varphi(Z_s)^2-\varphi(Z_0)^2\right|]\\ &\leq c\varepsilon^{3/2}\,\|\varphi\|_\infty\,\|\varphi\|_*(\mu,\mathbf{1}),\end{aligned}$$

We deduce that for $T\geq 0$, $\varepsilon\in(0,1]$,

$$\sum_{k=0}^{[T/\varepsilon]}|\kappa_k| \leq c\sqrt{\varepsilon}T\,\|\varphi\|_\infty\,\|\varphi\|_* \sup_{0\leq k\leq [T/\varepsilon]}(X^\varepsilon_{k\varepsilon},\mathbf{1}). \tag{26}$$

We deduce from Lemma 2, that $\sum_{k=0}^{[T/\varepsilon]}\kappa_k$ converges to 0 in L^2 as ε decreases to 0.

□

Lemma 6. *For every $\rho, T>0$, there is a compact set $K_{\rho,T}$ in $\mathbb{R}^d$ such that*

$$\sup_{0<\varepsilon\leq 1}\mathbb{P}(\sup_{0\leq k\leq T/\varepsilon}(X^\varepsilon_{k\varepsilon},\mathbf{1}_{K^c_{\rho,T}})>\rho)<\rho.$$

Proof. Let B_R denote the centered ball of $\mathbb{R}^d$ with radius R. Let g be a non negative function of class C^2 with bounded Lipschitz derivatives, defined on $\mathbb{R}^d$ such that $g=0$ on B_1, and $g=1$ outside B_2. Set $g_R(x)=g(x/R)$, with $R\geq 1$. We want to check that $\mathbb{E}[\sup_{0\leq k\leq [T/\varepsilon]}(X^\varepsilon_{k\varepsilon}, g_R)]$ converges to 0 as R increases to ∞ uniformly in $\varepsilon\in(0,1]$. From

$$(X^\varepsilon_{k\varepsilon}, g_R) = M(g_R)_{k\varepsilon} + \sum_{l=1}^{k}\eta_l + \varepsilon\sum_{l=1}^{k}(X^\varepsilon_{l\varepsilon}, \theta(X^\varepsilon_{l\varepsilon})A(X^\varepsilon_{l\varepsilon})g_R),$$

we deduce that

$$\sup_{0\leq k\leq [T/\varepsilon]} (X^\varepsilon_{k\varepsilon}, g_R) \leq \sup_{0\leq k\leq [T/\varepsilon]} M(g_R)_{k\varepsilon} + \sum_{k=1}^{[T/\varepsilon]} |\eta_k| + c\frac{T}{R} \sup_{0\leq k\leq [T/\varepsilon]} (X^\varepsilon_{k\varepsilon}, \mathbf{1}), \tag{27}$$

where c depends only on θ, b and σ. We used that

$$\| g_R \|_* \leq \frac{1}{R} \| g \|_* \leq c/R.$$

Using this inequality again, we deduce from (25), with φ replaced by g_R, and (16) respectively that the second and third terms of the right hand member converge to 0 in L^1 as R increases to $+\infty$ uniformly in $\varepsilon \in (0,1]$.

From Doob's inequality and the definition of κ (in Lemma 5), we get

$$\begin{aligned} \mathbb{E}[\sup_{0\leq k\leq [T/\varepsilon]} M(g_R)^2_{k\varepsilon}] &\leq 4\mathbb{E}[M(g_R)^2_{[T/\varepsilon]\varepsilon}] \\ &\leq 4(\mu_0, g_R)^2 + 4\mathbb{E}[\langle M(g_R)\rangle_{[T/\varepsilon]\varepsilon}] \\ &\leq 4(\mu_0, g_R)^2 + 4\mathbb{E}[\sum_{l=1}^{[T/\varepsilon]} \kappa_l] + 4\varepsilon \sum_{k=0}^{[T/\varepsilon]} \mathbb{E}[(X^\varepsilon_{k\varepsilon}, \theta(X^\varepsilon_{k\varepsilon}) g_R^2)]. \end{aligned} \tag{28}$$

We have $\lim_{R\to\infty} (\mu_0, g_R) = 0$. We deduce from (26) with φ replaced by g_R, and (16), that $\mathbb{E}[\sum_{l=1}^{[T/\varepsilon]} \kappa_l]$ converges to 0 as R increases to $+\infty$ uniformly in $\varepsilon \in (0,1]$. Since θ is bounded from above, we deduce from (13) and then (25) that, for $k \leq [T/\varepsilon]$,

$$\begin{aligned} \mathbb{E}[(X^\varepsilon_{k\varepsilon}, \theta(X^\varepsilon_{k\varepsilon}) g_R^2)] &\leq c\mathbb{E}[(X^\varepsilon_{k\varepsilon}, g_R^2)] \\ &\leq c\mathbb{E}[\sum_{l=1}^{k} |\eta^\varepsilon_l|] + c\varepsilon \sum_{l=0}^{k-1} \mathbb{E}[(X^\varepsilon_{l\varepsilon}, \theta(X^\varepsilon_{l\varepsilon}) A(X^\varepsilon_{l\varepsilon}) g_R^2)] + c(\mu_0, g_R^2) \\ &\leq c\sqrt{\varepsilon} \| g_R^2 \|_* + c\varepsilon \frac{1}{R} \sum_{l=0}^{k-1} \mathbb{E}[(X^\varepsilon_{l\varepsilon}, \mathbf{1})] + c(\mu_0, g_R^2) \\ &\leq c\sqrt{\varepsilon} \frac{1}{R} + c\varepsilon \frac{1}{R} k\mathbb{E}[\sup_{0\leq l\leq [T/\varepsilon]} (X^\varepsilon_{l\varepsilon}, \mathbf{1})] + c(\mu_0, g_R^2), \end{aligned}$$

where the constant c depends only on θ, b, σ and T. We deduce that $\mathbb{E}[(X^\varepsilon_{k\varepsilon}, \theta(X^\varepsilon_{k\varepsilon}) g_R^2)]$ converges to 0 as R increases to $+\infty$ uniformly in $\varepsilon \in (0,1]$.

We deduce from those results and the upper bound in (28), that $\mathbb{E}[\sup_{0\leq k\leq [T/\varepsilon]} M(g_R)^2_{k\varepsilon}]$ decreases to 0 as R increases to $+\infty$ uniformly in $\varepsilon \in (0,1]$. This implies thanks to (27) that $\mathbb{E}[\sup_{0\leq k\leq T/\varepsilon} (X^\varepsilon_{k\varepsilon}, \mathbf{1}_{B^c_{2R}})]$ decreases to 0 as R increases to ∞. In particular, for every $\rho, T > 0$, there exist $R > 1$ such that

$$\sup_{0<\varepsilon\leq 1} \mathbb{P}(\sup_{0\leq k\leq T/\varepsilon} (X^\varepsilon_{k\varepsilon}, \mathbf{1}_{B^c_{2R}}) > \rho) < \rho.$$

$\square$

4. Proof of Theorem 1

The proof will be done in 5 lemmas and follows [9]. Theorem 1 is a direct consequence of Lemma 10 and 12. Let $\varphi \in C^2$ be such that $\|\varphi\|_* < \infty$. To remember that $M(\varphi)$ depends on ε, we will write now $M^\varepsilon(\varphi)$ instead of $M(\varphi)$. Let $M^\varepsilon(\varphi) = (M^\varepsilon(\varphi)_t, t \geq 0)$ be the right continuous step function which is the extension of $(M^\varepsilon(\varphi)_{k\varepsilon}, k \in \mathbb{N})$.

Lemma 7. *The process $(\langle M^\varepsilon(\varphi)\rangle, \varepsilon \in (0,1])$ is C-tight as ε decreases to 0.*

Proof. Thanks to proposition VI.3.26 of [4], it is enough to check that for all $T > 0$, $\alpha > 0$ and $\eta > 0$, there exist $K > 0$ and $h > 0$, $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$,

$$\mathbb{P}(\sup_{t\leq T}\langle M^\varepsilon(\varphi)\rangle_t \geq K) \leq \alpha, \tag{29}$$

$$\mathbb{P}(\sup_{s\leq t\leq T, |t-s|\leq h}\langle M^\varepsilon(\varphi)\rangle_t - \langle M^\varepsilon(\varphi)\rangle_s \geq \eta) \leq \alpha. \tag{30}$$

Using (19) in Lemma 3 with $s = 0$ and $t = T$, we have

$$\begin{aligned}\mathbb{P}(\sup_{t\leq T}\langle M^\varepsilon(\varphi)\rangle_t \geq K) &\leq \frac{1}{K}\,\mathbb{E}[\langle M^\varepsilon(\varphi)\rangle_T] \\ &\leq \frac{1}{K}\,cT\mathbb{E}[\sup_{k\leq T/\varepsilon}(X^\varepsilon_{k\varepsilon}, 1)] \\ &\leq \frac{1}{K}\,cT\mathbb{E}[\sup_{k\leq T/\varepsilon}(X^\varepsilon_{k\varepsilon}, 1)^2]^{1/2}.\end{aligned}$$

Then (29) can be deduced from Lemma 2.

Notice that if $|t-s| \leq h$, then $|[t/\varepsilon]\varepsilon - [s/\varepsilon]\varepsilon| \leq h + \varepsilon$. Using again (19) in Lemma 3, we have

$$\begin{aligned}\mathbb{P}(\sup_{s\leq t\leq T, |t-s|\leq h}\langle M^\varepsilon(\varphi)\rangle_t - \langle M^\varepsilon(\varphi)\rangle_s \geq \eta) &\leq \mathbb{P}(c(h+\varepsilon)\,\|\varphi\|^2_\infty \sup_{k\leq T/\varepsilon}(X^\varepsilon_{k\varepsilon}, \mathbf{1}) \geq \eta) \\ &\leq \frac{c^2(h+\varepsilon)^2\,\|\varphi\|^4_\infty}{\eta^2}\mathbb{E}[\sup_{k\leq T/\varepsilon}(X^\varepsilon_{k\varepsilon}, \mathbf{1})^2].\end{aligned}$$

And (30) can be deduced from Lemma 2. □

Lemma 8. *The process $(M^\varepsilon(\varphi), \varepsilon \in (0,1])$ is C-tight as ε decreases to 0.*

Proof. We have already proved the C-tightness of $(\langle M^\varepsilon(\varphi)\rangle, \varepsilon \in (0,1])$. From theorem VI.4.13 of [4], we get that $(M^\varepsilon(\varphi), \varepsilon \in (0,1])$ is tight. To get the C-tightness, it is enough to check (see proposition VI.3.26 of [4]) that for all $T > 0$ and all $\eta > 0$,

$$\lim_{\varepsilon\to 0}\mathbb{P}(\sup_{k\leq T/\varepsilon}|M^\varepsilon(\varphi)_{(k+1)\varepsilon} - M^\varepsilon(\varphi)_{k\varepsilon}| \geq \eta) = 0.$$

We have:

$$\begin{aligned}\mathbb{P}(\sup_{k\leq T/\varepsilon}|M^\varepsilon(\varphi)_{(k+1)\varepsilon}-M^\varepsilon(\varphi)_{k\varepsilon}|\geq\eta) &\leq \frac{1}{\eta^4}\mathbb{E}[\sup_{k\leq T/\varepsilon}(M^\varepsilon(\varphi)_{(k+1)\varepsilon}-M^\varepsilon(\varphi)_{k\varepsilon})^4]\\ &\leq \frac{1}{\eta^4}\mathbb{E}[\sum_{k\leq T/\varepsilon}(M^\varepsilon(\varphi)_{(k+1)\varepsilon}-M^\varepsilon(\varphi)_{k\varepsilon})^4]\\ &\leq \frac{1}{\eta^4}c\,\|\varphi\|_\infty^4\,\varepsilon^2\frac{T}{\varepsilon}\mathbb{E}[1+\sup_{k\leq T/\varepsilon}(X^\varepsilon_{k\varepsilon},\mathbf{1})^2],\end{aligned}$$

where we used (18) of Lemma 3 for the last inequality. We conclude using Lemma 2.

□

Lemma 9. *The process $(X^\varepsilon(\varphi),\varepsilon\in(0,1])$ is C-tight as ε decreases to 0.*

Proof. From (13) we get, for $k\varepsilon\leq t<(k+1)\varepsilon$,

$$\begin{aligned}(X^\varepsilon_t,\varphi) &= (X^\varepsilon_{k\varepsilon},\varphi)\\ &= (\mu_0,\varphi)+M^\varepsilon(\varphi)_{k\varepsilon}+\varepsilon\sum_{l<k}(X^\varepsilon_{l\varepsilon},\theta(X^\varepsilon_{l\varepsilon})A(X^\varepsilon_{l\varepsilon})\varphi)+\sum_{l\leq k}\eta^\varepsilon_k\\ &= (\mu_0,\varphi)+M^\varepsilon(\varphi)_{k\varepsilon}+\Lambda^\varepsilon_t+Z^\varepsilon_t,\end{aligned}$$

where

$$\Lambda^\varepsilon_t=\int_0^{[t/\varepsilon]\varepsilon}(X^\varepsilon_u,\theta(X^\varepsilon_u)A(X^\varepsilon_u)\varphi)\,du \qquad\text{and}\qquad Z^\varepsilon_t=\sum_{l\leq k}\eta^\varepsilon_k.$$

Let us check that $(\Lambda^\varepsilon,\varepsilon\in(0,1])$ is C-tight as ε decreases to 0. Since $\Lambda^\varepsilon_0=0$, we have

$$\begin{aligned}\mathbb{P}(\sup_{0\leq s\leq T}|\Lambda^\varepsilon_s|\geq K) &\leq \frac{1}{K^2}\,\mathbb{E}[\sup_{0\leq s\leq T}(\int_0^{[s/\varepsilon]\varepsilon}(X^\varepsilon_u,\theta(X^\varepsilon_u)A(X^\varepsilon_u)\varphi)\;du)^2]\\ &\leq \frac{c}{K^2}\mathbb{E}[(\int_0^{[T/\varepsilon]\varepsilon}(X^\varepsilon_u,\mathbf{1})\;du)^2]\\ &\leq \frac{c}{K^2},\end{aligned}$$

thanks to Lemma 2. We also have for $0\leq s\leq t\leq T$, $h\geq 0$,

$$\begin{aligned}\mathbb{P}(\sup_{0\leq s\leq t\leq T,|t-s|\leq h}|\Lambda^\varepsilon_t-\Lambda^\varepsilon_s|\geq\eta) &= \mathbb{P}(\sup_{0\leq s\leq t\leq T,|t-s|\leq h}|\int_{[s/\varepsilon]\varepsilon}^{[t/\varepsilon]\varepsilon}(X^\varepsilon_u,\theta(X^\varepsilon_u)A(X^\varepsilon_u)\varphi)\,du|\geq\eta)\\ &\leq \frac{c}{\eta^2}\,\|\varphi\|_*^2(h+\varepsilon)^2\mathbb{E}[\sup_{k\leq T/\varepsilon}(X^\varepsilon_{k\varepsilon},1)^2]\\ &\leq \frac{c}{\eta^2}\,\|\varphi\|_*^2(h+\varepsilon)^2,\end{aligned}$$

thanks to Lemma 2. Thanks to proposition VI.3.26 of [4], those two inequalities imply that $(\Lambda^\varepsilon,\varepsilon\in(0,1])$ is C-tight. From lemma 4, we get that $\sup_{t\leq T}Z^\varepsilon_t$ converges to 0 in L^1 as ε decreases to 0. In particular it is C-tight. As a sum of C-tight processes, the family $(X^\varepsilon(\varphi),\varepsilon\in(0,1])$ is C-tight as ε decreases to 0.

□

Lemma 10. *The process family of process $(X^\varepsilon, \varepsilon \in (0,1])$ is C-tight as ε decreases to 0.*

This result is a consequence of the next theorem which is stated in [9], Lemma 9 and Lemma 6.

Let $C_b(\mathbb{R}^d) = \{f : \mathbb{R}^d \to \mathbb{R},\ f \text{ bounded and continuous}\}$. Let D_0 be a separating class in C_b in $\mathcal{M}_f$ (that is if μ and ν belongs to $\mathcal{M}_f$, if $\mu(f) = \nu(f)$ for all $\varphi \in D_0$, then $\mu = \nu$) containing 1 and which is closed under addition.

Theorem 11. *A sequence of càdlàg $\mathcal{M}_f$-valued process $(X^\varepsilon, \varepsilon \in (0,1])$ is C-tight as ε decreases to 0, in $D(\mathbb{R}+, \mathcal{M}_f)$ if and only if the following conditions hold:*

1. *$\forall \varphi \in D_0$, the process $(X^\varepsilon(\varphi), \varepsilon \in (0,1])$ is C-tight in $D(\mathbb{R}_+, \mathbb{R})$ as ε decreases to 0.*
2. *For every $\rho, T > 0$, there is a compact set $K_{\rho,T}$ in $\mathbb{R}^d$ such that*

$$\sup_{\varepsilon \in (0,1]} \mathbb{P}(\sup_{0 \le t \le T} (X_t^\varepsilon, \mathbf{1}_{K_{\rho,T}^c}) > \rho) < \rho.$$

Lemma 12. *Any limiting measure valued process $Y = (Y_t, t \ge 0)$ of $(X^\varepsilon, \varepsilon \in (0,1])$ as ε decreases to 0 , satisfies the martingale problem (MP) and has a continuous version.*

Proof. Let $(\varepsilon_n, n \in \mathbb{N})$ be a sequence decreasing to 0 such that $(X^{\varepsilon_n}, n \in \mathbb{N})$ converges in law to Y. Using Skorokhod's representation theorem, we may suppose that we have an a.s. convergence. Recall from (13) that

$$(X_t^{\varepsilon_n}, \varphi) = (\mu_0, \varphi) + M^{\varepsilon_n}(\varphi)_t + \int_0^{[t/\varepsilon_n]\varepsilon_n} (X_u^{\varepsilon_n}, \theta(X_u^{\varepsilon_n}) A(X_u^{\varepsilon_n})\varphi)\, du + \sum_{l \le t/\varepsilon_n} \eta_l^{\varepsilon_n}, \tag{31}$$

for any φ such that $\| \varphi \|_* < \infty$.

Using (25) as well as (26), and Lemma 2, we deduce that $\sup_{t \le T} \sum_{l \le t/\varepsilon_n} \eta_l^{\varepsilon_n}$ (resp. $\sup_{t \le T} \sum_{l \le t/\varepsilon_n} \kappa_l^{\varepsilon_n}$) converges to 0 in L^1 (resp. L^2) as $n \to \infty$. There exists a subsequence of $(\varepsilon_n, n \ge 0)$ such that those two convergences hold a.s. We still write $(\varepsilon_n, n \ge 0)$ for this subsequence. Since X^{ε_n} is C-tight, we get that Y is continuous and that a.s. for all $t \ge 0$, $X_t^{\varepsilon_n}$ converges to Y_t. In particular, since $\| \varphi \|_*$ is finite, this implies that a.s. for all $t \ge 0$, $(X_t^{\varepsilon_n}, \varphi)$ converges to (Y_t, φ) and $\int_0^{[t/\varepsilon_n]\varepsilon_n} (X_u^{\varepsilon_n}, \theta(X_u^{\varepsilon_n}) A(X_u^{\varepsilon_n})\varphi)\, du$ converges to $\int_0^t (Y_u, \theta(Y_u) A(Y_u)\varphi)\, du$. From (31) we deduce that $(M^{\varepsilon_n}(\varphi)_t, t \ge 0)$ converge a.s. to a continuous process say $(M(\varphi)_t, t \ge 0)$. And we have

$$(Y_t, \varphi) = (\mu_0, \varphi) + M(\varphi)_t + \int_0^t (Y_u, \theta(Y_u) A(Y_u)\varphi)\, du. \tag{32}$$

From Lemma 5, we have

$$\langle M^{\varepsilon_n}(\varphi) \rangle_t = 4 \int_0^{[t/\varepsilon_n]\varepsilon_n} (X_u^{\varepsilon_n}, \theta(X_u^{\varepsilon_n})\varphi^2)\, du + \sum_{k < [t/\varepsilon_n]\varepsilon_n} \kappa_k^{\varepsilon_n}.$$

In particular, $(\langle M^{\varepsilon_n}(\varphi) \rangle_t, t \ge 0)$ converge a.s. to

$$Q = (4 \int_0^t (Y_u, \theta(Y_u)\varphi^2)\, du, t \ge 0).$$

From (20) we deduce the martingale $M^{\varepsilon_n}(\varphi)$ (resp. $M^{\varepsilon_n}(\varphi)^2 - \langle M^{\varepsilon_n}(\varphi)\rangle$) is uniformly L^2 (resp. L^1). This implies that $M(\varphi)$ is an L^2 martingale and $M(\varphi)^2 - Q$ is an L^1 martingale (with respect to the filtration generated by $M(\varphi)$ and Q). Since $M(\varphi)$ and Q are continuous, we also get that $\langle M(\varphi)\rangle = Q$. To end the proof, we need to check that $M(\varphi)$ is a martingale with respect to the filtration generated by Y. Let $m \geq 1$, f be a bounded continuous function defined on $\mathcal{M}_f^m$. Let $0 \leq t_1 \leq \ldots \leq t_m \leq t \leq s$. Because of the uniform integrability of $M^{\varepsilon_n}(\varphi)$, we have that $\mathbb{E}[f(X^{\varepsilon_n}_{t_1}, \ldots, X^{\varepsilon_n}_{t_m})(M^{\varepsilon_n}(\varphi)_s - M^{\varepsilon_n}(\varphi)_t)]$ converges to $\mathbb{E}[f(Y_{t_1}, \ldots, Y_{t_m})(M(\varphi)_s - M(\varphi)_t)]$ as $n \to \infty$. Since $\mathbb{E}[f(X^{\varepsilon_n}_{t_1}, \ldots, X^{\varepsilon_n}_{t_m})(M^{\varepsilon_n}(\varphi)_s - M^{\varepsilon_n}(\varphi)_t)] = 0$, we deduce that $\mathbb{E}[f(Y_{t_1}, \ldots, Y_{t_m})(M(\varphi)_s - M(\varphi)_t)] = 0$. As this equality holds for any m, $0 \leq t_1 \leq \ldots \leq t_m \leq t \leq s$ and any bounded continuous function f, and since $M(\varphi)$ is adapted to the filtration generated by Y (thanks to formula (32)), we deduce that $M(\varphi)$ is a martingale with respect to the filtration generated by Y and that Q is its quadratic variation.

□

5. Appendix

We recall some moment formula for superprocesses under the excursion measure $\mathbb{N}$. Recall notations from section 2.1. Let φ and ψ be bounded measurable functions defined on $\mathbb{R}^d$.

Let Z be a diffusion with infinitesimal generator A started at point x under $\mathbb{E}_x$. Let $(\mathrm{P}_v, v \geq 0)$ denote the transition kernel of the diffusion Z. We have for $u > 0$,

$$\mathbb{N}_{x,A}[(X_u, \varphi)] = \mathbb{E}_x[\varphi(Z_u)] = \mathrm{P}_u(\varphi)(x). \tag{33}$$

We have for $u \geq v > 0$,

$$\mathbb{N}_{x,A}[(X_u, \varphi)(X_v, \psi)] = \mathbb{N}_{x,A}[(X_v, \varphi)(X_v, \mathrm{P}_{u-v}\psi)], \tag{34}$$

and

$$\mathbb{N}_{x,A}[(X_u, \varphi)(X_v, \psi)] = 4 \int_0^v dr\ \mathbb{E}[\mathrm{P}_{u-r}\varphi(Z_r)\mathrm{P}_{v-r}\psi(Z_r)]. \tag{35}$$

References

[1] J.-S. DHERSIN and L. SERLET. A stochastic calculus approach for the brownian snake. *Canad. J. of Math.*, 52(1):92–118, 2000.

[2] E. DYNKIN. Branching particle systems and superprocesses. *Ann. Probab.*, 19:1157–1194, 1991.

[3] S. N. ETHIER and T. G. KURTZ. *Markov processes.* Wiley, 1986.

[4] J. JACOD and A. N. SHIRYAEV. *Limit theorems for stochastic processes.* Springer-Verlag, 1987.

[5] J.-F. LE GALL. A path-valued Markov process and its connections with partial differential equations. In *Proceedings in First European Congress of Mathematics*, volume II, pages 185–212. Birkhäuser, Boston, 1994.

[6] J.-F. LE GALL. The Brownian snake and solutions of $\Delta u = u^2$ in a domain. *Probab. Th. Rel. Fields*, 102:393–432, 1995.

[7] E. PERKINS. Measure-valued branching-diffusions with spatial interactions. *Probab. Th. Rel. Fields*, 1992.

[8] E. PERKINS. *On the martingale problem for interactive measure-valued branching diffusions.* Number 549. Mem. Am. Math. Soc., 1995.

[9] E. A. PERKINS. Dawson-Watanbe superprocesses and measure-valued diffusions. Lecture at Saint-Flour, 1999.

[10] L. ROGERS and J. WALSH. Local time and stochastic area integrals. *Ann. Probab.*, 19(2):457–482, 1991.

[11] S. WATANABE. Killing operations in super-diffusions by brownian snakes. In *Trends in probability and related analysis (Taipei, 1998)*, pages 177–190, River Edge, 1999. World Sci. Publishing.

ENPC-CERMICS, 6 av. Blaise Pascal, Champs-sur-Marne, 77455 Marne La Vallée, France.
E-mail address: delmas@cermics.enpc.fr

UFR de Mathématiques et d'Informatique, Université René Descartes, 45 rue des Saints Pères, 75270 Paris Cedex 06, France
E-mail address: dhersin@math-info.univ-paris5.fr