Existence of solution for a micro-macro model of polymeric fluid : the FENE model

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1 Introduction and motivation

We continue here our endeavor, initiated in [9], to put the micro-macro models for polymeric fluid flows on a mathematically sound ground.

Let us recall for consistency that these models aim at circumventing the difficulty of finding a closure equation at the pure macroscopic level. In the case of non newtonian fluids such as polymeric fluids, such an equation links the stress tensor to the velocity field through, say, a partial differential equation or an integral relation. In order to build a micro-macro model, one goes down to the microscopic scale and makes use of kinetic theory to obtain a mathematical model for the evolution of the microstructures of the fluid, here the configurations of the polymer chains. We refer the reader to [9] or [12] for a more complete introduction to this type of models and to [1, 2, 6, 14] for a comprehensive survey of the physical background. Contrary to the purely macroscopic approach where the microscopic models are used to derive macroscopic constitutive equations, most of the time through some simplifying assumptions (closure assumptions) whose impact on the result is difficult to evaluate, the so-called micro-macro approach consists in keeping explicit track of both scales. In mathematical terms, this micro-macro approach translates

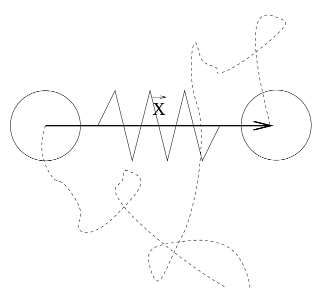

Figure 1: The polymer (in dashed line) is modelled by a "dumbbell" : two beads linked by a spring. The vector $\boldsymbol{X}$ is called the end-to-end vector.

into a coupled multiscale system of the following form (we consider here the simplest case : the so-called dumbbell model, where the polymer is modelled by two beads linked by a spring, see Figure 1) :

$$
\left\{
\begin{array}{l}
\rho\left(\dfrac{\partial \boldsymbol{u}}{\partial t}+\boldsymbol{u}.\nabla \boldsymbol{u}\right)=-\nabla p+\eta \Delta \boldsymbol{u}+\operatorname{div} \boldsymbol{\tau}, \\
\operatorname{div} \boldsymbol{u}=0, \\
\boldsymbol{\tau}=n \displaystyle\int(\boldsymbol{X} \otimes \boldsymbol{F}(\boldsymbol{X})) \psi(t, \boldsymbol{x}, \boldsymbol{X})\, d \boldsymbol{X}-n k_B T \mathrm{Id}, \\
\dfrac{\partial \psi}{\partial t}+\boldsymbol{u}.\nabla_{\boldsymbol{x}} \psi=-\operatorname{div}_{\boldsymbol{X}}\left(\left(\nabla_{\boldsymbol{x}} \boldsymbol{u}\, \boldsymbol{X}-\dfrac{2}{\zeta} \boldsymbol{F}(\boldsymbol{X})\right) \psi\right)+\dfrac{\sigma^2}{\zeta^2} \Delta_{\boldsymbol{X}} \psi,
\end{array}
\right.
\tag{1}
$$

where $\boldsymbol{u}(t,\boldsymbol{x})$ is the velocity of the fluid, $p(t,\boldsymbol{x})$ the pressure, $\boldsymbol{\tau}(t,\boldsymbol{x})$ the stress tensor, and $\psi(t,\boldsymbol{x},\boldsymbol{X})$ denotes the probability density function of the end-to-end vector $\boldsymbol{X}$ of the polymer at time t and at position $\boldsymbol{x}$. The other symbols are physical parameters : $\boldsymbol{F}(\boldsymbol{X})$ is the entropic force a representative polymer chain experiences, ρ and η respectively are the density and the viscosity of the ambiant fluid, n denotes the density of polymers, the coefficient σ is defined by $\sigma^2 = 2k_BT\zeta$ with T the temperature and ζ the friction coefficient of the beads within the fluid. It is to be noted that the Fokker-Planck equation on ψ holds at each macroscopic point $\boldsymbol{x}$.

Let us at once indicate that, from a physical point of view, the dumbbell model, for which the configuration space is $\mathbb{R}^3$ (that is, $\boldsymbol{X} \in \mathbb{R}^3$), is too crude to completely describe the evolution of the polymer chain. But this model serves as an efficient test problem for more sophisticated modelling strategies. In order to be more realistic, one has indeed to consider a model where the polymer is not just modelled by its end-to-end vector but by a chain of beads and springs, which leads to a system of the form (1), but with a Fokker-Planck equation set in a configuration space of dimension larger than 3. This highly complicates a direct numerical attack of the Fokker-Planck equation on ψ (there exists however such tentatives of direct attacks, see [17] and the references therein).

The main trend in the community of researchers performing numerical simulations of such complex flows is therefore to "replace" the Fokker-Planck equation by the underlying stochastic differential equation ruling the evolution of random variables whose density is ψ. Such an hybrid strategy mixing stochastic and deterministic aspects can be advantageously studied already in the setting of the simple dumbbell model. In the simple case of the dumbbell model, it indeed consists in turning (1) into the following mathematical system :

$$
\left\{
\begin{array}{l}
\rho\left(\dfrac{\partial \boldsymbol{u}}{\partial t}+\boldsymbol{u}.\nabla \boldsymbol{u}\right)=-\nabla p+\eta \Delta \boldsymbol{u}+\operatorname{div}(\boldsymbol{\tau}), \\
\operatorname{div}(\boldsymbol{u})=0, \\
\boldsymbol{\tau}=n \mathbb{E}(\boldsymbol{X} \otimes \boldsymbol{F}(\boldsymbol{X}))-n k_B T \mathrm{Id}, \\
d \boldsymbol{X}+\boldsymbol{u}.\nabla \boldsymbol{X} d t=\left(\nabla \boldsymbol{u} \boldsymbol{X}-\dfrac{2}{\zeta} \boldsymbol{F}(\boldsymbol{X})\right) d t+\dfrac{\sqrt{2} \sigma}{\zeta} d \boldsymbol{W}_t,
\end{array}
\right.
\tag{2}
$$

where $\boldsymbol{X}(t,\boldsymbol{x})$ is a stochastic process representing the end-to-end vector of the polymer modelled by a dumbbell (see Figure 1). The stochastic process $\boldsymbol{W}_t$ is a standard (multi-dimensional) Brownian motion and $\mathbb{E}$ denotes the expectation.

In our previous work (see [9]), we have made the (simple) mathematical analysis and the (more intricate) numerical analysis of this model when applied to a simple Couette flow (see Figure 2) and when considering a linear force in the dumbbell (model of hookean dumbbells : $\boldsymbol{F}(\boldsymbol{X}) = H\boldsymbol{X}$, with H a constant coefficient) (see also [7] for an other example of a mathematical analysis of a viscoelastic flow in this geometry). It then reduces to the

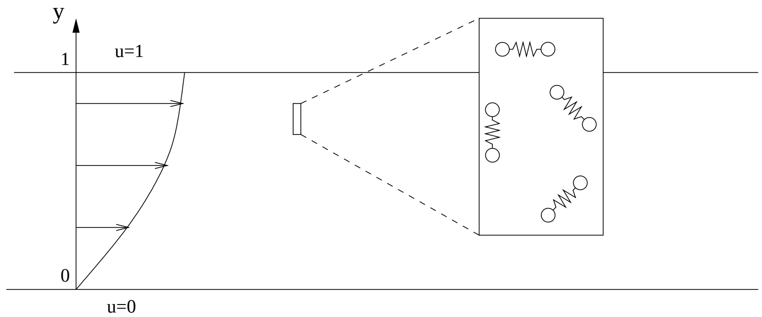

Figure 2: Velocity profile in a shear flow of a dilute solution of polymers.

system :

$$\begin{cases} \partial_t u - \partial_{yy} u = \partial_y \tau + f_{ext}, \\ \tau(t,y) = \mathbb{E}(X_t^y Y_t), \\ dX_t^y = \left(-\dfrac{X_t^y}{2} + \partial_y u Y_t \right) dt + dV_t, \\ dY_t = -\dfrac{Y_t}{2} dt + dW_t, \end{cases} \tag{3}$$

where (due to the simple geometry of the problem) $u = \boldsymbol{u}_x(y)$ and $\tau = \boldsymbol{\tau}_{xy}(y)$ are here valued in $\mathbb{R}$, while the space variable y varies in $\mathcal{O} = (0,1)$. In (3) and henceforth, we write all the equations in a non dimensional form and f_{ext} denotes an external force. The stochastic variables (X_t^y, Y_t) denote the components of the stochastic variable $\boldsymbol{X}_t$ introduced before. We have proved in [9] the well-posedness of the Cauchy problem by showing a global-in-time existence and uniqueness result. On the other hand, we have shown the convergence of the numerical approximation of the solution (finite difference in time, $\mathbb{P}1$ finite element in space, and Monte Carlo realizations) to the exact solution.

Despite its interest as a test problem for many mathematical and numerical techniques, the above hookean dumbbell model is somewhat limited since it can in fact be written under the form of a purely macroscopic model, namely the Oldroyd-B model, that we recall here in its differential form :

$$\boldsymbol{\tau} + \lambda \frac{\delta \boldsymbol{\tau}}{\delta t} = n k_B T \lambda (\nabla \boldsymbol{u} +^t \nabla \boldsymbol{u}), \tag{4}$$

with the upper convected derivative $\frac{\delta}{\delta t}$ defined by :

$$\frac{\delta \boldsymbol{\tau}}{\delta t} = \frac{\partial \boldsymbol{\tau}}{\partial t} + \boldsymbol{u}.\nabla \boldsymbol{\tau} - \boldsymbol{\tau}^t \nabla \boldsymbol{u} - \nabla \boldsymbol{u} \boldsymbol{\tau},$$

where $\lambda = \frac{\zeta}{4H}$ is a characteristic time.

In order to address more general situations, we here want to treat the case of a micro-macro model which cannot be written under the form of a macroscopic model, and therefore is genuinely micro-macro. An instance of this model (at least to the best of our knowledge, see [11] on this subject) is the so-called FENE model where the acronym FENE stands for Finite Extensible Nonlinear Elastic. In this model, the force within the spring has the following expression : $\boldsymbol{F}(\boldsymbol{X}) = \frac{H \boldsymbol{X}}{1 - ||\boldsymbol{X}||^2/(b k_B T/H)}$ (H and b being two constant coefficients). This model is more realistic from a physical point of view than the model of hookean dumbbell since it accounts for the finite extensibility of the real polymer. For example, this model exhibits shear-thinning or hysteretic behavior in elongational flows, contrary to the linear model of hookean dumbbells, and accordingly to experiment.

Like in [9], we only consider in the sequel the setting of a simple Couette flow. The FENE model then reads, in a non-dimensional form :

$$\partial_t u - \partial_{yy} u = \partial_y \tau + f_{ext}, \tag{5}$$

$$\tau = \mathbb{E}\left(\frac{X_t^y Y_t^y}{1 - \frac{(X_t^y)^2 + (Y_t^y)^2}{b}} \right), \tag{6}$$

$$\begin{cases} dX_t^y = \left(-\dfrac{1}{2} \dfrac{X_t^y}{1 - \frac{(X_t^y)^2 + (Y_t^y)^2}{b}} + \partial_y u \, Y_t^y \right) dt + dV_t, \\ dY_t^y = \left(-\dfrac{1}{2} \dfrac{Y_t^y}{1 - \frac{(X_t^y)^2 + (Y_t^y)^2}{b}} \right) dt + dW_t. \end{cases} \tag{7}$$

where the non-dimensional parameter $b > 0$ measures the finite extensibility of the polymer and is in practice of the order of 100 (see [14] page 217). The space variable y varies in $\mathcal{O} = (0, 1)$ and $t \in [0, T]$. The random variables are defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. The random process (V_t, W_t) is a $(\mathcal{F}_t)$-two-dimensional Brownian motion. For simplicity, the boundary conditions are taken homogenous : $u(t, y = 0) = u(t, y = 1) = 0$. The initial velocity is $u(t = 0, .) = u_0$, and (X_0, Y_0) is a $\mathcal{F}_0$-measurable random variable.

It is worth emphasizing the differences with respect to the hookean dumbbell model (3) (which can formally be recovered from (5-7) by letting b go to infinity) :

- No explicit expression of the stress in function of the velocity is known to date (the FENE model is not closed),
- Both components X_t^y and Y_t^y of the connecting vector modelling the polymer chain depend on the space variable,
- The drift term in (7) is nonlinear and singular.

Such differences make the mathematical analysis more delicate than that for the hookean dumbbell model. The purpose of the present article is to conduct such an analysis. We hope to be able to treat the numerical analysis of such a system in the future.

To the best of our knowledge, system (5-7) has never been analyzed mathematically. The only result concerning a problem close to (5-7) is due to M. Renardy in [15] where system (1) is analyzed and proved to admit a local-in-time solution in spaces of regular functions. The result applies to the case of a flow of polymeric inviscid fluid ($\eta = 0$) in $\mathbb{R}^3$ with spring forces slightly more explosive than the FENE force.

The article is organized as follows. In Section 2, we deal with the stochastic differential equation (7) (see also [8] for a more complete analysis of this stochastic differential equation). We first show the existence of a solution when $u = 0$ and then, using the Girsanov Theorem, we build a weak solution to (7) when the velocity u is arbitrarily given. Using Yamada-Watanabe Theorem, we next show that (7) admits a unique strong solution. The main difficulty in proving the existence of a solution to the stochastic differential equation comes from the singular nature of the drift. We shall however see that we can take benefit of this singular nature to obtain an *a priori* bound on the stochastic processes (which does not exist in the hookean case and must therefore be circumvented by *ad hoc* cut-off techniques, see [9]). We next consider the coupled system (5-7) and show some *a priori* estimates in Section 3. We use these estimates in Section 4 to prove our main result (stated in Theorem 1), namely a local-in-time existence and uniqueness result of the solution (u, X_t^y, Y_t^y) to the coupled system (5-7), being understood that (X_t^y, Y_t^y) is a strong solution (in the sense of probability theory) of (7) and u is a regular solution giving to (5) an almost everywhere sense (which requires a good regularity of the data : initial condition, boundary conditions, f_{ext}). We unfortunately are unable to extend this existence result to any arbitrary large time, nor to extend it to a less regular class of data. The numerical analysis of some discretization schemes used for the simulation of stochastic differential equations of type (7) is currently under study.

2 Existence of a solution to the stochastic differential equation

In this section, we consider the stochastic differential equation (7) with a given velocity u. More precisely, we fix y in $\mathcal{O}$, we set $g(t) = \partial_y u(y,t)$ for conciseness, and we suppose throughout this section that

$$g \in L^2_t. \tag{8}$$

We are interested in solving for $t \geq 0$ the following stochastic differential equation :

$$\begin{cases} dX^g_t = \left(-\frac{1}{2} \frac{X^g_t}{1 - \frac{(X^g_t)^2 + (Y^g_t)^2}{b}} + g(t)\, Y^g_t \right) dt + dV_t, \\ dY^g_t = \left(-\frac{1}{2} \frac{Y^g_t}{1 - \frac{(X^g_t)^2 + (Y^g_t)^2}{b}} \right) dt + dW_t, \end{cases} \tag{9}$$

with initial condition (X_0, Y_0). Throughout this paper we will suppose that (X_0, Y_0) is such that $\mathbb{P}\left(X_0^2 + Y_0^2 \geq b\right) = 0$. In this section, we consider that t varies in the whole of $\mathbb{R}_+$.

2.1 Notion of solution

Let us begin by giving a precise mathematical meaning to (9).

Definition 1 *We consider a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, a $(\mathcal{F}_t)$-two-dimensional Brownian motion (V_t, W_t) and a $\mathcal{F}_0$-measurable random variable (X_0, Y_0). We shall say that a $(\mathcal{F}_t)$-adapted process (X^g_t, Y^g_t) is a solution to (9) when : for $\mathbb{P}$-a.e. ω, $\forall t \geq 0$,*

$$\begin{cases} \displaystyle\int_0^t \left| \frac{1}{1 - \frac{(X^g_s)^2 + (Y^g_s)^2}{b}} \right| ds < \infty \text{ with the convention } \frac{1}{1 - \frac{x^2+y^2}{b}} = +\infty \text{ if } x^2 + y^2 = b, \\ X^g_t = X_0 + \displaystyle\int_0^t \left(-\frac{1}{2} \frac{X^g_s}{1 - \frac{(X^g_s)^2 + (Y^g_s)^2}{b}} + g(s) Y^g_s \right) ds + V_t, \\ Y^g_t = Y_0 + \displaystyle\int_0^t -\frac{1}{2} \frac{Y^g_s}{1 - \frac{(X^g_s)^2 + (Y^g_s)^2}{b}} ds + W_t. \end{cases} \tag{10}$$

Our purpose in this section is to show :

Proposition 1 *Assume that $b \geq 2$ and (8). There exists a unique $(\mathcal{F}_t)$-adapted process (X^g_t, Y^g_t) with values in $\mathcal{C}([0,\infty[, \mathbb{R}^2)$ solution to (9) in the sense of Definition 1. In addition, this solution is such that $\mathbb{P}\left(\exists t \geq 0, (X^g_t)^2 + (Y^g_t)^2 = b\right) = 0$ and (X^g_t, Y^g_t) is $\sigma\left(X_0, Y_0, (V_s, W_s)_{s \leq t}\right)$-adapted. Moreover, assuming $b > 4$ and $\left(\frac{1}{1 - \frac{X_0^2 + Y_0^2}{b}} \right)^p$ is integrable for some $p > 1$, we have the following expression of the stress (6) in function of the* *solution* (X_t^g, Y_t^g) *for* $g=0$ **henceforth denoted by** (X_t, Y_t) *:*

$$\mathbb{E}\left(\frac{X_t^g Y_t^g}{1-\frac{(X_t^g)^2+(Y_t^g)^2}{b}}\right) = \mathbb{E}\left(\left(\frac{X_t Y_t}{1-\frac{X_t^2+Y_t^2}{b}}\right)\mathcal{E}\left(\int_0^\bullet g(s) Y_s\, dV_s\right)_t\right), \tag{11}$$

where $\mathcal{E}\left(\int_0^\bullet g(s) Y_s\, dV_s\right)_t$ *is the exponential martingale :*

$$\mathcal{E}\left(\int_0^\bullet g(s) Y_s\, dV_s\right)_t = \exp\left(\int_0^t g(s) Y_s\, dV_s - \frac{1}{2}\int_0^t (g(s)Y_s)^2\, ds\right).$$

We begin by proving the uniqueness, next show the existence when $g=0$ and in a third step show the existence for a general g satisfying (8).

2.2 Uniqueness

Lemma 1 *Let* (X^g, Y^g) *and* $(\tilde{X}^g, \tilde{Y}^g)$ *be two solutions of (9) in the sense of Definition 1. Provided* (X^g, Y^g) *is such that* $\mathbb{P}\left(\exists t \geq 0, (X_t^g, Y_t^g) \geq b\right) = 0$*, then,*

$$\mathbb{P}\left(\forall t \geq 0, (X_t^g, Y_t^g) = (\tilde{X}_t^g, \tilde{Y}_t^g)\right) = 1.$$

Proof :

Let us consider the stopping time

$$\tau_n = \inf\left\{t, \max\left((X_t^g)^2 + (Y_t^g)^2, (\tilde{X}_t^g)^2 + (\tilde{Y}_t^g)^2\right) \geq b\left(1-\frac{1}{n}\right)\right\}.$$

Let set $F_x(x,y) = -\frac{1}{2}\frac{x}{1-\frac{x^2+y^2}{b}}$ and $F_y(x,y) = -\frac{1}{2}\frac{y}{1-\frac{x^2+y^2}{b}}$. These functions are Lipschitz continuous with constant K_n on the ball $B_n = \left\{(x,y),\, x^2+y^2 \leq b\left(1-\frac{1}{n}\right)\right\}$.
Let us now consider $P_t = X_t^g - \tilde{X}_t^g$ and $Q_t = Y_t^g - \tilde{Y}_t^g$. We have

$$\begin{aligned} P_t &= \int_0^t F_x(X_s^g, Y_s^g) - F_x(\tilde{X}_s^g, \tilde{Y}_s^g)\, ds + \int_0^t g(s) Q_s\, ds, \\ Q_t &= \int_0^t F_y(X_s^g, Y_s^g) - F_y(\tilde{X}_s^g, \tilde{Y}_s^g)\, ds. \end{aligned}$$

We can therefore write, for any $t \in (0, \tau_n)$:

$$|P_t| + |Q_t| \leq \int_0^t (2\,K_n + |g(s)|)\,(|P_s| + |Q_s|)\, ds.$$

Using Gronwall Lemma and the fact that $g \in L_t^1$, we deduce that, almost surely, for any $t \in (0, \tau_n)$, $P_t = 0$ and $Q_t = 0$. Hence (X_t^g, Y_t^g) and $(\tilde{X}_t^g, \tilde{Y}_t^g)$ coincide on $(0, \lim_{n\to\infty} \tau_n)$. As a consequence, $\tau_n = \inf\left\{t, (X_t^g)^2 + (Y_t^g)^2 \geq b\left(1-\frac{1}{n}\right)\right\}$ and by the assumption made on (X_t^g, Y_t^g), $\lim_{n\to\infty} \tau_n = +\infty$. ♢

Remark 1 *The proof makes a crucial use of the fact that (9) only differs from a system of ordinary differential equations by the simple addition of a Brownian motion.*

2.3 Existence when $g = 0$

The crucial lemma which will be used in the sequel states the existence of a (strong) solution to (9) when $g = 0$. We recall that this solution will be denoted in the following by (X_t, Y_t).

Lemma 2 *Assume that $b \geq 2$ and $g = 0$, then there exists a unique solution (X_t, Y_t) to (9) in the sense of Definition 1. This solution is such that $\mathbb{P}\left(\exists t \geq 0, (X_t)^2 + (Y_t)^2 = b\right) = 0$. In addition, for any $p \geq 1$, if b is such that $b > 2(p+1)$ and if the random variable $\left(\frac{1}{1 - \frac{X_0^2 + Y_0^2}{b}}\right)^p$ is integrable, then $t \longmapsto \mathbb{E}\left(\frac{1}{1 - \frac{X_t^2 + Y_t^2}{b}}\right)^p$ is locally bounded.*

Proof : We consider the following approximation of the stochastic differential equation (9), with $g = 0$:

$$\begin{cases} dX_t^n = \left(-\frac{1}{2}\frac{X_t^n}{\max\left(1 - \frac{(X_t^n)^2 + (Y_t^n)^2}{b}, \frac{1}{n}\right)}\right) dt + dV_t, \\ dY_t^n = \left(-\frac{1}{2}\frac{Y_t^n}{\max\left(1 - \frac{(X_t^n)^2 + (Y_t^n)^2}{b}, \frac{1}{n}\right)}\right) dt + dW_t, \end{cases} \tag{12}$$

and the stopping time :

$$\tau_n = \inf\left\{t, (X_t^n)^2 + (Y_t^n)^2 \geq b\left(1 - \frac{1}{n}\right)\right\}.$$

Using the fact that (12) has a unique strong solution (X_t^n, Y_t^n) on $[0, \tau_n)$ and setting $(X_t, Y_t) = (X_t^n, Y_t^n)$ on $[\tau_{n-1}, \tau_n)$, one obtains by continuation of the piecewise solutions a strong solution to (9) with $g = 0$ on $[0, \lim_{n \to \infty} \tau_n)$. Using Itô's formula, one finds that $R_t = (X_t)^2 + (Y_t)^2$ satisfies the following stochastic differential equation on $[0, \lim_{n \to \infty} \tau_n)$:

$$dR_t = \left(-\frac{R_t}{1 - \frac{R_t}{b}} + 2\right) dt + 2\left(X_t\, dV_t + Y_t dW_t\right).$$

Using Girsanov Theorem on (X_t^n, Y_t^n), one may next check that $\mathbb{P}(\exists t \in [0, \tau_n), (X_t^n)^2 + (Y_t^n)^2 = 0) = 0$ and therefore $\mathbb{P}(\exists t \in [0, \lim_{n \to \infty} \tau_n), R_t = 0) = 0$. The former equation may thus be written in the following form :

$$dR_t = \left(-\frac{R_t}{1 - \frac{R_t}{b}} + 2\right) dt + 2\sqrt{R_t} dB_t, \tag{13}$$

where B_t is a Brownian motion by Paul Lévy characterization. Let us now consider a scale function $s : (0, b) \longrightarrow \mathbb{R}$ such that :

$$\left(-\frac{x}{1 - \frac{x}{b}} + 2\right) s'(x) + 2\, x\, s''(x) = 0$$

which leads to

$$s'(x) = C(b - x)^{-b/2} x^{-1}.$$

We choose a primitive function s defined on $(0, b)$. This function s is increasing and is such that $\lim_{x \to b} s(x) = +\infty$, provided $b \geq 2$. Using Itô's formula, on $[0, \lim_{n \to \infty} \tau_n)$, we have

$$s(R_t) = s(R_0) + 2\int_0^t s'(R_s)\sqrt{R_s} dB_s. \tag{14}$$

Let us suppose first that $s(R_0)$ is integrable, so that $s(R_t)$ is a local martingale. Let k be a non-negative integer. We now introduce the stopping time

$$\sigma_k = \inf\left\{t < \lim_{n\to\infty} \tau_n, R_t \leq \frac{1}{k}\right\}, \text{ with the convention } \inf\{\emptyset\} = +\infty.$$

The random process $s(R_{t\wedge\tau_n\wedge\sigma_k})$ is a martingale. Thus we have for any t and for any n,

$$\mathbb{E}(s(R_{t\wedge\tau_n\wedge\sigma_k})) = \mathbb{E}(s(R_0))$$

which leads (using the fact that s is increasing) to

$$s\left(b\left(1-\frac{1}{n}\right)\right)\mathbb{P}(\tau_n \leq t\wedge\sigma_k) + s\left(\frac{1}{k}\right)(1-\mathbb{P}(\tau_n \leq t\wedge\sigma_k)) \leq \mathbb{E}(s(R_0)).$$

Taking first the limit $n \to \infty$, we obtain $\mathbb{P}(\lim_{n\to\infty}\tau_n \leq t\wedge\sigma_k) = 0$. Taking then the limit $k \to \infty$, we obtain $\mathbb{P}(\lim_{n\to\infty}\tau_n \leq t\wedge\lim_{k\to\infty}\sigma_k) = 0$. We know that $\mathbb{P}(\exists t \in [0, \lim_{n\to\infty}\tau_n), R_t = 0) = 0$, which implies $\mathbb{P}(\lim_{n\to\infty}\tau_n > \lim_{k\to\infty}\sigma_k) = 0$ and finally

$$\mathbb{P}\left(\lim_{n\to\infty}\tau_n \leq t\right) \leq \mathbb{P}\left(\lim_{n\to\infty}\tau_n \leq t\wedge\lim_{k\to\infty}\sigma_k\right) + \mathbb{P}\left(\lim_{n\to\infty}\tau_n > \lim_{k\to\infty}\sigma_k\right) = 0.$$

We have shown that $\mathbb{P}(\lim_{n\to\infty}\tau_n = \infty) = 1$, and we have therefore built a strong solution (X,Y) to (9) with $g = 0$ on $\mathbb{R}_+$. If $s(R_0)$ is not integrable, one has to use the same arguments as before on $\{\epsilon < X_0^2 + Y_0^2 < b-\epsilon\}$ where $\epsilon > 0$ (by multiplying (14) by $1_{\epsilon<X_0^2+Y_0^2<b-\epsilon}$) and conclude by letting ϵ go to 0.

If one considers another solution $(\tilde{X},\tilde{Y})$ of (9) with $g = 0$ in the sense of Definition 1, using Lemma 1, this solution is such that $\mathbb{P}\left(\forall t \geq 0, (X_t,Y_t) = (\tilde{X}_t,\tilde{Y}_t)\right) = 1$. This shows that (9) admits a unique strong solution.

Let us now turn to the integrability of $\left(\frac{1}{1-\frac{X_t^2+Y_t^2}{b}}\right)^p$ and let us consider first the case $p = 1$, assuming $\mathbb{E}\left(\frac{1}{1-\frac{R_0}{b}}\right) < \infty$. Using Itô's formula, it is easy to derive :

$$\mathbb{E}\left(\frac{1}{1-\frac{R_{t\wedge\tau_n}}{b}}\right) = \mathbb{E}\left(\frac{1}{1-\frac{R_0}{b}}\right) + \mathbb{E}\left(\int_0^{t\wedge\tau_n} \frac{2/b}{\left(1-\frac{R_s}{b}\right)^2} + \frac{(4-b)R_s/b^2}{\left(1-\frac{R_s}{b}\right)^3}\,ds\right).$$

Assuming $b > 4$, it is clear that $M = \sup_{x\in(0,b)}\left(\frac{2/b}{\left(1-\frac{x}{b}\right)^2} + \frac{(4-b)x/b^2}{\left(1-\frac{x}{b}\right)^3}\right) < \infty$ and one can then obtain

$$\mathbb{E}\left(\frac{1}{1-\frac{R_{t\wedge\tau_n}}{b}}\right) \leq \mathbb{E}\left(\frac{1}{1-\frac{R_0}{b}}\right) + M\,\mathbb{E}(t\wedge\tau_n).$$

This yields, for any $t > 0$,

$$\mathbb{E}\left(\frac{1}{1-\frac{R_t}{b}}\right) \leq \mathbb{E}\left(\frac{1}{1-\frac{R_0}{b}}\right) + M\,t.$$

For an exponent $p>1$, the same arguments show that $\left(\frac{1}{1-\frac{X_t^2+Y_t^2}{b}}\right)^p$ is integrable, provided $b>2(p+1)$ and that $\left(\frac{1}{1-\frac{X_0^2+Y_0^2}{b}}\right)^p$ is integrable. $\diamondsuit$

Remark 2 (On the assumption $b>2(p+1)$.) *We assume that $b\geq 2$ as in Lemma 2. Let*

$$\Pi(x,y)=\begin{cases} -\frac{b}{2}\ln\left(1-\frac{x^2+y^2}{b}\right) & \text{if } x^2+y^2<b,\\ +\infty & \text{otherwise,}\end{cases} \tag{15}$$

denote a potential of the FENE force. Setting $\boldsymbol{X}_t=(X_t,Y_t)$ and $\boldsymbol{W}_t=(V_t,W_t)$, we see that the stochastic differential equation (9) with $g=0$ has the following form :

$$d\boldsymbol{X}_t=-\frac{1}{2}\nabla\Pi(\boldsymbol{X}_t)\,dt+d\boldsymbol{W}_t. \tag{16}$$

Hence, one expects the probability measure with density

$$p_0(x,y)=\frac{\exp(-\Pi(x,y))}{\int_{\mathbf{R}^2}\exp(-\Pi(x,y))\,dx\,dy}=\frac{b+2}{2\pi b}\left(1-\frac{x^2+y^2}{b}\right)^{b/2}1_{x^2+y^2<b} \tag{17}$$

to be invariant. One can indeed prove this property by comparing (16) with stochastic differential equations where the potential Π is carefully regularized and for which the symmetry properties of the transition densities given by Rogers in [16] (see remark (ii) and line 2 p.161) hold. The choice of this invariant probability measure as the law for the initial random variable (X_0,Y_0) is natural from a physical point of view, since we consider here the start up of a shear flow : the fluid is therefore initially at rest (see also [3]). Notice that for this initial distribution, we have $\tau|_{t=0}=\mathbb{E}\left(\frac{X_0Y_0}{1-\frac{X_0^2+Y_0^2}{b}}\right)=0$. In addition, for this initial distribution, for any $t\geq 0$, (X_t,Y_t) has the density p_0 and therefore $\left(\frac{1}{1-\frac{X_t^2+Y_t^2}{b}}\right)^p$ is integrable as soon as $b>2(p-1)$ (and not only under the stronger assumption $b>2(p+1)$ made in Lemma 2). See [8] for more details.

Remark 3 (On the optimality of the assumption $b\geq 2$.) *The assumption $b\geq 2$ turns out to be a necessary condition to prevent (X_t,Y_t) from touching the boundary of the ball $B=B(0,\sqrt{b})=\{(x,y),x^2+y^2<b\}$ and therefore to have pathwise uniqueness of the solution to the stochastic differential equation (9) when $g=0$ in the sense of Definition 1. The function $\frac{1}{2}\Pi:\mathbb{R}^2\to]-\infty,+\infty]$, where Π is defined by (15), is a continuous convex function with domain B. Its subdifferential $\partial\left(\frac{1}{2}\Pi\right)$ is a maximal monotone operator on $\mathbb{R}^2$. According to [4],* **for any** *$b>0$, the multivalued stochastic differential equation*

$$d\boldsymbol{X}_t+\partial\left(\frac{1}{2}\Pi\right)(\boldsymbol{X}_t)\,dt\ni d\boldsymbol{W}_t$$

where $\boldsymbol{W}_t=(V_t,W_t)$ and with $\boldsymbol{X}_0=(X_0,Y_0)$ has a unique strong solution. This solution belongs to $\mathcal{C}(\mathbb{R}_+,\overline{B})$ and following the approach of [5] (see Lemmas 3.3, 3.4, 3.6 and 3.8), one can check that $\boldsymbol{X}_t=(X_t,Y_t)$ is a solution of the stochastic differential equation (9) when $g=0$ in the sense of Definition 1. In case $b\geq 2$, this solution is equal to the one given in Lemma 2 and $\mathbb{P}\left(\exists t\geq 0, X_t^2+Y_t^2=b\right)=0$. In case $0<b<2$, applying

Feller's test for explosions (see [10] pages 348-350) to the semi-martingale $R_t = ||\boldsymbol{X}_t||^2$ which satisfies (13), we check that $\mathbb{P}\left(\exists t \geq 0, X_t^2 + Y_t^2 = b\right) = 1$. In this case, using again results concerning multivalued stochastic differential equations, one can build a solution to (10) outside of the ball B, with initial condition on the boundary and with $g = 0$: this can be used to show that uniqueness in law and therefore pathwise uniqueness do not hold for (10). All these results are detailed in [8].

2.4 Existence in the general case

We now turn to the proof of Proposition 1 in the general case $g \neq 0$.

Lemma 2 provides us with a weak solution to the stochastic differential equation (9) when $g \in L_t^2$ by the Girsanov Theorem. Indeed, let us consider the solution (X_t, Y_t) defined in Lemma 2 in the probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. Under the probability $\mathbb{P}^g$ defined by

$$\left.\frac{d\mathbb{P}^g}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \mathcal{E}\left(\int_0^\bullet g(s) Y_s \, dV_s\right)_t ,$$

the process $(V_t^g, W_t^g) = (V_t - \int_0^t g(s) Y_s \, ds, W_t)$ is a Brownian motion and therefore $(X_t, Y_t, V_t^g, W_t^g, \mathbb{P}^g)$ is a weak solution of the stochastic differential equation (9).

By construction, this weak solution has its paths in $\mathcal{C}([0,T], B)$, where $B = B(0, \sqrt{b}) = \left\{(x,y), x^2 + y^2 < b\right\}$. On the other hand, we know that trajectorial uniqueness holds for such solutions in the ball (by Lemma 1). Therefore, by Yamada-Watanabe Theorem, we have the existence of a strong solution (X_t^g, Y_t^g) to (9) with its paths in $\mathcal{C}([0,T], B)$. Yamada-Watanabe Theorem also gives us uniqueness in law for the solution to (9).

Suppose we are now given another solution of (9) in the sense of Definition 1. By comparing this solution to the above strong solution (X_t^g, Y_t^g) and applying again Lemma 1, we obtain that this solution is equal to the one we have built. This also shows that any solution $(\tilde{X}^g, \tilde{Y}^g)$ of (9) in the sense of Definition 1 is such that $\mathbb{P}\left(\exists t \geq 0, (\tilde{X}_t^g)^2 + (\tilde{Y}_t^g)^2 = b\right) = 0$.

Let us now suppose that $b > 4$ and $\left(\frac{1}{1 - \frac{X_0^2 + Y_0^2}{b}}\right)^p$ is integrable for some $p > 1$. We want to show the equality (11). We need the following Lemma :

Lemma 3 *If $g \in L_t^2$, then we have, for any $1 \leq r < \infty$, if $b > 2(r+1)$, and provided that $\left(\frac{1}{1 - \frac{X_0^2 + Y_0^2}{b}}\right)^p$ is integrable for some $p > r$, for all t, $\left|\frac{X_t^g Y_t^g}{1 - \frac{(X_t^g)^2 + (Y_t^g)^2}{b}}\right|^r$ is integrable and, for any $\frac{b-2}{b-2(1+r)} \vee \frac{p}{p-r} < q < \infty$:*

$$\mathbb{E}\left(\left|\frac{X_t^g Y_t^g}{1 - \frac{(X_t^g)^2 + (Y_t^g)^2}{b}}\right|^r\right)^{1/r} \leq C_{q,r} \exp\left(\frac{q-1}{2r} b \int_0^t |g(s)|^2 \, ds\right),$$

where $C_{q,r}$ denotes a constant depending only on q, r, b and $\mathbb{E}\left(\left(\frac{1}{1 - \frac{X_0^2 + Y_0^2}{b}}\right)^{\frac{rq}{q-1}}\right)$.

Proof : Using Hölder inequality and the properties of the exponential martingale, we have :

$$\mathbb{E}\left|\frac{X_t^g Y_t^g}{1 - \frac{(X_t^g)^2 + (Y_t^g)^2}{b}}\right|^r = \mathbb{E}\left(\left|\frac{X_t Y_t}{1 - \frac{X_t^2 + Y_t^2}{b}}\right|^r \mathcal{E}\left(\int_0^\bullet g(s) Y_s \, dV_s\right)_t\right)$$

$$
\begin{aligned}
&\leq \mathbb{E}\left(\left|\frac{X_tY_t}{1-\frac{X_t^2+Y_t^2}{b}}\right|^{q'r}\right)^{1/q'} \mathbb{E}\left(\mathcal{E}\left(\int_0^\bullet g(s)Y_s\,dV_s\right)_t^q\right)^{1/q}\\
&\leq (C_{q,r})^r\mathbb{E}\left(\mathcal{E}\left(q\int_0^\bullet g(s)Y_s\,dV_s\right)_t \exp\left(\frac{q^2-q}{2}\int_0^t (g(s)Y_s)^2\,ds\right)\right)^{1/q}\\
&\leq (C_{q,r})^r \exp\left(\frac{q-1}{2}b\int_0^t |g(s)|^2\,ds\right)\mathbb{E}\left(\mathcal{E}\left(q\int_0^\bullet g(s)Y_s\,dV_s\right)_t\right)^{1/q}\\
&\leq (C_{q,r})^r \exp\left(\frac{q-1}{2}b\int_0^t |g(s)|^2\,ds\right)
\end{aligned}
$$

with $q' = \frac{q}{q-1}$ and $C_{q,r} = \sup_{t\in[0,T]} \mathbb{E}\left(\left(\frac{X_tY_t}{1-\frac{X_t^2+Y_t^2}{b}}\right)^{q'r}\right)^{1/(q'r)} < \infty$ by Lemma 2. $\diamondsuit$

Using this Lemma with $r=1$ (and a q such that $q > \frac{b-2}{b-4} \vee \frac{p}{p-1}$), one can show that $\dfrac{X_t^gY_t^g}{1-\frac{(X_t^g)^2+(Y_t^g)^2}{b}}$ is integrable and therefore that the equality (11) holds. This concludes the proof of Proposition 1.

3 Notion of solution and *a priori* estimates on the coupled system

We now consider the coupled system of equations (5-7). From now on, we suppose that t varies in a bounded interval $[0,T]$. The space variable y varies in $\mathcal{O}=(0,1)$. The notation $L_t^2(L_y^2)$ is a shortcut for $L^2([0,T],L^2(\mathcal{O}))$, for example.

3.1 Notion of solution

The notion of solution we shall deal with in the sequel is the following.

Definition 2 *Let us be given $u_0 \in H_y^1$, $f_{ext} \in L_t^2(L_y^2)$, together with a probabilized space $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$, (X_0,Y_0) a $\mathcal{F}_0$-measurable random variable and (V_t,W_t) a $(\mathcal{F}_t)$ two-dimensional Brownian motion. We shall say that $(u(t,y),X_t^y,Y_t^y)$ is a solution on the time interval $[0,\Theta)$ if $u \in L^\infty([0,\Theta],H_{0,y}^1)\cap L^2([0,\Theta],H_y^2)$ satisfies :*

$$
\partial_t u(t,y) - \partial_{yy}u(t,y) = \partial_y\mathbb{E}\left(\frac{X_t^yY_t^y}{1-\frac{(X_t^y)^2+(Y_t^y)^2}{b}}\right) + f_{ext}(t,y),
$$

in the sense of $\mathcal{D}'([0,\Theta)\times\mathcal{O})$ (at least), and for a.e. (y,ω), $\forall t\in(0,\Theta)$,

$$
\int_0^t \left|\frac{1}{1-\frac{(X_s^y)^2+(Y_s^y)^2}{b}}\right| ds < \infty, \text{ with the convention } \frac{1}{1-\frac{x^2+y^2}{b}} = +\infty \text{ if } x^2+y^2=b,
$$

$$
\begin{aligned}
X_t^y &= X_0 + \int_0^t \left(-\frac{1}{2}\frac{X_s^y}{1-\frac{(X_s^y)^2+(Y_s^y)^2}{b}} + \partial_y u\, Y_s^y\right) ds + V_t,\\
Y_t^y &= Y_0 + \int_0^t -\frac{1}{2}\frac{Y_s^y}{1-\frac{(X_s^y)^2+(Y_s^y)^2}{b}}\, ds + W_t.
\end{aligned}
$$

Remark 4 *One can notice that we require the solution u to the partial differential equation (5) is strong. This is due to the fact that our technique of proof requires an estimate on $\partial_y u$ in norm H^1_y in order to control the stress τ (see Section 3.2.2).*

Remark 5 *Since for a.e. $y \in \mathcal{O}$, $\partial_y u(.,y)$ is in $L^2([0,\Theta])$, we see that $(X^y_t, Y^y_t) = (X^{\partial_y u}_t, Y^{\partial_y u}_t)$, where $(X^{\partial_y u}_t, Y^{\partial_y u}_t)$ denotes the solution to (10) with $g = \partial_y u(.,y)$ (see Proposition 1).*

3.2 *A priori* estimates

In this section, we give some **formal** *a priori* estimates which will be used in the sequel to prove the existence of a solution to the coupled problem.

3.2.1 First energy estimate

The first *a priori* estimate expresses the conservation of the energy stored in the flow and in the dumbbells.

Lemma 4 (Global-in-time first energy estimate) *Let $(u(t,y), X^y_t, Y^y_t)$ be a solution of (5-7) on $[0,T)$ in the sense of Definition 2. Assume moreover $b > 6$ and $\left(\frac{1}{1-\frac{X_0^2+Y_0^2}{b}}\right)^p$ integrable for some $p > 2$. Then we have the following formal estimate :*

$$\frac{1}{2}\frac{d}{dt}\int_{\mathcal{O}} u^2 + \frac{d}{dt}\int_{\mathcal{O}} \mathbb{E}(\Pi(X^y_t, Y^y_t)) + \int_{\mathcal{O}} (\partial_y u)^2$$
$$+\frac{1}{2}\int_{\mathcal{O}} \mathbb{E}\left(\frac{(X^y_t)^2 + (Y^y_t)^2}{\left(1-\frac{(X^y_t)^2+(Y^y_t)^2}{b}\right)^2}\right) - \int_{\mathcal{O}} \mathbb{E}\left(\frac{1}{\left(1-\frac{(X^y_t)^2+(Y^y_t)^2}{b}\right)^2}\right) = \int_{\mathcal{O}} f_{ext} u, \quad (18)$$

where Π is the potential for the FENE force defined by (15). Under the additional assumptions $f_{ext} \in L^1_t(L^2_y)$ and $u_0 \in L^2_y$, this yields the following formal estimate on the solution :

$$||u||_{L^\infty_t(L^2_y)} + ||\partial_y u||_{L^2_t(L^2_y)} + ||\Pi(X^y, Y^y)||_{L^\infty_t(L^1_y(L^1_\omega))} + ||\Upsilon(X^y, Y^y)||_{L^2_t(L^2_y(L^2_\omega))} \leq C, \quad (19)$$

where $\Upsilon(x,y) = \frac{\sqrt{x^2+y^2}}{1-\frac{x^2+y^2}{b}}$ and C is a constant depending on T, $||u_0||_{L^2_y}$, $||f_{ext}||_{L^1_t(L^2_y)}$ and $\mathbb{E}(\Pi(X_0, Y_0))$.

Proof : Multiplying the equation (5) by u and integrating over $\mathcal{O}$, one obtains :

$$\frac{1}{2}\int_{\mathcal{O}} u(t,y)^2 - \frac{1}{2}\int_{\mathcal{O}} u_0(y)^2 + \int_0^t \int_{\mathcal{O}} (\partial_y u)^2 = -\int_0^t \int_{\mathcal{O}} \tau\, \partial_y u + \int_0^t \int_{\mathcal{O}} f_{ext} u. \quad (20)$$

Notice that this is the only formal operation that will be later on justified once the problem discretized : the following of the proof is completely rigorous.

A simple calculus shows that $\nabla\Pi = \frac{1}{1-\frac{x^2+y^2}{b}}(x,y)1_{x^2+y^2<b}$ and $\Delta\Pi = \frac{2}{\left(1-\frac{x^2+y^2}{b}\right)^2}1_{x^2+y^2<b}$.

Therefore, using Itô's formula, we have :

$$d(\Pi(X^y_t, Y^y_t)) = -\frac{1}{2}\Upsilon(X^y_t, Y^y_t)^2\, dt + \partial_y u \frac{X^y_t Y^y_t}{1-\frac{(X^y_t)^2+(Y^y_t)^2}{b}}\, dt + \frac{1}{\left(1-\frac{(X^y_t)^2+(Y^y_t)^2}{b}\right)^2}\, dt +$$
$$\frac{X^y_t}{1-\frac{(X^y_t)^2+(Y^y_t)^2}{b}}\, dV_t + \frac{Y^y_t}{1-\frac{(X^y_t)^2+(Y^y_t)^2}{b}}\, dW_t.$$

This calculus is justified by the fact that the random process (X_t^y, Y_t^y) does not touch the boundary of B (see Proposition 1). Integrating both in time and space and taking the expectation value, we therefore obtain :

$$\begin{aligned}\int_{\mathcal{O}} \mathbb{E}(\Pi(X_t^y, Y_t^y)) &= \int_{\mathcal{O}} \mathbb{E}(\Pi(X_0, Y_0)) - \frac{1}{2}\int_0^t \int_{\mathcal{O}} \mathbb{E}(\Upsilon(X_s^y, Y_s^y)^2)\, ds + \\ & \int_0^t \int_{\mathcal{O}} \partial_y u\, \tau + \int_0^t \int_{\mathcal{O}} \mathbb{E}\left(\frac{1}{\left(1 - \frac{(X_s^y)^2 + (Y_s^y)^2}{b}\right)^2}\right) ds. \end{aligned} \tag{21}$$

Notice that the expectations of the local martingales are null since we have assumed $b > 6$ and $\left(\frac{1}{1 - \frac{X_0^2 + Y_0^2}{b}}\right)^p$ integrable for some $p > 2$ so that, by Lemma 3, for a.e. $y \in \mathcal{O}$, $\mathbb{E}\left(\int_0^T \left(\frac{1}{1 - \frac{(X_s^y)^2 + (Y_s^y)^2}{b}}\right)^2 ds\right)$ has a finite value. By summing (20) and (21), one obtains the energy equality (18).

Estimate (19) is then obtained by using the energy equality. Indeed, the term $\Pi(X_t^y, Y_t^y)$ is positive and one can notice that the term $\int_{\mathcal{O}} \mathbb{E}\left(\frac{1}{\left(1 - \frac{(X_t^y)^2 + (Y_t^y)^2}{b}\right)^2}\right)$ can be bounded from above by the term $\frac{1}{2}\int_{\mathcal{O}} \mathbb{E}\left(\frac{(X_t^y)^2 + (Y_t^y)^2}{\left(1 - \frac{(X_t^y)^2 + (Y_t^y)^2}{b}\right)^2}\right)$ by writing :

$$\begin{aligned}\frac{1}{2}\frac{(X_t^y)^2 + (Y_t^y)^2}{\left(1 - \frac{(X_t^y)^2 + (Y_t^y)^2}{b}\right)^2} - \frac{1}{\left(1 - \frac{(X_t^y)^2 + (Y_t^y)^2}{b}\right)^2} &= \frac{1}{2}\frac{(X_t^y)^2 + (Y_t^y)^2}{\left(1 - \frac{(X_t^y)^2 + (Y_t^y)^2}{b}\right)^2} - \frac{1_{(X_t^y)^2 + (Y_t^y)^2 > 2+\epsilon}}{\left(1 - \frac{(X_t^y)^2 + (Y_t^y)^2}{b}\right)^2} \\ & \quad - \frac{1_{(X_t^y)^2 + (Y_t^y)^2 < 2+\epsilon}}{\left(1 - \frac{(X_t^y)^2 + (Y_t^y)^2}{b}\right)^2} \\ &\geq \frac{\epsilon}{2(2+\epsilon)}\frac{(X_t^y)^2 + (Y_t^y)^2}{\left(1 - \frac{(X_t^y)^2 + (Y_t^y)^2}{b}\right)^2} - \frac{b^2}{(b - (2+\epsilon))^2}\end{aligned}$$

with ϵ such that $b - 2 > \epsilon$. $\diamondsuit$

3.2.2 Second energy estimate

In order to show the second estimate, we have to use an expression of the stress τ which will give us regularity in the space variable y. In Proposition 1, we have shown that the stress τ has the following expression (assuming $b > 4$ and $\left(\frac{1}{1 - \frac{X_0^2 + Y_0^2}{b}}\right)^p$ is integrable for some $p > 1$, see (11)) :

$$\begin{aligned}\tau(t, y) &= \mathbb{E}\left(\frac{X_t^y Y_t^y}{1 - \frac{(X_t^y)^2 + (Y_t^y)^2}{b}}\right), \\ &= \mathbb{E}\left(\left(\frac{X_t Y_t}{1 - \frac{X_t^2 + Y_t^2}{b}}\right) \mathcal{E}\left(\int_0^\bullet \partial_y u(y) Y_s\, dV_s\right)_t\right).\end{aligned} \tag{22}$$

Diffrenciating (22) with respect to y enables us to convert the regularity of u to the one of $\partial_y \tau$, which provides us with the following local-in-time estimate.

Lemma 5 (Local-in-time second energy estimate) *Under the assumptions $b > 6$, $f_{ext} \in L^2_t(L^2_y)$, $u_0 \in H^1_y$ and provided that $\left(\frac{1}{1-\frac{X_0^2+Y_0^2}{b}}\right)^p$ is integrable for some $p > 2$, we have the following formal estimate on $[0, T']$, with $T' \in (0, T)$ depending on $||\partial_y u_0||_{L^2_y}$, on $||f_{ext}||_{L^2_t(L^2_y)}$, on b, and on $\mathbb{E}\left(\left(\frac{1}{1-\frac{X_0^2+Y_0^2}{b}}\right)^p\right)$:*

$$||u||_{L^\infty([0,T'],H^1_y)} + ||u||_{L^2([0,T'],H^2_y)} \leq C.$$

This also yields the following formal estimate on $\partial_t u$, on $[0, T']$:

$$||\partial_t u||_{L^2([0,T'],L^2_y)} \leq C.$$

In both cases, C is a constant depending on $||\partial_y u_0||_{L^2_y}$, $||f_{ext}||_{L^2([0,T'],L^2_y)}$, on b and on $\mathbb{E}\left(\left(\frac{1}{1-\frac{X_0^2+Y_0^2}{b}}\right)^p\right)$.

Proof : Multiplying (5) by $-\partial_{yy} u$ and integrating over $\mathcal{O}$, one obtains :

$$\frac{1}{2}\int_{\mathcal{O}}(\partial_y u(t,y))^2 - \frac{1}{2}\int_{\mathcal{O}}(\partial_y u_0)^2 + \int_0^t \int_{\mathcal{O}} (\partial_{yy} u)^2 = -\int_0^t \int_{\mathcal{O}} \partial_y \tau \partial_{yy} u - \int_0^t \int_{\mathcal{O}} f_{ext} \partial_{yy} u.$$

This yields

$$\int_{\mathcal{O}}(\partial_y u(t,y))^2 + \int_0^t \int_{\mathcal{O}} (\partial_{yy} u)^2 \leq A + 2\int_0^t \int_{\mathcal{O}} |\partial_y \tau||\partial_{yy} u|,$$

with $A = ||\partial_y u_0||^2_{L^2_y} + \int_0^T \int_{\mathcal{O}} |f_{ext}|^2$. Notice that this is a formal operation that will be later on justified once the problem discretized.

Using (22), we can derive :

$$\partial_y \tau = \mathbb{E}\left(\left(\frac{X_t Y_t}{1-\frac{X_t^2+Y_t^2}{b}}\right)\left(\int_0^t \partial_{yy} u Y_s \, dV_s - \int_0^t (\partial_{yy} u)(\partial_y u) Y_s^2 \, ds\right) \mathcal{E}\left(\int_0^\bullet \partial_y u Y_s \, dV_s\right)_t\right). \tag{23}$$

This can be shown in two steps, by first derivating with respect to y, for almost every ω, the random variable $\left(\frac{X_t Y_t}{1-\frac{X_t^2+Y_t^2}{b}}\right) \mathcal{E}\left(\int_0^\bullet \partial_y u(y) Y_s \, dV_s\right)_t$ and then by proving uniform integrability (in ω) on this derivative. To perform the first step, one can consider the random variable $\zeta_{t,h}$ defined by :

$$\zeta_{t,h} = \begin{cases} \dfrac{\int_0^t \partial_y u(y+h) Y_s \, dV_s - \int_0^t \partial_y u(y) Y_s \, dV_s}{h} & \text{if } h \neq 0 \\ \displaystyle\int_0^t \partial_{y,y} u(y) Y_s \, dV_s & \text{if } h = 0 \end{cases}$$

and prove that ζ is continuous by Kolmogorov Theorem (see Theorem 2.8 p. 53 of [10]). This is also done in a formal way, since it required regularity on u : this operation will be justified once the problem discretized. Notice that the following of the proof is now completely rigorous.

For the first term, one can obtain, by using the fact that Y_t is bounded (see Proposition 1) and Hölder inequality in ω, for any q such that $q > \frac{2(b-2)}{b-6} \vee \frac{2p}{p-2}$:

$$\left| \mathbb{E} \left(\left(\frac{X_t Y_t}{1 - \frac{X_t^2 + Y_t^2}{b}} \right) \left(\int_0^t \partial_{yy} u Y_s \, dV_s \right) \mathcal{E} \left(\int_0^\bullet \partial_y u Y_s \, dV_s \right)_t \right) \right| \leq$$

$$C_q \left(\mathbb{E} \left(\int_0^t \partial_{yy} u Y_s \, dV_s \right)^2 \right)^{1/2} \exp \left(\frac{q-1}{2} b \int_0^t (\partial_y u)^2 \, ds \right) \leq$$

$$C_q \sqrt{b} \left(\int_0^t |\partial_{yy} u|^2 \, ds \right)^{1/2} \exp \left(\frac{q-1}{2} b \int_0^t (\partial_y u)^2 \, ds \right).$$

For the second term, a similar argument shows :

$$\left| \mathbb{E} \left(\left(\frac{X_t Y_t}{1 - \frac{X_t^2 + Y_t^2}{b}} \right) \left(\int_0^t (\partial_{yy} u)(\partial_y u) Y_s^2 \, ds \right) \mathcal{E} \left(\int_0^\bullet \partial_y u Y_s \, dV_s \right)_t \right) \right| \leq$$

$$C_q' b \left(\int_0^t |\partial_{yy} u| |\partial_y u| \, ds \right) \exp \left(\frac{q-1}{2} b \int_0^t (\partial_y u)^2 \, ds \right).$$

We thus have (using, since we are in dimension one, $\partial_y u(r,y) \leq ||\partial_y u(r,.)||_{H^1_y}$) :

$$\begin{aligned}
&\int_0^t \int_{\mathcal{O}} |\partial_y \tau| |\partial_{yy} u| \, ds \leq \\
&\int_0^t \int_{\mathcal{O}} \exp \left(\frac{q-1}{2} b \int_0^s (\partial_y u)^2 \, dr \right) \left(C_q \sqrt{b} \left(\int_0^s |\partial_{yy} u|^2 \, dr \right)^{1/2} + C_q' b \left(\int_0^s |\partial_{yy} u| |\partial_y u| \, dr \right) \right) |\partial_{yy} u|(s) \, ds \\
&\leq \; C_q \sqrt{b} \int_0^t \exp \left(\frac{q-1}{2} b \int_0^s ||\partial_y u||^2_{H^1_y} \, dr \right) \left(\int_{\mathcal{O}} \left(\int_0^s |\partial_{yy} u|^2(r) \, dr \right)^{1/2} |\partial_{yy} u|(s) \right) ds \\
&\quad + C_q' b \int_0^t \exp \left(\frac{q-1}{2} b \int_0^s ||\partial_y u||^2_{H^1_y} \, dr \right) \left(\int_0^s ||\partial_y u||_{H^1_y}(r) \int_{\mathcal{O}} |\partial_{yy} u|(r) |\partial_{yy} u|(s) \, dr \right) ds.
\end{aligned}$$

So we obtain by the application of Cauchy Schwartz inequality (to the spatial integral) to both terms :

$$\begin{aligned}
&\int_0^t \int_{\mathcal{O}} |\partial_y \tau| |\partial_{yy} u| \, ds \leq \\
&\quad C_q \sqrt{b} \int_0^t \exp \left(\frac{q-1}{2} b \int_0^s ||\partial_y u||^2_{H^1_y}(r) \, dr \right) \left(\int_0^s ||\partial_{yy} u||^2_{L^2_y}(r) \, dr \right)^{1/2} ||\partial_{yy} u||_{L^2_y}(s) \, ds \\
&\quad + C_q' b \int_0^t \exp \left(\frac{q-1}{2} b \int_0^s ||\partial_y u||^2_{H^1_y}(r) \, dr \right) \left(\int_0^s ||\partial_y u||_{H^1_y}(r) ||\partial_{yy} u||_{L^2_y}(r) \, dr \right) ||\partial_{yy} u||_{L^2_y}(s) \, ds \\
&\leq \; \frac{1}{4} \int_0^t ||\partial_{yy} u||^2_{L^2_y}(s) \, ds \\
&\quad + C(q,b) \int_0^t \exp \left((q-1) b \int_0^s ||\partial_y u||^2_{H^1_y}(r) \, dr \right) \left(\int_0^s ||\partial_{yy} u||^2_{L^2_y}(r) \, dr \right) ds \\
&\quad + C'(q,b) \int_0^t \exp \left((q-1) b \int_0^s ||\partial_y u||^2_{H^1_y}(r) \, dr \right) \left(\int_0^s ||\partial_y u||_{H^1_y}(r) ||\partial_{yy} u||_{L^2_y}(r) \, dr \right)^2 ds.
\end{aligned}$$

We thus have shown the following inequality :

$$||\partial_y u||^2_{L^2_y}(t) + \frac{1}{2} \int_0^t ||\partial_{yy} u||^2_{L^2_y}(s) \, ds \leq$$

$$A+2C(q,b)\int_0^t \exp\left((q-1)b\int_0^s ||\partial_y u||^2_{H^1_y}(r)\,dr\right)\left(\int_0^s ||\partial_{yy}u||^2_{L^2_y}(r)\,dr\right)\,ds$$
$$+2C'(q,b)\int_0^t \exp\left((q-1)b\int_0^s ||\partial_y u||^2_{H^1_y}(r)\,dr\right)\left(\int_0^s ||\partial_y u||_{H^1_y}(r)||\partial_{yy}u||_{L^2_y}(r)\,dr\right)^2\,ds.$$

Let $f_1(t)=||\partial_y u||^2_{L^2_y}(t)$ and $f_2(t)=\int_0^t ||\partial_{yy}u||^2_{L^2_y}(s)\,ds$. We have :

$$\begin{aligned} f_1(t)+\frac{1}{2}f_2(t) \quad &\leq \quad A+2C(q,b)\int_0^t \exp\left(\alpha\left(\int_0^s f_1(r)\,dr+f_2(s)\right)\right)f_2(s)\,ds \\ &\quad +2C'(q,b)\int_0^t \exp\left(\alpha\left(\int_0^s f_1(r)\,dr+f_2(s)\right)\right)\left(\int_0^s f_1(r)\,dr+f_2(s)\right)f_2(s)\,ds, \end{aligned}$$

with $\alpha=(q-1)b$. Let $R(t)$ be the right hand side of the former equation. We can then write :

$$\begin{aligned} R'(t) \quad &= \quad 2\exp\left(\alpha\left(\int_0^t f_1(r)\,dr+f_2(t)\right)\right)\left(C(q,b)+C'(q,b)\left(\int_0^t f_1(r)\,dr+f_2(t)\right)\right)f_2(t) \\ &\leq \quad 4\exp\left(\alpha\left(\int_0^t R(r)\,dr+2R(t)\right)\right)\left(C(q,b)+C'(q,b)\left(\int_0^t R(r)\,dr+2R(t)\right)\right)R(t). \end{aligned}$$

By integrating in time, this leads to

$$R(t)\leq A+4\int_0^t \exp\left(\alpha\left(\int_0^s R(r)\,dr+2R(s)\right)\right)\left(C(q,b)+C'(q,b)\left(\int_0^s R(r)\,dr+2R(s)\right)\right)R(s)\,ds.$$

Since R is an increasing function, we have :

$$\begin{aligned} R(t) \quad &\leq \quad A+4\int_0^t \exp\left(\alpha\,(T+2)\,R(s)\right)\left(C(q,b)+C'(q,b)\,(T+2)\,R(s)\right)R(s)\,ds \\ &\leq \quad A+\int_0^t \exp\left(\beta R(s)\right)\left(c+c'R(s)\right)R(s)\,ds, \end{aligned}$$

with $\beta=\alpha\,(T+2)$, $c=4C(q,b)$ and $c'=4\,(T+2)\,C'(q,b)$. From this one can deduce that there exists $\gamma>0$ and $C>0$, (both depending on q, b, $\mathbb{E}\left(\left(\frac{1}{1-\frac{X_0^2+Y_0^2}{b}}\right)^p\right)$ and T) such that :

$$R(t)\leq A+C\int_0^t \exp(\gamma R(s))\,ds.$$

Let $H(t)$ denote the right hand side. It is easy to derive from this inequality the following estimate : $\forall t\in\left(0,\frac{1}{\gamma C}\exp(-\gamma A)\right)$,

$$R(t)\leq H(t)\leq \frac{1}{\gamma}\ln\left(\frac{1}{\exp(-\gamma A)-\gamma Ct}\right).$$

If we set $T'=\frac{1}{\gamma C}\left(\exp(-\gamma A)-\exp(-2\gamma A)\right)$, we have : for all $t\in(0,T')$,

$$R(t)\leq H(t)\leq 2A.$$

This leads to the following estimate : for all $t\in(0,T')$,

$$||\partial_y u||^2_{L^2_y}(t)+\frac{1}{2}\int_0^t ||\partial_{yy}u||^2_{L^2_y}(s)\,ds\leq 2A. \tag{24}$$

In order to obtain the estimate on $\partial_t u$, we observe that $\partial_t u = \partial_{yy} u + f_{ext} + \partial_y \tau$. We have already shown an estimation of $\partial_{yy} u$ in $L^2([0,T'], L^2_y)$ norm. Moreover, using the same argument as before, we can easily show that, for any function $v \in L^2([0,T'], L^2_y)$,

$$\left| \int_0^t \int_{\mathcal{O}} \partial_y \tau \, v \right| \leq C ||v||_{L^2([0,T'],L^2_y)},$$

where C is a constant depending on q, b, A, $\mathbb{E}\left(\left(\frac{1}{1-\frac{X_0^2+Y_0^2}{b}}\right)^p\right)$ and T'. This yields the estimate on $\partial_t u$. $\diamondsuit$

4 Existence of a solution to the coupled system

The aim of this section is to prove the following :

Theorem 1 (Local-in-time existence and uniqueness) *We assume that $b > 6$, $f_{ext} \in L^2_t(L^2_y)$, $u_0 \in H^1_y$ and $\left(\frac{1}{1-\frac{X_0^2+Y_0^2}{b}}\right)^p$ is integrable for some $p > 2$. Then there exists $T' \in (0,T)$ (depending on the data) such that the system (5-7) admits a unique solution $(u(t,y), X^y_t, Y^y_t)$ on $[0,T']$ in the sense given in Definition 2.*

Remark 6 *If one chooses some initial random variables (X_0, Y_0) distributed with the invariant density p_0 defined by (17), then Theorem 1 holds under the weaker assumption $b > 2$ (see Remark 2 and Lemma 5). We recall that none of these assumptions on b is restrictive in practice since b is physically of the order of 100.*

In the following, we assume $b > 6$, $f_{ext} \in L^2_t(L^2_y)$, $u_0 \in H^1_y$ and $\left(\frac{1}{1-\frac{X_0^2+Y_0^2}{b}}\right)^p$ is integrable for some $p > 2$.

In order to show the existence of a solution to the coupled system, we introduce the following variational formulation of (5) :
Find $u \in L^\infty_t(H^1_{0,y}) \cap L^2_t(H^2_y)$ such that for all $v \in H^1_{0,y}$,

$$\frac{d}{dt} \int_{\mathcal{O}} u \, v = -\int_{\mathcal{O}} \partial_y u \, \partial_y v - \int_{\mathcal{O}} \tau \, \partial_y v + \int_{\mathcal{O}} f_{ext} \, v, \tag{25}$$

together with

$$\tau = \mathbb{E}\left(\frac{X^y_t Y^y_t}{1-\frac{(X^y_t)^2+(Y^y_t)^2}{b}}\right), \tag{26}$$

$$X^y_t = X_0 + \int_0^t \left(-\frac{1}{2}\frac{X^y_s}{1-\frac{(X^y_s)^2+(Y^y_s)^2}{b}} + \partial_y u \, Y^y_s\right) ds + V_t, \tag{27}$$

$$Y^y_t = Y_0 + \int_0^t -\frac{1}{2}\frac{Y^y_s}{1-\frac{(X^y_s)^2+(Y^y_s)^2}{b}} \, ds + W_t. \tag{28}$$

The ordinary differential equation (25) is to be understood in $\mathcal{D}'([0,T))$. The stochastic differential equations (27-28) are to be understood in the sense of Definition 1. This problem is well defined. Indeed, Proposition 1 gives a strong solution to (27-28). Moreover, the term $\int_{\mathcal{O}} \tau \partial_y v$ has a meaning since Lemma 3 shows that τ is in $L^\infty_t(L^\infty_y)$. A solution to this variational problem is a solution to (5-7) in the sense of Definition 2.

4.1 Definition and resolution of the discretized problem

We introduce a Galerkin approximation of the variational problem (25 - 28). Let $(v_i)_{1\leq i\leq\infty} \in \mathcal{C}^\infty(\overline{\mathcal{O}}) \cap H^1_0(\mathcal{O})$ be such that $\{v_i\}$ is a basis of $H^1_0(\mathcal{O})$ and such that $\partial_{yy}v_i \in \mathrm{Vect}\{v_j, 1\leq j\leq i\}$ (take e.g. the eigenvectors of the Dirichlet laplacian on $\mathcal{O}$). We set $V_m = \mathrm{Vect}\{v_j, 1\leq j\leq m\}$. The problem we consider at the discrete level reads : *Find* $U^m \in L^\infty_t(\mathbb{R}^m)$ *such that the function* $u^m(t,y) = \sum_i U^m_i(t)v_i(y)$ *satisfies :*

$$\begin{aligned}
\frac{d}{dt}\int_{\mathcal{O}} u^m\, v_i &= -\int_{\mathcal{O}} \partial_y u^m\, \partial_y v_i - \int_{\mathcal{O}} \tau^m\, \partial_y v_i + \int_{\mathcal{O}} f_{ext}\, v_i, \ \textit{for } 1\leq i\leq m, &(29)\\
\tau^m &= \mathbb{E}\left(\frac{X^{y,m}_t Y^{y,m}_t}{1-\frac{(X^{y,m}_t)^2+(Y^{y,m}_t)^2}{b}}\right), &(30)\\
X^{y,m}_t &= X_0 + \int_0^t \left(-\frac{1}{2}\frac{X^{y,m}_s}{1-\frac{(X^{y,m}_s)^2+(Y^{y,m}_s)^2}{b}} + \partial_y u^m\, Y^{y,m}_s\right) ds + V_t, &(31)\\
Y^{y,m}_t &= Y_0 + \int_0^t -\frac{1}{2}\frac{Y^{y,m}_s}{1-\frac{(X^{y,m}_s)^2+(Y^{y,m}_s)^2}{b}}\, ds + W_t. &(32)
\end{aligned}$$

The initial condition $u^m(t=0)$ is $\Pi^m(u_0)$ where Π^m is the H^1-projection on V_m. The ordinary differential equation (29) is to be understood in $\mathcal{D}'([0,T))$. The stochastic differential equations (31-32) are to be understood in the sense of Definition 1. We know from Proposition 1 that the stress can be written in the following form :

$$\tau^m = \mathbb{E}\left(\left(\frac{X_tY_t}{1-\frac{X_t^2+Y_t^2}{b}}\right)\mathcal{E}\left(\int_0^\bullet \partial_y u^m Y_s\, dV_s\right)_t\right).$$

Notice first that the formal *a priori* estimates of Lemma 4 can now be derived rigorously on the discretized problem (since one can take $v = u^m$ as a test function in (29)) and show that if u^m is a solution of the problem (29-32) on $[0,T]$, then $||u^m||_{L^\infty([0,T],L^2_y)}$ is bounded which means that $||U^m(t)||_{L^\infty[0,T]} \leq M$ (where M depends on m). Notice also that if u^m is a solution of the problem (29-32) on $[0,\Theta]$, with $\Theta < T$, then we also have $||U^m(t)||_{L^\infty[0,\Theta]} \leq M$ with the same upper bound M (independant of Θ).

In the following, the dimension m is fixed and we omit the superscript m.

We now want to show that the nonlinear system (29-32) admits a solution. We introduce the matrices $A_{i,j} = \int_{\mathcal{O}} v_i v_j$, $B_{i,j} = \int_{\mathcal{O}} \partial_y v_i \partial_y v_j$, the field $\Xi = (v_1, ..., v_m)$ and the vector F_{ext} with components $\displaystyle\int_{\mathcal{O}} f_{ext} v_j$. We are going to construct a fixed point U of the following mapping which associates to any function $\Phi \in \mathcal{C}([0,T],\mathbb{R}^m)$ the function $F(\Phi) \in \mathcal{C}([0,T],\mathbb{R}^m)$ defined by :

$$F(\Phi)(t) = U_0 - A^{-1}\left(\int_0^t \left(B\Phi(s) - \int_{\mathcal{O}} \mathbb{E}\left(\frac{X_sY_s}{1-\frac{X_s^2+Y_s^2}{b}} Z(\Phi)_s\right)\partial_y\Xi + F_{ext}(s)\right) ds\right),$$

where

$$Z(\Phi)_s = \mathcal{E}\left(\int_0^\bullet \sum_i \Phi_i(r)\partial_y v_i Y_r\, dV_r\right)_s.$$

The initial condition U_0 is the vector with components $(\Pi^m(u_0))_j$.

First step :

First, we are going to show by the Picard fixed point theorem that the function F has a unique fixed point when restricted on $\mathcal{C}([0,\alpha], B(U_0,1))$ endowed with the uniform convergence topology, for some $\alpha \in (0,T)$ well chosen and only depending on $||F_{ext}||_{L^\infty_t}$, $\max_{1\le i\le m}||v_i||_{W^{1,\infty}_y}$, m, b and T (see inequalities (34) and (35) below). The ball $B(U_0,1)$ is defined by $B(U_0,1)=\{K\in\mathbb{R}^m, ||K-U_0||<1\}$. Notice first that we have, for any $\Phi\in\mathcal{C}([0,\alpha],B(U_0,1))$ and for any $t\in[0,\alpha]$:

$$\begin{aligned}
\mathbb{E}\left((Z(\Phi)_t)^2\right) &= \mathbb{E}\left(\left(\mathcal{E}\left(\int_0^\bullet \sum_i \Phi_i(r)\partial_y v_i Y_r\, dV_r\right)_t\right)^2\right)\\
&= \mathbb{E}\left(\exp\left(\int_0^t\left(\sum_i \Phi_i(r)\partial_y v_i Y_r\right)^2 dr\right)\mathcal{E}\left(2\int_0^\bullet \sum_i \Phi_i(r)\partial_y v_i Y_r\, dV_r\right)_t\right)\\
&\le \exp\left(C\alpha b\left(||\Phi||^2_{L^\infty[0,\alpha]}\right)\right)\mathbb{E}\left(\mathcal{E}\left(2\int_0^\bullet \sum_i \Phi_i(r)\partial_y v_i Y_r\, dV_r\right)_t\right)\\
&\le \exp\left(C\alpha b\left(||\Phi||^2_{L^\infty[0,\alpha]}\right)\right),
\end{aligned}$$

where C only depends on $\max_{1\le i\le m}||v_i||_{W^{1,\infty}_y}$. This means (since $\alpha<T$ and $\Phi\in\mathcal{C}([0,\alpha],B(U_0,1))$) :

$$||Z(\Phi)_t||_{L^\infty([0,\alpha],L^\infty_y(L^2_\omega))} \le \exp\left(CTb\left(1+||U_0||^2\right)\right). \tag{33}$$

Using this estimate (and Lemma 2), for any $\Phi\in\mathcal{C}([0,\alpha],B(U_0,1))$, we have the following estimate on $\Phi'=F(\Phi)$:

$$\begin{aligned}
||\Phi'(t)-U_0||_{L^\infty[0,\alpha]} &\le |||A^{-1}|||\left(|||B|||\,\alpha(1+||U_0||)+C\alpha||Z(\Phi)||_{L^\infty([0,\alpha],L^\infty_y(L^2_\omega))}+\alpha||F_{ext}||_{L^\infty_t}\right)\\
&\le C\alpha(1+||U_0||)\exp(C||U_0||^2),
\end{aligned}$$

where C is a constant depending only on $||F_{ext}||_{L^\infty_t}$, $\max_{1\le i\le m}||v_i||_{W^{1,\infty}_y}$, m, b and T (and not on $||U_0||$). Thus, $F\left(\mathcal{C}([0,\alpha],B(U_0,1))\right)\subset\mathcal{C}([0,\alpha],B(U_0,1))$ if we choose α small enough to ensure that :

$$C\alpha(1+||U_0||)\exp(C||U_0||^2)\le 1. \tag{34}$$

We next show that the function F restricted on $\mathcal{C}([0,\alpha],B(U_0,1))$ (with α small enough) is contracting. Let $\Phi^{1'}=F(\Phi^1)$, $Z^1_t=Z(\Phi^1)_t$ and $\Phi^{2'}=F(\Phi^2)$, $Z^2_t=Z(\Phi^2)_t$ where $\Phi^1\in B(U_0,1)$ and $\Phi^2\in B(U_0,1)$. Suppose moreover that (34) holds. We have

$$\begin{aligned}
||\Phi^{1'}(t)-\Phi^{2'}(t)||_{L^\infty[0,\alpha]} &\le |||A^{-1}|||\left(|||B|||\,\alpha||\Phi^1(t)-\Phi^2(t)||_{L^\infty[0,\alpha]}+C\alpha||Z^1_t-Z^2_t||_{L^\infty([0,\alpha],L^\infty_y(L^2_\omega))}\right)\\
&\le C'\alpha\left(||\Phi^1(t)-\Phi^2(t)||_{L^\infty[0,\alpha]}+||Z^1_t-Z^2_t||_{L^\infty([0,\alpha],L^\infty_y(L^2_\omega))}\right).
\end{aligned}$$

We now want to estimate $(Z^1_t-Z^2_t)$. We use the fact that $Z^k_t=1+\displaystyle\int_0^t\sum_i \Phi^k_i(r)\partial_y v_i\,Y_r\,Z^k_r\,dV_r$ $(k=1$ or $2)$:

$$\begin{aligned}
Z^1_t-Z^2_t &= \int_0^t\left(\sum_i \Phi^1_i(r)\partial_y v_i\,Z^1_r-\sum_i \Phi^2_i(r)\partial_y v_i\,Z^2_r\right)Y_r\,dV_r\\
&= \int_0^t\sum_i\left(\Phi^1_i(r)-\Phi^2_i(r)\right)\partial_y v_i\,Z^2_r Y_r\,dV_r+\int_0^t\left(\sum_i \Phi^1_i(r)\partial_y v_i\right)\left(Z^1_r-Z^2_r\right)Y_r\,dV_r.
\end{aligned}$$

From this and (33), we deduce that, $\forall t \in [0,\alpha]$,

$$\begin{aligned}
\mathbb{E}\left(\left(Z_t^1 - Z_t^2\right)^2\right) &\leq 2\mathbb{E}\left(\left(\int_0^t \sum_i \left(\Phi_i^1(r) - \Phi_i^2(r)\right) \partial_y v_i \, Z_r^2 Y_r \, dV_r\right)^2\right) \\
&\quad +2\mathbb{E}\left(\left(\int_0^t \left(\sum_i \Phi_i^1(r)\partial_y v_i\right) \left(Z_r^1 - Z_r^2\right) Y_r \, dV_r\right)^2\right), \\
&\leq 2\int_0^t \mathbb{E}\left(\left(\sum_i \left(\Phi_i^1(r) - \Phi_i^2(r)\right) \partial_y v_i \, Z_r^2 Y_r\right)^2\right) dr \\
&\quad +2\int_0^t \mathbb{E}\left(\left(\left(\sum_i \Phi_i^1(r)\partial_y v_i\right) \left(Z_r^1 - Z_r^2\right) Y_r\right)^2\right) dr, \\
&\leq 2bC'\alpha \exp\left(CTb\left(1 + ||U_0||^2\right)\right) ||\Phi^1 - \Phi^2||^2_{L^\infty[0,\alpha]} \\
&\quad +2bC'||\Phi^1||^2_{L^\infty[0,\alpha]} \int_0^t \mathbb{E}\left(\left(Z_r^1 - Z_r^2\right)^2\right) dr.
\end{aligned}$$

Using Gronwall Lemma, this yields an estimate :

$$||Z_t^1 - Z_t^2||_{L^\infty([0,\alpha], L_y^\infty(L_\omega^2))} \leq C'\alpha \exp(C'||U_0||^2)||\Phi^1 - \Phi^2||_{L^\infty[0,\alpha]},$$

where C' is a constant depending only on $||F_{ext}||_{L_t^\infty}$, $\max_{1\leq i\leq m} ||v_i||_{W_y^{1,\infty}}$, m, b and T. We finally have an inequality of the following type :

$$||\Phi^{1'}(t) - \Phi^{2'}(t)||_{L^\infty[0,\alpha]} \leq C'\alpha \exp(C'||U_0||^2)||\Phi^1(t) - \Phi^2(t)||_{L^\infty[0,\alpha]},$$

so that F is contracting if we have :

$$C'\alpha \exp(C'||U_0||^2) < 1. \tag{35}$$

At this stage, we have shown that for any initial condition U_0, there exists a solution $U \in \mathcal{C}([0,\alpha_0], \mathbb{R}^m)$ to the discrete problem on a time interval $[0,\alpha_0]$, with $\alpha_0 > 0$ such that (34) and (35) hold.

Second step (continuation) :
We can now start again the construction of a solution to (29-32) from the final point $U(\alpha_0)$ and $Z_{\alpha_0} = Z(U)_{\alpha_0} = \exp\left(\int_0^{\alpha_0} \sum_i U_i(r)\partial_y v_i Y_r \, dV_r - \frac{1}{2}\int_0^{\alpha_0} \left(\sum_i U_i(r)\partial_y v_i Y_r\right)^2 dr\right)$ using the same arguments as before. Notice that by the *a priori* estimate of Lemma 4, we have on the one hand $U(\alpha_0) \leq M$ and on the other hand $||Z_{\alpha_0}||_{L_y^\infty(L_\omega^2)} \leq \exp(C\alpha_0 b(1+M^2)) \leq \exp(CTb(1+M^2)) = M'$ (using (33)), with C only depending on $\max_{1\leq i\leq m} ||v_i||_{W_y^{1,\infty}}$. We now consider the mapping F^{α_0} which associates to any function $\Phi \in \mathcal{C}([\alpha_0,T], \mathbb{R}^m)$ the function $F^{\alpha_0}(\Phi) \in \mathcal{C}([\alpha_0,T], \mathbb{R}^m)$ defined by :

$$F^{\alpha_0}(\Phi)(t) = U(\alpha_0) - A^{-1}\left(\int_{\alpha_0}^t \left(B\Phi(s) - \int_{\mathcal{O}} \mathbb{E}\left(\frac{X_s Y_s}{1 - \frac{X_s^2 + Y_s^2}{b}} Z^{\alpha_0}(\Phi)_s\right) \partial_y \Xi + F_{ext}(s)\right) ds\right),$$

where $Z^{\alpha_0}(\Phi)_s = Z_{\alpha_0}\,\mathcal{E}\left(\int_{\alpha_0}^{\bullet} \sum_i \Phi_i(r)\partial_y v_i Y_r \, dV_r\right)_s$. The same arguments as before show that we can find a time interval $[\alpha_0, \alpha_0 + \alpha]$ (with $\alpha \in (0, T - \alpha_0)$) on which F^{α_0} has a fixed point. Indeed, what is important is just that $Z_{\alpha_0} \in L^\infty_y(L^2_\omega)$. This is for example the way one can estimate $Z^{\alpha_0}(\Phi)_t$, for any $t \in [\alpha_0, \alpha_0 + \alpha]$:

$$
\begin{aligned}
&\mathbb{E}\left((Z^{\alpha_0}(\Phi)_t)^2\right) = \mathbb{E}\left(\left(Z(\alpha_0)\,\mathcal{E}\left(\int_{\alpha_0}^{\bullet} \sum_i \Phi_i(r)\partial_y v_i Y_r\, dV_r\right)_t\right)^2\right) \\
&\quad = \mathbb{E}\left(Z^2_{\alpha_0}\,\mathbb{E}\left(\left(\mathcal{E}\left(\int_{\alpha_0}^{\bullet} \sum_i \Phi_i(r)\partial_y v_i Y_r\, dV_r\right)_t\right)^2 |\mathcal{F}_{\alpha_0}\right)\right) \\
&\quad = \mathbb{E}\left(Z^2_{\alpha_0}\,\mathbb{E}\left(\mathcal{E}\left(2\int_{\alpha_0}^{\bullet} \sum_i \Phi_i(r)\partial_y v_i Y_r\, dV_r\right)_t \exp\left(\int_{\alpha_0}^{t}\left(\sum_i \Phi_i(r)\partial_y v_i Y_r\right)^2 dr\right) |\mathcal{F}_{\alpha_0}\right)\right) \\
&\quad \leq \exp\left(C(\alpha-\alpha_0)b||\Phi||^2_{L^\infty[\alpha_0,\alpha_0+\alpha]}\right)\mathbb{E}\left(Z^2_{\alpha_0}\,\mathbb{E}\left(\mathcal{E}\left(2\int_{\alpha_0}^{\bullet} \sum_i \Phi_i(r)\partial_y v_i Y_r\, dV_r\right)_t |\mathcal{F}_{\alpha_0}\right)\right) \\
&\quad \leq \exp\left(C(\alpha-\alpha_0)b||\Phi||^2_{L^\infty[\alpha_0,\alpha_0+\alpha]}\right)\mathbb{E}\left(Z^2_{\alpha_0}\right).
\end{aligned}
$$

Going through the same arguments as before, one can thus show that F^{α_0}, when restricted to $\mathcal{C}([\alpha_0, \alpha_0+\alpha], B(U(\alpha_0), 1))$ is such that $F^{\alpha_0}\left(\mathcal{C}([\alpha_0, \alpha_0+\alpha], B(U(\alpha_0), 1))\right) \subset \mathcal{C}([\alpha_0, \alpha_0+\alpha], B(U(\alpha_0), 1))$ and is contracting, provided that α satisfies an inequality of the type :

$$C\alpha M'(1+M)\exp(CM^2) \leq 1$$

where C is a constant only depending on $||F_{ext}||_{L^\infty_t}$, $\max_{1\leq i\leq m}||v_i||_{W^{1,\infty}_y}$, m, b and T. We can choose

$$\alpha = \alpha_1 = \frac{1}{CM'(1+M)\exp(CM^2)}.$$

We have thus built a solution $U \in \mathcal{C}([0, \alpha_0+\alpha_1], \mathbb{R}^m)$ to the discrete problem on the interval $[0, \alpha_0+\alpha_1]$. The final points $U(\alpha_0+\alpha_1)$ and $Z(U)_{\alpha_0+\alpha_1}$ are again such that $||U(\alpha_0+\alpha_1)|| \leq M$ and $||Z(U)_{\alpha_0+\alpha_1}||_{L^\infty_y(L^2_\omega)} \leq M'$. This means that we can, by the same arguments, extend the solution on the time interval $[\alpha_0+\alpha_1, \alpha_0+2\alpha_1]$, and by a continuation argument, we can build a solution to (29-32) on the time interval $[0, T]$.

Remark 7 *This proves that any finite element approximation of the variational problem (25 - 28) has a solution on a time interval* $[0, T]$ *for any* $T > 0$.

Remark 8 *One can easily prove the uniqueness of a solution to the problem (25 - 28) on* $[0, T]$, *for example by adapting the proof of Lemma 6 to the finite dimensional case.*

4.2 Convergence of the discretized problem

We now turn to the convergence of the solution of the discretized problem. The formal *a priori* estimates of Lemma 5 can be derived rigorously on the discretized system. Indeed, one can take $v = -\partial_{y,y}u^m$ as a test function in (29) (see the special basis (v_i) we have chosen), and the expression (23) of the derivative of τ with respect to y is completely rigorous since $\forall t \in [0, T]$, $u^m(t, .) \in C^\infty(\overline{\mathcal{O}})$. Therefore, using Lemma 5, we know that there exists $T' > 0$ such that there exists a uniform bound on u^m in norm $L^\infty([0, T'], H^1_y) \cap L^2([0, T'], H^2_y)$ and on $\partial_t u^m$ in norm $L^2([0, T'], L^2_y)$.

Up to the extraction of a subsequence, we can suppose that there exists $u \in L^\infty([0, T'], H^1_y) \cap L^2([0, T'], H^2_y)$ such that :

- $u^m \rightharpoonup u$ weakly in $L^2([0,T'],H^2_y)$ and weakly-* in $L^\infty([0,T'],H^1_y)$,
- $\partial_t u^m \rightharpoonup \partial_t u$ weakly in $L^2([0,T'],L^2_y)$,
- $u^m \to u$ strongly in $L^2([0,T'],H^1_y)$.

For the third convergence, we use the standard fact that the injection $\{v \text{ s.t. } v \in L^2([0,T'],H^2_y \cap H^1_{0,y}),\, \partial_t v \in L^2([0,T'],L^2_y)\} \hookrightarrow L^2([0,T'],H^1_{0,y})$ is compact (see Theorem 5.1 p. 58 in [13]). We can also suppose that $u^m \longrightarrow u$ for almost every $(t,y) \in (0,t)\times\mathcal{O}$. We want to show the convergence of each of the terms of the following equation (for a fixed i) :

$$\frac{d}{dt}\int_{\mathcal{O}} u^m v_i + \int_{\mathcal{O}} \partial_y u^m \partial_y v_i = \int_{\mathcal{O}} \mathbb{E}\left(\left(\frac{X_t Y_t}{1-\frac{X_t^2+Y_t^2}{b}}\right)\mathcal{E}\left(\int_0^\bullet \partial_y u^m Y_s\, dV_s\right)_{T'}\right)\partial_y v_i + \int_{\mathcal{O}} f_{ext} v_i, \tag{36}$$

where we have used the following standard property of the exponential martingale : for any y fixed in $\mathcal{O}$, since $\partial_y u(t,y) \in L^2(0,\Theta)$, we have $\forall t \in (0,T')$,

$$\tau(t,y) = \mathbb{E}\left(\left(\frac{X_t Y_t}{1-\frac{X_t^2+Y_t^2}{b}}\right)\mathcal{E}\left(\int_0^\bullet \partial_y u(y) Y_s\, dV_s\right)_{T'}\right).$$

Using the above convergences, we easily pass to the limit in all terms of (36) but $\int_0^{T'}\int_{\mathcal{O}} \mathbb{E}\left(\left(\frac{X_t Y_t}{1-\frac{X_t^2+Y_t^2}{b}}\right)\mathcal{E}\left(\int_0^\bullet \partial_y u^m Y_s\, dV_s\right)_{T'} \partial_y v_i\, w\right)$, where $w \in C_0^\infty(0,T')$. Let us define the function $f^m(t,y,\omega) = \left(\frac{X_t Y_t}{1-\frac{X_t^2+Y_t^2}{b}}\right)\mathcal{E}\left(\int_0^\bullet \partial_y u^m Y_s\, dV_s\right)_{T'} \partial_y v_i\, w$. It is easy to see that, $\int_0^{T'} \partial_y u^m Y_s\, dV_s$ converges in $L^2_y(L^2_\omega)$ to $\int_0^{T'} \partial_y u Y_s\, dV_s$ and that $\frac{1}{2}\int_0^{T'} (\partial_y u^m Y_s)^2\, ds$ converges in $L^1_y(L^1_\omega)$ to $\frac{1}{2}\int_0^{T} (\partial_y u Y_s)^2\, ds$. We can therefore (extracting a subsequence) suppose that f^m converges for almost every (t,y,ω) towards $f(t,y,\omega) = \left(\frac{X_t Y_t}{1-\frac{X_t^2+Y_t^2}{b}}\right)\mathcal{E}\left(\int_0^\bullet \partial_y u Y_s\, dV_s\right)_{T'} \partial_y v_i\, w$. Moreover, we can find a uniform bound on the norm $L^2([0,T'],L^2_y(L^2_\omega))$ of f^m (using the same techniques as in Lemma 3). This shows that the family $(f^m)_{m\geq 1}$ is uniformly integrable and therefore that $\int_0^{T'}\int_{\mathcal{O}} \mathbb{E}(f^m) \longrightarrow \int_0^{T'}\int_{\mathcal{O}} \mathbb{E}(f)$.

Finally, one can prove by standard arguments (see e.g. [18] page 260) that $u(0) = u_0$ and this concludes the "existence part" of Theorem 1.

4.3 Uniqueness of the solution

Lemma 6 (Uniqueness of the solution) *The system (5-7) admits a unique solution on $[0,T']$ in the sense given in Definition 2.*

Proof : Let us consider two solutions (u,X,Y) and $(\tilde{u},\tilde{X},\tilde{Y})$. One easily obtains the following estimate on $w = u - \tilde{u}$, for any $0 < t < T'$:

$$\begin{aligned}
\frac{1}{2}\int_{\mathcal{O}} w^2(t) + \int_0^t\int_{\mathcal{O}} |\partial_y w|^2 &= -\int_0^t\int_{\mathcal{O}} (\tau-\tilde{\tau})\partial_y w \\
&\leq \frac{1}{2}\int_0^t\int_{\mathcal{O}} |\partial_y w|^2 + \frac{1}{2}\int_0^t\int_{\mathcal{O}} |\tau-\tilde{\tau}|^2,
\end{aligned}$$

where

$$\tau - \tilde{\tau} = \mathbb{E}\left(\left(\frac{X_t Y_t}{1 - \frac{X_t^2 + Y_t^2}{b}}\right)(Z(u) - Z(\tilde{u}))\right),$$

with $Z(u)_t = \mathcal{E}\left(\int_0^\bullet \partial_y u(y) Y_s \, dV_s\right)_t$. We have

$$\begin{aligned}
|\tau - \tilde{\tau}|^2 &= \mathbb{E}\left(\left(\frac{X_t Y_t}{1 - \frac{X_t^2 + Y_t^2}{b}}\right)(Z(u) - Z(\tilde{u}))\right)^2 \\
&\leq \mathbb{E}\left(\left(\frac{X_t Y_t}{1 - \frac{X_t^2 + Y_t^2}{b}}\right)^2\right) \mathbb{E}\left((Z(u) - Z(\tilde{u}))^2\right).
\end{aligned}$$

We know that

$$\begin{aligned}
Z(u)_t - Z(\tilde{u})_t &= \int_0^t \left(\partial_y u(y) Z(u)_r - \partial_y \tilde{u}(y) Z(\tilde{u})_r\right) Y_r dV_r \\
&= \int_0^t \left(\partial_y u(y) - \partial_y \tilde{u}(y)\right) Z(u)_r Y_r dV_r + \int_0^t \left(Z(u)_r - Z(\tilde{u})_r\right) \partial_y \tilde{u}(y) Y_r dV_r.
\end{aligned}$$

This yields :

$$\mathbb{E}\left((Z(u)_t - Z(\tilde{u})_t)^2\right) \leq 2b \int_0^t |\partial_y w(y)|^2 \mathbb{E}(Z(u)_r^2) \, dr + 2b \int_0^t |\partial_y \tilde{u}(y)|^2 \mathbb{E}\left((Z(u)_r - Z(\tilde{u})_r)^2\right) dr.$$

Using Gronwall Lemma and the fact that $\mathbb{E}(Z(u)_t^2) \leq \exp\left(b \int_0^{T'} |\partial_y u|^2 \, ds\right)$, this yields an estimate :

$$\mathbb{E}\left((Z(u)_t - Z(\tilde{u})_t)^2\right) \leq C \exp\left(C \int_0^{T'} |\partial_y \tilde{u}(y)|^2 + |\partial_y u(y)|^2\right) \int_0^t |\partial_y w(y)|^2.$$

We have finally, using the estimates of Lemma 5 :

$$\int_0^t \int_{\mathcal{O}} |\partial_y w|^2 \leq C \int_0^t \int_0^s \int_{\mathcal{O}} |\partial_y w|^2.$$

which shows that $w = 0$ by Gronwall Lemma.

In order to conclude this proof of Lemma 6, and therefore of Theorem 1, it remains to recall from Proposition 1 that the stochastic differential equation (7) admits a unique strong solution. $\diamondsuit$

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