Dual effect free stochastic controls

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Abstract

In stochastic optimal control, a key issue is the fact that "solutions" are searched for in terms of "feedback" over available information and, as a consequence, a major potential difficulty is the fact that present control may affect future available information. This is known as the *"dual effect"* of control.

Given a minimal framework (that is, an observation mapping from the product of a control set and of a random set towards an observation set), we define *open-loop lack of dual effect* as the property that the information provided by observations under *open-loop control laws* is fixed, whatever the open-loop control. Our main result consists in characterizing the maximal set of *closed-loop control laws* for which the information provided by observations closed with such a feedback remains also fixed.

We then address the multi-agent case. To obtain a comparable result, we are led to generalize the precedence and memory-communication binary relations introduced by Ho and Chu for the LQG problem, and to assume that the precedence relation is compatible with the memory-communication relation.

When the precedence relation induces an acyclic graph, we prove that, when open-loop lack of dual effect holds, the maximal set of *closed-loop control laws* for which the information provided by observations closed with such a feedback remains fixed is the set of feedbacks measurable with respect to this fixed information. We end by studying the dual effect for discrete time stochastic input-output systems with dynamic information structure, for which the same result holds.

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Contents

1 Introduction

In stochastic optimal control, a key issue is the fact that "solutions" are searched for in terms of "feedback" over available information (which is revealed in a *causal* way as time evolves). As a consequence, a major potential difficulty is the fact that present control may affect future available information. This is known as the *"dual effect"* of control: the control strategy has the (dual) purpose of addressing the problem of cost minimization directly, and, at the same time, of improving the situation of future decisions to be taken by enhancing the information available at the moment of making those decisions.

The most famous illustration of this difficulty is provided by the celebrated Witsenhausen's counterexample [11] which shows how a relatively simple Linear-Quadratic-Gaussian (LQG) stochastic optimal control problem may lead to very nonlinear "solutions" to the point that nobody, indeed, knows how to write down optimality conditions that would lead to a numerical resolution of this problem.

Witsenhausen's counterexample is based on a so-called "non classical information pattern", namely, in that specific case, the lack of *perfect recall*, or *memory*, of past observations. On the contrary, with a classical information pattern, the solution of the LQG problem is quite simple and this problem then enjoys many properties, and at the first rank of them, the lack of dual effect. That is, there is no way to affect the quality of future observations with the control law, even if *numerical values* of future observations *do depend* on past controls. Generally speaking, one way of talking about the "quality" of information provided by observations in mathematical terms is to consider the σ-field generated by those random variables. Thus, saying that information cannot be affected by control amounts to saying that those σ-fields are left invariant by the control.

Another notion also discovered about the LQG problem and discussed extensively in another paper by Witsenhausen [13] is the so-called *separation principle*. Whether the lack of dual effect is a prerequisite for some sort of separation principle to hold true is a subject we do not want to discuss in depth here. Our main motivation is the numerical resolution of stochastic optimal control problems, and, as we will try to explain it shortly in the next lines, the lack of dual effect is of paramount importance in this respect.

Essentially, we would like to distinguish between two points of view. On the one hand, when control variables are searched for as *functions of observations*, this very dependence expresses, by itself, the information structure of the problem. Then, the difficulty is rather that of manipulating such functions effectively, both because of the richness of such mathematical beings (the famous "curse of dimensionality") and because there are generally no reasons to restrain oneself to "well-behaved" functions that can be cast into nice mathematical "spaces".

On the other hand, control and observation variables may simply be considered as *random variables*, that is, indeed, as functions over a certain set Ω supplied with a basic σ-field and a probability measure. Then, one is faced with the problem of expressing that the control variables do not contain more information than the observation variables. This is achieved by saying that the former are *measurable* with respect to the latter. In the numerical handling of the problem, assuming that Ω is finite, the practical rule is to represent control variables as *piecewise* constant functions over the partitions of Ω determined by the observation variables (when the latter are constant over a subset, the former *must* also be constant over the same subset).

Then comes the crucial problem: can those partitions, defined by observations, be determined in advance, without reference to the solution itself, that is, without knowing (past) optimal controls? Here, we exactly touch the issue of whether or not the dual effect of control is present in the problem under consideration. In [3], it is shown that the lack of dual effect is a prerequisite to tackle stochastic optimal control problems by variational methods based on an approximation in terms of scenario trees. A sufficient condition for this lack of dual effect is given in that reference (which covers nontrivial cases when observation values do depend on past controls). It is our purpose in the present paper to revisit and enlarge this topic.

However, in [3], the property that partitions defined by observations are independent of controls is referred to as a *"separation principle"*. We decided to abandon this terminology here: maybe, the separation property refers more closely to a situation when controls are searched for as functions of observations and when a certain "factoring" of this function via a "filtering" problem and a "feedback design" problem occurs (see [13]).

In Section 2, we present a measurability pre-ordering tool for information structures (see [3]), with which we can properly define the dual effect and admissible feedbacks. The main result of this section consists in characterizing the no dual effect feedbacks set assuming no open-loop dual effect. We also relate the absence of dual effect to a form of "noise factoring" of the output functions together with an injectivity property.

In Section 3, we apply the general framework developped in the previous section to the multi-agent case (see [11, 12, 13, 14, 15, 7, 8]). We generalize the precedence and memory-communication binary relations introduced by Ho and Chu for the LQG problem in [7, 8]. Thus, assuming that the precedence relation is compatible with the memory-communication relation, we are able to prove that, when lack of dual effect holds for the set of open-loop feedbacks, it can then be extended to the set of admissible feedbacks that are measurable with respect to the fixed invariant partition resulting from the open-loop no dual effect. When the precedence relation induces an acyclic graph, we obtain a stronger conclusion: when lack of dual effect holds for the set of open-loop feedbacks, it can then be extended to the set of feedbacks that are measurable with respect to the fixed invariant partition resulting from the open-loop no dual effect. Here, such feedbacks are necessarily admissible whereas they were not before.

In Section 4, we present discrete time stochastic input-output systems with dynamic information structure as introduced in [3]. We relate the notions of causality and of perfect memory to the above precedence and memory-communication binary relations. We are thus able to apply the above result since, by causality, the precedence relation induces an acyclic graph.

2 Admissible feedback laws and dual effect

2.1 A minimal framework

The minimal framework is: an observation function $h : U \times \Omega \to Y$ with

- U, control set;
- Ω, random set;
- Y, observation set.

Examples of observation functions $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are

$$h(u, \omega) = u + \omega \;\text{ or }\; h(u, \omega) = \max(u, \omega)\,. \tag{1}$$

Observation functions often come from state space systems:

$$\left\{ \begin{array}{rcl} y_t &=& H_t(x_t, w_t)\,, \quad t = 0, \ldots, T\,, \\ x_{t+1} &=& F_{t+1}(x_t, u_t, v_{t+1})\,, \quad t = 0, \ldots, T-1\,. \end{array} \right. \tag{2}$$

Indeed, putting $u = (u_0, \ldots, u_{T-1})$ and $\omega = (x_0, w_0, v_1, w_1, \ldots, v_T, w_T)$, we can write $y = (y_0, \ldots, y_T) = h(u, \omega)$.

Definition 1
We define the set of all feedbacks

$$\Gamma \stackrel{\text{def}}{=} \{\gamma : \Omega \to U\}\,, \tag{3}$$

and the set $\perp_U$ of open-loop feedbacks (constant mappings),

$$\perp_U \stackrel{\text{def}}{=} \{\gamma \in \Gamma \mid \forall (\omega, \omega') \in \Omega^2\,, \quad \gamma(\omega) = \gamma(\omega')\}\,. \tag{4}$$

We wish to define a class of admissible feedbacks $\mathbb{F}^{\text{ad}}$ containing open-loop feedbacks:

$$\perp_U \subset \mathbb{F}^{\text{ad}} \subset \Gamma\,. \tag{5}$$

The definition of the set $\mathbb{F}^{\text{ad}}$ must capture the fact that a feedback is admissible if it depends on the observations. To this purpose, we need a tool to express that a feedback is measurable with respect to the observations.

However, the sets U, Ω and Y are not supposed to be measurable. Our motivation for this is double. First, we are ultimately interested in numerical applications, thus manipulating discrete sets for which no measurability concept is needed. Second, we feel that the introduction of σ-fields, by its technicalities, may hide the algebraic nature of the lack of dual effect. Indeed, we shall see that certain results are very tedious to express when one considers measurability issues. However, to stress the generality or the limits of our approach, we shall provide remarks all along the paper about what changes or not with a classical measurability framework.

Remark. In the classical measurability framework,

1. U, Ω and Y would be equipped with σ-fields, $\mathcal{U}$, $\mathcal{A}$ and $\mathcal{Y}$,
2. the mapping $h : (U,\mathcal{U}) \times (\Omega, \mathcal{A}) \to (Y, \mathcal{Y})$ would be supposed measurable,
3. Γ would consist only of measurable mappings from $(\Omega, \mathcal{A})$ to $(U,\mathcal{U})$.

2.2 A measurability pre-ordering tool for information structures

We revisit here parts of the mathematical framework developed in [3] by Carpentier, Cohen and Culioli.

To represent information structures (causality constraints, partial observations, etc.) and the notion of closed-loop strategies, one introduces a measurability pre-ordering, denoted $\preceq$, on mappings sharing the same domain (in fact on the partitions induced by such mappings). This is, roughly speaking, a ranking of these mappings according to their respective injectivity.

Definition 2
For any mapping $g\colon \Omega \to Y$, we denote by $\mathsf{part}(g)$ *the partition generated by g, i.e.*

$$\mathsf{part}(g) \stackrel{\text{def}}{=} \{g^{-1}(\{y\}), y \in g(\Omega)\} . \tag{6}$$

Note that, with the above definition, the partition $\mathsf{part}(g)$ never contains $\emptyset$.

Definition 3
Let $g_i\colon \Omega \to Y_i,\ i = 1, 2$.

We say that g_1 is measurable *with respect to g_2, and write $g_1 \preceq g_2$, if $\mathsf{part}(g_1) \supset \mathsf{part}(g_2)$, in the sense that every element of $\mathsf{part}(g_2)$ is included in an element of $\mathsf{part}(g_1)$ or, equivalently, every element of $\mathsf{part}(g_1)$ is the union of elements of $\mathsf{part}(g_2)$.*

We say that g_1 is equivalent to g_2, and write $g_1 \equiv g_2$, if $g_1 \preceq g_2$ and $g_2 \preceq g_1$.

Remark. This concept of measurability differs from the one in measure theory in that it refers to partitions and not to σ-fields. The difference is meaningless when the random set is discrete, and when we, as is the case here, make no use of probability measures.

In the classical measurability framework, $\mathsf{part}(g)$ would be replaced with $\sigma(g)$, *the σ-field generated by g*. We then say that g_1 is *measurable* with respect to g_2 if $\sigma(g_1) \subset \sigma(g_2)$, in the sense that every element of $\sigma(g_1)$ is also element of $\sigma(g_2)$; we write $g_1 \preceq g_2$. We say that g_1 is equivalent to g_2, and write $g_1 \equiv g_2$, if $\sigma(g_1) = \sigma(g_2)$.

The framework developped here may be seen as a particular case of classical measurability with the σ-fields $\mathcal{P}(\Omega)$ and $\mathcal{P}(\mathcal{Y})$ of all subsets of Ω and Y, rendering all mappings measurable.

Remark. We shall manipulate not only mappings with domain Ω, but also mappings with domain $U \times \Omega$. In this latter case, we shall write $\preceq_{U\times\Omega}$ and $\equiv_{U\times\Omega}$ to stress this fact.

The relation $\preceq$ induces a pre-ordering on the mappings sharing the same domain Ω (with the difficulty that such mappings do not form a set in the set theory). If we take the quotient with respect to the equivalence relation $\equiv$, we obtain a lattice in correspondence with the lattice of all partitions of Ω (see [9, p.138]).

Definition 4
The bottom $\perp$ *of the lattice of all partitions of* Ω *is* $\perp = \{\Omega\}$*, in correspondence with constant mappings over* Ω*.*

The top $\top$ *of the lattice of all partitions of* Ω *is* $\top = \{\{\omega\}, \omega \in \Omega\}$*, in correspondence with injective mappings over* Ω*.*

The sup (least upper bound) *operation on the lattice of partitions may be seen as an operation on mappings. If* $g_i \colon \Omega \to Y_i,\ i = 1, 2,\ g_1 \vee g_2$ *is any representative of the class containing the mapping*

$$\omega \in \Omega \mapsto (g_1(\omega), g_2(\omega)) \in Y_1 \times Y_2\,. \tag{7}$$

We shall frequently use the following property in the sequel:

$$\forall g_i \colon \Omega \to Y_i,\ i = 1, \ldots, 4\,, \quad g_1 \preceq g_2 \text{ and } g_3 \preceq g_4 \Rightarrow g_1 \vee g_3 \preceq g_2 \vee g_4\,. \tag{8}$$

Remark. When Ω is equipped with a σ-field $\mathcal{A}$, the set of all sub-σ-fields of $\mathcal{A}$ is a lattice with the *sup (least upper bound)*

$$\mathcal{Y}_1 \vee \mathcal{Y}_2 = \sigma(\mathcal{Y}_1 \cup \mathcal{Y}_2) \tag{9}$$

and *inf (greatest lower bound)*

$$\mathcal{Y}_1 \wedge \mathcal{Y}_2 = \mathcal{Y}_1 \cap \mathcal{Y}_2\,. \tag{10}$$

operations.

Remark. If we specify the image set U, we define

$$\top_U \stackrel{\text{def}}{=} \{\gamma \in \Gamma \mid \gamma \equiv \mathrm{id}_\Omega\}\,, \tag{11}$$

and we note that $\top_U$ consists of injective mappings from Ω to U.

We recall that $\perp_U$ has been defined in equation (4). We have:

$$\begin{cases} \forall \gamma \in \perp_U, & \mathsf{part}(\gamma) \in \perp\,, \\ \forall \gamma \in \top_U, & \mathsf{part}(\gamma) \in \top\,. \end{cases} \tag{12}$$

We now give a series of lemmas useful in the sequel.

2.3 Lemmas on measurability

Lemma 5

Let $g_i \colon \Omega \to Y_i,\ i = 1, 2$. The following conditions are equivalent characterizations of the fact that g_1 is measurable with respect to g_2:

1. $g_1 \preceq g_2$;
2. $\forall(\omega, \omega') \in \Omega^2, \quad g_2(\omega) = g_2(\omega') \Rightarrow g_1(\omega) = g_1(\omega')$;
3. *there exists a mapping $p \colon \operatorname{im} g_2 \to \operatorname{im} g_1$ such that $g_1 = p \circ g_2$ (as a mapping over $\operatorname{im} g_2$, p is uniquely defined).*

Proof. The equivalence of *1* and *2* is an immediate consequence of the definition of $\preceq$. Let us define the multi-application p as :

$$\forall y_2 \in \operatorname{im} g_2\,, \quad p(y_2) = \bigcup_{\omega \in g_2^{-1}(y_2)} g_1(\omega) \in \operatorname{im} g_1\ .$$

From *2*, we deduce that p is in fact a mapping ($p(y_2)$ contains a single element) satisfying *3*. The reverse implication is immediate. □

Remark. In the classical measurability framework, the lemma is now formulated as follows.

Suppose $(Y_i, \mathcal{Y}_i),\ i = 1, 2$ are $(\mathbb{R}^{n_i}, \mathcal{B}_{\mathbb{R}^{n_i}}),\ i = 1, 2$ (where $\mathcal{B}_{\mathbb{R}^{n_i}}$ stands for the Borel σ-field). Let $g_i \colon (\Omega, \mathcal{A}) \to (Y_i, \mathcal{Y}_i),\ i = 1, 2$ be measurable. g_1 is measurable with respect to g_2 ($g_1 \preceq g_2$) if and only if there exists a measurable (Borelian) mapping $p \colon \operatorname{im} g_2 \to \operatorname{im} g_1$ such that $g_1 = p \circ g_2$ (see [5, Ch. 1, p. 18]).

The former lemma being of paramount importance in the sequel of the paper, all forthcoming remarks concerning the classical measurability framework will assume that $(Y_i, \mathcal{Y}_i) = (\mathbb{R}^{n_i}, \mathcal{B}_{\mathbb{R}^{n_i}})$.

Lemma 6

Let $g_i \colon \Omega \to Y_i,\ i = 1, 2$. The following conditions are equivalent characterizations of the fact that g_1 is equivalent to g_2:

1. $g_1 \equiv g_2$;
2. $\forall(\omega, \omega') \in \Omega^2, \quad g_2(\omega) = g_2(\omega') \iff g_1(\omega) = g_1(\omega')$;
3. *there exists an injection $p \colon \operatorname{im} g_2 \to Y_1$ such that $g_1 = p \circ g_2$;*
4. *there exists a bijection $p \colon \operatorname{im} g_2 \to \operatorname{im} g_1$ such that $g_1 = p \circ g_2$.*

Proof. We have

$$g_1 \equiv g_2 \iff g_1 \preceq g_2 \text{ and } g_2 \preceq g_1 \iff \mathsf{part}(g_1) = \mathsf{part}(g_2)$$

if and only if, by Lemma 5, there exist a mapping $p\colon \operatorname{im} g_2 \to \operatorname{im} g_1$ such that $g_1 = p \circ g_2$ and a mapping $q\colon \operatorname{im} g_1 \to \operatorname{im} g_2$ such that $g_2 = q \circ g_1$.

This ends the proof since $p\colon \operatorname{im} g_2 \to \operatorname{im} g_1$ is a bijection if and only if there exist $q\colon \operatorname{im} g_1 \to \operatorname{im} g_2$ such that $p \circ q = \mathrm{id}_{\operatorname{im} g_1}$ and $q \circ p = \mathrm{id}_{\operatorname{im} g_2}$. □

Remark. In the classical measurability framework, the lemma is now formulated as follows.

Let $g_i\colon (\Omega, \mathcal{A}) \to (Y_i, \mathcal{Y}_i)$, $i = 1, 2$ be measurable. g_1 is equivalent to g_2 if and only if there exist a measurable mapping $p\colon \operatorname{im} g_2 \to \operatorname{im} g_1$ such that $g_1 = p \circ g_2$ and a measurable mapping $q\colon \operatorname{im} g_1 \to \operatorname{im} g_2$ such that $g_2 = q \circ g_1$.

Lemma 7

Let $g_i\colon \Omega \to Y_i$, $i = 1, 2$ and $f\colon Y_1 \times \Omega \to Y_3$. Assume that, for all $y_1 \in Y_1$, $f(y_1, \cdot\,) \preceq g_2(\cdot)$ and that $g_1(\cdot) \preceq g_2(\cdot)$. Then $f(g_1(\cdot), \cdot\,) \preceq g_2(\cdot)$.

Proof. By Lemma 5, there exist

1. a mapping $q\colon \operatorname{im} g_2 \to \operatorname{im} g_1$ such that $g_1 = q \circ g_2$,
2. for all $y_1 \in Y_1$, a mapping $p_{y_1}\colon \operatorname{im} g_2 \to \operatorname{im} Y_3$ such that $f(y_1, \cdot\,) = p_{y_1}(g_2(\cdot))$.

Introducing $\widetilde{p}\colon Y_1 \times \operatorname{im} g_2 \to \operatorname{im} Y_3$ defined by $\widetilde{p}(y_1, y_2) \stackrel{\text{def}}{=} p_{y_1}(y_2)$, we have $f(y_1, \omega) = \widetilde{p}(y_1, g_2(\omega))$. We deduce that

$$f(g_1(\omega), \omega) = \widetilde{p}(g_1(\omega), g_2(\omega)) = \widetilde{p}(q(g_2(\omega)), g_2(\omega))$$

so that, by Lemma 5, $f(g_1(\cdot), \cdot\,) \preceq g_2(\cdot)$. □

Remark. In the classical measurability framework, the lemma is now formulated as follows.

Let $g_i\colon (\Omega, \mathcal{A}) \to (Y_i, \mathcal{Y}_i)$, $i = 1, 2$ and $f\colon (Y_1, \mathcal{Y}_1) \times (\Omega, \mathcal{A}) \to Y_2$ be measurable.

Assume that $f(\cdot\,, \cdot\,) \preceq_{Y_1 \times \Omega} \mathrm{id}_{Y_1}(\cdot) \vee g_2(\cdot)$, where $\mathrm{id}_{Y_1}(\cdot)$ and $g_2(\cdot)$ are considered in a straightforward manner as mappings defined on $Y_1 \times \Omega$. Assume also that $g_1(\cdot) \preceq_\Omega g_2(\cdot)$ (as mappings defined on Ω). Then $f(g_1(\cdot), \cdot\,) \preceq_\Omega g_2(\cdot)$ (as mappings defined on Ω).

Lemma 8

Let $g_i\colon \Omega \to Y_i$, $i = 1, 2$ and $f\colon Y_1 \times \Omega \to Y_3$. Assume that, for all $y_1 \in Y_1$, $g_2(\cdot) \preceq f(y_1, \cdot\,)$, and that $g_1(\cdot) \preceq f(g_1(\cdot), \cdot\,)$. Then $g_2(\cdot) \preceq f(g_1(\cdot), \cdot\,)$.

Proof. We provide two proofs. The first one may be extended to the classical measurability framework, while the second cannot, but is much more intuitive.

By Lemma 5, there exist

1. a mapping $q\colon \operatorname{im} g_1 \to \operatorname{im} Y_3$ such that $g_1(\,\cdot\,) = q(f(g_1(\,\cdot\,),\,\cdot\,))$,
2. for all $y_1 \in Y_1$, a mapping $p_{y_1}\colon \operatorname{im} f(y_1,\,\cdot\,) \to \operatorname{im} g_2$ such that $g_2(\,\cdot\,) = p_{y_1}(f(y_1,\,\cdot\,))$.

Introducing $\widetilde{p}\colon \bigcup_{y_1 \in Y_1} \{y_1\} \times \operatorname{im} f(y_1,\,\cdot\,) \to \operatorname{im} g_2$ defined by $\widetilde{p}(y_1, y_3) \stackrel{\text{def}}{=} p_{y_1}(y_3)$, we have $g_2(\omega) = \widetilde{p}(y_1, f(y_1,\omega))$. We deduce that

$$g_2(\omega) = \widetilde{p}(y_1, f(y_1,\omega)) = \widetilde{p}(g_1(\omega), f(g_1(\omega),\omega)) = \widetilde{p}(q(f(g_1(\omega),\omega)), f(g_1(\omega),\omega))$$

so that, by Lemma 5, $g_2(\,\cdot\,) \preceq f(g_1(\,\cdot\,),\,\cdot\,)$. This ends the first proof.

Let $(\omega,\omega') \in \Omega^2$ be such that $f(g_1(\omega),\omega) = f(g_1(\omega'),\omega')$.
Since $g_1(\,\cdot\,) \preceq f(g_1(\,\cdot\,),\,\cdot\,)$, we have $g_1(\omega) = g_1(\omega')$. Putting $y_1 = g_1(\omega) = g_1(\omega')$, we thus get

$$f(y_1,\omega) = f(g_1(\omega),\omega) = f(g_1(\omega'),\omega') = f(y_1,\omega')\,.$$

On the other hand, we have $y_1 \in Y_1$, $g_2(\,\cdot\,) \preceq f(y_1,\,\cdot\,)$, so that $g_2(\omega) = g_2(\omega')$. Thus, by Lemma 5, we have proved that $g_2(\,\cdot\,) \preceq f(g_1(\,\cdot\,),\,\cdot\,)$. □

Remark. In the classical measurability framework, the lemma is now formulated as follows.

Let $g_i\colon (\Omega,\mathcal{A}) \to (Y_i,\mathcal{Y}_i)$, $i = 1,2$ and $f\colon (Y_1,\mathcal{Y}_1) \times (\Omega,\mathcal{A}) \to Y_2$ be measurable.

Assume that $g_2(\,\cdot\,) \preceq_{Y_1\times\Omega} \mathrm{id}_{Y_1}(\,\cdot\,) \vee f(\,\cdot\,,\,\cdot\,)$ as mappings defined on $Y_1 \times \Omega$, and that $g_1(\,\cdot\,) \preceq_\Omega f(g_1(\,\cdot\,),\,\cdot\,)$ (as mappings defined on Ω). Then $g_2(\,\cdot\,) \preceq_\Omega f(g_1(\,\cdot\,),\,\cdot\,)$ (as mappings defined on Ω).

2.4 Admissible feedbacks and observation after feedback

Since any feedback affects the available information, we introduce a notation for the observation after feedback.

Definition 9
For any $\gamma \in \Gamma$, the observation after feedback *$\eta^\gamma : \Omega \to Y$ is defined by*

$$\forall \omega \in \Omega\,, \quad \eta^\gamma(\omega) \stackrel{\text{def}}{=} h\big(\gamma(\omega),\omega\big)\,. \tag{13}$$

Definition 10
The set $\mathbb{F}^{\text{ad}}$ of admissible feedbacks *is*

$$\mathbb{F}^{\text{ad}} \stackrel{\text{def}}{=} \{\gamma \in \Gamma \mid \gamma \preceq \eta^\gamma\}\,. \tag{14}$$

This definition captures the fact that the feedback may depend only on the observations y, namely (by Lemma 5) that there exists a mapping $g : Y \to U$ such that the controls u produced by an admissible feedback are of the form $u = g(y)$.

Since any constant $\gamma \in \perp_U$ is measurable with respect to any mapping over Ω, we have

$$\perp_U \subset \mathbb{F}^{\text{ad}}\,, \tag{15}$$

meaning that open-loop feedbacks are admissible.

2.5 Definitions of open-loop dual effect and of the no dual effect feedbacks set

We wish here to formulate and characterize the *"dual effect"* of control (see [6, 2, 1, 10]). First, we define the dual effect through the action of *constant feedbacks* (open-loop control laws).

DEFINITION 11
There is no open-loop dual effect *for the stochastic controlled system with observation function* $h : U \times \Omega \to Y$ *if we have*

$$\forall (\gamma, \gamma') \in \perp_U \times \perp_U \,, \quad \eta^\gamma \equiv \eta^{\gamma'} \,. \tag{16}$$

We then denote by ζ *any mapping with domain* Ω *such that*

$$\forall \gamma \in \perp_U \,, \quad \eta^\gamma \equiv \zeta \,. \tag{17}$$

For instance, ζ *can be any mapping of the class of* η^γ *for* $\gamma \in \perp_U$. *We introduce*

$$\mathbb{F}^\zeta \stackrel{\text{def}}{=} \{\gamma \in \Gamma \mid \gamma \preceq \zeta\} \,. \tag{18}$$

PROPOSITION 12
There is no open-loop dual effect if and only if

$$\exists \zeta : \Omega \to \mathcal{Z} \,, \quad \exists p : U \times \mathcal{Z} \to Y \tag{19}$$

such that

$$\forall u \in U \,, \quad p(u, \cdot\,) : \mathcal{Z} \to Y \ \text{ is injective} \tag{20}$$

and that

$$\forall u \in U \,, \quad \forall \omega \in \Omega \,, \quad h(u, \omega) = p(u, \zeta(\omega)) \,. \tag{21}$$

Proof. This is a straightforward application of Lemma 6, for each $u \in U$. □

Remark. In the classical measurability framework, the definition of $\mathbb{F}^\zeta$ would remain the same, while the definition of *no open-loop dual effect* would be given by the characterization of the above Proposition, with ζ and p measurable.

Our main aim in this paper is to determine to what extent open-loop dual effect remains valid for a larger set of admissible closed loops. This motivates the following definition.

DEFINITION 13
Assuming no open-loop dual effect, the no dual effect feedbacks set *is*

$$\mathbb{F}^{\text{nde}} \stackrel{\text{def}}{=} \{\gamma \in \mathbb{F}^{\text{ad}} \mid \eta^\gamma \equiv \zeta\} \,. \tag{22}$$

We clearly have that

$$\forall (\gamma, \gamma') \in \mathbb{F}^{\text{nde}} \times \mathbb{F}^{\text{nde}} \,, \quad \eta^\gamma \equiv \eta^{\gamma'} \,, \tag{23}$$

and $\mathbb{F}^{\text{nde}}$ is the largest such set in $\mathbb{F}^{\text{ad}}$.

2.6 Characterization of the no dual effect feedbacks set

Our main result in this section is the following characterization of the no dual effect feedbacks set.

Proposition 14
Assuming no open-loop dual effect, we have that

$$\mathbb{F}^{\mathrm{nde}} = \mathbb{F}^{\mathrm{ad}} \cap \mathbb{F}^{\zeta} . \tag{24}$$

Proof. Let $\gamma \in \mathbb{F}^{\mathrm{nde}}$. On the one hand, we have, since $\gamma \in \mathbb{F}^{\mathrm{ad}}$, $\gamma(\,\cdot\,) \preceq \eta^{\gamma}(\,\cdot\,) = h\big(\gamma(\,\cdot\,),\,\cdot\,\big)$. On the other hand, we have, by assumption, $h\big(\gamma(\,\cdot\,),\,\cdot\,\big) = \eta^{\gamma}(\,\cdot\,) \equiv \zeta(\,\cdot\,)$. Thus, $\gamma(\,\cdot\,) \preceq h\big(\gamma(\,\cdot\,),\,\cdot\,\big) = \eta^{\gamma}(\,\cdot\,) \equiv \zeta(\,\cdot\,)$, so that $\gamma \in \mathbb{F}^{\mathrm{ad}} \cap \mathbb{F}^{\zeta}$. We have shown that

$$\mathbb{F}^{\mathrm{nde}} \subset \mathbb{F}^{\mathrm{ad}} \cap \mathbb{F}^{\zeta} .$$

Let $\gamma \in \mathbb{F}^{\mathrm{ad}} \cap \mathbb{F}^{\zeta}$.

1. On the one hand, we have, by definition of $\mathbb{F}^{\zeta}$, $\gamma \preceq \zeta$. On the other hand, we have, by assumption, $\forall u \in U$, $h(u,\,\cdot\,) \equiv \zeta(\,\cdot\,)$ and thus $\forall u \in U$, $h(u,\,\cdot\,) \preceq \zeta(\,\cdot\,)$. By Lemma 7, we deduce that $h\big(\gamma(\,\cdot\,),\,\cdot\,\big) \preceq \zeta(\,\cdot\,)$, that is $\eta^{\gamma} \preceq \zeta$.
2. On the one hand, we have, by definition of $\mathbb{F}^{\mathrm{ad}}$, $\gamma(\,\cdot\,) \preceq \eta^{\gamma}(\,\cdot\,) = h\big(\gamma(\,\cdot\,),\,\cdot\,\big)$. On the other hand, we have, by assumption, $\forall u \in U$, $h(u,\,\cdot\,) \equiv \zeta(\,\cdot\,)$ and thus $\forall u \in U$, $\zeta(\,\cdot\,) \preceq h(u,\,\cdot\,)$. A straightforward application of Lemma 8 then gives $\zeta(\,\cdot\,) \preceq h\big(\gamma(\,\cdot\,),\,\cdot\,\big)$, that is $\zeta \preceq \eta^{\gamma}$.

We thus have both $\eta^{\gamma} \preceq \zeta$ and $\zeta \preceq \eta^{\gamma}$, so that $\eta^{\gamma} \equiv \zeta$. We conclude that $\gamma \in \mathbb{F}^{\mathrm{nde}}$. We have thus shown that

$$\mathbb{F}^{\mathrm{ad}} \cap \mathbb{F}^{\zeta} \subset \mathbb{F}^{\mathrm{nde}} .$$

□

Remark. In the classical measurability framework, this proposition would be delicate to express, requiring in particular a technical definition of $\mathbb{F}^{\mathrm{nde}}$.

We now show by three examples that $\mathbb{F}^{\mathrm{ad}}$ and $\mathbb{F}^{\zeta}$ have no relationship in general.

1. Let $U = \Omega = \mathbb{R}$ and $h(u, \omega) = u$ for which no open-loop dual effect holds with any constant $\zeta \in \perp_U$. We thus have

$$\mathbb{F}^{\zeta} = \{\gamma \in \Gamma \mid \gamma \preceq \perp_U\} = \perp_U .$$

Since, for all $\gamma \in \Gamma$, $\eta^{\gamma} = \gamma$, we have

$$\mathbb{F}^{\mathrm{ad}} = \{\gamma \in \Gamma \mid \gamma \preceq \gamma\} = \Gamma .$$

This is thus a case where $\mathbb{F}^{\zeta} \subsetneq \mathbb{F}^{\mathrm{ad}}$.

2. Let $U = \Omega = \mathbb{R}$ and $h(u,\omega) = \omega$ for which no open-loop dual effect holds with $\zeta = \mathrm{id}_\Omega$ (or any injective mapping $\zeta \in \top_U$). We thus have

$$\mathbb{F}^\zeta = \{\gamma \in \Gamma \mid \gamma \preceq \mathrm{id}_\Omega\} = \Gamma\,.$$

Since, for all $\gamma \in \Gamma$, $\eta^\gamma = \mathrm{id}_\Omega$, we have

$$\mathbb{F}^{\mathrm{ad}} = \{\gamma \in \Gamma \mid \gamma \preceq \mathrm{id}_\Omega\} = \Gamma\,.$$

This is thus a case where $\mathbb{F}^\zeta = \mathbb{F}^{\mathrm{ad}}$.

3. Let $U = \Omega = \mathbb{R}$ and $h(u,\omega) = \omega - u$ for which no open-loop dual effect holds with $\zeta = \mathrm{id}_\Omega$ (or any injective mapping $\zeta \in \top_U$). We thus have

$$\mathbb{F}^\zeta = \{\gamma \in \Gamma \mid \gamma \preceq \mathrm{id}_\Omega\} = \Gamma\,.$$

On the other hand, $\mathrm{id}_\Omega \notin \mathbb{F}^{\mathrm{ad}}$ since $\eta^{\mathrm{id}_\Omega} = 0$. This is thus a case where $\mathbb{F}^{\mathrm{ad}} \subsetneq \mathbb{F}^\zeta$.

3 Dual effect for multi-agent stochastic input-output systems

Multi-agent decision problems have been studied for instance in [11, 12, 13, 14, 15, 7, 8]. We focus here on the modifications to bring to the above single-agent analysis in Section 2 in order to be able to define and characterize the lack of dual effect.

3.1 Multi-agent stochastic input-output systems

Let A be a finite set representing agents. Each agent $\alpha \in A$ is supposed to take only one decision $u_\alpha \in U_\alpha$, where U_α is the control set for agent α. We put

$$U_A \stackrel{\mathrm{def}}{=} \prod_{\alpha \in A} U_\alpha\,. \tag{25}$$

We denote

$$\Gamma_A \stackrel{\mathrm{def}}{=} \{\gamma : \Omega \to U_A\} = \{\gamma = (\gamma_\alpha)_{\alpha \in A} \mid \forall \alpha \in A\,,\ \gamma_\alpha : \Omega \to U_\alpha\}\,. \tag{26}$$

To each agent $\alpha \in A$ corresponds an observation function

$$h_\alpha : U_A \times \Omega \to Y_\alpha\,. \tag{27}$$

Definition 15
For any feedback $\gamma \in \Gamma_A$ and agent $\alpha \in A$, the observation of agent α after feedback *$\eta_\alpha^\gamma : \Omega \to Y_\alpha$ is defined by*

$$\forall \alpha \in A\,, \forall \omega \in \Omega\,, \quad \eta_\alpha^\gamma(\omega) \stackrel{\mathrm{def}}{=} h_\alpha\big(\gamma(\omega), \omega\big)\,. \tag{28}$$

In general, the observation available to agent α depends, through the feedback γ, upon the decisions of other agents.

Definition 16
The set of admissible feedbacks for the multi-agent stochastic input-output system $(h_\alpha)_{\alpha\in A}$ is

$$\mathbb{F}_A^{\mathrm{ad}} \stackrel{\mathrm{def}}{=} \{\gamma = (\gamma_\alpha)_{\alpha\in A} \in \Gamma_A \mid \forall\alpha\in A\,,\ \gamma_\alpha \preceq \eta_\alpha^\gamma\}\,. \tag{29}$$

The link with the single-agent case is obvious. If we put

$$U \stackrel{\mathrm{def}}{=} U_A = \prod_{\alpha\in A} U_\alpha\,, \quad Y \stackrel{\mathrm{def}}{=} \prod_{\alpha\in A} Y_\alpha\,, \quad h \stackrel{\mathrm{def}}{=} (h_\alpha)_{\alpha\in A}$$

we have that

$$\begin{cases} \Gamma_A & = & \Gamma & \text{as defined in equation (3),} \\ (\eta_\alpha^\gamma)_{\alpha\in A} & = & \eta^\gamma & \text{as defined in equation (13),} \\ \mathbb{F}_A^{\mathrm{ad}} & \subset & \mathbb{F}^{\mathrm{ad}} & \text{as defined in equation (14).} \end{cases}$$

Remark. Note that in general, $\mathbb{F}_A^{\mathrm{ad}} \subsetneq \mathbb{F}^{\mathrm{ad}}$, since by (8)

$$\forall\alpha\in A,\ \ \gamma_\alpha \preceq \eta_\alpha^\gamma \Rightarrow \gamma \equiv \bigvee_{\alpha\in A}\gamma_\alpha \preceq \bigvee_{\alpha\in A}\eta_\alpha^\gamma \equiv \eta^\gamma$$

but not the other way in general. Indeed, take $\Omega = \mathbb{R}^2$, $A = \{1,2\}$, $h_1(u,\omega) = \omega_1$, $h_2(u,\omega) = \omega_2$.

3.2 Precedence and memory-communication binary relations

Two binary relations between agents were introduced by Ho and Chu in [7, 8] for the multi-agent LQG problem. We generalize these relations here.

Precedence binary relation

Definition 17
For any $B \subset A$, let π_B denote the projection from U_A to U_B (see (25)). We also denote

$$\forall u\in U_A\,,\quad u_B \stackrel{\mathrm{def}}{=} \pi_B(u) = (u_\alpha)_{\alpha\in B}\ \textit{ for }\ u\in U_B\,. \tag{30}$$

For any $B \subset A$ and any family of mappings $(f_\alpha)_{\alpha\in A}$ taking values in U_A, we denote

$$f_B \stackrel{\mathrm{def}}{=} (f_\alpha)_{\alpha\in B}\,. \tag{31}$$

Remark. By abuse of notation, we will possibly consider u_B as an element of U_A using an embedding application from U_B into U_A. Accordingly, the projection π_B takes its values in U_A and we may write $\pi_B \circ \pi_B = \pi_B$.

DEFINITION 18
For any $\alpha \in A$, let

$$\mathcal{B}_\alpha \overset{\text{def}}{=} \{B \subset A \mid h_\alpha(\,\cdot\,,\,\cdot\,) \preceq_{U_A \times \Omega} \pi_B(\,\cdot\,) \vee \mathrm{id}_\Omega(\,\cdot\,)\}\,. \tag{32}$$

PROPOSITION 19
$\mathcal{B}_\alpha$ is stable by set intersection.

Proof. Since A is a finite set, we have to prove that $B \cap C \in \mathcal{B}_\alpha$ as soon as B and C are elements of $\mathcal{B}_\alpha$. Let $(B, C) \in \mathcal{B}_\alpha \times \mathcal{B}_\alpha$. By Lemma 5, there exist

- a mapping p such that $h_\alpha(\,\cdot\,,\,\cdot\,) = p\big(\pi_B(\,\cdot\,),\,\cdot\,\big)$,
- a mapping q such that $h_\alpha(\,\cdot\,,\,\cdot\,) = q\big(\pi_C(\,\cdot\,),\,\cdot\,\big)$,

and thus

$$p\big(\pi_B(u), \omega\big) = q\big(\pi_C(u), \omega\big),\ \ \forall (u, \omega) \in U_A \times \Omega\,.$$

Writing this equation for all u of the form $\pi_C(v)$, we obtain (recall here that the projection π_B is extended to take its values in U_A)

$$p\big(\pi_B \circ \pi_C(v), \omega\big) = q\big(\pi_C \circ \pi_C(v), \omega\big),\ \ \forall (v, \omega) \in U_A \times \Omega\,.$$

But $\pi_C \circ \pi_C = \pi_C$ so that

$$h_\alpha(\,\cdot\,,\,\cdot\,) = q\big(\pi_C(\,\cdot\,),\,\cdot\,\big) = p\big(\pi_B \circ \pi_C(\,\cdot\,),\,\cdot\,\big)\,.$$

From the very definition of π_B, we have

$$\pi_B \circ \pi_C = \pi_{B \cap C}\ ,$$

and thus $h_\alpha(\,\cdot\,,\,\cdot\,) \preceq_{U_A \times \Omega} \pi_{B \cap C}(\,\cdot\,) \vee \mathrm{id}_\Omega(\,\cdot\,)$. □

DEFINITION 20
Since $\mathcal{B}_\alpha$ is stable by set intersection, we define

$$[\alpha] \overset{\text{def}}{=} \cap_{B \in \mathcal{B}_\alpha} B\,. \tag{33}$$

In other words, $[\alpha] \subset A$ is the smallest subset $B \subset A$ such that

$$h_\alpha(\,\cdot\,,\,\cdot\,) \preceq_{U_A \times \Omega} \pi_B(\,\cdot\,) \vee \mathrm{id}_\Omega(\,\cdot\,)\,. \tag{34}$$

Definition 21
We define a precedence *binary relation $\mathcal{P}$ on A by*

$$\beta\,\mathcal{P}\,\alpha \iff \beta \in [\alpha] \tag{35}$$

and we say that β is a precedent *of α.*

In other words, if β is a *precedent* of α, then $h_\alpha(u,\omega)$ indeed depends upon u_β: the agent β influences the observation made by agent α. Since $h_\alpha(u,\omega)$ depends only on the components $u_{[\alpha]} = (u_\beta)_{\beta\in[\alpha]}$, by abuse of notation we shall write

$$h_\alpha(u,\omega) = h_\alpha(u_{[\alpha]},\omega) = h_\alpha(u_B,\omega), \quad \forall B \supset [\alpha]\,, \tag{36}$$

and we shall frequently use this latter relationship in the sequel.

Memory-communication binary relation

The following definition of memory-communication is inspired by [7].

Definition 22
We define a memory-communication *binary relation $\mathcal{M}$ on A by*

$$\forall \alpha \in A\,, \forall \beta \in A\,, \quad \beta\,\mathcal{M}\,\alpha \iff h_\beta(\,\cdot\,,\,\cdot\,) \preceq_{U_A\times\Omega} h_\alpha(\,\cdot\,,\,\cdot\,)\,, \tag{37}$$

and we say that β is remembered *by α. We introduce*

$$\forall \alpha \in A\,, \quad \|\alpha\| \overset{\text{def}}{=} \{\beta \in A \mid \beta\,\mathcal{M}\,\alpha\}\,. \tag{38}$$

When β is remembered by α, the observations made by agent β are part of those made by agent α. This is expressed by the following relationship

$$\forall \alpha \in A\,, \quad h_{\|\alpha\|}(\,\cdot\,,\,\cdot\,) \preceq_{U_A\times\Omega} h_\alpha(\,\cdot\,,\,\cdot\,)\,, \tag{39}$$

which results from

$$h_{\|\alpha\|}(\,\cdot\,,\,\cdot\,) = (h_\beta(\,\cdot\,,\,\cdot\,))_{\beta\in\|\alpha\|} \equiv_{U_A\times\Omega} \bigvee_{\beta\in\|\alpha\|} h_\beta(\,\cdot\,,\,\cdot\,) \preceq_{U_A\times\Omega} h_\alpha(\,\cdot\,,\,\cdot\,)\,.$$

by (8).

Properties

Definition 23
We say that the precedence binary relation $\mathcal{P}$ is included in (or compatible with) the memory-communication binary relation $\mathcal{M}$ if

$$\forall \alpha \in A\,, \forall \beta \in A\,, \quad \beta\,\mathcal{P}\,\alpha \Rightarrow \beta\,\mathcal{M}\,\alpha\,. \tag{40}$$

We denote this property by $\mathcal{P} \subset \mathcal{M}$.

Proposition 24
The following conditions are equivalent characterizations of the fact that the precedence binary relation $\mathcal{P}$ is included in the memory-communication binary relation $\mathcal{M}$:

$$\mathcal{P} \subset \mathcal{M} \iff \forall \alpha \in A\,, \quad [\alpha] \subset \|\alpha\|\,, \tag{41}$$

$$\mathcal{P} \subset \mathcal{M} \iff \forall \alpha \in A\,, \quad h_{[\alpha]}(\,\cdot\,,\,\cdot\,) \preceq_{U_A \times \Omega} h_{\alpha}(\,\cdot\,,\,\cdot\,)\,. \tag{42}$$

Proof. Equation (41) simply is a reformulation of the definition $\mathcal{P} \subset \mathcal{M}$.
It may be easily checked that

$$\forall B \subset A\,, \forall \alpha \in A\,, \quad B \subset \|\alpha\| \iff h_B(\,\cdot\,,\,\cdot\,) \preceq_{U_A \times \Omega} h_{\alpha}(\,\cdot\,,\,\cdot\,)\,. \tag{43}$$

Combined with equation (41), this gives equation (42). □

Remark. If the precedence binary relation $\mathcal{P}$ is included in the memory-communication binary relation $\mathcal{M}$, then it is clear by Lemma 5 that

$$\forall \alpha \in A\,, \forall \beta \in A\,, \quad \beta\,\mathcal{P}\,\alpha \Rightarrow \forall \gamma \in \Gamma_A\,,\, h_{\beta}(\gamma(\,\cdot\,),\,\cdot\,) \preceq_{\Omega} h_{\alpha}(\gamma(\,\cdot\,),\,\cdot\,)\,. \tag{44}$$

This latter property is taken as the definition of a *partially nested information structure* in [7, 8]. Note that the problem with this latter definition is the presence of any feedback γ. Our assumption is an "open-loop one" which does not require assumptions as to the closed-loop system.

Here are other properties of $\mathcal{M}$ and $\mathcal{P}$.

Proposition 25
The memory-communication *binary relation $\mathcal{M}$ is a pre-order on A such that*

$$\forall \alpha \in A\,, \forall \beta \in A\,, \quad \beta\,\mathcal{M}\,\alpha \iff \beta \in \|\alpha\| \Rightarrow [\beta] \subset [\alpha]\,. \tag{45}$$

Proof. $\mathcal{M}$ clearly is a pre-order on A (reflexive and transitive) since $\preceq$ is a pre-order on $U_A \times \Omega$.
Now, $\beta\,\mathcal{M}\,\alpha$ implies that

$$h_{\beta}(\,\cdot\,,\,\cdot\,) \preceq_{U_A \times \Omega} h_{\alpha}(\,\cdot\,,\,\cdot\,) \preceq_{U_A \times \Omega} \pi_{[\alpha]}(\,\cdot\,) \vee \mathrm{id}_{\Omega}(\,\cdot\,) \tag{46}$$

by definition of $[\alpha]$. Thus, from definition (33), we deduce that $[\beta] \subset [\alpha]$. □

Proposition 26
If the precedence binary relation $\mathcal{P}$ is included in the memory-communication binary relation $\mathcal{M}$, then $\mathcal{P}$ is transitive.

Proof. Let $(\alpha, \beta, \gamma) \in A^3$ be such that $\alpha \mathcal{P} \beta$ and $\beta \mathcal{P} \gamma$.

On the one hand, since $\alpha \mathcal{P} \beta$, by equation (35), we have $\alpha \in [\beta]$. On the other hand, by $\mathcal{P} \subset \mathcal{M}$ and equations (38) and (45), we get:

$$\beta \mathcal{P} \gamma \Rightarrow \beta \mathcal{M} \gamma \Rightarrow \beta \in \|\gamma\| \Rightarrow [\beta] \subset [\gamma] \,.$$

Thus, transitivity holds. Indeed, $\alpha \mathcal{P} \gamma$ is true since

$$\alpha \in [\beta] \subset [\gamma] \iff \alpha \mathcal{P} \gamma \,.$$

□

3.3 Definitions of open-loop dual effect and of the no dual effect feedbacks set

We now introduce a notion of open-loop dual effect adapted to the multi-agent case.

Definition 27
There is no open-loop dual effect *for the multi-agent stochastic controlled system with observation functions* $(h_\alpha)_{\alpha \in A}$ *if we have*

$$\forall (\gamma, \gamma') \in \perp_{U_A} \times \perp_{U_A} \,, \quad \forall \alpha \in A \,, \, \eta_\alpha^\gamma \equiv \eta_\alpha^{\gamma'} \,. \tag{47}$$

For all $\alpha \in A$*, we then denote by* ζ_α *any mapping with domain* Ω *such that*

$$\forall \gamma \in \perp_{U_A} \,, \quad \eta_\alpha^\gamma \equiv \zeta_\alpha \,. \tag{48}$$

For instance, ζ_α *can be any mapping of the class of* η_α^γ *for* $\gamma \in \perp_{U_A}$*. We introduce*

$$\mathbb{F}_A^\zeta \stackrel{\text{def}}{=} \{\gamma = (\gamma_\alpha)_{\alpha \in A} \in \Gamma_A \mid \forall \alpha \in A \,, \, \gamma_\alpha \preceq \zeta_\alpha\} \,. \tag{49}$$

Definition 28
Assuming no open-loop dual effect, the no dual effect feedback set *is*

$$\mathbb{F}_A^{\text{nde}} \stackrel{\text{def}}{=} \{(\gamma_\alpha)_{\alpha \in A} \in \Gamma_A \mid \forall \alpha \in A \,, \, \eta_\alpha^\gamma \equiv \zeta_\alpha\} \cap \mathbb{F}_A^{\text{ad}} \,. \tag{50}$$

Remark. With the notations of the remark following Definition 16, and with

$$\zeta \stackrel{\text{def}}{=} (\zeta_\alpha)_{\alpha \in A} \tag{51}$$

we have that

$$\mathbb{F}_A^{\text{nde}} \subset \mathbb{F}^{\text{nde}} \text{ as defined in equation (22).} \tag{52}$$

Note that, in general, $\mathbb{F}_A^{\text{nde}} \subsetneq \mathbb{F}^{\text{nde}}$ since by (8)

$$\forall \alpha \in A \,, \, \eta_\alpha^\gamma \equiv \zeta_\alpha \Rightarrow \eta^\gamma \equiv \bigvee_{\alpha \in A} \eta_\alpha^\gamma \equiv \bigvee_{\alpha \in A} \zeta_\alpha \equiv \zeta$$

but not the other way in general.

3.4 Characterization of the no dual effect feedbacks set when precedence implies memory-communication

Proposition 29
Assuming no open-loop dual effect, we have

$$\mathbb{F}_A^{\mathrm{nde}} \subset \mathbb{F}_A^{\mathrm{ad}} \cap \mathbb{F}_A^{\zeta} . \tag{53}$$

While the previous inclusion may be proved as in Proposition 14, the following one requires an additional assumption.

Proposition 30
Let us assume that no open-loop dual effect holds and that the precedence binary relation $\mathcal{P}$ is included in the memory-communication binary relation $\mathcal{M}$. Then

$$\mathbb{F}_A^{\mathrm{ad}} \cap \mathbb{F}_A^{\zeta} \subset \mathbb{F}_A^{\mathrm{nde}} . \tag{54}$$

Proof. We note that, by definition of $[\alpha]$ and by equation (36):

$$\forall \alpha \in A \, , \, \forall \gamma \in \Gamma_A \, , \quad \eta_\alpha^\gamma = \eta_\alpha^{\gamma_{[\alpha]}} . \tag{55}$$

Let $\gamma = (\gamma_\alpha)_{\alpha \in A} \in \mathbb{F}_A^{\mathrm{ad}} \cap \mathbb{F}_A^{\zeta}$, that is

$$\forall \alpha \in A \, , \, \gamma_\alpha \preceq \zeta_\alpha \ \text{ and } \ \gamma_\alpha \preceq \eta_\alpha^\gamma . \tag{56}$$

Let $\alpha \in A$ be fixed: we now prove that both $\eta_\alpha^\gamma \preceq \zeta_\alpha$ and $\zeta_\alpha \preceq \eta_\alpha^\gamma$.

Let us first show that $\eta_\alpha^\gamma \preceq \zeta_\alpha$.
By the no open-loop dual effect assumption, we have that, for any $u \in U_A$, by (8),

$$\gamma_{[\alpha]} = (\gamma_\beta)_{\beta \in [\alpha]} \equiv \bigvee_{\beta \in [\alpha]} \gamma_\beta \preceq \bigvee_{\beta \in [\alpha]} \zeta_\beta \preceq \bigvee_{\beta \in [\alpha]} h_\beta(u, \, \cdot \,) \equiv h_{[\alpha]}(u, \, \cdot \,) . \tag{57}$$

Since the precedence binary relation $\mathcal{P}$ is included in the memory-communication binary relation $\mathcal{M}$ and by equation (42), the right hand side is such that $h_{[\alpha]}(u, \, \cdot \,) \preceq h_\alpha(u, \, \cdot \,)$, and we deduce that

$$\gamma_{[\alpha]}(\, \cdot \,) \preceq h_\alpha(u, \, \cdot \,) \equiv \zeta_\alpha(\, \cdot \,) . \tag{58}$$

Combined with $h_\alpha(u, \, \cdot \,) = h_\alpha(u_{[\alpha]}, \, \cdot \,) \preceq \zeta_\alpha(\, \cdot \,)$, for all $u \in U_A$, this gives by Lemma 7,

$$\eta_\alpha^\gamma(\, \cdot \,) = \eta_\alpha^{\gamma_{[\alpha]}}(\, \cdot \,) = h_\alpha\big(\gamma_{[\alpha]}(\, \cdot \,), \, \cdot \,\big) \preceq \zeta_\alpha(\, \cdot \,) .$$

Now, let us prove that $\zeta_\alpha \preceq \eta_\alpha^\gamma$.

We have

$$\gamma_{[\alpha]} = (\gamma_\beta)_{\beta\in[\alpha]} \equiv \bigvee_{\beta\in[\alpha]} \gamma_\beta \preceq \bigvee_{\beta\in[\alpha]} \eta^\gamma_\beta \equiv \eta^\gamma_{[\alpha]} . \tag{59}$$

Since the precedence binary relation $\mathcal{P}$ is included in the memory-communication binary relation $\mathcal{M}$ and by equation (42), the right hand side is such that

$$\eta^\gamma_{[\alpha]}(\,\cdot\,) = h_{[\alpha]}\big(\gamma(\,\cdot\,),\,\cdot\,\big) \preceq h_\alpha\big(\gamma(\,\cdot\,),\,\cdot\,\big) = h_\alpha\big(\gamma_{[\alpha]}(\,\cdot\,),\,\cdot\,\big) .$$

We deduce that $\gamma_{[\alpha]}(\,\cdot\,) \preceq h_\alpha\big(\gamma_{[\alpha]}(\,\cdot\,),\,\cdot\,\big)$. Combined with $\zeta_\alpha(\,\cdot\,) \preceq h_\alpha(u,\,\cdot\,) = h_\alpha(u_{[\alpha]},\,\cdot\,)$, for all $u \in U_A$, this gives, by Lemma 8,

$$\zeta_\alpha(\,\cdot\,) \preceq h_\alpha(\gamma_{[\alpha]}(\,\cdot\,),\,\cdot\,) = h_\alpha(\gamma(\,\cdot\,),\,\cdot\,) = \eta^\gamma_\alpha(\,\cdot\,) .$$

□

3.5 Characterization of the no dual effect feedbacks set under additional assumption of acyclicity in the precedence relation

If we assume that the directed graph $\mathcal{G}(\mathcal{P})$ built from the binary relation $\mathcal{P}$ is acyclic (note that even simple loops $\alpha\mathcal{P}\alpha$ are forbidden), then Proposition 30 can be strengthened.

Proposition 31
Let $\mathbb{F}^{\text{nde}}$ be defined as in (22) and assume no open-loop dual effect.

If the precedence binary relation $\mathcal{P}$ is included in the memory-communication binary relation $\mathcal{M}$, then

$$\mathbb{F}^{\text{nde}}_A = \mathbb{F}^{\text{ad}}_A \cap \mathbb{F}^{\zeta}_A . \tag{60}$$

Moreover, if the directed graph built from the binary relation $\mathcal{P}$ is acyclic, then we also have

$$\mathbb{F}^{\text{nde}}_A = \{\gamma \in \Gamma_A \mid \forall \alpha \in A\,,\, \gamma_\alpha \preceq \zeta_\alpha\} . \tag{61}$$

Proof. The first equation (60) is a simple consequence of Propositions 29 and 30.

The rest of the proof is based on induction on a particular ordering of the set of agents. Indeed, we know from graph theory that the directed graph $\mathcal{G}(\mathcal{P})$ is acyclic if and only if it is possible to perform a topological sort of this graph [4, p.485], i.e. a linear ordering of the agents such that if $\alpha\mathcal{P}\beta$ then α appears before β in the ordering (we must point out here that this ordering is not unique, it is just used here as a tool which provide a relevant ordering for the induction proof). In other words, there exists an ordering $A = \{\alpha_0, \ldots, \alpha_{T-1}\}$ (we assume that $T = \text{card } A \geq 2$) such that

$$[\alpha_0] = \emptyset \text{ and } \forall i = 1, ..., T-1\,, \quad [\alpha_i] \subset \{\alpha_0, \ldots, \alpha_{i-1}\} . \tag{62}$$

Using the ordering of agents $\alpha_0, \ldots, \alpha_{T-1}$, we now prove by induction that

$$\Big(\forall i \in \{0, \ldots, T-1\} \;\; \gamma_{\alpha_i} \preceq \zeta_{\alpha_i}\Big) \Rightarrow \Big(\forall i \in \{0, \ldots, T-1\} \;\; \gamma_{\alpha_i}(\,\cdot\,) \preceq h_{\alpha_i}\big(\gamma(\,\cdot\,),\,\cdot\,\big)\Big) .$$

Let the induction assumption be

$$H(i)\ \Big(\forall s\in\{0,\dots,i\}\ \gamma_{\alpha_s}\preceq\zeta_{\alpha_s}\Big)\Rightarrow\Big(\forall s\in\{0,\dots,i\}\ \gamma_{\alpha_s}(\,\cdot\,)\preceq h_{\alpha_s}\big(\gamma(\,\cdot\,),\,\cdot\,\big)\Big)\,.$$

Suppose that $\gamma_{\alpha_0}(\,\cdot\,)\preceq\zeta_{\alpha_0}(\,\cdot\,)\equiv h_{\alpha_0}(u,\,\cdot\,)$. Here, h_{α_0} is independent of u, since agent α_0 has no predecessor for the $\mathcal{P}$ relation ($[\alpha_0]=\emptyset$). Thus $\gamma_{\alpha_0}(\,\cdot\,)\preceq h_{\alpha_0}(\gamma(\,\cdot\,),\,\cdot\,)=h_{\alpha_0}(u,\,\cdot\,)\equiv\zeta_{\alpha_0}(\,\cdot\,)$, and $H(0)$ is true.

Assume that $H(i-1)$ is true. Suppose that $\forall s\in\{0,\dots,i\}\ \gamma_{\alpha_s}\preceq\zeta_{\alpha_s}$. Then,

$$\gamma_{[\alpha_i]}(\,\cdot\,)=\bigvee_{\beta\in[\alpha_i]}\gamma_\beta(\,\cdot\,)\preceq\bigvee_{\beta\in[\alpha_i]}h_\beta\big(\gamma(\,\cdot\,),\,\cdot\,\big)$$

since $[\alpha_i]\subset\{\alpha_0,\dots,\alpha_{i-1}\}$.

Now, since $\mathcal{P}$ is compatible with $\mathcal{M}$, we have that $[\alpha_i]\subset\|\alpha_i\|$, and thus

$$\bigvee_{\beta\in[\alpha_i]}h_\beta\big(\gamma(\,\cdot\,),\,\cdot\,\big)\preceq\bigvee_{\beta\in\|\alpha_i\|}h_\beta\big(\gamma(\,\cdot\,),\,\cdot\,\big)=h_{\|\alpha_i\|}\big(\gamma(\,\cdot\,),\,\cdot\,\big)\preceq h_{\alpha_i}\big(\gamma(\,\cdot\,),\,\cdot\,\big)$$

by equation (39).

By equations (36), we have

$$h_{\alpha_i}\big(\gamma(\,\cdot\,),\,\cdot\,\big)=h_{\alpha_i}\big(\gamma_{[\alpha_i]}(\,\cdot\,),\,\cdot\,\big)$$

Thus, combining the three above relationships, we get:

$$\gamma_{[\alpha_i]}(\,\cdot\,)\preceq h_{\alpha_i}\big(\gamma_{[\alpha_i]}(\,\cdot\,),\,\cdot\,\big)\,.$$

On the other hand, we have by assumption

$$\gamma_{\alpha_i}(\,\cdot\,)\preceq\zeta_{\alpha_i}(\,\cdot\,)\equiv h_{\alpha_i}(u_{[\alpha_i]},\,\cdot\,)\,.$$

A straightforward application of Lemma 8 then gives

$$\gamma_{\alpha_i}(\,\cdot\,)\preceq h_{\alpha_i}\big(\gamma(\,\cdot\,),\,\cdot\,\big)\,.$$

Thus, $H(i)$ is true. □

4 Dual effect for discrete time stochastic input-output systems with dynamic information structure

We can now simply treat the sequential case.

4.1 Discrete time stochastic input-output systems with dynamic information structure

The sequential case is a particular case of multi-agent stochastic input-output systems for which A represents time

$$A = \{0, \dots, T-1\} \text{ where } T \geq 2. \tag{63}$$

With the notations of Section 3, we have that

$$\prod_{t=0}^{T-1} U_t = U_A = U_{\{0,\dots,T-1\}} \,.$$

We are given a family $\{h_t\}_{t=0,\dots,T-1}$ of *observation functions* for $t = 0, \dots, T-1$:

$$h_t : U_A \times \Omega \to Y_t \,. \tag{64}$$

We now have a notion of causality.

Definition 32
For $t = 0, \dots, T-1$, the t-th prefix operator *ρ_t on U_A is $\pi_{\{0,\dots,t\}}$, that is*

$$\rho_t : U_A \to U_{\{0,\dots,t\}}, \quad \rho_t(u) = \rho_t(u_0, \dots, u_{T-1}) \stackrel{\text{def}}{=} (u_0, \dots, u_t) \,. \tag{65}$$

Observation causality *is the property that*

$$\forall t \in \{0, \dots, T-1\} \,, \quad h_t(\,\cdot\,,\,\cdot\,) \preceq_{U_A \times \Omega} \rho_{t-1}(\,\cdot\,) \vee \mathrm{id}_\Omega(\,\cdot\,) \,, \tag{66}$$

with the convention that $\rho_{-1}(\,\cdot\,) \vee \mathrm{id}_\Omega(\,\cdot\,) = \mathrm{id}_\Omega(\,\cdot\,)$.

Definition 33
Open-loop perfect memory *is the property that*

$$\forall t \in \{0, \dots, T-2\} \,, \quad h_t(\,\cdot\,,\,\cdot\,) \preceq_{U_A \times \Omega} h_{t+1}(\,\cdot\,,\,\cdot\,) \,. \tag{67}$$

Remark. Observe that, here, we only care about causality with respect to control, but not with respect to noise (this latter property is, for instance, valid for state-space systems like (2)). This is because, from the mathematical point of view, only the former kind of causality is needed.

Proposition 34
Observation causality implies that the directed graph built from the binary relation $\mathcal{P}$ is acyclic.

Observation causality and open-loop perfect memory imply that the precedence binary relation $\mathcal{P}$ is included in the memory-communication binary relation $\mathcal{M}$.

Proof. With the notations of section 3, observation causality means that

$$\forall t \in \{0,\dots,T-1\}\,, \quad [t] \subset \{0,\dots,t-1\}\,. \tag{68}$$

This is precisely a topological ordering as introduced in the proof of Proposition 31: the directed graph built from the binary relation $\mathcal{P}$ is thus acyclic.

Open-loop perfect memory may be stated as

$$\forall t \in \{0,\dots,T-1\}, \quad h_{\{0,\dots,t\}}(\,\cdot\,,\,\cdot\,) \equiv \bigvee_{s\in\{0,\dots,t\}} h_s(\,\cdot\,,\,\cdot\,) \preceq_{U_A\times\Omega} h_{t+1}(\,\cdot\,,\,\cdot\,)\,, \tag{69}$$

which may be also expressed as

$$\{0,\dots,t-1\} \subset \|t\|\,.$$

Hence $\forall t \in \{0,\dots,T-1\}\,, \quad [t] \subset \|t\|$. The proof is complete by equation (41). □

4.2 Characterization of the no dual effect feedbacks set

In the sequential case, the description of the no dual effect feedbacks set is more precise than in Proposition 30.

Proposition 35

Let us assume that no open-loop dual effect holds (in the sense of Definition 27) and that the family $\{h_t\}_{t=0,\dots,T-1}$ of observation functions is causal and has open-loop perfect memory. Let us define as in (50)

$$\mathbb{F}_A^{\text{nde}} \overset{\text{def}}{=} \Big\{(\gamma_t)_{t\in A} : \Omega \to U_A \mid \forall t \in \{0,\dots,T-1\}, \gamma_t \preceq \eta_t^\gamma \text{ and } \eta_t^\gamma \equiv \zeta_t\Big\}\,.$$

Then

$$\mathbb{F}_A^{\text{nde}} = \{(\gamma_t)_{t\in A} : \Omega \to U_A \mid \forall t \in \{0,\dots,T-1\},\ \gamma_t \preceq \zeta_t\}\,.$$

Proof. From Proposition 34, we know that the precedence binary relation $\mathcal{P}$ is included in the memory-communication binary relation $\mathcal{M}$ and that the graph of the binary relation $\mathcal{P}$ is acyclic.

A straightforward application of Proposition 31 completes the proof. □

Remark. Define *closed-loop perfect memory* as the property that, for all $t = 0,\dots,T-1$,

$$\forall \gamma = (\gamma_t)_{t\in A} : \Omega \to U_A, \quad h_t(\gamma(\,\cdot\,),\,\cdot\,) \preceq_\Omega h_{t+1}(\gamma(\,\cdot\,),\,\cdot\,)\,. \tag{70}$$

then open-loop perfect memory does imply closed-loop perfect memory.

On the contrary, a weaker form of open-loop perfect memory

$$\forall u \in U_A,\ \ h_t(u,\,\cdot\,) \preceq_\Omega h_{t+1}(u,\,\cdot\,)$$

does not imply closed-loop perfect memory. This can directly be seen with the following example: let $h_0(\omega) = \omega$ and $h_1(u,\omega) = u - \omega$; then $h_0(\cdot) \preceq h_1(u,\cdot)$ for all u; whereas, for $\gamma(\omega) = \omega$, this γ is admissible since $\gamma \preceq h_0$, but obviously $h_0(\,\cdot\,) \not\preceq h_1(\gamma(\,\cdot\,),\,\cdot\,)$ since the latter is the zero mapping.

5 Conclusion

In this paper, the notion of dual effect in stochastic optimal control (in which decisions are based on observations) is revisited using first a minimal framework in which ingredients are

- the observation set,
- the noise set,
- an observation mapping from the product of control and noise sets to the observation set,
- a feedback law from the noise to the control set.

Dual effect is defined as the capacity of admissible control strategies to affect the quality of observations. This quality is characterized by a partial ordering of partitions of the noise set generated by the "observation mappings after feedback" or "closed-loop observations" (that is, when control is generated by a feedback law in the observation mapping). This partial ordering expresses the measurability conditions found in standard formulations of stochastic optimal control problems.

The lack of dual effect is thus defined as the invariance of those partitions with respect to a given subset of admissible feedbacks. When lack of dual effect holds for the set of open-loop feedbacks (open-loop lack of dual effect), our goal is to identify the maximal subset of admissible feedbacks for which lack of dual effect still holds.

We introduce a general definition of admissible feedbacks and we prove that, when lack of dual effect holds for the set of open-loop feedbacks, it can then be extended to the set of admissible feedbacks that are measurable with respect to the fixed invariant partition resulting from the open-loop no dual effect.

We study multi-agents problems with an appropriate definition of admissible feedbacks. Following Ho and Chu in [7, 8], we introduce a *precedence* relation and a *memory-communication* relation on the set of agents.

When lack of dual effect holds for the set of open-loop feedbacks (open-loop lack of dual effect), our goal is to identify the maximal subset of admissible feedbacks for which lack of dual effect still holds.

We prove that, when lack of dual effect holds for the set of open-loop feedbacks, it can then be extended to the set of admissible feedbacks that are measurable with respect to the fixed invariant partition resulting from the open-loop no dual effect if we assume that the precedence relation is compatible with the memory-communication relation. These assumptions depend only on the open-loop system and are thus more easy to check than the *partially nested information structure* of Ho and Chu which requires properties on the closed-loop system.

When the precedence relation induces an acyclic graph, we obtain a stronger conclusion: when lack of dual effect holds for the set of open-loop feedbacks, it can then be extended to the set of feedbacks that are measurable with respect to the fixed invariant partition resulting from the open-loop no dual effect. Here, such feedbacks are necessarily admissible whereas they were not before. As an application, we treat the case of discrete time stochastic input-output systems with dynamic information structure, namely *observation causality* and *open-loop perfect memory.*

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