

Mathematical analysis of a nonlinear parabolic equation arising in the modelling of non-newtonian flows

Eric Cancès^(a), Isabelle Catto^(b) and Yousra Gati^(a)

(a) CERMICS, Ecole Nationale des Ponts et Chaussées and INRIA,
6 & 8 avenue Blaise Pascal, Cité Descartes, 77455 Marne-la-Vallée Cedex 2, France.

(b) CEREMADE, UMR CNRS 7534, Université Paris IX-Dauphine,
Place du Maréchal de Lattre de Tassigny, F-75775 Paris Cedex 16, France.

June 6, 2003

Abstract

The mathematical properties of a nonlinear parabolic equation arising in the modelling of non-newtonian flows are investigated. The peculiarity of this equation is that it may degenerate into a hyperbolic equation (in fact a linear advection equation). Depending on the initial data, at least two situations can be encountered: the equation may have a unique solution in a convenient class, or it may have infinitely many solutions.

1 Introduction

Modelling the flow of complex fluids is a very intricate problem which is far from being solved up to now. Besides studies which aim at improving phenomenological rheological models (purely macroscopic constitutive laws), only a few attempts are made to recover the rheological behavior of a complex fluid from elementary physical processes arising in its microstructure.

The mesoscopic model which has been proposed by Hébraud and Lequeux in [3] deals with simple shear flows of concentrated suspensions. It is obtained by dividing the material in a large number of mesoscopic elements (“blocks”) with a given shear stress σ (σ is a real number; it is in fact an extra-diagonal term of the stress tensor in convenient coordinates) and by considering the evolution of the probability density $p(t, \sigma)$ which represents the distribution of stress in the assembly of blocks. Under various assumptions on the evolution of the stresses of the blocks which will be described below, the equation for the probability density $p(t, \sigma)$ for a block to be under stress σ at time t may be written as:

$$\left\{ \begin{array}{l} \partial_t p = -b(t) \partial_\sigma p + D(p(t)) \partial_{\sigma\sigma}^2 p - \frac{\chi_{\mathbb{R} \setminus [-\sigma_c, \sigma_c]}(\sigma)}{T_0} p + \frac{D(p(t))}{\alpha} \delta_0(\sigma) \quad \text{on } (0; T) \times \mathbb{R}; \quad (1.1a) \\ p \geq 0; \quad (1.1b) \\ p(0, \sigma) = p_0(\sigma), \quad (1.1c) \end{array} \right.$$

where for $f \in L^1(\mathbb{R})$, we denote by

$$D(f) = \frac{\alpha}{T_0} \int_{|\sigma| > \sigma_c} f(\sigma) d\sigma .$$

In equation (1.1a), $\chi_{\mathbb{R} \setminus [-\sigma_c, \sigma_c]}$ denotes the characteristic function of the open set $\mathbb{R} \setminus [-\sigma_c, \sigma_c]$ and δ_0 the Dirac delta function on \mathbb{R} . Each term arising in the above equation (1.1a) (HL equation in short) has a clear physical interpretation. When a block is sheared, the stress of this block evolves with a variation rate $b(t) = G_0 \dot{\gamma}(t)$ proportional to the shear rate $\dot{\gamma}(t)$ (G_0 is an elasticity constant); in this study, the shear rate $\dot{\gamma}(t)$, and therefore the function $b(t)$, are assumed to be in $L^2_{\text{loc}}(\mathbb{R}^+)$. When the modulus of the stress overcomes a critical value σ_c , the block becomes unstable and may relax into a state with zero stress after a characteristic relaxation time T_0 . This phenomenon induces a rearrangement of the blocks and is modelled through the diffusion term $D(p(t)) \partial_{\sigma\sigma}^2 p$. The diffusion coefficient $D(p(t))$ is assumed to be proportional to the amount of stress which has to be redistributed by time unit and the positive parameter α is supposed to represent the “mechanical fragility” of the material.

In all that follows, the parameters α , T_0 and σ_c are positive, and the initial data p_0 in (1.1c) is a given probability density; that is

$$p_0 \geq 0, \quad p_0 \in L^1(\mathbb{R}), \quad \int_{\mathbb{R}} p_0 = 1 . \quad (1.2)$$

We will be looking for solutions $p = p(t, \sigma)$ in $C_t^0(L_\sigma^1 \cap L_\sigma^2)$ such that σp belongs to $L_t^\infty(L_\sigma^1)$ to the nonlinear parabolic partial differential equation (1.1). The subscript σ refers to integration over \mathbb{R} with respect to σ , whereas the subscript t refers to time integration on $[0, T]$ for any $T > 0$. Note that the average stress in the material is given by

$$\tau(t) = \int_{\mathbb{R}} \sigma p(t, \sigma) d\sigma , \quad (1.3)$$

and therefore the above condition on σp ensures that the average stress is an essentially bounded function of time.

Actually in practice, the shear rate is not uniform in the flow and in order to better describe the coupling between the macroscopic flow and the evolution of the microstructure we introduce and study in a second paper [1] a micro-macro model. In this model p is also a function of the macroscopic space variables and the average stress defined by (1.3) is inserted into the macroscopic equation governing the velocity of the macroscopic flow.

In order to lighten the notation and without loss of generality we assume from now on that $\sigma_c = 1$ and $T_0 = 1$. This amounts to changing the time and stress scales.

The main difficulties one encounters in the mathematical analysis come from the nonlinearity in the diffusion term and also and even more from the fact that the parabolic equation may degenerate when the viscosity coefficient $D(p)$ vanishes, and this will be shown to may appear only when $D(p_0) = 0$. This difficulty is illustrated on a simplified example just below and also in Section 5 where we discuss the existence of stationary solutions in the case when the shear rate b is a constant.

Let us first of all look at the following simplified model which already includes the difficulties we are going to face to in the study of equation (1.1). We consider the equation :

$$\begin{cases} \partial_t u &= D(u(t)) \partial_{\sigma\sigma}^2 u ; \\ u(0, \sigma) &= \frac{1}{2} \chi_{]-1,1[}(\sigma) , \end{cases} \quad (1.4)$$

where $\chi_{]-1,1[}$ is the characteristic function of the interval $] - 1, 1[$. The initial condition is on purpose chosen in such a way that $D(u(t = 0)) = 0$. The function $u = \frac{1}{2} \chi_{]-1,1[}(\sigma)$ is a stationary solution to this equation and for this solution $D(u(t))$ is identically zero. But it is not the unique solution to (1.4) in $C_t^0(L_\sigma^2) \cap L_t^\infty(L_\sigma^1)$. It is indeed possible to construct a so-called *vanishing viscosity solution* for which $D(u(t)) > 0$ for all $t > 0$, and there are actually infinitely many solutions to this equation. (This statement is obtained as a corollary of Lemma 4.2 in Section 4 below.)

As far as equation (1.1) is concerned, we show that, in the case when $D(p_0) = 0$ and $b \equiv 0$, we may have either a unique or infinitely many solutions, depending on the initial data (see Proposition 4.1 in Section 4).

On the other hand, we are able to prove the following existence and uniqueness result in the non-degenerate case when $D(p_0) > 0$:

Theorem 1.1 *Let the initial data p_0 satisfy the conditions*

$$p_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad p_0 \geq 0, \quad \int_{\mathbb{R}} p_0 = 1 \quad \text{and} \quad \int_{\mathbb{R}} |\sigma| p_0 < +\infty, \quad (1.5)$$

and assume that

$$D(p_0) > 0 .$$

Then, for every $T > 0$, there exists a unique solution p to the system (1.1) in $L_t^\infty(L_\sigma^1 \cap L_\sigma^2) \cap L_t^2(H_\sigma^1)$. Moreover, $p \in L_{t,\sigma}^\infty \cap C_t^0(L_\sigma^1 \cap L_\sigma^2)$, $\int_{\mathbb{R}} p(t, \sigma) d\sigma = 1$ for all $t > 0$, $D(p) \in C_t^0$ and for every $T > 0$ there exists a positive constant $\nu(T)$ such that

$$\min_{0 \leq t \leq T} D(p(t)) \geq \nu(T) .$$

Besides $\sigma p \in L_t^\infty(L_\sigma^1)$ so that the average stress $\tau(t)$ is well-defined by (1.3) in L_t^∞ .

The first step toward the existence proof of solutions to (1.1) will consist in the study of so-called vanishing viscosity approximations, which are the unique solutions to the family of equations

$$\begin{cases} \partial_t p_\varepsilon = -b(t) \partial_\sigma p_\varepsilon + (D(p_\varepsilon(t)) + \varepsilon) \partial_{\sigma\sigma}^2 p_\varepsilon - \chi_{\mathbb{R} \setminus [-1,1]} p_\varepsilon + \frac{D(p_\varepsilon(t))}{\alpha} \delta_0(\sigma) ; & (1.6a) \\ p_\varepsilon \geq 0 ; & (1.6b) \\ p_\varepsilon(0, \cdot) = p_0 & (1.6c) \end{cases}$$

(recall that we have rescaled the time and stress units to get $T_0 = 1$ and $\sigma_c = 1$). Section 2 below is devoted to the proof of the following

Proposition 1.1 (Existence and uniqueness of vanishing viscosity approximations)

Let $T > 0$ be given. We assume that the initial data satisfies the same conditions (1.5) as in the statement of the theorem. Then, for every $T > 0$ and $0 < \varepsilon \leq 1$, there exists a unique solution p_ε to (1.6) in $L_t^\infty(L_\sigma^1 \cap L_\sigma^2) \cap L_t^2(H_\sigma^1)$. Moreover, $p_\varepsilon \in L_{t,\sigma}^\infty \cap C_t^0(L_\sigma^1 \cap L_\sigma^2)$, $D(p_\varepsilon) \in C_t^0$,

$$\int_{\mathbf{R}} p_\varepsilon = 1, \quad (1.7)$$

$$0 \leq p_\varepsilon \leq \|p_0\|_{L_\sigma^\infty} + \sqrt{\frac{\alpha}{\pi}} \sqrt{T}, \quad (1.8)$$

and for every $T > 0$, there exist positive constants $C_1(T, p_0)$, $C_2(T, p_0)$ and $C_3(T, p_0)$ which are independent of ε such that

$$\sup_{0 \leq t \leq T} \int_{\mathbf{R}} |\sigma| p_\varepsilon \leq C_1(T, p_0), \quad (1.9)$$

$$\sup_{0 \leq t \leq T} \int_{\mathbf{R}} p_\varepsilon^2 \leq C_2(T, p_0), \quad (1.10)$$

and

$$\int_0^T (\varepsilon + D(p_\varepsilon)) \int_{\mathbf{R}} |\partial_\sigma p_\varepsilon|^2 \leq C_3(T, p_0). \quad (1.11)$$

Theorem (1.1) is then proved in Section 3 while the degenerate case is investigated in Section 4. Lastly, the description of stationary solutions in the constant shear rate case is carried out in Section 5.

2 The vanishing viscosity approximation

This section is devoted to the proof of Proposition 1.1.

We begin with the following :

Lemma 2.1 (Uniqueness) *Let p_0 satisfy (1.2). Then for every $T > 0$ and $0 < \varepsilon$, there exists at most one solution p_ε to (1.6) in $L_t^\infty(L_\sigma^1 \cap L_\sigma^2) \cap L_t^2(H_\sigma^1)$. Moreover, $p_\varepsilon \in C_t^0(L_\sigma^2)$ (thus, the initial condition makes sense) and*

$$\int_{\mathbf{R}} p_\varepsilon = 1, \quad (2.1)$$

for almost every t in $[0, T]$.

Proof of Lemma 2.1: We begin with proving that every solution to (1.6) in $L_t^\infty(L_\sigma^1 \cap L_\sigma^2) \cap L_t^2(H_\sigma^1)$ satisfies (2.1). We fix $R > 1$ and we consider a cut-off C^2 function $\phi_R = \phi_R(\sigma)$ with compact support which is equal to 1 when $0 \leq |\sigma| \leq R$ and to 0 when $|\sigma| \geq 2R$ and such that

$$|\phi_R'| \leq \frac{C}{R}, \quad (2.2)$$

where here and below C denotes a positive constant that is independent of R . Notice that ϕ' is equal to 0 on $]-\infty, -2R]$, on $[-R, R]$ and on $[2R, +\infty[$.

Now, we multiply (1.6a) by ϕ_R and integrate over $[0, t] \times \mathbb{R}$ to obtain

$$\begin{aligned} \int_{\mathbb{R}} p_\varepsilon(t) \phi_R - \int_{\mathbb{R}} p_0 \phi_R &= - \int_0^t b(s) \int_{\mathbb{R}} \partial_\sigma p_\varepsilon(s) \phi_R - \int_0^t (D(p_\varepsilon(s)) + \varepsilon) \int_{\mathbb{R}} \partial_\sigma p_\varepsilon(s) \phi_R' \\ &\quad - \int_0^t \int_{|\sigma|>1} p_\varepsilon(s) \phi_R + \frac{1}{\alpha} \int_0^t D(p_\varepsilon(s)) \phi_R(0). \end{aligned}$$

We bound from above the terms on the right-hand side as follows. First, we have

$$\begin{aligned} \left| \int_0^t b(s) \int_{\mathbb{R}} \partial_\sigma p_\varepsilon(s) \phi_R \right| &\leq \int_0^t |b(s)| \int_{\mathbb{R}} p_\varepsilon(s) |\phi_R'| \\ &\leq \frac{C}{R} \int_0^t |b(s)| \int_{R \leq |\sigma| \leq 2R} p_\varepsilon(s) \leq \frac{C}{R}, \end{aligned}$$

thanks to (2.2) and using that $p_\varepsilon \in L_t^\infty(L_\sigma^1)$ and $b \in L_t^1$. Next,

$$\begin{aligned} \int_0^t (D(p_\varepsilon) + \varepsilon) \left| \int_{\mathbb{R}} \partial_\sigma p_\varepsilon \phi' \right| &\leq (\varepsilon + \alpha \|p_\varepsilon\|_{L_t^\infty(L_\sigma^1)}) \int_0^t \|\partial_\sigma p_\varepsilon\|_{L_\sigma^2} \|\phi_R'\|_{L_\sigma^2} \\ &\leq \frac{C\sqrt{t}}{R^{1/2}} \|\partial_\sigma p_\varepsilon\|_{L_{t,\sigma}^2} \leq \frac{C}{R^{1/2}}, \end{aligned}$$

thanks again to (2.2), Cauchy-Schwarz' inequality and since $\partial_\sigma p_\varepsilon$ is in $L_{t,\sigma}^2$. Finally,

$$\begin{aligned} 0 \leq \frac{1}{\alpha} \int_0^t D(p_\varepsilon) - \int_0^t \int_{|\sigma|>1} p_\varepsilon \phi_R &= \int_0^t \int_{|\sigma|>1} p_\varepsilon (1 - \phi_R) \\ &\leq \int_0^t \int_{|\sigma|>R} p_\varepsilon, \end{aligned}$$

and the right-hand side goes to 0 as R goes to infinity since p_ε is in $L_t^\infty(L_\sigma^1)$. All this together yields

$$\int_{\mathbb{R}} p_\varepsilon(t) = \lim_{R \rightarrow +\infty} \int_{\mathbb{R}} p_\varepsilon(t) \phi_R = \lim_{R \rightarrow +\infty} \int_{\mathbb{R}} p_0 \phi_R = \int_{\mathbb{R}} p_0 = 1,$$

for almost every t in $[0, T]$. In particular, this implies that $D(p_\varepsilon) \leq \alpha$.

Let us now argue by contradiction by assuming that there exist two solutions p_1 and p_2 to (1.6a) corresponding to the same initial data p_0 . By subtracting the equations satisfied by p_1 and p_2 respectively, we obtain

$$\begin{cases} \partial_t q = -b(t) \partial_\sigma q + D(q) \partial_{\sigma\sigma}^2 p_1 + (D(p_2) + \varepsilon) \partial_{\sigma\sigma}^2 q - \chi_{\mathbb{R} \setminus [-1,1]} q + \frac{D(q)}{\alpha} \delta_0(\sigma); \\ q(0, \sigma) = 0, \end{cases} \quad (2.3)$$

where $q = p_1 - p_2$. We multiply (2.3) by q and integrate over \mathbb{R} with respect to σ to obtain, after integrations by parts,

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} q^2 + (D(p_2) + \varepsilon) \int_{\mathbb{R}} |\partial_\sigma q|^2 + \int_{|\sigma|>1} q^2 = \frac{D(q)}{\alpha} q(t, 0) - D(q) \int_{\mathbb{R}} \partial_\sigma p_1 \partial_\sigma q. \quad (2.4)$$

We first remark that since $\int_{\mathbf{R}} p_1 = \int_{\mathbf{R}} p_2 = 1$ thanks to (2.1), we get

$$|D(q)| = \alpha \left| \int_{|\sigma| < 1} q \right| \leq \alpha \sqrt{2} \|q\|_{L^2_\sigma},$$

with the help of Cauchy-Schwarz' inequality. Next, using the Sobolev embedding of $H^1(\mathbf{R})$ into $L^\infty(\mathbf{R})$, we bound from above the terms on the right-hand side in the following way :

$$\begin{aligned} & \left| \frac{D(q)}{\alpha} q(t, 0) - D(q) \int_{\mathbf{R}} \partial_\sigma p_1 \partial_\sigma q \right| \\ & \leq \sqrt{2} \|q\|_{L^2_\sigma} \|q\|_{L^\infty_\sigma} + \sqrt{2} \alpha \|q\|_{L^2_\sigma} \int_{\mathbf{R}} |\partial_\sigma p_1 \partial_\sigma q| \\ & \leq \sqrt{2} \|q\|_{L^2_\sigma} \left(\|q\|_{L^2_\sigma}^2 + \|\partial_\sigma q\|_{L^2_\sigma}^2 \right)^{\frac{1}{2}} + \sqrt{2} \alpha \|q\|_{L^2_\sigma} \|\partial_\sigma p_1\|_{L^2_\sigma} \|\partial_\sigma q\|_{L^2_\sigma} \\ & \leq \frac{1}{\varepsilon} \|q\|_{L^2_\sigma}^2 + \frac{\alpha^2}{\varepsilon} \|q\|_{L^2_\sigma}^2 \|\partial_\sigma p_1\|_{L^2_\sigma}^2 + \frac{\varepsilon}{2} \|q\|_{L^2_\sigma}^2 + \varepsilon \|\partial_\sigma q\|_{L^2_\sigma}^2. \end{aligned}$$

Therefore, comparing with (2.4) we deduce

$$\frac{1}{2} \frac{d}{dt} \|q\|_{L^2_\sigma}^2 \leq \left(\frac{1}{\varepsilon} + \frac{\alpha^2}{\varepsilon} \|\partial_\sigma p_1\|_{L^2_\sigma}^2 + \frac{\varepsilon}{2} \right) \|q\|_{L^2_\sigma}^2.$$

Finally, by applying the Gronwall lemma, we prove that $\|q\|_{L^2_\sigma}^2 \leq 0$, thus $q = 0$. The uniqueness of the solution follows. \diamond

Remark 2.1 *The same proof shows that if there exists a solution to (1.1) in $L_t^\infty(L_\sigma^1 \cap L_\sigma^2) \cap L_t^2(H_\sigma^1)$ such that $\inf_{0 \leq t \leq T} D(p(t)) > 0$, then it is unique in this space.*

We now turn to the existence part in the statement of Proposition 1.1. From now on we fix a positive constant $\varepsilon \leq 1$. The proof of Proposition 1.1 will be carried out by the Schauder fixed point theorem. For given positive constants $M (\geq \varepsilon)$ and R , we introduce $\mathcal{D}_{\varepsilon, M}$ and Y_R two closed convex subsets of respectively L_t^2 and $L_{t, \sigma}^2$ as follows:

$$\begin{aligned} \mathcal{D}_{\varepsilon, M} &= \{a \in L_t^2; \varepsilon \leq a \leq M\} \\ Y_R &= \{p \in L_{t, \sigma}^2; p \geq 0, \sup_{0 \leq t \leq T} \int_{\mathbf{R}} |\sigma| p \leq R\}. \end{aligned}$$

To simplify notation we denote

$$\begin{cases} \varphi_\eta(x) &= \frac{1}{\sqrt{2\pi} \eta} \exp\left(-\frac{x^2}{2\eta^2}\right) \quad \text{if } \eta > 0; \\ \varphi_0 &= \delta_0. \end{cases}$$

We first prove the following

Proposition 2.1 *Let $T > 0$ and let $p_0 \in L^2(\mathbf{R})$ such that $p_0 \geq 0$. Then, for every a in $\mathcal{D}_{\varepsilon, M}$ and q in Y_R , there exists a unique solution p in $L_t^\infty(L_\sigma^2) \cap L_t^2(H_\sigma^1)$ to*

$$\begin{cases} \partial_t p(t, \sigma) = -b(t) \partial_\sigma p(t, \sigma) + a(t) \partial_{\sigma\sigma}^2 p(t, \sigma) - \chi_{\mathbf{R} \setminus [-1, 1]}(\sigma) p(t, \sigma) + \frac{D(q)}{\alpha} \delta_0(\sigma); & (2.5a) \\ p(0, \sigma) = p_0(\sigma). & (2.5b) \end{cases}$$

Moreover, $p \in C_t^0(L_\sigma^2)$, p is non-negative and

$$p_- \leq p \leq p_+, \quad (2.6)$$

with

$$p_-(t, \sigma) = e^{-t} \int_{-\infty}^{+\infty} p_0(\sigma') \varphi_{\sqrt{2 \int_0^t a}}(\sigma - \sigma' - \chi(t)) d\sigma' \quad (2.7)$$

and

$$\begin{aligned} p_+(t, \sigma) &= \int_{-\infty}^{+\infty} p_0(\sigma') \varphi_{\sqrt{2 \int_0^t a}}(\sigma - \sigma' - \chi(t)) d\sigma' + \\ &+ \frac{1}{\alpha} \int_0^t D(q(s)) \varphi_{\sqrt{2 \int_s^t a}}(\sigma - \chi(t) + \chi(s)) ds, \end{aligned} \quad (2.8)$$

where $\chi(t) = \int_0^t b(s) ds$. In addition,

i. If $p_0 \in L^\infty(\mathbb{R})$, then p is in $L_{t,\sigma}^\infty$ and

$$0 \leq p \leq \|p_0\|_{L^\infty} + \frac{R \sqrt{T}}{\sqrt{\pi} \sqrt{\varepsilon}}. \quad (2.9)$$

ii. If $\int_{\mathbb{R}} |\sigma| p_0 < +\infty$ (thus $p_0 \in L^1(\mathbb{R})$), then $|\sigma| p \in L_t^\infty(L_\sigma^1)$. More precisely, we have

$$\begin{aligned} \sup_{0 \leq t \leq T} \int_{\mathbb{R}} |\sigma| p &\leq \int_{\mathbb{R}} |\sigma| p_0 + \sqrt{T} \|b\|_{L^2(0,T)} \|p_0\|_{L^1} + \frac{2R}{3} T^{3/2} \|b\|_{L^2(0,T)} \\ &+ \frac{2}{\sqrt{\pi}} (MT)^{1/2} \|p_0\|_{L^1} + \frac{4R \sqrt{M}}{3 \sqrt{\pi}} T^{3/2}. \end{aligned} \quad (2.10)$$

Moreover $p \in C_t^0(L_\sigma^1)$ and $D(p) \in C_t^0$.

Proof of Proposition 2.1: Let us first observe that for every q in Y_R , $D(q) \in L_t^\infty$ since

$$0 \leq D(q(t)) \leq \alpha \int_{|\sigma| > 1} |\sigma| q \leq \alpha R, \quad (2.11)$$

for almost every t in $[0, T]$. Therefore the source term $D(q(t)) \delta_0(\sigma)$ in (2.5a) is in $L_t^\infty(H_\sigma^{-1})$ and the existence and the uniqueness of a solution $p \in C_t^0(L_\sigma^2) \cap L_t^2(H_\sigma^1)$ to the system (2.5) is well-known (see for example [2]). In particular, the initial condition makes sense. Owing to the fact that the source term is non-negative, the proof that $p \geq 0$ is also standard (see again [2]).

We now check the pointwise inequality (2.6).

This is ensured by the maximum principle with observing that p_- and p_+ given respectively by (2.7) and (2.8) are the unique solutions to the systems

$$\begin{cases} \partial_t p_- &= -b \partial_\sigma p_- + a \partial_{\sigma\sigma}^2 p_- - p_-; \\ p_-(0, \sigma) &= p_0(\sigma), \end{cases} \quad (2.12)$$

and

$$\begin{cases} \partial_t p_+ &= -b \partial_\sigma p_+ + a \partial_{\sigma\sigma}^2 p_+ + \frac{D(q)}{\alpha} \delta_0(\sigma); \\ p_+(0, \sigma) &= p_0(\sigma), \end{cases} \quad (2.13)$$

respectively. We now turn to the proof of statement *i*. and assume that p_0 belongs to $L^\infty(\mathbb{R})$. Then, using the two facts that for every $\nu > 0$, $\int_{\mathbb{R}} \varphi_\nu = 1$ and $\varphi_\nu \leq \frac{1}{\sqrt{2\pi}\nu}$, (2.9) is easily deduced from $p \leq p_+$ with the help of (2.11) and since $a \geq \varepsilon$.

Suppose now that $\int_{\mathbb{R}} |\sigma| p_0 < +\infty$. This together with the assumption $p_0 \in L^2(\mathbb{R})$, guarantees that $p_0 \in L^1(\mathbb{R})$ (see also below). Using (2.6) again, we now have

$$\begin{aligned}
\int_{\mathbb{R}} |\sigma| p &\leq \int_{\mathbb{R}} |\sigma| p_+ \\
&\leq \int_{\mathbb{R}} \int_{\mathbb{R}} p_0(\sigma') |\sigma| \varphi_{\sqrt{2\int_0^t a}}(\sigma - \chi(t) - \sigma') d\sigma d\sigma' \\
&\quad + \frac{1}{\alpha} \int_0^t D(q(s)) \left(\int_{\mathbb{R}} |\sigma| \varphi_{\sqrt{2\int_s^t a}}(\sigma - \chi(t) + \chi(s)) d\sigma \right) ds \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} p_0(\sigma') |\sigma + \sigma' + \chi(t)| \varphi_{\sqrt{2\int_0^t a}}(\sigma) d\sigma d\sigma' \\
&\quad + \frac{1}{\alpha} \int_0^t D(q(s)) \left(\int_{\mathbb{R}} |\sigma + (\chi(t) - \chi(s))| \varphi_{\sqrt{2\int_s^t a}}(\sigma) d\sigma \right) ds \\
&\leq \int_{\mathbb{R}} |\sigma| p_0(\sigma) d\sigma + |\chi(t)| \|p_0\|_{L^1} + \frac{1}{\alpha} \int_0^t |\chi(t) - \chi(s)| D(q(s)) ds \\
&\quad + \frac{2}{\sqrt{\pi}} \left(\int_0^t a \right)^{1/2} \|p_0\|_{L^1} + \frac{2}{\alpha\sqrt{\pi}} \int_0^t D(q(s)) \left(\int_s^t a \right)^{1/2} ds, \tag{2.14}
\end{aligned}$$

since $\int_{\mathbb{R}} |\sigma| \varphi_\nu(\sigma) d\sigma = (2/\pi)^{1/2} \nu$ and $\int_{\mathbb{R}} \varphi_\nu = 1$. With the help of (2.11) and observing that $|\chi(t) - \chi(s)| \leq \sqrt{t-s} \|b\|_{L^2(0,T)}$, we then deduce (2.10).

We now use this bound to check that $p \in C_t^0(L_\sigma^1)$ and $D(p) \in C_t^0$. Indeed, for any t , any sequence t_n in $[0, T]$ which converges to t and $A > 1$, we have

$$\begin{aligned}
\int_{\mathbb{R}} |p(t_n) - p(t)| &= \int_{|\sigma| \leq A} |p(t_n) - p(t)| + \int_{|\sigma| \geq A} |p(t_n) - p(t)| \\
&\leq \sqrt{2A} \left(\int_{\mathbb{R}} |p(t_n) - p(t)|^2 \right)^{1/2} + \frac{1}{A} \int_{\mathbb{R}} |\sigma| (|p(t_n)| + |p(t)|) \\
&\leq \sqrt{2A} \left(\int_{\mathbb{R}} |p(t_n) - p(t)|^2 \right)^{1/2} + \frac{2}{A} \sup_{0 \leq t \leq T} \int_{\mathbb{R}} |\sigma| |p(t)|. \tag{2.15}
\end{aligned}$$

For any fixed A the first term in the right-hand side goes to 0 as n goes to infinity since $p \in C_t^0(L_\sigma^2)$ and then the second term is arbitrarily small as A goes to infinity. The same argument yields the continuity of $D(p(t))$ with respect to t . ◇

The following proposition aims at checking the required assumptions to apply the Schauder fixed point theorem.

Proposition 2.2 *Let $T_f > 0$ be given. We assume that*

$$p_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad p_0 \geq 0, \quad \int_{\mathbb{R}} p_0 = 1 \quad \text{and} \quad \int_{\mathbb{R}} |\sigma| p_0 < +\infty. \tag{2.16}$$

Let $0 < \varepsilon \leq 1$, $R = 1 + \int_{\mathbb{R}} |\sigma| p_0$ and $M = 1 + 2\alpha$. We define

$$T_c = \frac{9}{25} \left[\|b\|_{L^2(0, T_f)} + \frac{2\sqrt{1+2\alpha}}{\sqrt{\pi}} \right]^{-2}. \quad (2.17)$$

Then, for every $T \leq \min(\frac{1}{R}; T_c)$, the function $\mathcal{T} : (a; q) \mapsto (D(p) + \varepsilon; p)$, with p being the solution to the system (2.5), maps $\mathcal{D}_{\varepsilon, M} \times Y_R$ into itself. Moreover \mathcal{T} is continuous and $\mathcal{T}(\mathcal{D}_{\varepsilon, M} \times Y_R)$ is relatively compact in $L^2(0, T) \times L^2_{t, \sigma}$.

Proof of Proposition 2.2:

Step 1: \mathcal{T} is well-defined.

According to Proposition 2.1, p is in $C_t^0(L_\sigma^1)$ and $D(p) \in C_t^0$. We now prove that with our choice for M (which ensures that $\varepsilon + D(p_0) \leq 1 + \alpha \leq M$), $D(p) + \varepsilon \in \mathcal{D}_{\varepsilon, M}$. For this, we again use the inequality $p \leq p_+$, the definition (2.8) of p_+ , the rough estimate $\int_{|\sigma| > 1} \varphi_\nu \leq \int_{\mathbb{R}} \varphi_\nu = 1$ and (2.11) to obtain

$$\sup_{0 \leq t \leq T} D(p(t)) \leq \sup_{0 \leq t \leq T} D(p_+(t)) \leq \alpha + \alpha R T \leq 2\alpha,$$

for $T \leq \frac{1}{R}$. It only remains now to check that $\sup_{0 \leq t \leq T} \int_{\mathbb{R}} |\sigma| p \leq R$. We thus go back to (2.10) and observe that this condition holds provided

$$T \leq \max\{t > 0; \|b\|_{L^2(0, T_f)} \sqrt{t} \left(1 + \frac{2R}{3}t\right) + \frac{2\sqrt{M}t}{\sqrt{\pi}} + \frac{4R\sqrt{M}t^{3/2}}{3\sqrt{\pi}} \leq 1\}.$$

Since we already have demanded that $t \leq T \leq \frac{1}{R}$ a sufficient condition is then

$$\sqrt{T} \left[\frac{5}{3} \|b\|_{L^2(0, T_f)} + \frac{10\sqrt{1+2\alpha}}{3\sqrt{\pi}} \right] \leq 1,$$

which reduces to $T \leq T_c$ with T_c given by (2.17).

Our next step will consist in establishing *a priori* bounds on p in $L_t^\infty(L_\sigma^2) \cap L_t^2(H_\sigma^1)$.

Step 2: *A priori bounds.*

If we multiply equation (2.5a) by p and integrate by parts over \mathbb{R} with respect to σ we easily obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} p^2 + a \int_{\mathbb{R}} |\partial_\sigma p|^2 \leq \frac{D(q)}{\alpha} p(t, 0).$$

Since from the Sobolev embedding of $H^1(\mathbb{R})$ into $L^\infty(\mathbb{R})$ and the bound (2.11) on $D(q)$ we get

$$\begin{aligned} \left| \frac{D(q)}{\alpha} p(t, 0) \right| &\leq R \|p\|_{L^\infty} \\ &\leq R \left(\|p\|_{L_\sigma^2}^2 + \|\partial_\sigma p\|_{L_\sigma^2}^2 \right)^{\frac{1}{2}} \\ &\leq \frac{R^2}{2\varepsilon} + \frac{\varepsilon}{2} \|p\|_{L_\sigma^2}^2 + \frac{\varepsilon}{2} \|\partial_\sigma p\|_{L_\sigma^2}^2, \end{aligned}$$

we may write

$$\frac{1}{2} \frac{d}{dt} \|p\|_{L_\sigma^2}^2 + (a - \frac{\varepsilon}{2}) \|\partial_\sigma p\|_{L_\sigma^2}^2 \leq \frac{R^2}{2\varepsilon} + \frac{\varepsilon}{2} \|p\|_{L_\sigma^2}^2. \quad (2.18)$$

We recall that $a \geq \varepsilon$ and we apply the Gronwall lemma to obtain

$$\sup_{0 \leq t \leq T} \|p\|_{L_\sigma^2}^2 \leq e^{\varepsilon T} \left(\|p_0\|_{L_\sigma^2}^2 + \frac{T R^2}{\varepsilon} \right). \quad (2.19)$$

We now return to (2.18) and integrate it over $[0; T]$ to obtain

$$\varepsilon \|\partial_\sigma p\|_{L_{t,\sigma}^2}^2 \leq \|p_0\|_{L_\sigma^2}^2 (1 + \varepsilon T e^{\varepsilon T}) + \frac{T R^2}{\varepsilon} (1 + \varepsilon T e^{\varepsilon T}). \quad (2.20)$$

Step 3: *The function \mathcal{T} is continuous.*

We consider a sequence $(a_n; q_n)$ in $\mathcal{D}_{\varepsilon, M} \times Y_R$ such that a_n converges to a strongly in L_t^2 and q_n converges to q strongly in $L_{t,\sigma}^2$, and we denote $\mathcal{T}(a_n; q_n) = (D(p_n) + \varepsilon; p_n)$. We have to prove that p_n converges strongly to p in $L_{t,\sigma}^2$ and $D(p_n)$ converges to $D(p)$ strongly in L_t^2 , with $(D(p) + \varepsilon; p) = \mathcal{T}(a; q)$.

In virtue of (2.19) and (2.20), the sequence p_n is bounded in $L_t^\infty(L_\sigma^2) \cap L_t^2(H_\sigma^1)$. Then, $\partial_\sigma p_n$ is bounded in $L_t^\infty(H_\sigma^{-1})$ and $\partial_{\sigma\sigma}^2 p_n$ is bounded in $L_t^2(H_\sigma^{-1})$. Since $a_n \partial_{\sigma\sigma}^2 p_n$ is bounded in $L_t^2(H_\sigma^{-1})$, $b \in L_t^2$ and $D(q_n) \delta_0$ is bounded in $L_t^2(H_\sigma^{-1})$, $\partial_t p_n$ is bounded in $L_t^2(H_\sigma^{-1})$. This together with the fact that p_n is bounded in $L_t^2(H_\sigma^1)$ implies that, up to a subsequence, p_n converges strongly towards p in $L_t^2(L_{\text{loc},\sigma}^2)$ (the convergence being weak in $L_t^2(H_\sigma^1)$) thanks to a well-known compactness result [4]. In particular, p_n converges to p almost everywhere. Thus $p \geq 0$ and by the Fatou's lemma, $\int_{\mathbf{R}} |\sigma| p \leq R$ almost everywhere on $[0; T]$. Hence p belongs to Y_R . We are going to show that the convergence is actually strong in $L_{t,\sigma}^2$.

In virtue of (2.9) in Proposition 2.1, we dispose of a uniform *a priori* bound on p_n in $L_{t,\sigma}^\infty$ (hence also on p). For the strong convergence in $L_{t,\sigma}^2$ we then argue as follows. For any fixed positive real number K , we have

$$\begin{aligned} \int_0^T \int_{\mathbf{R}} |p_n - p|^2 &\leq \int_0^T \int_{|\sigma| \leq K} |p_n - p|^2 + \int_0^T \int_{|\sigma| > K} |p_n - p|^2 \\ &\leq \int_0^T \int_{|\sigma| \leq K} |p_n - p|^2 + (\|p_n\|_{L_{t,\sigma}^\infty} + \|p\|_{L_{t,\sigma}^\infty}) \frac{2RT}{K}, \end{aligned}$$

owing to the fact that p_n and p belong to a bounded subset of $Y_R \cap L_{t,\sigma}^\infty$. We then conclude by letting n next K go to infinity.

We now prove that $D(p_n)$ converges to $D(p)$ strongly in L_t^2 . We shall actually prove that $D(p_n)$ converges to $D(p)$ strongly in L_t^1 and then use the fact that $D(p_n)$ is bounded in L_t^∞ , in virtue of (2.11) and because p_n lies in Y_R . Let us fix $K > 1$. Then, we have

$$\begin{aligned} \frac{1}{\alpha} \int_0^T |D(p_n) - D(p)| &= \int_0^T \left| \int_{|\sigma| > 1} (p_n - p) \right| \\ &\leq \int_0^T \int_{1 < |\sigma| < K} |p_n - p| + \frac{1}{K} \int_0^T \int_{|\sigma| > K} |\sigma| (|p_n| + |p|) \\ &\leq \int_0^T \int_{1 < |\sigma| < K} |p_n - p| + \frac{2RT}{K}, \end{aligned} \quad (2.21)$$

because p and p_n belong to Y_R . Since p_n converges to p strongly in $L_t^1(L_{loc,\sigma}^1)$, we conclude that $D(p_n)$ converges to $D(p)$ in L_t^1 by letting n next K go to infinity in (2.21).

In order to pass to the limit in the equation satisfied by p_n (thereby proving that $(D(p) + \varepsilon; p) = \mathcal{T}(a; q)$), we now observe that the strong convergence of q_n to q in $L_{t,\sigma}^2$, together with the argument in (2.21) above shows that $D(q_n)$ converges to $D(q)$ strongly in L_t^2 . It is then easily proved that p is a weak solution to (1.6a) and since p is in $L_t^2(H_\sigma^1)$ it is the unique solution to (2.5) corresponding to a and q . In particular, the whole sequence p_n converges and not only a subsequence.

Step 4: $\mathcal{T}(\mathcal{D}_\varepsilon \times Y_R)$ is relatively compact.

Let $(D(p_n) + \varepsilon; p_n) = \mathcal{T}(a_n; q_n)$ be a sequence in $\mathcal{T}(\mathcal{D}_{\varepsilon,M} \times Y_R)$. We have to prove that we may extract a subsequence which converges strongly in $L_t^2 \times L_{t,\sigma}^2$. Exactly as for the proof of the continuity, the *a priori* estimates (2.19) and (2.20) ensure that the sequence p_n is bounded in $L_t^\infty(L_\sigma^2) \cap L_t^2(H_\sigma^1)$. Since $|\sigma|p_n$ is bounded $L_t^\infty(L_\sigma^1)$, we can mimic the argument in Step 3 above to deduce that up to a subsequence the sequence p_n converges to some p in Y_R strongly in $L_{t,\sigma}^2$ and that $D(p_n)$ converges to $D(p)$ strongly in L_t^2 . \diamond

We are now in position to conclude the proof of Proposition 1.1.

Let $T_f > 0$ and $0 < \varepsilon \leq 1$ being given. We are going to prove the existence of a unique solution on $[0; T_f]$.

Being given an initial data p_0 which satisfies (1.5), existence of a solution p_ε is ensured from Proposition 2.2 by applying the Schauder fixed point theorem on “short” time interval $[0; T_1]$ with $T_1 = \min(\frac{\alpha}{R_1}, T_c)$ and where $R_1 = 1 + \int_{\mathbf{R}} |\sigma| p_0$. This solution is uniquely defined in virtue of Lemma 2.1 and we know from (2.1) that $\int_{\mathbf{R}} p_\varepsilon(T_1) = 1$. Moreover from Proposition 2.1 $p_\varepsilon(T_1) \in L_\sigma^\infty$ and by construction $\int_{\mathbf{R}} |\sigma| p_\varepsilon(T_1) \leq R_1$. Therefore $p_\varepsilon(T_1)$ satisfies the same conditions (2.16) as p_0 . Then, repeating the same argument we may build a solution to (1.6) with initial data $p_\varepsilon(T_1)$ on $[T_1; T_2]$ with $T_2 = \min(\frac{1}{R_2}, T_c)$, where $R_2 = R_1 + 1 = \int_{\mathbf{R}} |\sigma| p_0 + 2$. Thanks to the uniqueness result (Lemma 2.1), if we now glue this solution to p_ε at $t = T_1$ we obtain the unique solution to (1.6) on $[0; T_1 + T_2]$. It is now clearly seen that for any integer $n \geq 1$ we may build a solution to (1.6) on $[0; \sum_{1 \leq k \leq n} T_k]$ with $T_k = \min\left(\left(k + \int_{\mathbf{R}} |\sigma| p_0\right)^{-1}; T_c\right)$. Since $\sum_{1 \leq k \leq n} T_k$ obviously goes to $+\infty$ together with n , existence (and uniqueness) of the solution p_ε to (1.6) is obtained on every time interval.

For the proof of (1.8) we argue as for the proof of (2.9) in Proposition 2.1. Defining p_ε^+

as in (2.8) with a replaced by $D(p_\varepsilon) + \varepsilon$ and $D(q)$ by $D(p_\varepsilon)$ we obtain

$$\begin{aligned}
0 &\leq p_\varepsilon \leq p_\varepsilon^+ \\
&\leq \|p_0\|_{L^\infty} + \frac{1}{\alpha\sqrt{\pi}} \int_0^t \frac{D(p_\varepsilon(s))}{2\sqrt{\varepsilon + \int_s^t D(p_\varepsilon)}} ds \\
&\leq \|p_0\|_{L^\infty} + \frac{1}{\alpha\sqrt{\pi}} \left[\sqrt{\varepsilon + \int_0^t D(p_\varepsilon)} - \sqrt{\varepsilon} \right] \\
&\leq \|p_0\|_{L^\infty} + \frac{1}{\alpha\sqrt{\pi}} \sqrt{\int_0^t D(p_\varepsilon)} \\
&\leq \|p_0\|_{L^\infty} + \frac{\sqrt{\alpha}\sqrt{T}}{\sqrt{\pi}}.
\end{aligned}$$

Then

$$\int_{\mathbf{R}} p_\varepsilon^2 \leq \|p_\varepsilon\|_{L^\infty} \int_{\mathbf{R}} p_\varepsilon,$$

from which (1.10) follows gathering together (2.1) and (1.8) and, with the notation of the proposition,

$$C_2(T, p_0) = \|p_0\|_{L^\infty} + \frac{\sqrt{\alpha}\sqrt{T}}{\sqrt{\pi}}.$$

The proof of (1.9) follows the same lines as the proof of (2.14). Indeed, we again use the pointwise inequality $p_\varepsilon \leq p_\varepsilon^+$ and replace $D(q)$ by $D(p_\varepsilon) (\leq \alpha)$ and a by $D(p_\varepsilon) + \varepsilon (\leq \alpha + 1)$ in (2.14) and use (2.16) to deduce

$$\sup_{0 \leq t \leq T} \int_{\mathbf{R}} |\sigma| p_\varepsilon \leq \int_{\mathbf{R}} |\sigma| p_0 + \sqrt{T} \left(\frac{2\sqrt{1+\alpha}}{\sqrt{\pi}} + \|b\|_{L^2(0,T)} \right) + \frac{2}{3} T^{3/2} \left(1 + \frac{2\sqrt{1+\alpha}}{\sqrt{\pi}} \right), \quad (2.22)$$

whence (1.9) with $C_1(T, p_0)$ being the quantity in the right-hand side of (2.22).

In order to prove (1.11), we multiply (1.6a) by p_ε , and we integrate by parts over \mathbf{R} with respect to σ to obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbf{R}} p_\varepsilon^2 + (D(p_\varepsilon) + \varepsilon) \int_{\mathbf{R}} |\partial_\sigma p_\varepsilon|^2 + \int_{|\sigma|>1} p_\varepsilon^2 = \frac{D(p_\varepsilon)}{\alpha} p_\varepsilon(t, 0). \quad (2.23)$$

We use the L^∞ bound (1.8) to bound the right-hand side and we integrate (2.23) with respect to t over $[0; T]$ to deduce (1.11) with

$$C_3(T, p_0) = \|p_0\|_{L^\infty} \left(\frac{1}{2} + T \right) + \frac{\sqrt{\alpha}}{\sqrt{\pi}} T^{3/2},$$

using that $\|p_0\|_{L^2_\sigma}^2 \leq \|p_0\|_{L^\infty} \int_{\mathbf{R}} p_0$. ◇

3 The non degenerate case: $D(p_0) > 0$

The main result of this section corresponds to the statement of Theorem 1.1 and fully describes the issue of existence and uniqueness of solutions to the HL equation (1.1) in the non-degenerate case. It is summarized in the following :

Proposition 3.1 *Let p_0 satisfy (1.5). We assume that $D(p_0) > 0$. Then, the HL equation (1.1) has a unique solution p in $C_t^0(L_\sigma^2) \cap L_t^2(H_\sigma^1)$ and p is the limit (in $L_{t,\text{loc}}^2(L_\sigma^2) \cap C_{t,\text{loc}}^0(L_\sigma^2)$) of (p_ϵ) when ϵ goes to 0 where p_ϵ is the vanishing viscosity solution whose existence and uniqueness is ensured by Proposition 1.1. Moreover, $p \in L_{t,\sigma}^\infty \cap C_t^0(L_\sigma^1)$, $\sigma p \in L_t^\infty(L_\sigma^1)$ and $\int_{\mathbb{R}} p = 1$. Furthermore, $D(p) \in C_t^0$ and for every $T > 0$ there exists a positive constant $\nu(T)$ such that*

$$\min_{0 \leq t \leq T} D(p(t)) \geq \nu(T) . \quad (3.1)$$

We begin with proving the following :

Lemma 3.1 *We assume that p_0 satisfies (1.5). Then, if $D(p_0) > 0$, $D(p_\epsilon)(t) > 0$ for every $t \in [0, T]$, with p_ϵ being the unique solution to (1.6) provided by Proposition 1.1 and, actually, for every $T > 0$ there exists a positive constant $\nu(T)$ such that*

$$\min_{0 \leq t \leq T} D(p_\epsilon(t)) \geq \nu(T) , \quad (3.2)$$

for every $0 < \epsilon \leq 1$.

Remark 3.1 *Note that this bound from below is independent of ϵ , but it comes out from the proof that it depends on p_0 and on the shear b .*

Proof of Lemma 3.1: The proof relies on the bound from below in (2.6) that we integrate over $|\sigma| > 1$ to obtain

$$D(p_\epsilon(t)) \geq \alpha \int_{|\sigma| > 1} p_\epsilon^- \geq \alpha e^{-t} \int_{\mathbb{R}} p_0(\sigma') \left(\int_{|\sigma| > 1} \varphi_{\sqrt{2 \int_0^t (D(p_\epsilon) + \epsilon)}}(\sigma - \sigma' - \chi(t)) d\sigma \right) d\sigma' . \quad (3.3)$$

Let us define $K_\chi = [-1 - \chi(t), 1 - \chi(t)]$. The function $\sigma \mapsto \varphi_{\sqrt{2 \int_0^t (D(p_\epsilon) + \epsilon)}}(\sigma - \sigma' - \chi(t))$ is a Gaussian probability density with mean $\sigma' + \chi(t)$ and squared width $2 \int_0^t (D(p_\epsilon) + \epsilon)$. Therefore, for every $\sigma' \in \mathbb{R} \setminus K_\chi$, we have

$$\int_{|\sigma| > 1} \varphi_{\sqrt{2 \int_0^t (D(p_\epsilon) + \epsilon)}}(\sigma - \sigma' - \chi(t)) d\sigma \geq \frac{1}{2} ,$$

which implies

$$(3.3) \geq \frac{\alpha}{2} e^{-T} \int_{\mathbb{R} \setminus K_\chi} p_0 = \frac{\alpha}{2} e^{-T} \int_{|\sigma + \chi(t)| > 1} p_0 .$$

In the zero shear case ($b \equiv 0$, thus $\chi \equiv 0$) the proof is over and

$$\min_{0 \leq t \leq T} D(p(t)) \geq \frac{1}{2} e^{-T} D(p_0) .$$

In the general case, a strictly positive bound from below is available as long as the support of p_0 is not contained in K_χ . We thus define

$$t^* = \inf \left\{ t > 0; \int_{|\sigma+\chi(t)|>1} p_0 = 0 \right\}. \quad (3.4)$$

Then $0 < t^*$ (t^* possibly even infinite), the support of p_0 is contained in $[-1-\chi(t^*), 1-\chi(t^*)]$, and for every $T < \frac{t^*}{2}$, (3.2) holds for some positive constant $\nu_1(T)$ defined by

$$\nu_1(T) = \frac{\alpha}{2} e^{-T} \min_{0 \leq t \leq T} \int_{|\sigma+\chi(t)|>1} p_0. \quad (3.5)$$

It is worth emphasizing that this quantity is independent of ε . If $t^* = +\infty$, the proof is over and $\nu(T) = \nu_1(T)$ fits. Let us now examine the case when $t^* < +\infty$ and $T \geq \frac{t^*}{2}$.

We go back to (3.3), take t in $[\frac{t^*}{2}; T]$ and denote $x = \int_0^t (D(p_\varepsilon) + \varepsilon)$ for shortness. Then

$$\begin{aligned} D(p_\varepsilon(t)) &\geq \alpha e^{-T} \int_{-1-\chi(t^*)}^{1-\chi(t^*)} p_0(\sigma') \left(\int_{|\sigma|>1} \varphi_{\sqrt{2x}}(\sigma - \sigma' - \chi(t)) d\sigma \right) d\sigma' \\ &= \alpha e^{-T} \int_{-1-\chi(t^*)}^{1-\chi(t^*)} p_0(\sigma') \left(\int_{|\sigma|>1} \frac{e^{-(\sigma-\sigma'-\chi(t))^2/4x}}{2\sqrt{\pi}\sqrt{x}} d\sigma \right) d\sigma' \\ &= \frac{\alpha}{\sqrt{\pi}} e^{-T} \int_{-1-\chi(t^*)}^{1-\chi(t^*)} p_0(\sigma') \left(\int_{-\infty}^{-1+\sigma'+\chi(t)} \frac{e^{-\sigma^2/4x}}{2\sqrt{x}} d\sigma + \int_{1+\sigma'+\chi(t)}^{+\infty} \frac{e^{-\sigma^2/4x}}{2\sqrt{x}} d\sigma \right) d\sigma' \\ &= \frac{\alpha}{\sqrt{\pi}} e^{-T} \int_{-1-\chi(t^*)}^{1-\chi(t^*)} p_0(\sigma') \left(\int_{\frac{1+\sigma'+\chi(t)}{2\sqrt{x}}}^{+\infty} e^{-t^2} dt + \int_{\frac{1-\sigma'-\chi(t)}{2\sqrt{x}}}^{+\infty} e^{-t^2} dt \right) d\sigma' \\ &\geq \frac{\alpha}{\sqrt{\pi}} e^{-T} \left(\int_{-1-\chi(t^*)}^{1-\chi(t^*)} p_0(\sigma') d\sigma' \right) \left(\int_{\frac{2-\chi(t^*)+\chi(t)}{\sqrt{2t^*}\nu_1(t^*/2)}}^{+\infty} e^{-t^2} dt + \int_{\frac{2+\chi(t^*)-\chi(t)}{\sqrt{2t^*}\nu_1(t^*/2)}}^{+\infty} e^{-t^2} dt \right) \\ &\geq \frac{\alpha}{\sqrt{\pi}} e^{-T} \min_{t^*/2 \leq t \leq T} \left(\int_{\frac{2-\chi(t^*)+\chi(t)}{\sqrt{2t^*}\nu_1(t^*/2)}}^{+\infty} e^{-t^2} dt + \int_{\frac{2+\chi(t^*)-\chi(t)}{\sqrt{2t^*}\nu_1(t^*/2)}}^{+\infty} e^{-t^2} dt \right), \end{aligned} \quad (3.6)$$

since $\int_{-1-\chi(t^*)}^{1-\chi(t^*)} p_0 = 1$ and $x \geq \int_0^{t^*/2} D(p_\varepsilon) \geq t^* \nu_1(t^*/2)/2$ thanks to (3.5). The proof of Lemma 3.1 then follows by defining

$$\nu(T) = \min(\nu_1(T); \nu_2(T)),$$

with $\nu_1(T)$ given by (3.5) and $\nu_2(T)$ being the positive quantity in the right-hand side of (3.6), that is

$$\nu_2(T) = \frac{\alpha}{\sqrt{\pi}} e^{-T} \min_{t^*/2 \leq t \leq T} \left(\int_{\frac{2-\chi(t^*)+\chi(t)}{\sqrt{2t^*}\nu_1(t^*/2)}}^{+\infty} e^{-t^2} dt + \int_{\frac{2+\chi(t^*)-\chi(t)}{2\sqrt{2t^*}\nu_1(t^*/2)}}^{+\infty} e^{-t^2} dt \right).$$

◇

Proof of Proposition 3.1:

We first go back to the proof of the bound (1.11) on $\partial_\sigma p_\varepsilon$ and more precisely we look at (2.23) and observe that in virtue of (3.1)

$$\nu(T) \int_0^T \int_{\mathbf{R}} |\partial_\sigma p_\varepsilon|^2 \leq C_3(T, p_0). \quad (3.7)$$

Let now ε_n denote any sequence in $[0, 1]$ which goes to 0 as n goes to infinity. To shorten the notation we denote by p_n instead of p_{ε_n} the corresponding sequence of solutions to (1.6). With the above bound (3.7) on p_n and (1.10), we know that p_n is bounded in $L_t^2(H_\sigma^1)$ independently of n . Moreover thanks to (2.1) and (1.8) p_n is bounded in $L_t^\infty(L_\sigma^1 \cap L_\sigma^\infty)$ and we also dispose of a uniform bound on $\int_{\mathbf{R}} |\sigma| p_n$ in virtue of (1.9). Therefore arguing exactly as in the proof of Proposition 2.2 (Step 4) where we have proved that the mapping \mathcal{T} is relatively compact in $L_t^2 \times L_{t,\sigma}^2$ we show that p_n converges to some p strongly in $L_{t,\sigma}^2$ and $D(p_n)$ converges to $D(p)$ in L_t^2 . Then p is a solution to the initial problem (1.1) in $L_t^2(H_\sigma^1) \cap L_t^\infty(L_\sigma^1 \cap L_\sigma^\infty)$, $\int_{\mathbf{R}} p = 1$ and $\int_{\mathbf{R}} |\sigma| p < +\infty$. Moreover,

$$\inf_{0 \leq t \leq T} D(p(t)) \geq \nu(T).$$

This non-degeneracy condition on the viscosity coefficient ensures that there is at most one solution to (1.1) in $L_t^2(H_\sigma^1) \cap L_t^\infty(L_\sigma^2)$ (this follows by an obvious adaptation of the proof of Lemma 2.1 to this case). Therefore the limiting function p is uniquely defined and does not depend on the sequence ε_n . Moreover the whole sequence p_n converges to this unique limit and not only a subsequence. \diamond

As a conclusion of this subsection let us make the following comment which is a byproduct of Proposition 3.1. Let p be a solution to (1.1) in $C_t^0(L_\sigma^1 \cap L_\sigma^2)$, then as soon as $D(p(t))$ is positive for some time t it remains so afterwards since the solution can be continued in a unique way beginning from time t .

4 The degenerate case : $D(p_0) = 0$

Throughout this section we assume that $D(p_0) = 0$ and therefore the support of p_0 is included in $[-1; +1]$. Assume that we dispose of a solution to (1.1) in $C_t^0(L_\sigma^1 \cap L_\sigma^2)$. We may define $t_* \in \mathbb{R}^+ \cup \{+\infty\}$ by

$$t_* = \max \left\{ t > 0; \int_0^t D(p) = 0 \right\}. \quad (4.1)$$

According to the comment at the end of the previous section for every $t > t_*$, $D(p(t)) > 0$ while $D(p(t)) = 0$ for all t in $[0; t_*]$. On $[0; t_*]$, the HL equation (1.1) reads

$$\begin{cases} \partial_t p &= -b(t) \partial_\sigma p; \\ p &\geq 0; \\ p(0, \cdot) &= p_0; \\ D(p(t)) &= 0. \end{cases}$$

The above system reduces to

$$\begin{cases} p(t, \sigma) &= p_0(\sigma - \chi(t)); \\ D(p(t)) &= 0, \quad \text{for all } t \text{ in } [0; t_*]. \end{cases} \quad (4.2)$$

The second equation in (4.2) is compatible with the first one as long as

$$\int_{|\sigma+\chi(t)|>1} p_0 = 0, \quad \text{for all } t \text{ in } [0; t_*] .$$

Therefore there exists a maximal time interval $[0; T_c]$ on which the HL equation may reduce to a mere transport equation and this is for an intrinsic time T_c (possibly infinite) defined by

$$T_c = \inf \left\{ t > 0; \int_{|\sigma+\chi(t)|>1} p_0 > 0 . \right\} . \quad (4.3)$$

Note that T_c is completely determined by the data p_0 and b . If $T_c = +\infty$, the steady state $p(t, \sigma) = p_0(\sigma - \chi(t))$ is a solution of the HL equation for all time. We shall now exhibit circumstances under which it is not the unique solution. For convenience, we restrict ourselves to the case when $b \equiv 0$ (we then have obviously $T_c = +\infty$).

For $p_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that $p_0 \geq 0$, let us denote by F_{p_0} the function from \mathbb{R}^+ to \mathbb{R}^+ defined by $F_{p_0}(0) = D(p_0)$ and by

$$\forall x > 0, \quad F_{p_0}(x) = \alpha \int_{|\sigma|>1} \left(\int_{\mathbb{R}} p_0(\sigma') \varphi_{\sqrt{2x}}(\sigma - \sigma') d\sigma' \right) d\sigma .$$

Proposition 4.1 *Let p_0 satisfy (1.5) and be such that $D(p_0) = 0$, then*

i. If F_{p_0} satisfies

$$\int_0^1 \frac{dx}{F_{p_0}(x)} = +\infty, \quad (4.4)$$

then $p(t, \sigma) = p_0(\sigma)$ is the unique solution to (1.1) in $C_t^0(L_\sigma^2)$;

ii. Otherwise, (1.1) has an infinite number of solutions in $C_t^0(L_\sigma^2)$. The set of solutions to (1.1) is made of the steady state $p(t, \sigma) = p_0(\sigma)$ and of the functions $(q_{t_0})_{t_0 \geq 0}$ defined by

$$q_{t_0}(t, \sigma) = \begin{cases} p_0(\sigma) & \text{if } t \leq t_0 \\ q(t - t_0, \sigma) & \text{if } t > t_0 \end{cases}$$

where q is the unique solution to (1.1) in $C_t^0(L_\sigma^2)$ such that $D(q) > 0$ on $]0, +\infty[$. Besides,

$$p_\epsilon \xrightarrow{\epsilon \rightarrow 0} q \quad \text{strongly in } L_{t, \text{loc}}^2(L_\sigma^2). \quad (4.5)$$

Lemma 4.1 *Let $p_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that*

$$p_0 \geq 0, \quad \int_{\mathbb{R}} p_0 = 1, \quad D(p_0) = 0.$$

The function F_0 is in $C^0([0, +\infty[) \cap C^\infty(]0, +\infty[)$, and is positive on $]0, +\infty[$. In addition, $F'_{p_0} > 0$ on $]0, +\infty[$.

Proof of Lemma 4.1: It is easy to check that $F_{p_0} \in C^0([0, +\infty[) \cap C^\infty(]0, +\infty[)$, and that $F_{p_0} > 0$ on $]0, +\infty[$. Since $D(p_0) = 0$, the function p_0 is supported in $[-1, 1]$. Thus, for any $x > 0$

$$\begin{aligned}
F_{p_0}(x) &= \alpha \int_{|\sigma|>1} \left(\int_{\mathbf{R}} p_0(\sigma') \varphi_{\sqrt{2x}}(\sigma - \sigma') d\sigma' \right) d\sigma \\
&= \alpha \int_{-1}^1 p_0(\sigma') \left(\int_{|\sigma|>1} \frac{e^{-(\sigma-\sigma')^2/4x}}{2\sqrt{\pi}\sqrt{x}} d\sigma \right) d\sigma' \\
&= \alpha \int_{-1}^1 p_0(\sigma') \left(\int_{-\infty}^{-1+\sigma'} \frac{e^{-\sigma^2/4x}}{2\sqrt{\pi}\sqrt{x}} d\sigma + \int_{1+\sigma'}^{+\infty} \frac{e^{-\sigma^2/4x}}{2\sqrt{\pi}\sqrt{x}} d\sigma \right) d\sigma' \\
&= \alpha \frac{1}{\sqrt{\pi}} \int_{-1}^1 p_0(\sigma') \left(\int_{\frac{1+\sigma'}{2\sqrt{x}}}^{+\infty} e^{-t^2} dt + \int_{\frac{1-\sigma'}{2\sqrt{x}}}^{+\infty} e^{-t^2} dt \right) d\sigma'. \tag{4.6}
\end{aligned}$$

It follows that for any $x > 0$,

$$F'_{p_0}(x) = \alpha \frac{1}{\sqrt{\pi}} \int_{-1}^1 p_0(\sigma') \left(\frac{1+\sigma'}{4x^{3/2}} e^{-\frac{(1+\sigma')^2}{4x}} + \frac{1-\sigma'}{4x^{3/2}} e^{-\frac{(1-\sigma')^2}{4x}} \right) d\sigma' > 0.$$

◇

Lemma 4.2 *Let $\gamma \geq 0$ and $p_0 \in L^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$ such that*

$$p_0 \geq 0, \quad \int_{\mathbf{R}} p_0 = 1, \quad \int_{\mathbf{R}} |\sigma| p_0 < +\infty, \quad D(p_0) = 0.$$

Let us consider the problem

$$\begin{cases} \partial_t w = D(w(t)) \partial_{\sigma\sigma}^2 w - \gamma w \\ w(0, \sigma) = p_0(\sigma). \end{cases} \tag{4.7}$$

- i. If F_{p_0} satisfies (4.4) then $p(t, \sigma) = p_0(\sigma)$ is the unique solution to (4.7) in $C_t^0(L_\sigma^2)$;*
- ii. Otherwise, (4.7) has an infinite number of solutions in $C_t^0(L_\sigma^2)$. The set of solutions to (4.7) is made of the steady state $w(t, \sigma) = p_0(\sigma)$ and of the functions $(v_{t_0})_{t_0 \geq 0}$ defined by*

$$v_{t_0}(t, \sigma) = \begin{cases} p_0(\sigma) & \text{if } t \leq t_0 \\ v(t - t_0, \sigma) & \text{if } t > t_0 \end{cases}$$

where v is the unique solution to (4.7) in $C_t^0(L_\sigma^2)$ such that $D(v) > 0$ on $]0, +\infty[$.

Corollary 4.1 *The initial data $p_0 = \frac{1}{2}\chi_{]-1,1[}$ fulfills the assumptions of the above lemma and $\int_0^1 \frac{dx}{F_{p_0}(x)} < +\infty$. Therefore there are infinitely many solutions to the equation (1.4) in the introduction.*

Proof of Corollary 4.1: The only point to be checked is that $\int_0^1 \frac{dx}{F_{p_0}(x)} < +\infty$. With the standard notation $\operatorname{erfc}(z) \equiv \int_z^{+\infty} e^{-t^2} dt$, and by using (4.6) and symmetry considerations, simple calculations yield

$$\begin{aligned} F_{p_0}(x) &= \frac{2\alpha\sqrt{x}}{\sqrt{\pi}} \int_0^{\frac{1}{\sqrt{x}}} \operatorname{erfc}(\sigma) d\sigma \\ &= \frac{2\alpha}{\sqrt{\pi}} \left[\operatorname{erfc}\left(\frac{1}{\sqrt{x}}\right) - \frac{1}{2}\sqrt{x}e^{-\frac{1}{x}} + \frac{1}{2}\sqrt{x} \right]. \end{aligned}$$

Since $\operatorname{erfc}(z) \sim \frac{1}{2}e^{-z^2}/z$ for z going to $+\infty$, $F_{p_0}(x) \sim \frac{\alpha}{\sqrt{\pi}}\sqrt{x}$ near 0 and the integrability of $1/F_{p_0}$ on $[0; 1]$ follows. \diamond

Proof of Lemma 4.2

Let us consider a non-negative function $D \in C^0([0, +\infty[, \mathbb{R}^+)$. The unique solution in $C_t^0(L_\sigma^2)$ of the problem

$$\begin{cases} \partial_t w_D = D(t) \partial_{\sigma\sigma}^2 w_D - \gamma w_D; \\ w_D(0, \sigma) = p_0(\sigma), \end{cases} \quad (4.8)$$

is given by

$$w_D(t, \sigma) = \begin{cases} e^{-\gamma t} p_0(\sigma) & \text{if } t \leq t^*; \\ e^{-\gamma t} \int_{\mathbb{R}} p_0(\sigma') \varphi_{\sqrt{2 \int_0^t D(s) ds}}(\sigma - \sigma') d\sigma' & \text{if } t > t^*, \end{cases} \quad (4.9)$$

where $t^* = \inf \left\{ t > 0, \int_0^t D > 0 \right\}$. Any solution to (4.7) thus satisfies $w = w_{D(w)}$ and therefore

$$\begin{aligned} D(w(t)) &= D(w_{D(w)}(t)) \\ &= \alpha \int_{|\sigma|>1} w_{D(w)}(t, \sigma) d\sigma \\ &= \alpha e^{-\gamma t} \int_{|\sigma|>1} \left(\int_{\mathbb{R}} p_0(\sigma') \varphi_{\sqrt{2 \int_0^t D(w(s)) ds}}(\sigma - \sigma') d\sigma' \right) d\sigma \\ &= e^{-\gamma t} F_{p_0} \left(\int_0^t D(w(s)) ds \right). \end{aligned}$$

It follows that the function $D(w)$ is solution in $C^0([0, +\infty[)$ to the nonlinear integral equation

$$y(t) = e^{-\gamma t} F_{p_0} \left(\int_0^t y(s) ds \right). \quad (4.10)$$

On the other hand, if $D \in C^0([0, +\infty[)$ is solution to (4.10) it is easy to check that the function w_D defined by (4.9) is solution to (4.8).

If condition (4.4) is fulfilled, equation (4.10) has a unique solution in $C^0([0, +\infty[)$ (the constant function equal to zero) and the steady state $w(t, \cdot) = p_0$ thus is the unique solution to (4.7)

in $C_t^0(L_\sigma^2)$; otherwise, the set of solutions to (4.10) is made of the steady state $w(t, \cdot) = p_0$ and of the family $(y_{t_0})_{t_0 \geq 0}$ with

$$y_{t_0}(t) = \begin{cases} 0 & \text{if } t \leq t_0 \\ z(t - t_0) & \text{if } t > t_0 \end{cases}$$

where the function z is defined on $[0, +\infty[$ by

$$\int_0^{z(t)} \frac{dx}{F(x)} = \begin{cases} \frac{1-e^{-\gamma t}}{\gamma}, & \text{if } \gamma > 0; \\ t, & \text{otherwise.} \end{cases}$$

Statement *ii.* is obtained by denoting by v the solution to (4.8) associated with the function $z(t)$. \diamond

Proof of Proposition 4.1:

The solution p_ϵ to equation (1.6) satisfies the inequalities

$$p_\epsilon^-(t, \sigma) \leq p_\epsilon(t, \sigma) \leq p_\epsilon^+(t, \sigma) \quad \text{almost everywhere}$$

where p_ϵ^- and p_ϵ^+ are defined in $C_t^0(L_\sigma^2)$ by

$$\begin{cases} \partial_t p_\epsilon^- = (D(p_\epsilon(t)) + \epsilon) \partial_{\sigma\sigma}^2 p_\epsilon^- - p_\epsilon^-; \\ p_\epsilon^-(0, \sigma) = p_0(\sigma); \end{cases} \quad \begin{cases} \partial_t p_\epsilon^+ = (D(p_\epsilon(t)) + \epsilon) \partial_{\sigma\sigma}^2 p_\epsilon^+ + \frac{D(p_\epsilon)}{\alpha} \delta_0; \\ p_\epsilon^+(0, \sigma) = p_0(\sigma). \end{cases}$$

Therefore on the one hand

$$D(p_\epsilon(t)) \geq D(p_\epsilon^-(t)) = e^{-t} F_{p_0} \left(\int_0^t (D(p_\epsilon) + \epsilon) \right) \quad (4.11)$$

and on the other hand

$$\begin{aligned} D(p_\epsilon(t)) \leq D(p_\epsilon^+(t)) &= F_{p_0} \left(\int_0^t (D(p_\epsilon) + \epsilon) \right) + \int_0^t \frac{D(p_\epsilon)(s)}{\alpha} \left(\int_{|\sigma|>1} \varphi \sqrt{2 \int_s^t (D(p_\epsilon) + \epsilon)} \right) ds \\ &\leq F_{p_0} \left(\int_0^t (D(p_\epsilon) + \epsilon) \right) + \frac{1}{\alpha} \int_0^t D(p_\epsilon)(s) ds. \end{aligned}$$

If (4.4) is not fulfilled, using (4.11) and the property that F_{p_0} is strictly increasing on $[0, +\infty[$, we obtain that

$$D(p_\epsilon) \geq z(t)$$

where $z(t)$ is the function defined in the proof of Lemma 4.2. As for any $0 < t_0 \leq T$, there exists $\eta > 0$ such that $z(t) \geq \eta$ on $[t_0, T]$ the same reasoning as in the non-degenerate case leads to the conclusion that (p_ϵ) converges up to an extraction to p in $\mathcal{D}'(]0, +\infty[\times \mathbb{R})$ and in $L^2([t_0, T], L^2(\mathbb{R}))$ for any $0 < t_0 < T < +\infty$, p being a solution to (1.1) in $C^0(]0, +\infty[, L_\sigma^2)$ such that $D(p) > 0$ on $]0, +\infty[$. \diamond

5 Steady states

Throughout this section the shear rate b is assumed to be a given constant and we are looking for solutions in $L^1(\mathbb{R})$ to the following system :

$$\begin{cases} -b \partial_\sigma p + D(p) \partial_{\sigma\sigma}^2 p - \chi_{\mathbb{R} \setminus [-1,1]} p + \frac{D(p)}{\alpha} \delta_0(\sigma) = 0 & \text{on } (0; T) \times \mathbb{R} ; \end{cases} \quad (5.1)$$

$$\begin{cases} p \geq 0 , \int_{\mathbb{R}} p = 1 ; \end{cases} \quad (5.2)$$

$$\begin{cases} D(p) = \alpha \int_{|\sigma|>1} p(\sigma) d\sigma . \end{cases} \quad (5.3)$$

Our main results are summarized in the following :

Proposition 5.1

- i* If $b \equiv 0$, any probability density which is compactly supported in $[-1; +1]$ is a solution to (5.1) which satisfies $D(p) = 0$. If $\alpha \leq \frac{1}{2}$, these are the only stationary solutions (and there are infinitely many), whereas when $\alpha > \frac{1}{2}$ there exists a unique stationary solution corresponding to a positive value of D , which is explicitly given by (5.4) and (5.6) below. This solution is even and with exponential decay at infinity.
- ii* If $b \neq 0$, for any $\alpha > 0$, there exists a unique stationary solution to (5.1), and it corresponds to a positive value for D , which is implicitly given by (5.7) and (5.8) below. This solution has exponential decay at infinity.

Remark 5.1 *The statement in the above proposition is already pointed out by Hébraud and Lequeux [3].*

Proof of Proposition 5.1:

The case when $b \equiv 0$

We first observe that any non-negative function p which is normalized in $L^1(\mathbb{R})$ and with support in $[-1; +1]$ is a solution to the system (5.1) since in that case all terms in equation (5.1) cancel. We now examine the issue of existence of solutions of (5.1) such that $D(p) > 0$. For simplicity we denote $D = D(p)$. For given constant $D > 0$, it is very easy to calculate explicitly the solutions of (5.1) on each of the three regions $\sigma < -1$, $\sigma \in [-1; +1]$ and $\sigma > 1$. Using compatibility conditions on \mathbb{R} and the fact that p has to be in $L^1(\mathbb{R})$ one obtains :

$$p(\sigma) = \begin{cases} \frac{\sqrt{D}}{2\alpha} e^{(1+\sigma)/\sqrt{D}} & \text{if } \sigma \leq -1 , \\ \frac{1}{2\alpha} \sigma + \frac{\sqrt{D} + 1}{2\alpha} & \text{if } -1 \leq \sigma \leq 0 , \\ -\frac{1}{2\alpha} \sigma + \frac{\sqrt{D} + 1}{2\alpha} & \text{if } 0 \leq \sigma \leq 1 , \\ \frac{\sqrt{D}}{2\alpha} e^{(1-\sigma)/\sqrt{D}} & \text{if } 1 \leq \sigma . \end{cases} \quad (5.4)$$

The compatibility condition $D = D(p)$ happens to be then automatically satisfied and the normalization constraint $\int_{\mathbb{R}} p = 1$ imposes that D solves

$$D + \sqrt{D} = \alpha - \frac{1}{2}. \quad (5.5)$$

Since $D \geq 0$, we immediately reach a contradiction when $\alpha < \frac{1}{2}$, whereas when $\alpha > \frac{1}{2}$ equation (5.5) admits a unique positive solution; namely

$$D = -\frac{1}{2} + \frac{\sqrt{4\alpha - 1}}{2}. \quad (5.6)$$

The case when $b \neq 0$

First of all, we observe that if $D = 0$ every term in equation (5.1) but $b \partial_{\sigma} p$ vanish. Thus p has to be a non-zero constant which is in contradiction with $p \in L^1(\mathbb{R})$. So necessarily $D > 0$. For given positive constant D , we then solve (5.1) as above and obtain

$$p(\sigma) = \begin{cases} a_1 e^{\beta^+ \sigma} & \text{if } \sigma \leq -1, \\ a_2 e^{\frac{b}{D} \sigma} + a_2 - \frac{D}{b\alpha} & \text{if } -1 \leq \sigma \leq 0, \\ (a_2 - \frac{D}{b\alpha}) e^{\frac{b}{D} \sigma} + a_2 & \text{if } 0 \leq \sigma \leq 1, \\ a_1 e^{\beta^- \sigma} & \text{if } 1 \leq \sigma, \end{cases} \quad (5.7)$$

with

$$\beta^{\pm} = \frac{b}{2D} \pm \frac{1}{2} \sqrt{\frac{b^2 + 4D}{D^2}},$$

$$a_1 = \frac{e^{\frac{1}{2} \sqrt{\frac{b^2 + 4D}{D^2}}}}{\alpha (\beta^+ e^{b/2D} - \beta^- e^{-b/2D})},$$

and

$$a_2 = \frac{D \beta^+ e^{b/2D}}{\alpha b (\beta^+ e^{b/2D} - \beta^- e^{-b/2D})}.$$

It is tedious but easy to check that this function always fulfills the self-consistency condition $D = D(p)$ and that the normalization condition $\int_{\mathbb{R}} p = 1$ reads

$$\frac{D}{b} \frac{(1 + \beta^+) + (\beta^- - 1) e^{-b/D}}{\beta^+ - \beta^- e^{-b/D}} + D = \alpha. \quad (5.8)$$

For any $b > 0$ (the negative values of b are dealt with by replacing σ by $-\sigma$), the left-hand side of (5.8) is a continuous function which goes to $+\infty$ when D goes to infinity and goes to zero when D goes to 0. This already ensures the existence of at least one steady state for any $\alpha > 0$. Moreover, setting $z = \frac{b^2}{D}$ (for example) we may rewrite the left-hand side of (5.8) as

$$f(z) = \frac{b^2}{z} + \frac{2b^2}{z} \left[\frac{1 + \frac{1}{2b} z \coth(z/2b) + \frac{1}{2b} (z^2 + 4z)^{1/2}}{z + (z^2 + 4z)^{1/2} \coth(z/2b)} \right].$$

Next we check that the function f is monotone decreasing (thus, the left-hand side of (5.8) is increasing with respect to D), whence the uniqueness result. \diamond

Acknowledgements. We would like to thank Philippe Coussot for pointing out the Hébraud-Lequeux equation to us. We also warmly thank Claude Le Bris for stimulating discussions.

References

- [1] E. CANCÈS, I. CATTO AND Y. GATI, in preparation.
- [2] M. CHIPOT, *Elements of nonlinear analysis*, Birkhäuser, Basel, 2000.
- [3] P. HÉBRAUD AND F. LEQUEUX, *Mode coupling theory for the pasty rheology of soft glassy materials*, Phys. Rev. Lett., **81**(14), pp. 2934–2937, 1998.
- [4] J.-L. LIONS, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris, 1969.