A probabilistic approach for nonlinear equations involving the fractional Laplacian and a singular operator

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Abstract

We consider a class of nonlinear integro-differential equations involving a fractional power of the Laplacian and a nonlocal quadratic nonlinearity represented by a singular integral operator. Initially, we introduce cut-off versions of this equation, replacing the singular operator by its Lipschitz continuous regularizations. In both cases we show the local existence and global uniqueness in $L^1 \cap L^p$. Then we associate with each regularized equation a stable-process-driven nonlinear diffusion; the law of this nonlinear diffusion has a density which is a global solution in L^1 of the cut-off equation. In the next step we remove the cut-off and show that the above densities converge in a certain space to a solution of the singular equation. In the general case, the result is local, but under a more stringent balance condition relating the dimension, the power of the fractional Laplacian and the degree of the singularity, it is global and gives global existence for the original singular equation. Finally, we associate with the singular equation a nonlinear singular diffusion and prove propagation of chaos to the law of this diffusion for the related cut-off interacting particle systems. Depending on the nature of the singularity in the drift term, we obtain either a strong pathwise result or a weak convergence result.

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1 Introduction

We consider nonlinear integro-differential evolution equations involving a fractional Laplacian and a nonlinear singular integral operator. More precisely, we will assume in this paper that

$$\alpha \in (1,2),$$

and study the initial value problem for nonlinear and nonlocal evolution equation

$$\partial_t u = -(-\Delta)^{\alpha/2} u - \nabla \cdot (uB(u)), \tag{1.1}$$

$$u(0,x) = u_0(x), \tag{1.2}$$

where function $u : I\!R_+ \times I\!R^d \mapsto I\!R$ and operator $-(-\Delta)^{\alpha/2}$ is a fractional power of the Laplacian in $I\!R^d$ which is defined either as a Fourier multiplier operator

$$-(-\Delta)^{\alpha/2} v(x) = \mathcal{F}^{-1}\Big(|\xi|^{\alpha}\mathcal{F}(v)(\xi)\Big)(x), \tag{1.3}$$

with $\mathcal{F}$ denoting the Fourier transform, or, equivalently, as a Markov process jump operator

$$-(-\Delta)^{\alpha/2} v(x) = K \int_{I\!R^d} \Big(v(x+y) - v(x) - \nabla v(x)\cdot y \mathbf{1}_{|y|\le 1}\Big) \frac{dy}{|y|^{d+\alpha}}, \tag{1.4}$$

where $K = K_{\alpha,d}$ is a constant. The integral operator $B(u)$ is defined by the formula

$$B(u)(x) = \int_{I\!R^d} b(x-y)u(y)\,dy \tag{1.5}$$

where kernel $b : I\!R^d \mapsto I\!R^d$ is assumed to be continuously differentiable on $(I\!R^d)^*$ and to satisfy the following potential estimates

$$|b(x)| \le C|x|^{\beta-d} \quad , \quad |Db(x)| \le C|x|^{\gamma-d}, \tag{1.6}$$

for some $0 < \beta < d$ and $0 \le \gamma < d$; Db denotes the matrix of derivatives of b. The initial condition $u(0,x) = u_0(x)$ will be assumed to be a function in $L^1(I\!R^d) \cap L^p(I\!R^d)$ with $p > 1$ to be specified later. The L^1 assumption is essential for our probabilistic interpretation to be applicable.

Linear evolution problems involving fractional Laplacians have long been extensively studied in mathematical and physical literature; in the latter under the name *anomalous diffusion* (see, e.g., [18], and [19]). Our motivation to investigate *nonlinear* evolution problems involving fractional Laplacians was the study [14] of growing fractal interfaces in the presence of self-similar hopping surface diffusion which expanded the classical Kardar-Parisi-Zhang (KPZ) model. Equation (1.1) is an extension of models considered in [14] and analysis of its global and exploding solutions can be found in [5]. Related issues of Lévy diffusion-driven conservation laws were considered in [7].

A significant observation is that, for $d = 2$, $\alpha = 2$, and the operator $B(u)(x) = \int b(x-y)u(y)\,dy$ determined by the Biot-Savart kernel

$$b(z) = \frac{(-z_2, z_1)}{2\pi|z|^2},$$

equation (1.1) becomes the vorticity equation for the two-dimensional Navier-Stokes system. Selecting, for $d = 3$,

$$b(z) = C\frac{z}{|z|^3}$$

leads to models of Brownian diffusion of electric charge carriers or self-gravitating particles (depending on the sign of constant C, see [3], [4]). These examples motivate potential estimates for kernel B imposed by (1.6).

In this paper, solutions of equation (1.1) are understood to be *weak solutions*, that is functions $u(t,x) = u_t(x)$ such that, for each space-time smooth test function $\psi(s,x)$ on $I\!R_+ \times I\!R^d$,

$$\begin{aligned} &\int_{I\!R^d} \psi(t,x)u_t(x)\,dx - \int_{I\!R^d} \psi(0,x)u_0(x)\,dx \\ =\; &\int_0^t \int_{I\!R^d} \Big[\frac{\partial}{\partial s}\psi(s,x) - (-\Delta)^{\alpha/2}\psi(s,x) + B(u_s)(x).\nabla\psi(s,x)\Big] u_s(x)\,dx\,ds. \end{aligned} \tag{1.7}$$

If we denote by p_t^α the convolution kernel of the Lévy semigroup $\exp(-t(-\Delta)^{\alpha/2})$ in $I\!R^d$ then, for a smooth function ϕ defined on $I\!R^d$, and fixed t, the test function $\psi(s,x) = p_{t-s}^\alpha \star \phi(x)$ satisfies equation

$$\frac{\partial}{\partial s}\psi(s,x) - (-\Delta)^{\alpha/2}\psi(s,x) = 0.$$

In this case, equation (1.7) can be rewritten in the form

$$\begin{aligned} &\int_{I\!R^d} \phi(x)u_t(x)\,dx \\ =\; &\int_{I\!R^d} p_t^\alpha \star \phi(x)u_0(x)\,dx + \int_0^t \int_{I\!R^d} B(u_s)(x).\nabla_x p_{t-s}^\alpha \star \phi(x)u_s(x)\,dx\,ds. \end{aligned}$$

For a C^2-function ϕ with compact support, properties of kernel p_t^α recalled in Subsection 2.3 combined with Fubini's Theorem give that, almost surely, weak solutions u_t satisfy the crucial *mild equation*

$$u_t(x) = p_t^\alpha \star u_0 - \int_0^t \nabla p_{t-s}^\alpha \star (B(u_s)u_s)ds. \tag{1.8}$$

The contents of this paper can be outlined as follows: initially, we introduce cut-off versions of this equation, replacing the singular operator by its Lipschitz continuous regularizations. In Section 3 we show, for both equation (1.1), and its regularized version, a local existence and global uniqueness result in $L^1 \cap L^p$. Our tool is a fixed point theorem. This result significantly strengthens results obtained in [5].

In Section 4, we associate with the regularized equation a stable-process-driven nonlinear diffusion; the law of this nonlinear diffusion has a density which is a global solution in L^1 of the cut-off equation. Then we remove the cut-off and show that the above densities converge in a certain space to a solution of the singular equation. In the general case, the result is local, but under a more stringent balance condition relating the dimension, the power of the fractional Laplacian and the degree of the singularity, it is global and gives global existence for the original singular equation. Then we construct *pathwise* (strong) interacting particle approximations for solutions of the regularized equations, driven by certain Lévy processes. In fact, these Lévy processes are symmetric stable processes with index α, where $\alpha/2$ is the power of the Laplacian which appears in the original equation.

In section 5, we associate with the solution of the original singular equation a nonlinear singular diffusion. When the cut-off parameter tends to 0 and the size of the system tends to infinity in a suitable way we prove a propagation of chaos result for the associated particle systems. Depending on the nature of the singularity in the drift term, we obtain either a strong pathwise result or a weak convergence result.

For classical propagation of chaos results for quasilinear partial differential equations see [21] (and [22]). In [6], the authors obtained only a weak convergence result and assumed that $\gamma > 0$; we allow the important case $\gamma = 0$ although our results are then weaker. In the special case of the two-dimensional Navier-Stokes equation results similar to those of the present paper were obtained in [16] and [17]. Our methods combine tools used in [6], [16], and [17].

In the reminder of the paper we adopt the following notation: For each $p \geq 1$, the space $L^p(\mathbb{R}^d)$ will be denoted by L^p and its norm by $\|.\|_p$. For any function g belonging to $L^1 \cap L^p$, we will introduce the norm

$$|||g|||_p = \|g\|_1 + \|g\|_p. \tag{1.9}$$

For each $T > 0$ and $p \geq 1$, $\mathbf{F}_{p,T}$ denotes the Banach space $L^\infty([0,T], L^1 \cap L^p)$ endowed with the norm

$$|||u|||_{p,T} = \sup_{t \leq T} |||u_t|||_p \ .$$

C_b^k stands for the space of functions continuous and bounded together with their derivatives up to order k.

Constants C appearing in the paper are generic and may change from line to line.

2 Analytic and Functional Preliminaries

In this section we gather background information about analytic and functional properties of the semigroup generated by the fractional Laplacian, about the singular kernel b and its cut-off (regularized) version b_ε. These facts will be needed in the following sections. We start by recalling the classical Young inequality and a generalization of Gronwall's Lemma proved in Fontbona [11].

If $1 \leq p, q, r \leq \infty$ and $1/p + 1/q = 1/r + 1$ then, for $f \in L^p$ and $g \in L^q$, the convolution function $f \star g$ belongs to L^r and

$$\|f \star g\|_r \quad \leq \quad \|f\|_p \|g\|_q. \tag{2.1}$$

Lemma 2.1 *Let $g : [0,T] \to I\!R_+$ be a bounded measurable function and suppose that there are positive constants C, A and θ such that, for all $t \leq T$,*

$$g(t) \leq A + C \int_0^t (t-s)^{\theta-1} g(s)\, ds.$$

Then,

$$\sup_{t \leq T} g(t) \leq C_T A,$$

where constant C_T does not depend on A.

2.1 Convolution kernel of the Lévy semigroup

The basic self-similarity property and decay estimates for the density of the Lévy semigroup and its gradient are summarized in the following Lemma (see, e.g., [5] Lemma 5.3):

Lemma 2.2 *Denote by p_t^α the convolution kernel of the Lévy semigroup $\exp(-t(-\Delta)^{\alpha/2})$ in $I\!R^d$. Then*

$$p_t^\alpha(x) = t^{-d/\alpha} p_1^\alpha(xt^{-1/\alpha}), \quad \text{and} \quad 0 \leq p_1^\alpha(x) \leq C_{d,\alpha}(1 + |x|^{d+\alpha})^{-1}, \tag{2.2}$$

for some $C_{d,\alpha} > 0$. Moreover,

$$|\nabla p_1^\alpha(x)| \leq \tilde{C}_{d,\alpha} |x|^{d-1+\alpha}(1 + |x|^{d+\alpha})^{-2} \tag{2.3}$$

for some $\tilde{C}_{d,\alpha} > 0$.

The Young inequality (2.1) then immediately gives

Corollary 2.3 *If $m \geq q \geq 1$, and $f \in L^q$, then*

$$\|p_t^\alpha \star f\|_m \quad \leq \quad C t^{-\frac{d}{\alpha}(\frac{1}{q} - \frac{1}{m})} \|f\|_q; \tag{2.4}$$

$$\|\nabla p_t^\alpha \star f\|_m \quad \leq \quad C t^{-\frac{d}{\alpha}(\frac{1}{q} - \frac{1}{m}) - \frac{1}{\alpha}} \|f\|_q. \tag{2.5}$$

2.2 Singular kernel b and operator B

The singular kernel b and the associated convolution operator B were introduced in (1.5-6). The boundedness properties of operator B and its derivative can then be summarized as follows:

Proposition 2.4 *For each function g belonging to $L^1 \cap L^p$ with $p > d/\beta$, function $B(g)$ belongs to L^∞, and*

$$\|B(g)\|_\infty \le C\||g\||_p. \tag{2.6}$$

If in addition $\gamma > 0$, and $p > d/\gamma$, then $DB(g)$ is equal to $Db * g$, belongs to L^∞, and satisfies

$$\|D(B(g))\|_\infty \le C\||g\||_p.$$

In particular, in this case $B(g)$ is a Lipschitz function.

Proof If q is the conjugate of p, $1/p + 1/q = 1$, then

$$p > \frac{d}{\beta} \quad \Leftrightarrow \quad q < \frac{d}{d-\beta}.$$

Consequently,

$$\begin{aligned}
\|B(g)\|_\infty &\le C\Big(\int_{\{|x-y|\le 1\}} + \int_{\{|x-y|>1\}}\Big)\frac{1}{|x-y|^{d-\beta}}g(y)\,dy \\
&\le C\Big(\|g\|_p\Big(\int_{\{|x-y|\le 1\}}\frac{1}{|x-y|^{q(d-\beta)}}\,dy\Big)^{1/q} + \|g\|_1\Big) \\
&\le C(\|g\|_p + \|g\|_1)
\end{aligned} \tag{2.7}$$

since $q(d-\beta) < d$. The same argument employing (1.6), but with β replaced by γ, leads to the second part of Proposition 2.3, if we prove that

$$D(B(g)) = Db * g. \tag{2.8}$$

To obtain this identity, we first introduce cut-off functions

$$B_h(g)(x) = \int \mathbf{1}_{\{|x-y|>h\}} b(x-y)g(y)dy$$

where $h > 0$ and prove by dominated convergence theorem that

$$D(B_h(g))(x) = \int \mathbf{1}_{\{|x-y|>h\}} Db(x-y)g(y)dy.$$

For $h \to 0$, functions $B_h(g)$ and $D(B_h(g))$ converge uniformly to $B(g)$ and $D(B(g))$, respectively. Hence the function $D(B(g))$ is continuous and $D(B(g)) = Db * g$. ♠

2.3 Cut-off kernel b_ε and operator B_ε

Let $\eta : I\!R \mapsto I\!R_+$ be an even, increasing function in $C_b^{d+1-\min(\lfloor\beta\rfloor-1,\lfloor\gamma\rfloor)}$, such that

$$\begin{aligned}
\eta(x) &= 1, \quad \text{for } |x| \ge 1, \quad \|\eta\|_\infty \le 1\ , \\
\eta(0) &= 0, \quad \text{and} \quad \eta^{(k)}(0) = 0, \quad \text{for all } k \le d - \min(\lfloor\beta\rfloor - 1, \lfloor\gamma\rfloor).
\end{aligned}$$

where $\lfloor\beta\rfloor$ denotes the integer part of β. In particular, $\|\eta'\|_\infty < +\infty$. For each $\varepsilon > 0$, the *cut-off kernel* b_ε is defined as follows:

$$b_\varepsilon(x) = \eta\left(\frac{|x|}{\varepsilon}\right) b(x). \tag{2.9}$$

Taking into account (1.6) one can immediately verify that b_ε is such that

$$\begin{aligned} b_\varepsilon(x) &= b(x), \quad \text{for } |x| \geq \varepsilon, \quad \text{and} &(2.10)\\ |b_\varepsilon(x)| &\leq |b(x)|, \quad \text{and} \qquad |Db_\varepsilon(x)| \leq C\Big(\frac{1}{|x|^{d-\beta+1}} + \frac{1}{|x|^{d-\gamma}}\Big), &(2.11)\end{aligned}$$

for all $x \in I\!R^d$.

In the remainder of the paper, K_ε will denote an upper bound for $|b_\varepsilon|$ and L_ε a Lipschitz constant for b_ε. Properties of η imply that, for $\varepsilon \to 0$, one can choose K_ε behaving as $C\varepsilon^{-d+\beta}$, and L_ε as $C\varepsilon^{-d+\min(\beta-1,\gamma)}$.

In view of (2.11), operator B_ε associated with cut-off kernel b_ε enjoys the following boundedness property: for $p > d/\beta$ and each function $g \in L^1 \cap L^p$,

$$\sup_{\varepsilon>0} \|B_\varepsilon(g)\|_\infty \leq C\||g|\|_p, \tag{2.12}$$

where constant C is independent of ε. The rate of approximation of operator B by cut-off operators B_ε is described in the following

Proposition 2.5 *Let $p > d/\beta$ and $f \in L^p$. Then, for each $\varepsilon' > \varepsilon > 0$, and each $x \in I\!R^d$,*

$$\begin{aligned} \Big|\int_{I\!R^d}(b_{\varepsilon'}(x-y) - b_\varepsilon(x-y))f(y)dy\Big| &\leq C(\varepsilon')^{\frac{d-q(d-\beta)}{q}}\|f\|_p;\\ \Big|\int_{I\!R^d}(b_\varepsilon(x-y) - b(x-y))f(y)dy\Big| &\leq C\varepsilon^{\frac{d-q(d-\beta)}{q}}\|f\|_p, \end{aligned} \tag{2.13}$$

where $1/p + 1/q = 1$, and C is a constant independent of $\varepsilon, \varepsilon'$, and x.

Proof By (2.10), we have

$$\begin{aligned} &\Big|\int_{I\!R^d}(b_{\varepsilon'}(x-y) - b_\varepsilon(x-y))f(y)\,dy\Big|\\ \leq\; &\int_{|x-y|\leq\varepsilon'}(|b_{\varepsilon'}(x-y)| + |b_\varepsilon(x-y)|)|f(y)|\,dy\\ \leq\; &2C\Big(\int_{|x-y|\leq\varepsilon'}|x-y|^{-(d-\beta)q}dy\Big)^{\frac{1}{q}}\|f\|_p\\ \leq\; &2C\|f\|_p(\varepsilon')^{\frac{d-q(d-\beta)}{q}}\Big(\int_{|u|\leq 1}|u|^{-q(d-\beta)}\,du\Big)^{\frac{1}{q}}. \end{aligned} \tag{2.14}$$

The last integral is finite since $q < d/(d-\beta)$. The second inequality in Proposition 2.5 can be obtained by a similar argument. ♠

3 Existence and Uniqueness

In this section we obtain a local existence and global uniqueness result for solutions of mild equation (1.8). Our results extend those proved in [5] and remove restrictions on parameters d, α and β present in that paper. Throughout this section we assume that u_0 belongs to $L^1 \cap L^p$, for some p such that $d/\beta < p \leq \infty$.

3.1 Local existence

The following version of Banach's fixed point theorem can be found in Cannone [8], Lemma 1.2.6:

Lemma 3.1 *Let $(\mathcal{X}, \|.\|)$ be a Banach space and $B : \mathcal{X} \times \mathcal{X} \mapsto \mathcal{X}$ be a bilinear mapping such that, for each $x_1, x_2 \in \mathcal{X}$,*

$$\|B(x_1, x_2)\| \leq \eta \|x_1\| \|x_2\|.$$

Then, for each $y \in \mathcal{X}$ satisfying inequality $4\eta\|y\| < 1$, equation

$$x = y + B(x, x)$$

admits a unique solution x in the ball $\{z \in \mathcal{X} : \|z\| \leq R\}$ of radius $R = (1 - \sqrt{1 - 4\eta\|y\|})/(2\eta)$. Moreover, this solution satisfies inequality $\|x\| \leq 2\|y\|$.

An application of the above fixed point lemma permits us to prove the following local existence result for the singular mild equation and cut-off mild equation:

Theorem 3.2 *Suppose that $u_0 \in L^1 \cap L^p$ for a p such that $d/\beta < p \leq \infty$. Then, for some $T_0 > 0$, there exists a function $u \in \mathbf{F}_{p,T_0}$ such that, for every $t \in [0, T_0]$,*

$$u_t \quad = \quad p_t^\alpha \star u_0 - \int_0^t \nabla p_{t-s}^\alpha \star (B(u_s) u_s) ds, \tag{3.1}$$

and, for each $\varepsilon > 0$, there exists a function $u^\varepsilon \in \mathbf{F}_{p,T_0}$ such that, for every $t \in [0, T_0]$,

$$u_t^\varepsilon \quad = \quad p_t^\alpha \star u_0 - \int_0^t \nabla p_{t-s}^\alpha \star (B_\varepsilon(u_s^\varepsilon) u_s^\varepsilon) ds. \tag{3.2}$$

Moreover,

$$|||u|||_{p,T_0} \leq 2|||u_0|||_p \ ; \quad \sup_{t \leq T_0} \|B(u_t)\|_\infty \leq C|||u|||_{p,T_0} \leq C|||u_0|||_p, \tag{3.3}$$

$$\sup_{\varepsilon > 0} |||u^\varepsilon|||_{p,T_0} \leq 2|||u_0|||_p \ ; \ \sup_{\varepsilon > 0} \sup_{t \leq T_0} \|B_\varepsilon(u_t^\varepsilon)\|_\infty \leq C|||u_0|||_p. \tag{3.4}$$

Proof We apply fixed point Lemma 3.1 with $\mathcal{X} = \mathbf{F}_{p,T}$ defined in the Introduction and endowed with the norm $|||u|||_{p,T} = \sup_{t\le T} |||u_t|||_p$. Then the Young inequality (2.1) implies that the bilinear mapping

$$B(u,v)(t,x) = \int_0^t \nabla p^\alpha_{t-s} \star (B(v_s)u_s)ds,$$

defined for $u, v \in \mathbf{F}_{p,T}$, satisfies, by Proposition 2.4, inequality

$$|||B(u,v)|||_{p,T} \le CT^{1-\frac{1}{\alpha}} |||u|||_{p,T} |||v|||_{p,T}. \tag{3.5}$$

Denoting

$$U_0(t,x) = p^\alpha_t \star u_0(x),$$

we easily see that

$$|||U_0|||_{p,T} \le |||u_0|||_p. \tag{3.6}$$

¿From Lemma 3.1 we deduce that if T_0 is such that

$$|||u_0||| < \frac{1}{4CT_0^{1-\frac{1}{\alpha}}},$$

then there exists a unique solution u which satisfies inequality

$$|||u|||_{p,T_0} \le 2|||u_0|||_p.$$

The second estimation in (3.3) follows by Proposition 2.4.

Observe that, if B is replaced by B_ε, (3.5) remains true with the same constant independent of ε. This fact follows directly from the definition of b_ε. So the reasoning that worked for b also applies to b_ε and the same conclusion, with the same time T_0, holds. In particular, one has

$$\sup_{\varepsilon>0} |||u^\varepsilon|||_{p,T_0} \le 2|||u_0|||_p,$$

and, by (2.12),

$$\sup_{\varepsilon>0} \sup_{t\le T_0} \|B_\varepsilon(u^\varepsilon_t)\|_\infty \le 2C|||u_0|||_p. \tag{3.7}$$

This concludes the proof of Theorem 3.2. ♠

As a corollary to the above local existence theorem we obtain a result about the convergence of solutions of the cut-off equations to the solution of the original singular mild equation.

Corollary 3.3 *Under notation and assumptions of Theorem 3.2 we have that*

$$|||u^\varepsilon - u|||_{p,T_0} \le C\,\varepsilon^{\frac{d-q(d-\beta)}{q}}, \tag{3.8}$$

where q is the conjugate of p, so that $\lim_{\varepsilon\to 0} |||u^\varepsilon - u|||_{p,T_0} = 0$. *We deduce that*

$$\sup_{t\le T_0} \|B_\varepsilon(u^\varepsilon_t) - B(u_t)\|_\infty \le C\,\varepsilon^{\frac{d-q(d-\beta)}{q}}. \tag{3.9}$$

In both cases, the constant C depends on T_0 and $|||u_0|||_{p,T_0}$.

Proof In view of (3.1), (3.2) and Corollary 2.2, we have, for $t \leq T_0$,

$$
\begin{aligned}
\|u_t - u_t^\varepsilon\|_p \leq & \int_0^t \|\nabla p_{t-s}^\alpha \star [(B(u_s)u_s) - (B_\varepsilon(u_s^\varepsilon)u_s^\varepsilon)]\|_p \, ds \\
\leq & \int_0^t (t-s)^{-\frac{1}{\alpha}} \|(B(u_s)u_s) - (B_\varepsilon(u_s^\varepsilon)u_s^\varepsilon)\|_p \, ds \\
\leq & \int_0^t (t-s)^{-\frac{1}{\alpha}} \|(B(u_s) - B_\varepsilon(u_s)))u_s \\
& \qquad + (B_\varepsilon(u_s) - B_\varepsilon(u_s^\varepsilon))u_s + B_\varepsilon(u_s^\varepsilon)(u_s - u_s^\varepsilon)\|_p \, ds \\
\leq & \; C\varepsilon^{\frac{d-q(d-\beta)}{q}} |||u|||_{p,T_0}^2 \\
& \qquad + C\Big(|||u|||_{p,T_0} + |||u^\varepsilon|||_{p,T_0}\Big) \int_0^t (t-s)^{-\frac{1}{\alpha}} |||u_s - u_s^\varepsilon|||_p \, ds, \qquad (3.10)
\end{aligned}
$$

with the last inequality following from Proposition 2.5. Similar bound obtains for $\|u_t - u_t^\varepsilon\|_1$. Finally, we get that

$$
|||u_t - u_t^\varepsilon|||_p \leq C\varepsilon^{\frac{d-q(d-\beta)}{q}} |||u_0|||_p^2 + C|||u_0|||_p \int_0^t (t-s)^{-\frac{1}{\alpha}} |||u_s - u_s^\varepsilon|||_p \, ds,
$$

which together with the generalized Gronwall's Lemma 2.1 implies that

$$
\sup_{t \leq T_0} |||u_t - u_t^\varepsilon|||_p \leq C(T_0, |||u_0|||_p) \, \varepsilon^{\frac{d-q(d-\beta)}{q}}.
$$

Now, by Propositions 2.4 and 2.5, and the first inequality in (3.4), we get

$$
\begin{aligned}
\sup_{s \leq T_0} \|B(u_s) - B_\varepsilon(u_s^\varepsilon)\|_\infty & \leq \sup_{s \leq T_0} \|B(u_s) - B(u_s^\varepsilon)\|_\infty + \sup_{s \leq T_0} \|B(u_s^\varepsilon) - B_\varepsilon(u_s^\varepsilon)\|_\infty \\
& \leq C|||u_s - u_s^\varepsilon|||_{p,T_0} + C\varepsilon^{\frac{d-q(d-\beta)}{q}} |||u_0|||_p. \qquad (3.11)
\end{aligned}
$$

Then (3.8) allows us to get (3.9), concluding thus the proof of the corollary. ♠

3.2 Global uniqueness

The next theorem asserts the global uniqueness for the singular mild equation with initial data in $L^1 \cap L^p$, and for the cut-off mild equation with the initial data in L^1.

Theorem 3.4 *(a) Let $u_0 \in L^1 \cap L^p$ and $d/\beta < p \leq \infty$. Then, for all $T > 0$, the singular mild equation (3.1) has at most one solution u in $\mathbf{F}_{p,T}$ satisfying the initial condition $u(0,.) = u_0$.*

(b) Let $u_0 \in L^1$ and $\varepsilon > 0$. Then, for all $T > 0$, the cut-off mild equation (3.2) has at most one solution u in $\mathbf{F}_{1,T} = L^\infty([0,T], L^1)$ satisfying the initial condition $u(0,.) = u_0$.

Proof We only prove the first assertion; the second one can be obtained in a similar fashion utilizing the fact that, for $g \in L^1$, $\|B_\varepsilon(g)\|_\infty \leq K_\varepsilon \|g\|_1$.

If u and v are two solutions of (3.1) in $\mathbf{F}_{p,T}$ then, using Proposition 2.4, we obtain that

$$\begin{aligned} |||u_t - v_t|||_p &\leq C \int_0^t (t-s)^{-\frac{1}{\alpha}} |||B(u_s)u_s - B(v_s)v_s|||_p \, ds \\ &\leq C(|||u|||_{p,T} + |||v|||_{p,T}) \int_0^t (t-s)^{-\frac{1}{\alpha}} |||u_s - v_s|||_p \, ds. \end{aligned} \tag{3.12}$$

Now, an application of the generalized Gronwall's Lemma 2.1 concludes the proof of the theorem. ♠

Putting together Theorem 3.2 and 3.4 we finally obtain

Corollary 3.5 *Let $u_0 \in L^1 \cap L^p$ and $d/\beta < p \leq \infty$. Then there exists a $T_0 > 0$ such that the singular mild equation (3.1) has a unique solution u in $\mathbf{F}_{p,T_0}$ satisfying the initial condition $u(0,.) = u_0$.*

4 A probabilistic model for the cut-off equation

In this section we construct a McKean-style nonlinear process for the cut-off weak equation

$$\partial_t u = -(-\Delta)^{\alpha/2} u - \nabla \cdot (u B_\varepsilon(u)), \tag{4.1}$$

$$u(0,x) = u_0(x), \tag{4.2}$$

and then use it to develop an interacting particle system whose empirical measure converges to the solution of the cut-off equation as the system's size increases to infinity. Throughout this section $\varepsilon > 0$ is fixed and the cut-off convolution kernel b_ε which replaces kernel b in (1.1) is that of Subsection 2.3.

4.1 The nonlinear process

Let us begin with an observation that, although our methods are probabilistic, they can handle situations where the initial condition u_0 is a general function in L^1 rather than just a probability density, i.e., a nonnegative and normalized function in L^1. As we shall see later on, this can be accomplished by writing the initial condition in the form

$$u_0(x) = h(x) \frac{|u_0|(x)}{\|u_0\|_1}, \quad \text{where } h(x) = \frac{u_0(x)}{|u_0|(x)} \|u_0\|_1, \tag{4.3}$$

an idea due to Jourdain [13]. Function h (defined with the convention $\frac{0}{0} = 0$) is a measurable function bounded by $\|u_0\|_1$.

Our first step is to employ probabilistic tools to construct a solution of (4.1). For this purpose we will adapt the pathwise approach of Sznitman [21]. In view of (1.4), the underlying process corresponding to the fractional Laplacian with index $\alpha \in (1,2)$ is a process with jumps. Thus we select as path space the Skorohod space $\mathbb{D}([0,T],\mathbb{R}^d)$, where $T>0$ is a fixed terminal time. Denote by $\mathcal{P}_T$ the space of probability measures on $\mathbb{D}([0,T],\mathbb{R}^d)$.

Definition 4.1 Let $P\in\mathcal{P}_T$ and h be the function defined in (4.3). The family $(\tilde{P}_t)_{t\geq 0}$ is said to be a *weighted version* of the time-marginals of P if, for each Borel set $A\in\mathcal{B}(\mathbb{R}^d)$, and each $t\geq 0$,

$$\tilde{P}_t(A)=E_P\Big(1_A(X_t)h(X_0)\Big) \tag{4.4}$$

Remark 4.2 Note that, for $t=0$, we obtain that $\tilde{P}_0(dx)=u_0(x)dx$. Moreover, the total variation of the signed measure $\tilde{P}_t$ is smaller than $\|u_0\|_1$. Finally, if P_t has a density with respect to the Lebesgue measure, then so does $\tilde{P}_t$.

Now we can formally introduce the nonlinear process we are interested in.

Definition 4.3 Consider an $\mathbb{R}^d$-valued random variable X_0 with law $|u_0|(x)/\|u_0\|_1$, and an independent symmetric α-stable process $(S_t)_{t\geq 0}$. An $\mathbb{R}^d$-valued càdlàg process $(X^\varepsilon_t)_{t\in[0,T]}$ is said to be a *cut-off nonlinear process associated with (4.1)* if it is a solution of the *cut-off nonlinear stochastic differential equation* (SDE_ε)

$$X^\varepsilon_t=X_0+S_t+\int_0^t\int_{\mathbb{R}^d}b_\varepsilon(X^\varepsilon_s-y)\tilde{P}^\varepsilon_s(dy)ds, \tag{4.5}$$

where P^ε_s is the law of X^ε_s, and $\tilde{P}^\varepsilon_s$ is its weighted version.

Proposition 4.4 *If process X^ε is a solution of (SDE_ε) then, for each $t>0$, the weighted version $\tilde{P}^\varepsilon_t$ of its law has a density $v^\varepsilon_t(x)$ which is a solution of (3.2). Moreover, for each $t\in[0,T]$,*

$$\|v^\varepsilon_t\|_1\leq\|u_0\|_1. \tag{4.6}$$

Proof Recall the standard decomposition of an α-stable process S:

$$S_t=\int_{(0,t]\times\mathbb{R}^d}y1_{0<|y|<1}\tilde{N}(ds,dy)+\int_{(0,t]\times\mathbb{R}^d}y1_{|y|\geq 1}N(ds,dy) \tag{4.7}$$

where $N(ds,dy)$ is a Poisson point measure with intensity $\nu(dy)ds=K|y|^{-d-\alpha}dy\,ds$ and where $\tilde{N}(ds,dy)=N(ds,dy)-\nu(dy)\,ds$ is the compensated martingale measure of N. Then, introducing $\psi\in C_b^{1,2}(\mathbb{R}^+\times\mathbb{R}^d)$ and using Itô's formula, we obtain that

$$\begin{aligned}\psi(t,X^\varepsilon_t) &= \psi(0,X_0)\\ &\quad+\int_0^t\Big(-(-\Delta)^{\alpha/2}\psi(s,X^\varepsilon_s)+b_\varepsilon\star\tilde{P}^\varepsilon_s(X^\varepsilon_s).\nabla\psi(s,X^\varepsilon_s)+\frac{\partial}{\partial s}\psi(s,X^\varepsilon_s)\Big)ds\\ &\quad+\text{ a martingale .}\end{aligned}$$

Multiplying terms in the above equality $h(X_0)$ and taking expectations, we obtain that

$$\begin{aligned}\int_{I\!\!R^d} \psi(t,x)\tilde{P}^\varepsilon_t(dx) &= \int_{I\!\!R^d} \psi(0,x)u_0(x)dx + \int_0^t \int_{I\!\!R^d} \Big(-(-\Delta)^{\alpha/2}\psi(s,x) \\ &\quad + b_\varepsilon \star \tilde{P}^\varepsilon_s(x).\nabla\psi(s,x) + \frac{\partial}{\partial s}\psi(s,x)\Big)\, \tilde{P}^\varepsilon_s(dx)\, ds. \qquad (4.8)\end{aligned}$$

Next, consider $\phi \in C^2_b(I\!\!R^d)$, fix $t>0$ and take $\psi(s,x) = p^\alpha_{t-s}\star\phi(x)$. Then ψ is sufficiently smooth to satisfy (4.8). Since, by construction, the total mass of $\tilde{P}^\varepsilon_t$ is at most $\|u_0\|_1$ and b_ε is bounded, by Fubini's theorem,

$$\begin{aligned}\int_{I\!\!R^d} \phi(x)\,\tilde{P}^\varepsilon_t(dx) &= \int_{I\!\!R^d} p^\alpha_t \star \phi(x)u_0(x)\,dx \\ &\quad + \int_0^t \int_{I\!\!R^d} b_\varepsilon \star \tilde{P}^\varepsilon_s(x).\nabla p^\alpha_{t-s}\star\phi(x)\,\tilde{P}^\varepsilon_s(dx)\,ds \\ &= \int_{I\!\!R^d} p^\alpha_t \star \phi(x)u_0(x)\,dx \\ &\quad + \int_{I\!\!R^d} \phi(y)dy\Big(\int_0^t \int_{I\!\!R^d} b_\varepsilon \star \tilde{P}^\varepsilon_s(x).\nabla p^\alpha_{t-s}(x-y)\,\tilde{P}^\varepsilon_s(dx)\,ds\Big).\end{aligned}$$

Since the preceding equality is available for each smooth function ϕ, for each time t, measure $\tilde{P}^\varepsilon_t$ has a density v^ε_t with respect to the Lebesgue measure. Moreover, v^ε is a solution of (3.2) and, by construction, it satisfies bound $\|v^\varepsilon_s\|_1 \le \|u_0\|_1$, for each time $s \le T$. ♠

Analysis of the cut-off nonlinear stochastic differential equation (SDE_ε) will be facilitated by the following result:

Lemma 4.5 *Let X_0 and S be as given in Definition 4.3. If $a : I\!\!R^d \mapsto I\!\!R^d$ is a bounded Lipschitz function then the stochastic differential equation*

$$X_t = X_0 + S_t + \int_0^t a(X_s)\,ds$$

has a unique (pathwise and in law) solution belonging to $I\!\!D([0,T], I\!\!R^d)$.

Proof Since a function in $I\!\!D([0,T], I\!\!R^d)$ is bounded on $[0,T]$ we can use a fixed point theorem in $L^\infty([0,T], I\!\!R^d)$, "$\omega$ by ω", to show that $X_t(\omega)$ is well defined in $I\!\!D([0,T], I\!\!R^d)$ and pathwise unique. Furthermore, an application of the Yamada-Watanabe Theorem gives the uniqueness in law. ♠

Our study of the existence and uniqueness problem for the stochastic differential equation (SDE_ε) will be done in topology induced by a version of the *Vaserstein distance* $\rho_T(P,Q)$ (see, e.g., [9]) defined on the subspace $\hat{\mathcal{P}}_T = \{P \in \mathcal{P}_T : P_0 = |u_0|/\|u_0\|_{L^1}\ \}$ of $\mathcal{P}_T$ by the formula

$$\rho_T(P,Q) = \inf \int_{I\!\!D_T\times I\!\!D_T} \Big(\sup_{t\le T} |x(t)-y(t)| \wedge 1\Big)\, R(dx,dy),$$

where the infimum is taken over measures R with marginals P and Q, and such that $R(x(0) = y(0)) = 1$. The metric space $(\hat{\mathcal{P}}_T, \rho_T)$ is complete.

For $m \in \hat{\mathcal{P}}_T$, we denote by $\psi(m)$ the law of the process X^m defined by equation

$$X_t^m = X_0 + S_t + \int_0^t \int_{I\!R^d} b_\varepsilon(X_s^m - y)\tilde{m}_s(dy)ds, \tag{4.9}$$

where X_0 and S have been given in Definition 4.3, with $\tilde{m}_s$ being the weighted version of the marginal of m at time s. Since b_ε is Lipchitz continuous, the mapping $x \mapsto \int_{I\!R^d} b_\varepsilon(x-y)\tilde{m}_s(dy)$ is Lipschitz continuous, uniformly for $s \in [0,T]$. In view of Lemma 4.5, process X^m is well defined pathwise. Moreover, with a proof modelled on that given in Sznitman [21], we have

Lemma 4.6 *The mapping ψ sends $\hat{\mathcal{P}}_T$ into $\hat{\mathcal{P}}_T$ and satisfies inequality*

$$\rho_t(\psi(m^1), \psi(m^2)) \leq C \int_0^t \rho_s(m^1, m^2)ds. \tag{4.10}$$

Proof We start with a coupling argument. For $m^1, m^2 \in \hat{\mathcal{P}}_T$ let X^{m^1} and X^{m^2} be defined by (4.9) with m respectively replaced by m^1 and m^2. We have

$$\begin{aligned}
\rho_t(\psi(m^1), \psi(m^2)) \quad & \leq E\left[(\sup_{s\leq t} |X_s^{m^1} - X_s^{m^2}|) \wedge 1\right] \\
& \leq E\left[\left(\int_0^t \left|b_\varepsilon \star \tilde{m}_s^1(X_s^{m^1}) - b_\varepsilon \star \tilde{m}_s^2(X_s^{m^2})\right| ds\right) \wedge 1\right].
\end{aligned}$$

Furthermore,

$$\begin{aligned}
E\left[\left|b_\varepsilon \star \tilde{m}_s^1(X_s^{m^1}) - b_\varepsilon \star \tilde{m}_s^2(X_s^{m^2})\right|\right] \ \leq \ & E\left[\left|b_\varepsilon \star \tilde{m}_s^1(X_s^{m^1}) - b_\varepsilon \star \tilde{m}_s^1(X_s^{m^2})\right|\right] \\
& + E\left[\left|b_\varepsilon \star \tilde{m}_s^1(X_s^{m^2}) - b_\varepsilon \star \tilde{m}_s^2(X_s^{m^2})\right|\right],
\end{aligned}$$

with

$$\begin{aligned}
& E\left[\left|b_\varepsilon \star \tilde{m}_s^1(X_s^{m^1}) - b_\varepsilon \star \tilde{m}_s^1(X_s^{m^2})\right|\right] \\
\leq \quad & E\left[\int |b_\varepsilon(X_s^{m^1} - y) - b_\varepsilon(X_s^{m^2} - y)|\, \tilde{m}_s^1(dy)\right] \\
\leq \quad & (L_\varepsilon \vee 2K_\varepsilon) E[\|u_0\|_1\, |X_s^{m^1} - X_s^{m^2}| \wedge 1] \ \leq (L_\varepsilon \vee 2K_\varepsilon)\|u_0\|_1\, E[\sup_{u\leq s} |X_u^{m^1} - X_u^{m^2}| \wedge 1],
\end{aligned}$$

and, with (Y^1, Y^2) denoting the canonical process on $I\!D_T \times I\!D_T$, and R with marginals m^1 and m^2 and such that $R(Y^1(0) = Y^2(0)) = 1$,

$$\begin{aligned}
& E\left[\left|b_\varepsilon \star \tilde{m}_s^1(X_s^{m^2}) - b_\varepsilon \star \tilde{m}_s^2(X_s^{m^2})\right|\right] \\
= \quad & E\left|\int_{I\!R^d} b_\varepsilon(X_s^{m^2} - y_1)\, \tilde{m}_s^1(dy_1) - \int_{I\!R^d} b_\varepsilon(X_s^{m^2} - y_2)\, \tilde{m}_s^2(dy_2)\right|
\end{aligned}$$

$$
\begin{aligned}
&= E\left|\int_{\mathbb{D}_T} b_\varepsilon(X_s^{m^2}-Y_s^1)h(Y_0^1)\,m^1(dY^1)-\int_{\mathbb{D}_T} b_\varepsilon(X_s^{m^2}-Y_s^2)h(Y_0^2)\,m^2(dY^2)\right|\\
&= E\left|\int_{\mathbb{D}_T\times\mathbb{D}_T}\Big(b_\varepsilon(X_s^{m^2}-Y_s^1)-b_\varepsilon(X_s^{m^2}-Y_s^2)\Big)h(Y_0^1)\,R(dY^1,dY^2)\right|\\
&\le E\int_{\mathbb{D}_T\times\mathbb{D}_T}\|u_0\|_1\,(L_\varepsilon\vee 2K_\varepsilon)\,(|Y_s^1-Y_s^2|\wedge 1)\,R(dY^1,dY^2))\\
&\le (L_\varepsilon\vee 2K_\varepsilon)\,\|u_0\|_1\int_{\mathbb{D}_T\times\mathbb{D}_T}E[\sup_{u\le s}|Y_s^1-Y_s^2|\wedge 1]\,R(dY^1,dY^2))\\
&\le C(L_\varepsilon\vee 2K_\varepsilon)\rho_s(m^1,m^2).
\end{aligned}
$$

Thus we finally obtain that, for $t\le T$,

$$
\begin{aligned}
&E\left[\sup_{s\le t}|X_s^{m^1}-X_s^{m^2}|\wedge 1\right]\\
\le\quad & C(L_\varepsilon\vee 2K_\varepsilon)\left(\int_0^t E\left[\sup_{u\le s}|X_u^{m^1}-X_u^{m^2}|\wedge 1\right]ds+\int_0^t\rho_s(m^1,m^2)\,ds\right),
\end{aligned}
$$

so that, by Gronwall's Lemma,

$$
E\left[\sup_{s\le t}|X_s^{m^1}-X_s^{m^2}|\wedge 1\right]\le C\int_0^t\rho_s(m^1,m^2)\,ds
$$

which concludes the proof of the Lemma.

♠

Using Gronwall's Lemma and completeness of $\hat{\mathcal{P}}_T$, we can now deduce, still following [21], the following existence and uniqueness results for the cut-off nonlinear process associated with equation (4.1) and introduced in Definition 4.3. The result also yields the existence of a bounded solution for the cut-off mild equation:

Proposition 4.7 *(a) For a given X_0 and S as in Definition 4.3, the nonlinear stochastic differential equation (SDE_ε) (4.5) has a unique solution.*

(b) The cut-off mild equation (3.2) has a solution in $L^\infty([0,T],L^1(\mathbb{R}^d))$.

Proof The weak uniqueness is immediately obtained from inequality (4.10). Indeed, a weak solution of (4.5) is exactly a fixed point for the mapping ψ. To prove the existence of such a solution, we iterate (4.10) starting with a measure $m\in\hat{\mathcal{P}}_T$ and prove that the sequence $(\psi^n(m))_n$ is Cauchy in the complete space $\hat{\mathcal{P}}_T$. This gives its convergence to a probability measure P^ε which is a weak solution of (4.5).

Consider now the drift term $\int_{\mathbb{R}^d} b_\varepsilon(.-y)\tilde{P}_s^\varepsilon(dy)$ which is a bounded and Lipschitz-continuous function. Then, by Lemma 4.5, for a given X_0 and S, there exists a unique (pathwise and in law) solution X^ε to the following stochastic differential equation $(SDE.P^\varepsilon)$:

$$
X_t^\varepsilon = X_0+S_t+\int_0^t\int_{\mathbb{R}^d} b_\varepsilon(X_s^\varepsilon-y)\tilde{P}_s^\varepsilon(dy)ds. \tag{4.11}
$$

Since both, P^ε and the law of X^ε, are weak solutions of $(SDE.P^\varepsilon)$, they have to be equal which proves part (a) of the Proposition.

Part (b) follows from Proposition 4.4, since the density v^ε of $\tilde{P}^\varepsilon$ is a solution of (3.2) such that, for all $\varepsilon > 0$, and $t \le T$,

$$\|v_t^\varepsilon\|_1 \le \|u_0\|_1. \tag{4.12}$$

♠

At the end of this section, the above Proposition, Theorem 3.2 and Theorem 3.4 will yield some uniform estimates and some convergence results for the sequence $v_t^\varepsilon, t \in [0, T_0]$. Under a more stringent condition on d, α, β, which we call the *Balance Condition*, we will obtain global results, the main goal of this section.

Proposition 4.8 *(a) Let $p > d/\beta$. If $u_0 \in L^1 \cap L^p$ then, for each $\varepsilon > 0$,*

$$\sup_\varepsilon \sup_{t \le T_0} \|v_t^\varepsilon\|_p < +\infty.$$

(b) Assume moreover that $d - (\alpha - 1) < \beta$. Then for any $T > 0$, we have

$$\sup_\varepsilon \sup_{t \le T} \|v_t^\varepsilon\|_p < +\infty.$$

Proof For any $T > 0$ and $v \in L^\infty([0,T], L^p)$, let us denote by $\|v\|_{p,T}$ the norm $\sup_{t\le T} \|v_t\|_p$.

(a) Since by the above proposition, $v^\varepsilon \in \mathbf{F}_{1,T}$ for all $T > 0$, Theorem 3.4 implies that function v^ε coincides on $[0, T_0]$ with the unique function u^ε defined in Theorem 3.2, and then

$$\sup_\varepsilon \sup_{t \le T_0} \|v_t^\varepsilon\|_p < +\infty.$$

(b) Let $\eta = \alpha - 1$. One has $0 < \eta < 1$. First, let us consider r such that $d/\beta < r < d/(d - \eta)$, which is possible since $d - \eta < \beta$ and assume that $u_0 \in L^1 \cap L^r$. First observe that, for each t, function v_t^ε belongs to L^r and that $\sup_\varepsilon \|v^\varepsilon\|_{r,T} < +\infty$.

Indeed, using (3.2), (4.12) and the fact that b_ε is bounded by K_ε, we have

$$\begin{aligned} \|v_t^\varepsilon\|_r &\le \|p_t^\alpha \star u_0\|_r + \int_0^t \|\nabla p_{t-s}^\alpha \star (B_\varepsilon(v_s^\varepsilon) v_s^\varepsilon)(x)\|_r \, ds \\ &\le \|u_0\|_r + \int_0^t C(t-s)^{-\frac{d}{\alpha}(1-\frac{1}{r})-\frac{1}{\alpha}} \|B_\varepsilon(v_s^\varepsilon) v_s^\varepsilon\|_1 \, ds \\ &\le \|u_0\|_r + C K_\varepsilon \|u_0\|_1^2 \int_0^t (t-s)^{-\frac{d}{\alpha}(1-\frac{1}{r})-\frac{1}{\alpha}} \, ds, \end{aligned} \tag{4.13}$$

with the second inequality following from (2.5). The last quantity is finite since the integrand's exponent is greater than -1 as $r < d/(d - \eta)$. Hence $v^\varepsilon \in \mathbf{F}_{r,T}$. Now using the second inequality above and (2.12), since $r > d/\beta$, we obtain that

$$\begin{aligned} \sup_{s \le t} \|v_s^\varepsilon\|_r &\le \|u_0\|_r + \int_0^t C(t-s)^{-\frac{d}{\alpha}(1-\frac{1}{r})-\frac{1}{\alpha}} \|B_\varepsilon(v_s^\varepsilon) v_s^\varepsilon\|_1 ds \\ &\le \|u_0\|_r + C\|u_0\|_1 \int_0^t (t-s)^{-\frac{d}{\alpha}(1-\frac{1}{r})-\frac{1}{\alpha}} (\sup_{\tau \le s} \|v_\tau^\varepsilon\|_r + \|u_0\|_1) \, ds \end{aligned}$$

where the constant C does not depend on ε and we conclude by an application of Gronwall's Lemma 2.1.

Second, note that if $r' > r$ is such that $d/r - d/r' < \eta$ and $u_0 \in L^{r'}$ then function v_t^ε belongs to $L^{r'}$ and, moreover, $\sup_\varepsilon \|v^\varepsilon\|_{r',T} < +\infty$. Indeed, in view of the first part of the proof we know that $\sup_\varepsilon \|v^\varepsilon\|_{r,T} < +\infty$. Since $\sup_\varepsilon \|v^\varepsilon\|_{1,T} \leq \|u_0\|_1$, we deduce that $\sup_\varepsilon |||v^\varepsilon|||_{r,T} < +\infty$ and, by (2.12), as $r > d/\beta$, we have $\sup_\varepsilon \|B_\varepsilon(v^\varepsilon)\|_{\infty,T} \leq C|||v^\varepsilon|||_{r,T} < +\infty$. Therefore

$$
\begin{aligned}
\|v_t^\varepsilon\|_{r'} &\leq \|p_t^\alpha \star u_0\|_{r'} + \int_0^t \|\nabla p_{t-s}^\alpha \star (B_\varepsilon(v_s^\varepsilon)v_s^\varepsilon)(x)\|_{r'}\, ds \\
&\leq \|u_0\|_{r'} + \int_0^t C(t-s)^{-\frac{d}{\alpha}(\frac{1}{r}-\frac{1}{r'})-\frac{1}{\alpha}} \|B_\varepsilon(v_s^\varepsilon)v_s^\varepsilon\|_r\, ds \quad \text{by (2.5)} \\
&\leq \|u_0\|_{r'} + C\sup_\varepsilon \|B_\varepsilon(v_s^\varepsilon)\|_{\infty,T} \sup_\varepsilon \|v^\varepsilon\|_{r,T} \int_0^t (t-s)^{-\frac{d}{\alpha}(\frac{1}{r}-\frac{1}{r'})-\frac{1}{\alpha}}\, ds < +\infty
\end{aligned}
$$

as $d/r - d/r' < \eta$.

Let us now consider any $p > d/\beta$. Then either $p < d/(d-\eta)$ and the result is obtained by the first part of the proof, or $p \geq d/(d-\eta)$. In the latter case, we will use a recursive argument to show that $v_t^\varepsilon \in L^p$ and that $\sup_\varepsilon \|v^\varepsilon\|_{p,T} < +\infty$. Observe that since $u_0 \in L^1 \cap L^p$, u_0 belongs to all $L^r, 1 \leq r \leq p$.

Since $d-\eta < \beta$, there exists $\lambda \in]0,1[$ such that $d-\lambda\eta < \beta$. Consider $r_0 = d/(d-\lambda\eta)$. Then $d/\beta < r_0 < d/(d-\eta)$ and r_0 is less than p. So $u_0 \in L^{r_0}$ and then the same is true for v_t^ε thanks to the first part of the proof, and $\sup_\varepsilon \|v^\varepsilon\|_{r_0,T} < +\infty$. We now define

$$
r_k = \frac{d}{d - (k+2\lambda)\frac{\eta}{2}}.
$$

For $1 \leq k \leq N = \lfloor \frac{2d}{\eta} - 2\lambda \rfloor$, $r_k \in I\!R_+ \cup \{+\infty\}$ is greater than r_{k-1} and such that $d/r_k = d/r_{k-1} - \eta/2$. Using inductively the second part of the proof, we obtain that if $1 \leq k \leq N$, as soon as $u_0 \in L^{r_k}$, $\sup_\varepsilon \|v^\varepsilon\|_{r_k,T} < +\infty$.

If $r_N < p$, then, since $2d/\eta - 2\lambda < N+1$, we get

$$
\frac{d}{r_N} = d - (N+2\lambda)\frac{\eta}{2} < \frac{\eta}{2} < \eta + \frac{d}{p}
$$

so that, in view of the second part of the proof, if $u_0 \in L^p$, then $v^\varepsilon \in \mathbf{F}_{p,T}$, and $\sup_\varepsilon \|v^\varepsilon\|_{p,T} < +\infty$.

If $r_N \geq p$, then, for some $k < N$, we have $1/r_k \geq 1/p > 1/r_{k+1}$, so that

$$
\frac{d}{r_k} - \frac{d}{p} < \frac{d}{r_k} - \frac{d}{r_{k+1}} = \frac{\eta}{2},
$$

and the proof of the proposition is now complete, again by application of the second part of the proof. ♠

Theorem 4.9 *Suppose that $u_0 \in L^1 \cap L^p$ with $d/\beta < p \leq \infty$.*

(a) As ε tends to 0, the sequence (v^ε) converges in $\mathbf{F}_{p,T_0}$ to the function u defined in Corollary 3.5, and

$$|||v^\varepsilon - u|||_{p,T_0} + \sup_{t \leq T_0} \|B_\varepsilon(v_t^\varepsilon) - B(u_t)\|_\infty \leq C\varepsilon^{\frac{d-q(d-\beta)}{q}}.$$

(b) Assume that α, β, and d satisfy the Balance Condition, i.e. $d - (\alpha - 1) < \beta$. Then, for each $T > 0$, the sequence (v^ε) is Cauchy in $\mathbf{F}_{p,T}$, and converges in this space to a function v. Moreover, v is a solution of the singular mild equation (3.1) and

$$|||v^\varepsilon - v|||_{p,T} + \sup_{t \leq T} \|B_\varepsilon(v_t^\varepsilon) - B(v_t)\|_\infty \leq C\varepsilon^{\frac{d-q(d-\beta)}{q}}.$$

Proof (a) Since v^ε coincides on $[0, T_0]$ with u^ε defined in Theorem 3.2, the conclusion follows immediately from Corollary 3.3.

(b) Let $\varepsilon' > \varepsilon > 0$. Exactly as in the proof of Corollary 3.3, using additionally Proposition 4.8, one shows that

$$|||v_t^{\varepsilon'} - v_t^\varepsilon|||_p \quad \leq \quad C_1(\varepsilon')^{\frac{d-q(d-\beta)}{q}} + C_2 \int_0^t (t-s)^{-\frac{1}{\alpha}} |||v_s^{\varepsilon'} - v_s^\varepsilon|||_p \, ds, \tag{4.14}$$

with q being the conjugate of p. We conclude now by an application of Gronwall's Lemma 2.1. The sequence (v^ε) is Cauchy in $\mathbf{F}_{p,T}$ and thus converges in this space to a function v belonging to $\mathbf{F}_{p,T}$. It is immediate to see that v is a solution of (3.1). The last assertion is obtained like in the proof of Corollary 3.3. ♠

Note that the above global existence result for the singular mild equation (3.1) generalizes Theorem 2.2 in [6].

Remark 4.10 In view of Theorem 4.9 and Subsection 3.2, for α, β and d satisfying the Balance Condition $d-(\alpha-1) < \beta$, and for each $T > 0$, and $u_0 \in L^1 \cap L^p$ with $d/\beta < p \leq \infty$, we have thus succeeded in constructing via a probabilistic approach a unique solution u of the singular mild equation (3.1) in $\mathbf{F}_{p,T}$.

4.2 The interacting particle system

In this section we prove a "strong" propagation of chaos result for the nonlinear process considered in Section 4.1. A "weak" result of this type has been obtained in [6].

Let $X_0^i, i = 1, 2, \ldots$, be independent copies of X_0 and $S^i, i = 1, 2, \ldots$, independent copies of S, independent of the initial conditions. Consider independent copies $\bar{X}^{i,\varepsilon}, i = 1, 2, \ldots$, of the nonlinear process determined by nonlinear stochastic differential equation

$$\bar{X}_t^{i,\varepsilon} = X_0^i + S_t^i + \int_0^t b_\varepsilon \star v_s^\varepsilon(\bar{X}_s^{i,\varepsilon})\, , ds, \tag{4.15}$$

where the weighted version of the law of $\bar{X}_t^{i,\varepsilon}$ has density v_t^ε.

For each $n \in I\!N^*$, and $1 \leq i \leq n$, consider the following interacting particle system constructed on the same probability space:

$$\begin{aligned} X_t^{i,n,\varepsilon} &= X_0^i + S_t^i + \int_0^t \frac{1}{n}\sum_{j=1}^n b_\varepsilon(X_s^{i,n,\varepsilon} - X_s^{j,n,\varepsilon})h(X_0^j)ds \\ &= X_0^i + S_t^i + \int_0^t \int_{I\!R^d} b_\varepsilon(X_s^{i,n,\varepsilon} - y)\tilde{\mu}_s^{n,\varepsilon}(dy)ds \end{aligned} \tag{4.16}$$

where $\tilde{\mu}_s^{n,\varepsilon}$ is the weighted version of the empirical measure

$$\tilde{\mu}_s^{n,\varepsilon} = \frac{1}{n}\sum_{j=1}^n h(X_0^j)\delta_{X_s^{j,n,\varepsilon}}. \tag{4.17}$$

Since kernel b_ε is Lipschitz continuous and function h is bounded, the standard argument provides a proof of the existence and uniqueness result for system (4.16).

Theorem 4.11 *For each $1 \leq i \leq n$,*

$$E(\sup_{t\leq T}|X_t^{i,n,\varepsilon} - \bar{X}_t^{i,\varepsilon}|) \leq \frac{C_1 K_\varepsilon}{\sqrt{n}L_\varepsilon}\exp(2\|u_0\|_1 L_\varepsilon T). \tag{4.18}$$

Remark 4.12 Described in Subsection 2.3 behaviour of L_ε and K_ε as functions of ε implies that

$$E(\sup_{t\leq T}|X_t^{i,n,\varepsilon} - \bar{X}_t^{i,\varepsilon}|) \leq \frac{C_1\varepsilon^{\max(1,\beta-\gamma)}}{\sqrt{n}}\exp(C_2\|u_0\|_1(\varepsilon^{-d+\min(\beta-1,\gamma)})). \tag{4.19}$$

Proof of Theorem 4.11 The proof is an adaptation of the proof of Proposition 2.2 in [12]. Since,

$$\begin{aligned} X_t^{i,n,\varepsilon} - \bar{X}_t^{i,\varepsilon} = &\int_0^t \frac{1}{n}\sum_{j=1}^n b_\varepsilon(X_s^{i,n,\varepsilon} - X_s^{j,n,\varepsilon})h(X_0^j)\,ds - \int_0^t b_\varepsilon \star v_s^\varepsilon(\bar{X}_s^{i,\varepsilon})\,ds \\ = &\int_0^t \frac{1}{n}\sum_{j=1}^n h(X_0^j)\left[b_\varepsilon(X_s^{i,n,\varepsilon} - X_s^{j,n,\varepsilon}) - b_\varepsilon(\bar{X}_s^{i,\varepsilon} - \bar{X}_s^{j,\varepsilon})\right]\,ds \\ &+ \int_0^t \left[\frac{1}{n}\sum_{j=1}^n h(X_0^j)b_\varepsilon(\bar{X}_s^{i,\varepsilon} - \bar{X}_s^{j,\varepsilon}) - b_\varepsilon \star v_s^\varepsilon(\bar{X}_s^{i,\varepsilon})\right]\,ds \end{aligned}$$

we have

$$\begin{aligned} \sup_{s\leq t}|X_t^{i,n,\varepsilon} - \bar{X}_t^{i,\varepsilon}| \leq &\int_0^t \frac{1}{n}\sum_{j=1}^n \|h\|_\infty\, L_\varepsilon\left(|X_s^{i,n,\varepsilon} - \bar{X}_s^{i,\varepsilon}| + |X_s^{j,n,\varepsilon} - \bar{X}_s^{j,\varepsilon}|\right)\,ds \\ &+ \int_0^t \left|\frac{1}{n}\sum_{j=1}^n h(X_0^j)b_\varepsilon(\bar{X}_s^{i,\varepsilon} - \bar{X}_s^{j,\varepsilon}) - b_\varepsilon \star v_s^\varepsilon(\bar{X}_s^{i,\varepsilon})\right|\,ds \end{aligned}$$

Since the sequence $(X^{i,n,\varepsilon}, \bar{X}^{i,\varepsilon})_{1\leq i\leq n}$ is exchangeable,

$$E\Big[\sup_{s\leq t}|X_t^{i,n,\varepsilon} - \bar{X}_t^{i,\varepsilon}|\Big] \leq 2\|h\|_\infty\, L_\varepsilon \int_0^t \sup_{u\leq s}|X_s^{i,n,\varepsilon} - \bar{X}_s^{i,\varepsilon}|\, ds$$

$$+\int_0^t \left[E\left(\frac{1}{n}\sum_{j=1}^n h(X_0^j)b_\varepsilon(\bar{X}_s^{i,\varepsilon} - \bar{X}_s^{j,\varepsilon}) - b_\varepsilon \star v_s^\varepsilon(\bar{X}_s^{i,\varepsilon})\right)^2\right]^{1/2} ds$$

The above expectation can be evaluated as follows:

$$\begin{aligned}
&E\left(\frac{1}{n}\sum_{j=1}^n h(X_0^j)b_\varepsilon(\bar{X}_s^{i,\varepsilon} - \bar{X}_s^{j,\varepsilon}) - b_\varepsilon \star v_s^\varepsilon(\bar{X}_s^{i,\varepsilon})\right)^2 \\
= \quad & \frac{1}{n^2}\sum_{j,k=1}^n E\Big[\Big(h(X_0^j)b_\varepsilon(\bar{X}_s^{i,\varepsilon} - \bar{X}_s^{j,\varepsilon}) - b_\varepsilon \star v_s^\varepsilon(\bar{X}_s^{i,\varepsilon})\Big) \\
&\qquad \times \Big(h(X_0^k)b_\varepsilon(\bar{X}_s^{i,\varepsilon} - \bar{X}_s^{k,\varepsilon}) - b_\varepsilon \star v_s^\varepsilon(\bar{X}_s^{i,\varepsilon})\Big)\Big]. \qquad (4.20)
\end{aligned}$$

Observe that $\bar{X}^{i,\varepsilon}, 1 \leq i \leq n$, are independent so that, for $i \neq j$, and since the weighted law of $X_s^{j,\varepsilon}$ is equal to $v_s^\varepsilon(y)dy$,

$$E\Big[h(X_0^j)b_\varepsilon(\bar{X}_s^i - \bar{X}_s^j)\Big|\bar{X}_s^l, l\neq j\Big] = \quad \int b_\varepsilon(\bar{X}_s^i - y)v_s^\varepsilon(y)\,dy = b_\varepsilon \star v_s^\varepsilon(\bar{X}_s^i).$$

If $j \neq k$ then at least one of the two indices is different from i; suppose it is j. Then

$$\begin{aligned}
&E\Big[\Big(h(X_0^j)b_\varepsilon(\bar{X}_s^{i,\varepsilon} - \bar{X}_s^{j,\varepsilon}) - b_\varepsilon \star v_s^\varepsilon(\bar{X}_s^{i,\varepsilon})\Big) \\
&\qquad \times \Big(h(X_0^k)b_\varepsilon(\bar{X}_s^{i,\varepsilon} - \bar{X}_s^{k,\varepsilon}) - b_\varepsilon \star v_s^\varepsilon(\bar{X}_s^{i,\varepsilon})\Big)\Big] \\
= \quad & E\Big[E\Big[h(X_0^j)b_\varepsilon(\bar{X}_s^i - \bar{X}_s^j) - b_\varepsilon \star v_s^\varepsilon(\bar{X}_s^i)\Big|\bar{X}_s^l, l\neq j\Big] \\
&\qquad \times \Big(h(X_0^k)b_\varepsilon(\bar{X}_s^{i,\varepsilon} - \bar{X}_s^{k,\varepsilon}) - b_\varepsilon \star v_s^\varepsilon(\bar{X}_s^{i,\varepsilon})\Big)\Big] = 0.
\end{aligned}$$

Consequently, remembering that $\|h\|_\infty = \|u_0\|_1$, we get that the expression (4.20) is equal to

$$\frac{1}{n^2}\sum_{j=1}^n E\Big[\Big(h(X_0^j)b_\varepsilon(\bar{X}_s^i - \bar{X}_s^j) - b_\varepsilon \star v_s^\varepsilon(\bar{X}_s^i)\Big)^2\Big] \leq \frac{4}{n}K_\varepsilon^2\|u_0\|_1^2,$$

where K_ε is an upper bound for b_ε. Finally, we obtain that

$$E\left[\sup_{s\leq t}|X_s^{i,n,\varepsilon} - \bar{X}_s^{i,\varepsilon}|\right] \leq 2\|u_0\|_1 L_\varepsilon \int_0^t E\left[\sup_{s\leq t}|X_s^{i,n,\varepsilon} - \bar{X}_s^{i,\varepsilon}|\right] ds + \frac{2tK_\varepsilon\|u_0\|_1}{\sqrt{n}}.$$

We conclude with the help of Gronwall's Lemma that

$$E(\sup_{t\leq T}|X_t^{i,n,\varepsilon} - \bar{X}_t^{i,\varepsilon}|) \leq \frac{CK_\varepsilon}{L_\varepsilon\sqrt{n}}\exp(2\|u_0\|_1 L_\varepsilon T).$$

♠

5 Convergence of the cut-off model to the singular model

In this section T will be selected to be equal to T_0 defined in Section 3 in the general case; it can be selected to be any positive number under the Balance Condition $d-(\alpha-1)<\beta$ found in Proposition 4.8. Function u will denote either, as before, the unique solution of the singular mild equation (3.1) on $[0,T_0]$ (see Corollary 3.5), or, under the Balance Condition, the unique solution of the singular mild equation (3.1) on $[0,T]$ obtained in Theorem 4.9 (then $u=v$, see Remark 4.10).

Let us also introduce the associated stochastic differential equation on $[0,T]$,

$$X_t = X_0 + S_t + \int_0^t B(u_s)(X_s)ds, \tag{5.1}$$

where X_0 has density $|u_0(x)|/\|u_0\|_1$ and S is an independent symmetric stable process with index α. According to the following Proposition, any weak solution of this equation is a singular nonlinear process :

Proposition 5.1 *Assume that X_t is a solution of equation (5.1) on $[0,T]$. Then, for each $t\le T$, the weighted version $\tilde{P}_t$ of the law of X_t has a density with respect to the Lebesgue measure equal to u_t.*

Proof The proof now follows the familiar pattern. Using Itô's formula with appropriate test functions like in the proof of Proposition 4.4, one can prove that, for each t, $\tilde{P}_t$ is absolutely continuous with respect to the Lebesgue measure and that its density w_t solves the "linear" evolution equation

$$w_t \quad = \quad p_t^\alpha \star u_0 - \int_0^t \nabla p_{t-s}^\alpha \star (B(u_s)w_s)ds. \tag{5.2}$$

This equation has a unique solution in $L^\infty([0,T],L^1)$. Since $w\in L^\infty([0,T],L^1)$, and since u, as a solution of (3.1), is also a solution of (5.2), we conclude that $u=w$. ♠

5.1 The case $\gamma>0$

The present subsection considers the case $\gamma>0$, for which we will obtain pathwise convergence results. Here T can be selected to be any positive number only under the reinforced Balance Condition $d-(\alpha-1)<\min(\beta,\gamma)$, and we assume that $u_0\in L^1\cap L^p$ with $d/\min(\beta,\gamma)<p$.

As in the previous section, we consider independent variables (X_0^i,S^i) with the same law as (X_0,S). For $\varepsilon>0$ and $i\ge 1$, we also introduce the solution $\bar{X}^{i,\varepsilon}$ of the cut-off nonlinear stochastic differential equation (4.15). It follows from Proposition 2.4 that if $\gamma>0$, and $g\in L^p$, with $p>d/\min(\beta,\gamma)$, then $B(g)$ is a bounded Lipschitz function with a Lipschitz constant less than $C|||g|||_p$. Hence, for the solution $u\in\mathbf{F}_{p,T}$ of the singular mild equation (3.1), the function $x\to B(u_s)(x)$ is bounded and Lipschitz continuous uniformly for $s\in[0,T]$. Therefore, by Lemma 4.5, for $i\ge 1$, equation (5.1) with (X_0,S) replaced by (X_0^i,S^i) has a unique solution X^i.

The next theorem shows the pathwise convergence result of the cut-off nonlinear processes to the corresponding singular nonlinear processes.

Theorem 5.2 *Assume that $d-(\alpha-1) < min(\beta,\gamma)$, and let $p > d/\min(\beta,\gamma)$, $u_0 \in L^1 \cap L^p$. Then, for a deterministic constant C independent of ε, for each $T > 0$ and each $i \geq 1$,*

$$\sup_{t \leq T} |\bar{X}^{i,\varepsilon}_t - \bar{X}^i_t| \leq C\varepsilon^{\frac{d-q(d-\beta)}{q}}, \tag{5.3}$$

where $1/p + 1/q = 1$. In particular, for each sequence (ε_n) tending to 0 as n tends to infinity, the sequence (P^{ε_n}) converges on $\hat{\mathcal{P}}_T$ to the law P^u.

Proof Let us recall that according to Proposition 2.4, $DB(u^\varepsilon)$ is bounded by a constant times $|||u^\varepsilon|||_{p,T}$ and, in particular, $B(u^\varepsilon)$ is Lipschitz continuous with a constant bounded by $|||u^\varepsilon|||_{p,T}$. Assume that $u_0 \in L^p$. Then, using Proposition 2.5,

$$\begin{aligned}
|\bar{X}^{i,\varepsilon}_t - \bar{X}^i_t| \leq \quad & \left| \int_0^t \left(\int_{I\!R^d} b_\varepsilon(\bar{X}^{i,\varepsilon}_s - y) v^\varepsilon_s(y) dy - \int_{I\!R^d} b(\bar{X}^i_s - y) u_s(y) dy \right) ds \right| \\
\leq \quad & \int_0^t \int_{I\!R^d} \left| (b_\varepsilon(\bar{X}^{i,\varepsilon}_s - y) - b(\bar{X}^{i,\varepsilon}_s - y)) v^\varepsilon_s(y) \right| dyds \\
& + \int_0^t \left| \int_{I\!R^d} (b(\bar{X}^{i,\varepsilon}_s - y) - b(\bar{X}^i_s - y)) v^\varepsilon_s(y) dy \right| ds \\
& + \int_0^t \int_{I\!R^d} \left| b(\bar{X}^i_s - y)(v^\varepsilon_s(y) - u_s(y)) \right| dyds \\
\leq \quad & C\varepsilon^{\frac{d-q(d-\beta)}{q}} |||v^\varepsilon|||_{p,T} + C|||v^\varepsilon|||_{p,T} \int_0^t \left| \bar{X}^{i,\varepsilon}_s - \bar{X}^i_s \right| ds + C|||v^\varepsilon - u|||_{p,T}.
\end{aligned} \tag{5.4}$$

We conclude the proof by Proposition 4.8, Theorem 4.9 and Gronwall's Lemma. ♠

5.2 The general case $\gamma \geq 0$

In the general case $\gamma \geq 0$ we are only able to prove a weak result.

Proposition 5.3 *Let $p > d/\beta$ and $u_0 \in L^1 \cap L^p$. For each sequence (ε_n) tending to 0 as n tends to infinity, the sequence (P^{ε_n}) is uniformly tight on $\hat{\mathcal{P}}_T$.*

Proof Since each P^{ε_n} is a weak solution of (4.5) with ε replaced by ε_n, by the standard argument we know that the sequence (P^{ε_n}) is uniformly tight if and only if the sequence $(\int_0^t b_{\varepsilon_n} \star v^{\varepsilon_n}(X^{\varepsilon_n}_s)\, ds)$ satisfies the Aldous criterion (see, [1]). The criterion is satisfied since Proposition 4.8, inequality $\sup_n \sup_{t\in[0,T]} \|v^{\varepsilon_n}_t\|_1 \leq \|u_0\|_1$ and (2.12) imply that

$$\sup_n \sup_{t\in[0,T]} \|B_{\varepsilon_n}(v^{\varepsilon_n}_t)\|_\infty < +\infty.$$

♠

Theorem 5.4 *Let $p > d/\beta$ and $u_0 \in L^1 \cap L^p$. For each sequence (ε_n) tending to 0 as n tends to infinity, the sequence (P^{ε_n}) converges in $\hat{\mathcal{P}}_T$ to the unique weak solution of equation (5.1).*

Proof Since the sequence (P^{ε_n}) is uniformly tight in $\hat{\mathcal{P}}_T$, there exists a subsequence which, for convenience, will also be denoted by (P^{ε_n}), which converges to $Q \in \hat{\mathcal{P}}_T$. We will prove that Q is a weak solution of equation (5.1). We can characterize each P^{ε_n} as the unique solution of the nonlinear martingale problem related to (SDE_{ε_n}). The law P^{ε_n} is the unique probability measure on $\hat{\mathcal{P}}_T$ such that $P_0^{\varepsilon_n} = |u_0(x)|dx/\|u_0\|_1$ and if X is the canonical process on $I\!D([0,T], I\!R^d)$ then, for each $\phi \in C_b^2(I\!R^d)$,

$$\phi(X_t) - \phi(X_0) - \int_0^t \Big(- (-\Delta)^{\alpha/2}\phi(X_s) + b_{\varepsilon_n} \star \tilde{P}_s^{\varepsilon_n}(X_s).\nabla\phi(X_s)\Big) ds \tag{5.5}$$

is a P^{ε_n}-martingale. Here $\tilde{P}_s^{\varepsilon_n}(dx) = u_s^{\varepsilon_n}(x)dx$, and u^{ε_n} is the unique solution of (3.2). Then, for each integer k, for each $0 < s_1, ..., s_k \le s < t$, and for each continuous bounded function $G(x_1, ..., x_k)$, one has

$$\begin{aligned} E^{\varepsilon_n}\bigg(\Big[\phi(X_t) - \phi(X_s) \quad & - \int_s^t \Big(- (-\Delta)^{\alpha/2}\phi(X_r) \\ & + b_{\varepsilon_n} \star u_r^{\varepsilon_n}(X_r).\nabla\phi(X_r)\Big) dr\Big] G(X_{s_1}, ..., X_{s_k})\bigg) = 0, \end{aligned} \tag{5.6}$$

where E^{ε_n} denotes the expectation under P^{ε_n}.

We would like to prove that

$$\begin{aligned} E^{Q}\bigg(\Big[\phi(X_t) - \phi(X_s) \quad & - \int_s^t \Big(- (-\Delta)^{\alpha/2}\phi(X_r) \\ & + b \star u_r(X_r).\nabla\phi(X_r)\Big) dr\Big] G(X_{s_1}, ..., X_{s_k})\bigg) = 0 \end{aligned} \tag{5.7}$$

which, together with the argument used in the preceding subsection, will establish that Q is a weak solution of equation (5.1).

The mapping

$$F(X) = \phi(X_t) - \phi(X_s) - \int_s^t \Big(- (-\Delta)^{\alpha/2}\phi(X_r) + b \star u_r(X_r).\nabla\phi(X_r)\Big) dr$$

is not continuous since the projections $X \to X_t$ are not continuous for the Skorohod topology. However, $X \to X_t$ is Q-almost surely continuous for all t outside the at most countable set $D_Q = \{t > 0 : \ Q(X_t \neq X_{t^-}) > 0\}$. Then F is Q-almost surely continuous if $s, t, s_1, ..., s_k$ are not in D_Q. Let us prove (5.7) for $s, t, s_1, ..., s_k \notin D_Q$, which suffices to characterize Q as a weak solution of (5.1). One has

$$E^Q(F(X)) = \lim_{n \to +\infty} E^{\varepsilon_n}(F(X))$$

and (5.6) will imply (5.7) if $\lim_{n\to+\infty} A_n = 0$, where

$$A_n = E^{\varepsilon_n}\Bigg(\Big(\int_s^t (b_{\varepsilon_n} \star u_r^{\varepsilon_n}(X_r) - b \star u_r(X_r)).\nabla\phi(X_r)\,dr\Big)G(X_{s_1},...,X_{s_k})\Bigg).$$

Since

$$\begin{aligned} |A_n| &\leq CE^{\varepsilon_n}\Big(\int_s^t |b_{\varepsilon_n} \star u_r^{\varepsilon_n}(X_r) - b \star u_r(X_r)|\,dr\Big) \\ &\leq C\int_0^T \|B_{\varepsilon_n}(u_r^{\varepsilon_n}) - B(u_r)\|_\infty\,dr, \end{aligned} \tag{5.8}$$

we conclude by Theorem 4.9 that Q satisfies (5.7).

Finally, it remains to check that there exists at most one weak solution of (5.1) belonging to $\hat{\mathcal{P}}_T$. Adapting Theorem 4.2, p. 184, in Ethier-Kurtz [10] to the non-homogeneous case (as in Stroock-Varadhan [20], Theorem 6.2.3, p. 147, for the diffusion case), we obtain that it is enough to check that, for any $(s,x) \in [0,T] \times I\!R^d$, uniqueness holds for the time-marginals of any solution $Q^{s,x} \in \mathcal{P}_{T-s}$ of the martingale problem : $Q_0^{s,x} = \delta_x$ and, for any $\phi \in C_b^2(I\!R^d)$,

$$\phi(X_t) - \phi(X_0) - \int_0^t \Big(-(-\Delta)^{\alpha/2}\phi(X_r) + b \star u_{s+r}(X_r).\nabla\phi(X_r)\Big)dr \tag{5.9}$$

is a $Q^{s,x}$-martingale. By Itô's formula for appropriate test functions we can verify that for each $t \in]0,T-s]$, $Q_t^{s,x}$ has a density q_t and that $t \in]0,T-s] \to q_t$ solves the evolution equation

$$q_t = p_t^\alpha \star \delta_x - \int_0^t \nabla p_{t-r}^\alpha \star (B(u_{s+r})q_r)dr \tag{5.10}$$

for which uniqueness holds in $L^\infty(]0,T],L^1)$. Hence the time marginals are unique and (5.1) has no more than one weak solution, which concludes the proof. ♠

5.3 Propagation of chaos

All of the above results can now be put together to yield the following propagation of chaos result:

Theorem 5.5 *Let (ε_n) be a sequence converging to 0 so that, with constant C_1 from Theorem 4.11,*

$$\frac{C_1 K_{\varepsilon_n}}{\sqrt{n}L_{\varepsilon_n}}\exp\Big(2\|u_0\|_1 L_{\varepsilon_n} T\Big) \to 0$$

as n tends to infinity. Then, for each fixed integer k, the laws of $(X^{1,n,\varepsilon_n},...,X^{k,n,\varepsilon_n})$ converge in the space of probability measures on the path space $I\!D([0,T],I\!R^d)$ to the product measure $P^{\otimes k}$, where P denotes the law of the unique weak solution of equation (5.1).

Remark 5.6 Note that (4.19) provides a more precise behaviour of ε_n as a function of n.

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