A posteriori discontinuous Galerkin error estimates for transient convection-diffusion equations

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Abstract

A posteriori error estimates are derived for unsteady convection-diffusion equations discretized with the non-symmetric interior penalty and the local discontinuous Galerkin methods. First, an error representation formula in a user specified output functional is derived using duality techniques. Then, an $L^2_t(L^2_x)$ a posteriori estimate consisting of elementwise residual-based error indicators is obtained by eliminating the dual solution. Numerical experiments are performed to assess the convergence rate of the various error indicators on a model problem.

Key words: a posteriori error estimates, duality techniques, non-symmetric interior penalty, local discontinuous Galerkin, convection-diffusion equations

1 Introduction

Adaptive finite element methods based on discontinuous approximation spaces have been under rapid development recently, notably because of their flexibility in both local mesh subdivision and local polynomial degree variation. The inherent flexibility of discontinuous Galerkin (DG) methods allows for the general construction of temporal and spatial non-uniformities, more so than conventional continuous finite element techniques, though at a higher computational expense. Furthermore, they are locally conservative, allow for nonconforming grids, and successfully handle the difficulties associated with high gradient solutions. Cockburn et al. [1] contains a thorough survey of modern implementations in various applications.

While an extensive body of work with a priori error analysis exists for these methods applied to transient convection-diffusion equations (see [1–5] and references therein), their a posteriori error analysis and implementation is significantly less developed. A posteriori estimators rely on the derivation of computable bounds on the error and may be used to signify where refinement in spatial quantities or polynomial degree may be adaptively modified. They can be particularly useful in applications where solution gradients vary in orders of magnitude across spatial domains, such as those arising in convection-dominated transport.

A posteriori error estimators for DG methods have focused primarily on steady-state equations of elliptic and hyperbolic type. Recent work by Becker et al. [6] and Karakasian and Pascal [7] establishes energy norm estimates for elliptic equations. Houston et al. [8] derive computable upper bounds on a natural DG energy norm for incompressible Stokes flows. We mention work by Rivière and Wheeler [9] who utilize a standard elliptic duality technique to derive L^2 estimates. The use of a duality argument also extends to hyperbolic problems to derive estimates of functional quantities of interest, leading to adaptivity based on more physically meaningful quantities than the energy or L^2 norm. Such error bounds for first order hyperbolic problems were derived by Larson and Barth in [10]. Süli and collaborators [11,12] also derive and implement various error bounds for general linear and nonlinear target functionals of the solution within an adaptive framework.

There are considerably fewer papers that are concerned with a posteriori error estimation for DG methods applied to transient problems. Adjerid et al. [13] and Flaherty et al. [14] exploit superconvergence results to construct asymptotically correct estimates of spatial discretization errors for unsteady linear and nonlinear hyperbolic conservation laws. This application was also explored by Hartmann and Houston [15] where they employ duality techniques to derive estimates based on functional quantities of interest and demonstrate that "weighted" a posteriori error indicators can lead to sharper bounds and more efficient meshes than corresponding "unweighted" indicators: estimates based on the elimination of the dual solution in the analysis. Results for transient convection-diffusion operators remain sparse; we mention the work of Sun and Wheeler [16], where an explicit $L^2(L^2)$ and target functional estimates are derived for a symmetric discretization of the diffusion operator. Formal $L^2(L^2)$ and target functional estimates of a non-symmetric interior penalty formulation and the related "local" discontinuous Galerkin formulation remain unexplored in the literature.

In this paper, we focus our attention on the derivation of an explicit error estimator for the transient convection-diffusion problem

$$\phi\partial_t c + \nabla \cdot (uc - D\nabla c) = \phi f \quad \text{on } \Omega, \quad t \ge 0, \tag{1}$$
$$(uc - D\nabla c) \cdot n = (u\hat{g}) \cdot n \quad \text{on } \partial\Omega_{\text{in}}, \quad t \ge 0, \tag{2}$$
$$(-D\nabla c) \cdot n = 0 \quad \text{on } \partial\Omega_{\text{out}}, \, t \ge 0, \tag{3}$$
$$c(x,0) = c_0(x) \quad \text{on } \Omega, \tag{4}$$

defined on polygonal bounded domain $\Omega \in \mathbb{R}^d$, $d = 2$ or 3, with unit outward normal n to Lipschitz boundary $\partial\Omega$. Let $\partial\Omega = \partial\Omega_{\text{in}} \cup \partial\Omega_{\text{out}}$ be partitioned into disjoint inflow and outflow boundary portions $\partial\Omega_{\text{in}} = \{x \in \partial\Omega : u \cdot n < 0\}$ and $\partial\Omega_{\text{out}} = \{x \in \partial\Omega : u \cdot n \ge 0\}$, respectively. In typical porous media applications, $c(x,t)$ represents the concentration of some chemical component, $\phi(x)$ is the effective porosity of the medium and is bounded above and below by positive constants, $u(x,t)$ is the Darcy velocity, $D(x,u,t)$ is a diffusion/dispersion tensor assumed to be uniformly positive definite (but not necessarily symmetric), and $f(x,t)$ is a source term. We will assume the Darcy velocity field vector u is given and satisfies the continuity equation $\nabla \cdot u = 0$.

Our approach to derive a posteriori error estimates is based on the use of a duality argument and Galerkin orthogonality and is similar to techniques used by Rivière and Wheeler in [9] for elliptic equations. The particular non-symmetric discontinuous Galerkin method (NIPG) we consider for discretizing the diffusion operator was originally developed by Oden, Baumann and Babuška in [17], and extended by Rivière et al. by adding interior penalty terms in the formulation to weakly enforce inner element continuity; see [18] and references therein. We also consider the local discontinuous Galerkin method (LDG) developed by Cockburn and Shu [2].

2 Non-symmetric Interior Penalty Galerkin (NIPG)

Let $\{T_h\}_{h>0}$ denote a family of finite element subdivisions of domain Ω partitioned into open disjoint elements Ω_e such that $\bar{\Omega} = \cup_{\Omega_e \in T_h} \bar{\Omega}_e$. We denote by $H^s(\Omega)$ the standard Sobolev spaces equipped with the usual norms $\|\cdot\|^2_{H^s(\Omega)}$. For a time-space function u, the notation $u \in L^2_t(H^s_x)$ (resp. $u \in C^k_t(H^s_x)$) means that the function $t \mapsto u(t,\cdot) \in H^s(\Omega)$ is in $L^2(0,T)$ (resp. $C^k(0,T)$) where T is given. Define $\mathbb{P}^k(\Omega_e)$ to be the set of polynomials of degree less than or equal to k on Ω_e and consider the finite element space $V_h = \{v \in L^2(\Omega) : \forall \Omega_e \in T_h, \ v|_{\Omega_e} \in \mathbb{P}^k(\Omega_e)\}$.

We will use the standard L^2 inner product notation $(\cdot,\cdot)_R$ for domains $R \in \mathbb{R}^d$, and the notation $\langle\cdot,\cdot\rangle_R$ to denote integration over $(d-1)$-dimensional manifolds. Let F_h be the set of faces belonging to elements $\Omega_e \in T_h$ and partition F_h into $F^{\text{i}} \cup F^{\partial}_{\text{in}} \cup F^{\partial}_{\text{out}}$, where F^{i} denotes the set of interior faces, F^{∂}_{in}

the set of those located on $\partial\Omega_{\text{in}}$, and $F^{\partial}_{\text{out}}$ the set of those located on $\partial\Omega_{\text{out}}$. For a face $F \in F^{\text{i}}$ shared by elements Ω_{e1} and Ω_{e2} with respective unit outward normals n_1 and n_2, define the average and (vector-valued) jump of $v \in V_h$ as $\{v\} = \frac{1}{2}(v_1 + v_2)$ and $[v] = (v_1 n_1 + v_2 n_2)$, respectively, where $v_1 = v|_{\Omega_{e1}}$ and $v_2 = v|_{\Omega_{e2}}$. Define the upwind value $v^{\uparrow} = v_1$ when $u \cdot n_1 > 0$, else $v^{\uparrow} = v_2$. Similarly, for a function $w \in [V_h]^d$, define the average and (scalar-valued) jump as $\{w\} = \frac{1}{2}(w_1 + w_2)$ and $[w] = (w_1 \cdot n_1 + w_2 \cdot n_2)$, respectively.

The NIPG formulation consists of seeking $u_h \in C^1_t(V_h)$ such that $\forall v \in V_h$ and $\forall t \geq 0$,

$$(\phi \partial_t c_h, v)_\Omega + a_{\text{NIPG}}(c_h, v) = (\phi f, v)_\Omega - (u\hat{g} \cdot n, v)_{\partial\Omega_{\text{in}}}, \tag{5}$$

with the initial condition $(c_0 - c_h(0,\cdot), v)_\Omega = 0$, $\forall v \in V_h$. The bilinear form a_{NIPG} is given by

$$\begin{aligned} a_{\text{NIPG}}(c_h, v) = &- \sum_{\Omega_e \in T_h} (uc_h - D\nabla c_h, \nabla v)_{\Omega_e} + \sum_{F \in F^{\partial}_{\text{out}}} \langle uc_h \cdot n, v \rangle_F \\ &+ \sum_{F \in F^{\text{i}}} \Big(\langle uc_h^{\uparrow}, [v] \rangle_F - \langle \{D\nabla c_h\}, [v] \rangle_F + \langle \{D\nabla v\}, [c_h] \rangle_F + \langle \sigma_F [c_h], [v] \rangle_F \Big), \end{aligned} \tag{6}$$

where $\sigma_F = \frac{\sigma_0}{|F|}$, σ_0 is a positive constant, and $|F|$ the $(d-1)$-dimensional measure of F. Let the error in the solution be defined as $e_c = c - c_h$. Our goal is to control the error in the functional

$$\Psi(e_c) = \int_0^T (\psi_1(e_c), e_c)_\Omega \, \mathrm{d}t + (\psi_2(e_c(T,\cdot)), e_c(T,\cdot))_\Omega, \tag{7}$$

where ψ_1 and ψ_2 are user specified functions. Let ξ satisfy the adjoint equation

$$\phi \partial_t \xi + \nabla \cdot (u\xi + D^T \nabla \xi) = \psi_1(e_c) \quad \text{on } \Omega, \quad t \leq T, \tag{8}$$

$$(u\xi + D^T \nabla \xi) \cdot n = 0 \quad \text{on } \partial\Omega_{\text{out}}, \ t \leq T, \tag{9}$$

$$(-D^T \nabla \xi) \cdot n = 0 \quad \text{on } \partial\Omega_{\text{in}}, \ t \leq T, \tag{10}$$

$$\xi(x, T) = -\psi_2(e_c(T,\cdot)) \quad \text{on } \Omega. \tag{11}$$

We first derive an error representation formula.

Theorem 2.1 *Assume that the solution c to (1)–(4) and the solution ξ to (8)–(11) are both in $L^2_t(H^2_x) \cap C^0_t(L^2_x)$. Assume the diffusion/dispersion tensor D to be continuous. Then,*

$$\begin{aligned} \Psi(e_c) = &- \int_0^T (R_{\text{eqn}}, \xi - \xi^*)_\Omega \, \mathrm{d}t - (R_{t=0}, (\xi - \xi^*)(0,\cdot))_\Omega \\ &+ \int_0^T \Big(\sum_{F \in F^{\partial}_{\text{in}}} \langle R_{\text{in}}, \xi - \xi^* \rangle_F + \sum_{F \in F^{\partial}_{\text{out}}} \langle R_{\text{out}}, \xi - \xi^* \rangle_F \Big) \, \mathrm{d}t \\ &+ \int_0^T \sum_{F \in F^{\text{i}}} \Big(\langle R_{[c_h]}, D^T \nabla \xi + D \nabla \xi^* \rangle_F - \langle R_{[D\nabla c_h]} - u \cdot R_{[c_h]}, \xi - \xi^* \rangle_F \Big) \, \mathrm{d}t, \end{aligned} \tag{12}$$

where ξ^* is arbitrary in $V_h \cap C^0(\overline{\Omega})$ and where we have introduced the residuals

$$
\begin{aligned}
&R_{\text{eqn}} = \phi f - \phi \partial_t c_h + \nabla \cdot (u c_h - D \nabla c_h), && (13)\\
&R_{[c_h]} = [c_h], \quad R_{[D\nabla c_h]} = [D \nabla c_h], && (14)\\
&R_{\text{in}} = u\hat{g} \cdot n - (u c_h - D\nabla c_h) \cdot n, \quad R_{\text{out}} = -D\nabla c_h \cdot n, && (15)\\
&R_{t=0} = c_0 - c_{h,0}. && (16)
\end{aligned}
$$

Proof. Using (7), (8), and (11), we infer

$$\Psi(e_c) = -\int_0^T (\phi \partial_t e_c, \xi)_\Omega \, \mathrm{d}t - (R_{t=0}, \xi(0,\cdot))_\Omega + \int_0^T (\nabla \cdot (u\xi + D^T \nabla \xi), e_c)_\Omega \, \mathrm{d}t.$$

Integrate by parts the diffusion contribution to the last term and use (10) to obtain

$$
\begin{aligned}
(\nabla \cdot (D^T \nabla \xi), e_c)_\Omega = &- \sum_{\Omega_e \in T_h} (\nabla \xi, D \nabla e_c)_{\Omega_e} + \sum_{F \in F^{\mathrm{i}}} \langle D^T \nabla \xi, [e_c] \rangle_F \\
&+ \sum_{F \in F^\partial_{\text{out}}} \langle D^T \nabla \xi \cdot n, e_c \rangle_F.
\end{aligned}
$$

Let ξ^* be arbitrary in $V_h \cap C^0(\overline{\Omega})$. Using Galerkin orthogonality, we obtain

$$
\begin{aligned}
\Psi(e_c) = &- \int_0^T (\phi \partial_t e_c, \xi - \xi^*)_\Omega \, \mathrm{d}t - (R_{t=0}, (\xi - \xi^*)(0,\cdot))_\Omega \\
&+ \int_0^T \sum_{\Omega_e \in T_h} (u e_c - D \nabla e_c, \nabla(\xi - \xi^*))_{\Omega_e} \, \mathrm{d}t \\
&+ \int_0^T \Big(\sum_{F \in F^{\mathrm{i}}} \langle D^T \nabla \xi, [e_c] \rangle_F + \sum_{F \in F^\partial_{\text{out}}} \langle D^T \nabla \xi \cdot n, e_c \rangle_F \Big) \, \mathrm{d}t \\
&+ \int_0^T \Big(\sum_{F \in F^{\mathrm{i}}} \langle \{ D \nabla \xi^* \}, [e_c] \rangle_F + \sum_{F \in F^\partial_{\text{out}}} \langle u e_c \cdot n, \xi^* \rangle_F \Big) \, \mathrm{d}t.
\end{aligned}
$$

Integrate by parts the term in the second line of the above equation to infer

$$
\begin{aligned}
&\sum_{\Omega_e \in T_h} (u e_c - D \nabla \mathrm{e}_c, \nabla(\xi - \xi^*))_{\Omega_e} = - \sum_{\Omega_e \in T_h} (\nabla \cdot (u e_c - D \nabla \mathrm{e}_c), \xi - \xi^*)_{\Omega_e} \\
&+ \sum_{F \in F^{\mathrm{i}}} \langle [u e_c - D \nabla e_c], \xi - \xi^* \rangle_F + \sum_{F \in F^\partial_{\text{in}} \cup F^\partial_{\text{out}}} \langle (u e_c - D \nabla e_c) \cdot n, \xi - \xi^* \rangle_F.
\end{aligned}
$$

Using (9), we readily deduce the error representation formula (12). □

From the error representation formula (12), it is possible to infer a residual-based a posteriori error estimate where the dual solution has been eliminated using the Cauchy–Schwarz inequality, local approximation properties of the finite element space V_h, and a global stability result for the dual problem. Set $\psi_1(e_c) = e_c$ and $\psi_2(e_c) = 0$ so that $\Psi(e_c) = \|e_c\|^2_{L^2_t(L^2_x)}$. Assume that the resulting dual problem (8)–(11) satisfies the stability estimate

$$\max_{0\le t\le T} \|\xi(\cdot,t)\|_{\Omega}^2 + \int_0^T \|\xi\|_{H^2(\Omega)}^2 \,\mathrm{d}t \le C \int_0^T \|e_c\|_{L^2(\Omega)}^2 \,\mathrm{d}t. \tag{17}$$

Furthermore, assume that the following approximation properties proven in [19] for $d = 2$ also hold for $d = 3$. For element Ω_e in T_h and $\phi \in H^s(\Omega_e)$, there exists a constant C depending on s but independent of ϕ, k, and element diameter h_e and a sequence $\phi_h^* \in \mathbb{P}^k(\Omega_e)$, such that for $0 \le q \le s$ and for $\mu = \min(k+1, s)$,

$$\|\phi - \phi_h^*\|_{H^q(\Omega_e)} \le C \frac{h_e^{\mu-q}}{k^{s-q}} \|\phi\|_{H^s(\Omega_e)} \qquad s \ge 0, \tag{18}$$

$$\|\phi - \phi_h^*\|_{H^r(\partial\Omega_e)} \le C \frac{h_e^{\mu-r-1/2}}{k^{s-r-1/2}} \|\phi\|_{H^s(\Omega_e)} \quad s > \frac{1}{2} + \delta,\ \delta = 0, 1. \tag{19}$$

Corollary 2.1 *With the above assumptions, an $L_t^2(L_x^2)$ a posteriori error estimate holds for the formulation (5) of the form*

$$\|e_c\|_{L_t^2(L_x^2)}^2 \le C \int_0^T \sum_{\Omega_e \in T_h} \eta_e^2 \,\mathrm{d}t, \tag{20}$$

with elementwise error indicators

$$\begin{aligned}
\eta_e^2 &= \frac{h_e^4}{k^4} \|R_{\mathrm{eqn}}\|_{L^2(\Omega_e)}^2 + \frac{h_e^4}{k^4} \|R_{t=0}\|_{L^2(\Omega_e)}^2 \\
&+ \sum_{F \in \partial\Omega_e \cap \Omega} \left(\frac{\tilde{h}_e^3}{k^3} \|R_{[D\nabla c_h]} + u \cdot R_{[c_h]}\|_{L^2(F)}^2 + \frac{\tilde{h}_e^2}{k^2} \|D\|_{L_\infty}^2 \|R_{[c_h]}\|_{L^2(F)}^2 \right. \\
&\left. + \|D\|_{L^\infty}^2 \|R_{[c_h]}\|_{L^2(F)}^2 \right) + \sum_{F \in \partial\Omega_e \cap \partial\Omega_{\mathrm{in}}} \frac{\tilde{h}_e^3}{k^3} \|R_{\mathrm{in}}\|_{L^2(F)}^2 + \sum_{F \in \partial\Omega_e \cap \partial\Omega_{\mathrm{out}}} \frac{\tilde{h}_e^3}{k^3} \|R_{\mathrm{out}}\|_{L^2(F)}^2,
\end{aligned} \tag{21}$$

for $\tilde{h}_e$ the maximal element diameter over all elements with the common face F and C a constant independent of h_F.

Proof. Use the error representation formula (12) together with the stability estimate (17) and the approximation results (18)–(19). The only term requiring special attention is the first one in the third line of (12). Writing $D^T\nabla\xi + D\nabla\xi^* = (D + D^T)\nabla\xi - D\nabla(\xi - \xi^*)$, the second term yields the last term in the second line of (21). The quantity $\|\nabla\xi\|_F$ is estimated by $\|\xi\|_{H^2(\Omega_e)}$ where Ω_e is an element to which F belongs. Using the stability estimate (17), this yields the first term in the third line of (21). □

Convergence orders of the various contributions to (21) are assessed numerically in Section 4. Note also that with some additional algebra, it is possible to derive an error representation formula and an a posteriori error estimate when the diffusion tensor is discontinuous across mesh interfaces.

3 Local Discontinuous Galerkin (LDG)

The LDG method consists of seeking c_h, $\tilde{z}_h$, $z_h \in C_t^1(V_h)$ such that $\forall v \in V_h$, $\forall w \in [V_h]^d$, $\forall \tilde{w} \in [V_h]^d$, and $\forall t \geq 0$,

$$\begin{aligned}(\phi \partial_t c_h, v)_\Omega - \sum_{\Omega_e \in T_h} (u c_h + z_h, \nabla v)_{\Omega_e} + \sum_{F \in F^{\mathrm{i}}} \langle u c_h^{\uparrow} + \{z_h\}, [v] \rangle_F \\ + \sum_{F \in F^{\partial}_{\mathrm{out}}} \langle u c_h \cdot n, v \rangle_F = (\phi f, v)_\Omega - (u \hat{g} \cdot n, v)_{\partial \Omega_{\mathrm{in}}},\end{aligned} \tag{22}$$

$$(\tilde{z}_h, w)_\Omega - \sum_{\Omega_e \in T_h} (c_h, \nabla \cdot w)_{\Omega_e} + \sum_{F \in F^{\mathrm{i}}} \langle \{c_h\}, [w] \rangle_F + (c_h, w \cdot n)_{\partial\Omega} = 0, \tag{23}$$

$$(D \tilde{z}_h, \tilde{w})_\Omega - (z_h, \tilde{w})_\Omega = 0, \tag{24}$$

with the same initial condition as before. Again, let ξ satisfy the dual problem (8)–(11).

Theorem 3.1 *Assume that the solution c to (1)–(4) and the solution ξ to (8)–(11) are both in $L_t^2(H_x^2) \cap C_t^0(L_x^2)$. Assume the diffusion/dispersion tensor D to be continuous and piecewise-linear in space. Then,*

$$\begin{aligned}\Psi(e_c) = &- \int_0^T (R_{\mathrm{eqn}}, \xi - \xi^*)_\Omega \, \mathrm{d}t - (R_{t=0}, (\xi - \xi^*)(0, \cdot))_\Omega \\ &+ \int_0^T \Big(\sum_{F \in F^{\partial}_{\mathrm{in}}} \langle R_{\mathrm{in}}, \xi - \xi^* \rangle_F + \sum_{F \in F^{\partial}_{\mathrm{out}}} \langle R_{\mathrm{out}}, \xi - \xi^* \rangle_F \Big) \, \mathrm{d}t \\ &+ \int_0^T \sum_{F \in F^{\mathrm{i}}} \Big(\langle R_{[c_h]}, D^T \nabla \xi \rangle_F - \langle R_{[D \nabla c_h]} - u \cdot R_{[c_h]}, \xi - \xi^* \rangle_F \Big) \, \mathrm{d}t,\end{aligned} \tag{25}$$

where ξ^* is arbitrary in $V_h \cap C^0(\overline{\Omega})$ and where the residuals R_{eqn}, $R_{[c_h]}$, $R_{[D\nabla c_h]}$, R_{in}, R_{out}, and $R_{t=0}$ are defined in (13)–(16).

Proof. The proof is similar to that of Theorem 2.1. The main difference is that Galerkin orthogonality now yields

$$\begin{aligned}\Psi(e_c) = &- \int_0^T (\phi \partial_t e_c, \xi - \xi^*)_\Omega \, \mathrm{d}t - (R_{t=0}, (\xi - \xi^*)(0, \cdot))_\Omega \\ &+ \int_0^T \Big(\sum_{\Omega_e \in T_h} (u e_c, \nabla(\xi - \xi^*))_{\Omega_e} - (D \nabla e_c, \nabla \xi)_{\Omega_e} - (e_z, \nabla \xi^*)_{\Omega_e} \Big) \, \mathrm{d}t \\ &+ \int_0^T \Big(\sum_{F \in F^{\mathrm{i}}} \langle D^T \nabla \xi, [e_c] \rangle_F + \sum_{F \in F^{\partial}_{\mathrm{out}}} \langle D^T \nabla \xi \cdot n, e_c \rangle_F + \sum_{F \in F^{\partial}_{\mathrm{out}}} \langle u e_c \cdot n, \xi^* \rangle_F \Big) \, \mathrm{d}t,\end{aligned}$$

where $e_z = z - z_h$ and $z = -D \nabla c$. Similarly, set $e_{\tilde{z}} = \tilde{z} - \tilde{z}_h$ where $\tilde{z} = -\nabla c$. Owing to the assumption on D, we infer $D^T \nabla \xi^* \in [V_h]^d$ and hence

$$(e_z, \nabla \xi^*)_{\Omega_e} = (e_{\tilde{z}}, D^T \nabla \xi^*)_{\Omega_e} = -(\nabla e_c, D^T \nabla \xi^*)_{\Omega_e} = -(D \nabla e_c, \nabla \xi^*)_{\Omega_e},$$

whence the error representation formula (25) readily follows. $\square$

Corollary 3.1 *With the assumptions of Corollary 2.1, an $L^2_t(L^2_x)$ a posteriori error estimate holds for the formulation (22)–(24) of the form (20) with elementwise error indicators given by (21).*

4 Numerical Results

In this section we present numerical results to illustrate the convergence order of the various terms in the a posteriori error estimates. For the sake of brevity, we consider the NIPG error estimates (21). As a model problem, consider a 1D convection-diffusion equation posed over domain $\Omega = (0, 4\pi)$ with initial data $u_0(x) = \sin(x)$, source term $f = 0$, inflow data $\hat{g}(t) = -e^{-Dt}\sin(ut)$, diffusion coefficient $D = 1$, and advection velocity $u = 1$. The simulation time is set to $T = 0.5$. Since the diffusion length scale can be estimated as $\delta = (DT)^{\frac{1}{2}} = 0.7$, we infer that the restriction of the solution to the interval $(0, 2\pi)$ is approximately given by $c(t, x) = e^{-Dt}\sin(x - ut)$.

Numerical experiments are performed on two series of meshes: a series of uniform meshes with step size $h = 2^{-p}\frac{\pi}{8}$ $(0 \le p \le 3)$ and a series of non-uniform meshes which are constructed from the uniform meshes by setting the step size alternatively to $\frac{h}{2}$ and $\frac{3h}{2}$ for adjacent cells. Problem (5) is discretized in time using an explicit Euler method and a time step of 2.5×10^{-5}. Results are presented in Tables 1 and 2. We evaluate the quantities

$$T_1 = \left(\sum_{x_j \in (0,2\pi)} [c_h(x_j)]^2 \right)^{\frac{1}{2}}, \quad T_2 = \left(\sum_{x_j \in (0,2\pi)} [c'_h(x_j)]^2 \right)^{\frac{1}{2}},$$

$$T_3 = R_{in}(t = T, x = 0), \qquad T_4 = \left(\sum_{\Omega_e \in (0,2\pi)} \|R_{t=0}\|^2_{L^2(\Omega_e)} \right)^{\frac{1}{2}},$$

where the x_j's denotes the mesh vertices. On the uniform meshes, superconvergence is obtained so that the upper bound in (21) scales as h^2. On the non-uniform mesh, the first term in the third equation of (21) dominates the upper bound, yielding a convergence order of $h^{\frac{3}{2}}$ approximately. This estimate is compatible with standard a priori estimates for convection-diffusion equations.

p	T_1	order	T_2	order	T_3	order	T_4	order
0	2.98e-2	–	7.73e-1	–	2.53e-2	–	7.18e-3	–
1	1.79e-2	2.2	5.51e-1	.49	5.97e-3	2.1	1.80e-3	2.0
2	3.36e-3	2.4	3.90e-1	.50	1.22e-3	2.3	4.50e-3	2.0
3	6.07e-4	2.5	2.76e-1	.50	2.03e-4	2.6	1.13e-3	2.0

Table 1
Convergence tests on uniform meshes

p	T_1	order	T_2	order	T_3	order	T_4	order
0	4.94e-2	–	7.84e-1	–	5.78e-2	–	1.40e-2	–
1	1.95e-2	1.3	5.54e-1	.49	1.37e-2	2.1	3.51e-3	2.0
2	7.82e-3	1.3	3.92e-1	.50	3.30e-3	2.1	8.79e-4	2.0
3	1.04e-3	1.4	2.78e-1	.50	7.29e-4	2.2	2.20e-4	2.0

Table 2
Convergence tests on non-uniform meshes

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