

DISLOCATION DYNAMICS WITH A MEAN CURVATURE TERM: SHORT TIME EXISTENCE AND UNIQUENESS

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Abstract In this paper, we study a new model for dislocation dynamics with a mean curvature term. We prove a short time existence and uniqueness result for our model. We also prove a Lipschitz a priori estimate for a class of second order operator and an estimate of the modulus of continuity in time of the solution by using a regularization of the initial condition. As far as we know, these two ideas, which are crucial for our argument to work, are new.

Keywords: Dislocation dynamics, non-local equations, mean curvature motion, viscosity solutions.

1 Introduction

1 Physical motivation

Plastic deformation is mainly due to the movement of linear defects called dislocations, whose typical length in metallic alloys is of the order of $10^{-6}m$ and thickness of the order of $10^{-9}m$. In the face centered cubic structure, dislocations move in well defined crystallographic planes. Since the beginning of the 90's, the research field of dislocation is enjoying a new boom, in particular thanks to the power of computers which now makes possible to simulate dislocations in a 3D domain. In certain models in 3D, dislocations are described, for example, by the interconnection of straight dislocation segments, with only a few given orientations (see for example Devincere, Kubin [18]).

More recently, a new approach has been introduced: *phase field model of dislocations* (see for example Rodney, Le Bouar, Finel [26] and Garroni, Müller [22]). One of the advantages of this method is that the possible topological changes during the dislocation movement are automatically taken into account. For a more complete introduction on dislocations and for

more references, we refer to Alvarez, Hoch, Le Bouar, Monneau [3] and [4].

Here, we consider a new model which contains a mean curvature term. In order to model the movement of a dislocation γ in its crystallographic plane, we assume that Γ is the edge of a smooth bounded set $\Omega \subset \mathbb{R}^2$ and we define:

$$\rho = \begin{cases} 1 & \text{in } \Omega, \\ 0 & \text{in } \mathbb{R}^2/\Omega. \end{cases} \quad (1)$$

Then, we consider the energy associated to this dislocation of the form:

$$\mathcal{E} = \int_{\mathbb{R}^2} -\frac{1}{2}(c_0 \star \rho)\rho + \int_{\Gamma} \gamma(\vec{n}),$$

where \vec{n} is the normal to the curve, the kernel c_0 depends only on the space variables, \star denotes the convolution in space and γ is an energy of tension line (see Remark 1.1 for an explicit example of c_0 and γ). In the following section, we derive in a heuristic way the displacement velocity of the curve Γ in the normal direction by computing the first variation of the energy. This gives a velocity $c = (c_0 \star \rho) + \lambda(\vec{n})\kappa$, where κ is the mean curvature, $\lambda = \gamma + \gamma''$ and $\frac{\partial \rho}{\partial t} = c|D\rho|$. We can then reformulate the problem by a ‘‘level set’’ equation on the set $\{u \leq 0\}$ of a smooth function u which then satisfies:

$$u_t = (c_0 \star [u] + \lambda(\vec{n})\kappa)|Du|,$$

with:

$$\begin{cases} \rho = [u] = \begin{cases} 1 & \text{if } u > 0 \\ 0 & \text{if } u \leq 0 \end{cases}, \\ \vec{n} = \frac{Du}{|Du|}, \\ \kappa = \operatorname{div} \left(\frac{Du}{|Du|} \right). \end{cases} \quad (2)$$

2 Description of the problem

We describe the model in an heuristic way. Let us consider an orthonormal basis (e_1, e_2, e_3) of \mathbb{R}^3 and we denote the coordinates by $x = (x_1, x_2, x_3)$. The energy of the dislocation along the line is singular. To solve this problem, Brown [11], [12] then Barnett [9] and Gavazza, Barnett [10] propose to surround the dislocation Γ by a tube T_ϵ of size ϵ and to consider the energy of the form:

$$\mathcal{E} = \int_{\mathbb{R}^3 \setminus T_\epsilon} \frac{1}{2} \Lambda e^{class} \cdot e^{class} + \int_{T_\epsilon} \gamma_0(\vec{n}), \quad (3)$$

where Λ represents the elastic coefficients, γ_0 is an energy of tension line and e^{class} is a deformation and is solution of:

$$\begin{cases} \operatorname{div}(\Lambda e^{class}) = 0, \\ \operatorname{inc}(e^{class}) = -\operatorname{inc}(\rho \delta_0(x_3) e^0) \text{ where } e^0 = \frac{1}{2}(b \otimes e_3 + e_3 \otimes b), \\ e^{class}(x) \rightarrow 0 \text{ when } |x| \rightarrow \infty. \end{cases} \quad (4)$$

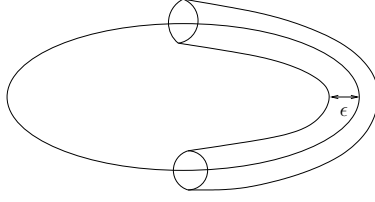


Figure 1: The cutoff tube of radius ϵ .

Here, Γ belongs to the plane (e_1, e_2) , the vector e_3 is the vector normal to the plane, $b \in \mathbb{R}^3$ is a constant vector (called Burgers vector) associated to the dislocation line and the operator of incompatibility inc is defined for a field $e = (e_{ij}) \in S^3$, the set of symmetric 3×3 matrix, by:

$$(\text{inc}(e))_{ij} = \sum_{i_1, j_1=1}^3 \varepsilon_{i i_1 i_2} \varepsilon_{j j_1 j_2} \partial_{i_1} \partial_{j_1} e_{i_2 j_2}, \quad i, j = 1, 2, 3$$

where we note as usual

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } (ijk) \text{ is a positive permutation of } (123), \\ -1 & \text{if } (ijk) \text{ is a negative permutation of } (123), \\ 0 & \text{if two indices are the same.} \end{cases}$$

The solution e^{class} of (4) satisfies $e^{\text{class}} \sim \frac{1}{r}$, for r small, where r is the distance to the dislocation (cf Alvarez et al. [3] for a description of the physical model). Finally, the cutoff tube is represented by the figure 1. We consider an approximate model of this one where the field e is given by $e = \chi \star e^{\text{class}}$, with χ a regularizing core function (connected to ϵ) to be adjusted, and the energy (3) is replaced by the following energy:

$$\mathcal{E} = \int_{\mathbb{R}^3} \frac{1}{2} \Lambda e \cdot e + \eta \int_{\Gamma} \gamma_0(\vec{n}),$$

where η is to be adjusted and connected to ϵ . We set $\gamma(\vec{n}) = \eta \gamma_0(\vec{n})$. In order to compute the dislocation dynamics, we compute the first variation of the energy (see Alvarez et al. [3]). We defined $\Gamma_\delta(s) = \Gamma(s) + \delta h(s) \cdot \vec{n}_\Gamma(s)$. Then, the following holds

$$-\left. \frac{d\mathcal{E}(\Gamma_\delta)}{d\delta} \right|_{\delta=0} = \int_{\Gamma} c \cdot h,$$

with $c = c_0 \star \rho + \lambda(\vec{n}) \kappa$, where $c_0 = c_0(x_1, x_2)$ only depends on Λ and χ , $\lambda(\vec{n}) = (\gamma(\vec{n}) + \gamma''(\vec{n}))$. Thus, the evolution is postulated to be $\frac{\partial \rho}{\partial t} = c |D\rho|$ with ρ defined in (1) and $\Gamma = \partial\Omega$.

REMARK 1.1 (Explicit example for c_0 and γ)

If $b = |b|e_1$, then, for the isotropic elasticity, one can give the value of c_0 :

$$\hat{c}_0(\xi) = \frac{-\mu b^2}{2} e^{-\zeta \sqrt{\xi_1^2 + \xi_2^2}} \left(\frac{\xi_1^2 + \frac{1}{1-\nu} \xi_2^2}{\sqrt{\xi_1^2 + \xi_2^2}} \right)$$

and the form of γ :

$$\gamma(\vec{n}) = C \left(n_1^2 + \frac{1}{1-\nu} n_2^2 \right),$$

where $\vec{n} = (n_1, n_2)$ is the normal vector to the curve, C is a prefactor (which depends on the Burgers vector and elasticity coefficients), $\zeta > 0$ is a physical parameter (depending on the material), ν is the Poisson ratio and μ is the Lamé coefficient. We refer to Alvarez et al. [3] section 6 for the expression of c_0 and to Hirth, Lothe [23] chapter 6 and 7 for the form of γ .

REMARK 1.2 Formally, we have:

$$\frac{d\mathcal{E}}{dt} = \int_{\Gamma} -c^2 \leq 0.$$

The model we study is pertinent with respect to the one studying in Alvarez et al. [3], since we add a mean curvature term which better approximates the energy near the dislocation. We found this term in Gavazza, Barnett [10] and Brown [11].

3 Main result

The goal of the paper is to prove short time existence and uniqueness of the function u . Since the Hamiltonian intervening in the equation is not continuous (and even not defined when $|Du|$ is zero), a natural framework for the study is the theory of viscosity solutions (for a good introduction to this theory, we refer to Barles [6], [7], Crandall, Ishii, Lions [15], Crandall, Lions [16], [17], Ishii [24] and Ishii, Lions [25] and for an introduction to viscosity solution for evolving fronts, we refer to Ambrosio [5], Barles, Soner, Souganidis [8], Chen, Giga, Goto [13], Evans [20], Evans, Spruck [21] and Souganidis [28]). We consider the following problem : find $u(x, t)$ solution of

$$\begin{cases} u_t = (c_0 \star [u])|Du| - F(Du, D^2u) \text{ in } \mathbb{R}^n \times (0, T), \\ u(x, t = 0) = u_0(x) \text{ in } \mathbb{R}^n, \end{cases} \quad (5)$$

where $[u]$ is the characteristic function of the set $\{u > 0\}$ (see (2)). Moreover, we assume

$$c_0 \in L_{\text{int}}^{\infty}(\mathbb{R}^n) \cap BV(\mathbb{R}^n), \quad (6)$$

where $BV(\mathbb{R}^n)$ is the space of bounded variations functions and

$$L_{\text{int}}^{\infty}(\mathbb{R}^n) = \{f : \mathbb{R}^n \rightarrow \mathbb{R} : \|f\|_{L_{\text{int}}^{\infty}(\mathbb{R}^n)} < \infty\}$$

with

$$\|f\|_{L_{\text{int}}^{\infty}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \|f\|_{L^{\infty}(Q(x))}$$

and $Q(x)$ is the unit square centered at x :

$$Q(x) = \left\{ x' \in \mathbb{R}^n : |x_i - x'_i| \leq \frac{1}{2} \right\}.$$

The assumptions (HF) on the operator F are the next ones:

(i) The operator F is elliptic, ie, $\forall X, Y \in S^n, \forall p \in \mathbb{R}^n$,

$$\text{if } X \leq Y \text{ then } F(p, X) \geq F(p, Y), \quad (7)$$

where S^n (the set of symmetric $n \times n$ matrix) is equipped with its natural order.

(ii) F is locally bounded on $\mathbb{R}^n \times S^n$, continuous on $\mathbb{R}^n \setminus \{0\} \times S^n$ and

$$F^*(0, 0) = F_*(0, 0) = 0, \quad (8)$$

where F^* (resp. F_*) is the upper-semicontinuous (usc) envelope (resp. lower semicontinuous (lsc) envelope) of F , ie the smallest usc function $\geq F$ (resp. the greatest lsc function $\leq F$).

(iii) F is geometric, ie

$$F(\nu p, \nu A + \mu p \otimes p) = \nu F(p, A), \quad \forall \nu > 0, \mu \in \mathbb{R}, A \in S^n. \quad (9)$$

The main result is:

THEOREM 1.3 (Short time existence and uniqueness)

Let $u_0 : \mathbb{R}^n \rightarrow \mathbb{R}$ be a lipshitz continuous function on \mathbb{R}^n such that

$$|Du_0| < B_0 \quad \text{in } \mathbb{R}^n \quad (10)$$

and

$$\frac{\partial u_0}{\partial x_n} > b_0 > 0 \quad \text{in } \mathbb{R}^n. \quad (11)$$

Let c_0 satisfying (6). Then, under the assumptions (HF), there exists

$$T^* = \inf \left\{ \frac{1}{|c_0|_{BV(\mathbb{R}^n)}} \ln \left(1 + \frac{b_0}{2B_0} \right), \frac{b_0}{B_0} \frac{1}{16\|c_0\|_{L^\infty_{\text{int}}(\mathbb{R}^n)}}, \frac{1}{|c_0|_{BV(\mathbb{R}^n)}} \ln \left(1 + \frac{b_0}{B_0} \frac{|c_0|_{BV(\mathbb{R}^n)}}{8\|c_0\|_{L^\infty_{\text{int}}}} \right) \right\}$$

such that there exists a unique viscosity solution of the problem (5) in $\mathbb{R}^n \times [0, T^*]$. Moreover, the solution satisfies

$$|Du(x, t)| < 2B_0 \quad \text{on } \mathbb{R}^n \times [0, T^*], \quad (12)$$

$$\frac{\partial u}{\partial x_n}(x, t) > b_0/2 > 0 \quad \text{on } \mathbb{R}^n \times [0, T^*]. \quad (13)$$

REMARK 1.4 This theorem gives, in particular, in the case of dimension two, and for

$$F(p, X) = -\text{tr} \left(\left(I - \frac{p \otimes p}{|p|^2} \right) X \right) \left(\lambda \left(\frac{p}{|p|} \right) \right), \quad (14)$$

with $\lambda > 0$ and smooth, short time existence and uniqueness for dislocation dynamics with a mean curvature term.

REMARK 1.5 *A short time existence and uniqueness result for the problem:*

$$\begin{cases} u_t = (c_0 \star [u] + c_1)|Du| \text{ in } \mathbb{R}^2 \times (0, T), \\ u(x, t = 0) = u_0(x) \text{ in } \mathbb{R}^2, \end{cases}$$

is proved by Alvarez, Hoch, Le Bouar, Monneau [3], [4] and Alvarez, Carlini, Monneau, Rouy [2] and a long time existence and uniqueness result for positive velocity is proved by Alvarez, Cardaliaguet, Monneau [1]. Our problem is more general than this one since we add a second order term.

For the proof of this theorem, we will use a fix point method. To do that, we will need to prove a lipschitz a priori estimate for second order operator of the form $H(x, t, Du) + F(Du, D^2u)$ with smooth initial condition. We will also prove an estimate of the modulus of continuity in time of the solution by using a regularization of the initial condition. As far as we know, these two ideas, which are crucial for our argument to work, are new.

4 Organization of the article

In section 2, we give some preliminary results on a local problem. First, in section 2.1, we recall the definition of viscosity solutions and we give an existence and uniqueness result for the local problem. Then, in section 2.2, we give some results on the regularity of the solution of the local problem. In section 3, we prove Theorem 1.3 for the non-local problem. Finally, we give, in appendix, the proof of the parabolic Ishii Lemma used in section 2.

2 Preliminary results for a local problem

Given $T > 0$, we consider the following problem:

$$\begin{cases} u_t + G(x, t, Du, D^2u) = 0 \text{ in } (0, T) \times \mathbb{R}^n, \\ u(x, t = 0) = u_0(x) \text{ in } \mathbb{R}^n, \end{cases} \quad (15)$$

with the following assumptions (H_0):

- (i) $G(x, t, p, X) = -c(x, t)|p| + F(p, X)$ and F satisfies the assumptions (HF),
- (ii) $c : \mathbb{R}^n \times (0, T) \rightarrow \mathbb{R}$ is bounded, lipschitz in space (we note L_c its lipschitz constant) and uniformly continuous in time (we note ω_c its modulus of continuity, defined by: $\forall x \in \mathbb{R}^n, \forall s, t \in [0, T), |c(x, t) - c(x, s)| \leq \omega_c(|t - s|)$),
- (iii) u_0 is lipschitz (we note B_0 its lipschitz constant).

1 Existence and uniqueness for the problem (15)

We define the following sets:

$$USC(\mathbb{R}^n \times [0, T)) = \{u : \mathbb{R}^n \times [0, T) \rightarrow \mathbb{R} \text{ locally bounded, upper semicontinuous}\}$$

$$LSC(\mathbb{R}^n \times [0, T)) = \{u : \mathbb{R}^n \times [0, T) \rightarrow \mathbb{R} \text{ locally bounded, lower semicontinuous}\}$$

We then define the solutions of (15) in the following way:

DEFINITION 2.1 (Viscosity subsolution, supersolution and solution)

A function $u \in USC(\mathbb{R}^n \times [0, T))$ is a viscosity subsolution of (15) if it satisfies:

(i) $u(x, t = 0) \leq u_0(x)$ in \mathbb{R}^n ,

(ii) for every $(x_0, t_0) \in \mathbb{R}^n \times (0, T)$ and for every test function $\Phi : (\mathbb{R}^n \times (0, T)) \rightarrow \mathbb{R}$, C^1 in time and C^2 in space, that is tangent from above to u at (x_0, t_0) , the following holds:

$$\frac{\partial \Phi}{\partial t}(x_0, t_0) + G_*(x_0, t_0, D\Phi, D^2\Phi) \leq 0.$$

A function $v \in LSC(\mathbb{R}^n \times [0, T))$ is a viscosity supersolution of (15) if it satisfies:

(i) $v(x, t = 0) \geq u_0(x)$ in \mathbb{R}^n ,

(ii) for every $(x_0, t_0) \in \mathbb{R}^n \times (0, T)$ and for every test function $\Phi : (\mathbb{R}^n \times [0, T)) \rightarrow \mathbb{R}$, C^1 in time and C^2 in space, that is tangent from below to v at (x_0, t_0) , the following holds:

$$\frac{\partial \Phi}{\partial t}(x_0, t_0) + G^*(x_0, t_0, D\Phi, D^2\Phi) \geq 0.$$

A function $u \in C^0(\mathbb{R}^n \times [0, T))$ is a viscosity solution of (15) if, and only if, it is a sub and a supersolution of (15).

REMARK 2.2 The condition ϕ is C^1 in time and C^2 in space means that ϕ is differentiable in time, twice differentiable in space and $\phi, \phi_t, D_x\phi, D_x^2\phi$ are continuous in space and time.

We give another equivalent definition. In order to do that, we define the parabolic sub and superdifferential. If $u : \mathbb{R}^n \times (0, T) \rightarrow \mathbb{R}$, then \mathcal{P}^+u is defined by $(a, p, X) \in \mathbb{R} \times \mathbb{R}^n \times S^n$ belongs to $\mathcal{P}^+u(x, t)$ if $(x, t) \in \mathbb{R} \times (0, T)$ and

$$u(y, s) \leq u(x, t) + a(s - t) + \langle p, y - x \rangle + \frac{1}{2} \langle X(y - x), y - x \rangle + o(|s - t| + |y - x|^2)$$

as $\mathbb{R} \times (0, T) \ni (y, s) \rightarrow (x, t)$. Similarly, $\mathcal{P}^-u = -\mathcal{P}^+(-u)$. We also define the two following sets:

$$\bar{\mathcal{P}}^+u(x, t) = \left\{ \begin{array}{l} (a, p, X) \in \mathbb{R} \times \mathbb{R}^n \times S^n, \exists (x_n, t_n, a_n, p_n, X_n) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times S^n \\ \text{such that } (a_n, p_n, X_n) \in \mathcal{P}^+u(x_n, t_n) \\ \text{and } (x_n, t_n, u(x_n, t_n), a_n, p_n, X_n) \rightarrow (x, t, u(x, t), a, p, X) \end{array} \right\}$$

The set $\bar{\mathcal{P}}^-u(x, t)$ is defined in a similar way. We then have the following definition for the solutions of (15), which is equivalent to the definition 2.1 (see Crandall et al. [15]):

DEFINITION 2.3 (Equivalent definition for viscosity solutions)

A function $u \in USC(\mathbb{R}^n \times [0, T])$ is a viscosity subsolution of (15) if it satisfies:

(i) $u(x, t = 0) \leq u_0(x)$ in \mathbb{R}^n ,

(ii) for every $(x, t) \in \mathbb{R}^n \times (0, T)$ and for every $(a, p, X) \in \mathcal{P}^+u(x, t)$, we have:

$$a + G_*(x, t, p, X) \leq 0.$$

A function $v \in LSC(\mathbb{R}^n \times [0, T])$ is a viscosity supersolution of (15) if it satisfies:

(i) $v(x, t = 0) \geq u_0(x)$ in \mathbb{R}^n ,

(ii) for every $(x, t) \in \mathbb{R}^n \times (0, T)$ and for every $(a, p, X) \in \mathcal{P}^-v(x, t)$, we have:

$$a + G_*(x, t, p, X) \geq 0.$$

A function $u \in C^0(\mathbb{R}^n \times [0, T])$ is a viscosity solution of (15) if, and only if, it is a sub and a supersolution of (15).

Assumption (C) We say that an usc function w satisfies the compactness assumption (C) if for every $(z, s) \in \mathbb{R}^n \times \mathbb{R}_*^+$, there exists $r > 0$ such that, for every $M > 0$, there exists C such that,

$$\left. \begin{array}{l} |(x, t) - (z, s)| \leq r \\ (\tau, p, X) \in \bar{\mathcal{P}}^+w(x, t) \\ |w(x, t)| + |p| + |X| \leq M \end{array} \right\} \Rightarrow \tau \leq C.$$

We recall the parabolic Ishii lemma, proved in appendix (section 4):

LEMMA 2.4 (Parabolic Ishii)

Let U and V be open sets of \mathbb{R}^n , $u \in USC(U \times \mathbb{R}^+)$ and $v \in LSC(V \times \mathbb{R}^+)$. Let $\phi : U \times V \times \mathbb{R}^+ \rightarrow \mathbb{R}$ of class C^2 . Assume that $(x, y, t) \rightarrow u(x, t) - v(y, t) - \phi(x, y, t)$ reaches a local maximum in $(\bar{x}, \bar{y}, \bar{t}) \in U \times V \times \mathbb{R}_*^+$. We note $\tau = \partial_t \phi(\bar{x}, \bar{y}, \bar{t})$, $p_1 = D_x \phi(\bar{x}, \bar{y}, \bar{t})$, $p_2 = -D_y \phi(\bar{x}, \bar{y}, \bar{t})$ and $A = D^2 \phi(\bar{x}, \bar{y}, \bar{t})$. Assume also that u and $-v$ satisfy the assumption (C). Then, for every $\alpha > 0$ such that $\alpha A < I$, there exists $\tau_1, \tau_2 \in \mathbb{R}$ and $X, Y \in S^n$ such that:

$$\begin{aligned} & \tau = \tau_1 - \tau_2, \\ & (\tau_1, p_1, X) \in \bar{\mathcal{P}}^+u(\bar{x}, \bar{t}), \quad (\tau_2, p_2, Y) \in \bar{\mathcal{P}}^-v(\bar{y}, \bar{t}), \\ & \frac{-1}{\alpha} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq (I - \alpha A)^{-1} A. \end{aligned}$$

REMARK 2.5 The assumption (C) is satisfied by u and $-v$ as soon as u is a subsolution and v is a supersolution of a parabolic equation.

We also recall the fundamental property of geometric equations:

LEMMA 2.6 (Fundamental property of geometric equations)

Let $\theta : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous, non decreasing scalar function and u be a subsolution (respectively a supersolution) of (15), then $\theta(u)$ is also a subsolution (resp. a supersolution).

For the proof of this Lemma, we refer to Soner [27] (Theorem 1.11).

We now prove the following comparison principle:

THEOREM 2.7 (Comparison principle)

Let $u \in USC(\mathbb{R}^n \times [0, T])$ be a subsolution and $v \in LSC(\mathbb{R}^n \times [0, T])$ be a supersolution of (15). Assume that $u_0(x) = u(0, x) \leq v(0, x) = v_0(x)$ in \mathbb{R}^n , then, under the assumptions (H_0) , $u \leq v$ in $\mathbb{R}^n \times [0, T]$.

Proof of theorem 2.7

Let us consider for the moment that u and v are bounded. Moreover, we remark that for $\gamma > 0$, $\tilde{u} = u - \frac{\gamma}{T-t}$ is a subsolution of (15) and satisfies, in the viscosity sense:

$$\tilde{u}_t + G_*(x, t, D\tilde{u}, D^2\tilde{u}) \leq \frac{-\gamma}{(T-t)^2} \leq \frac{-\gamma}{T^2}.$$

Indeed, $D\tilde{u} = Du$, $D^2\tilde{u} = D^2u$ and $\tilde{u}_t = u_t - \frac{\gamma}{(T-t)^2}$. So:

$$\tilde{u}_t + G_*(x, t, D\tilde{u}, D^2\tilde{u}) = -\frac{\gamma}{(T-t)^2} + u_t + G_*(x, t, Du, D^2u) \leq \frac{-\gamma}{(T-t)^2}.$$

This is written in a formal way but it is easy to show it by using test functions.

Since $u \leq v$ follows from $\tilde{u} \leq v$ in the limit $\gamma \rightarrow 0$, it will simply suffice to prove the comparison principle under the additional assumptions:

$$\begin{cases} u_t + G_*(x, t, Du, D^2u) \leq \frac{-\gamma}{T^2} < 0, \\ \lim_{t \rightarrow T} u(t, x) = -\infty. \end{cases} \quad (16)$$

We set:

$$M = \sup_{(x,t) \in \mathbb{R}^n \times [0, T]} \{u(x, t) - v(x, t)\}.$$

We want to show that $M \leq 0$. Assume $M > 0$. So, there exists (x^*, t^*) such that $u(x^*, t^*) - v(x^*, t^*) > 0$. Then, we duplicate the variables in space by considering:

$$\bar{M} = \sup_{(x,y,t) \in \mathbb{R}^n \times \mathbb{R}^n \times [0, T]} \left\{ u(x, t) - v(y, t) - \frac{|x-y|^4}{4\epsilon} - \frac{\alpha}{2} (|x|^2 + |y|^2) \right\}.$$

We remark that $\bar{M} \geq u(x^*, t^*) - v(x^*, t^*) - \alpha|x^*|^2$, and so $\bar{M} > 0$ for α small enough. Thanks to the term $\frac{\alpha}{2}(|x|^2 + |y|^2)$, this supremum is reached (because u and v are bounded). We then note $(\bar{x}, \bar{y}, \bar{t})$ a point of maximum. We will now use the following lemma:

LEMMA 2.8 (Passing to the limit in α and ϵ)

We set $M' = \lim_{h \rightarrow 0} \sup_{|y-x| \leq h, t \in [0, T]} (u(x, t) - v(y, t))$. Then, the following holds:

- (i) $\lim_{\alpha \rightarrow 0} \alpha \bar{x} = \lim_{\alpha \rightarrow 0} \alpha \bar{y} = 0$,
- (ii) $\lim_{\epsilon \rightarrow 0} |\bar{x} - \bar{y}|^4 = 0$,
- (iii) $\lim_{\epsilon \rightarrow 0} \lim_{\alpha \rightarrow 0} \bar{M} = M'$,
- (iv) $\lim_{\epsilon \rightarrow 0} \lim_{\alpha \rightarrow 0} \frac{1}{\epsilon} |\bar{x} - \bar{y}|^4 = 0$,
- (v) $\lim_{\epsilon \rightarrow 0} \lim_{\alpha \rightarrow 0} \alpha (|\bar{x}|^2 + |\bar{y}|^2) = 0$.

We finish the proof of the theorem before proving this lemma.

Then, we distinguish two cases:

1st case: $\forall \epsilon > 0, \exists \alpha \in (0, \epsilon)$ such that $\bar{t} = 0$. Then, there exists sequences $\epsilon_n \rightarrow 0$ and $\alpha_n \rightarrow 0$ such that $\bar{t}_n = 0$ and:

$$0 < \bar{M} \leq u(\bar{x}_n, 0) - v(\bar{y}_n, 0) \leq u_0(\bar{x}_n) - u_0(\bar{y}_n) \leq B_0(|\bar{x}_n - \bar{y}_n|),$$

where B_0 is the lipschitz constant of u_0 . We then obtain a contradiction because $|\bar{x}_n - \bar{y}_n| \rightarrow 0$ (see the item 2. of Lemma 2.8).

2nd case: Assume that there exists $\epsilon > 0$ such that for every $\alpha \in (0, \epsilon)$, we have $\bar{t} > 0$. We can choose ϵ small enough (else, we apply the argument of the first case), ie, such that:

$$\frac{|\bar{x} - \bar{y}|^4}{\epsilon} \leq \frac{\gamma}{2T^2 L_c}. \quad (17)$$

We set $\tilde{u}(x, t) = u(x, t) - \frac{\alpha}{2}|x|^2$ and $\tilde{v}(x, t) = v(x, t) + \frac{\alpha}{2}|x|^2$. Then, we have:

$$\bar{M} = \sup_{(x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times [0, T]} \left\{ \tilde{u}(x, t) - \tilde{v}(y, t) - \frac{|x - y|^4}{4\epsilon} \right\}.$$

Then, we consider the test function $\psi(x, y, t) = \frac{|x - y|^4}{4\epsilon}$ and we set $\bar{p} = \bar{x} - \bar{y}$. We use the parabolic Ishii Lemma. With the same notations, the following holds:

$$\tau = 0,$$

$$p_1 = \frac{|\bar{p}|^2 \bar{p}}{\epsilon} = p_2,$$

$$A = \frac{2}{\epsilon} |\bar{p}|^2 \begin{pmatrix} Z & -Z \\ -Z & Z \end{pmatrix}, \text{ avec } Z = \frac{I}{2} + \frac{\bar{p} \otimes \bar{p}}{|\bar{p}|^2}.$$

and for every β such that $\beta A < I$, there exists $X, Y \in S^n$ and two reals τ_1 and τ_2 such that:

$$\begin{aligned} \tau_1 - \tau_2 &= 0, \\ \left(\tau_1, \frac{|\bar{p}|^2 \bar{p}}{\epsilon}, X \right) &\in \bar{\mathcal{P}}^+ \tilde{u}(\bar{x}, \bar{t}), \\ \left(\tau_2, \frac{|\bar{p}|^2 \bar{p}}{\epsilon}, Y \right) &\in \bar{\mathcal{P}}^- \tilde{v}(\bar{y}, \bar{t}), \\ \frac{-1}{\beta} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} &\leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq (I - \beta A)^{-1} A. \end{aligned}$$

The last inequality implies that $X \leq Y$. We use the following lemma:

LEMMA 2.9 (Matrix estimates)

We have the following estimates on the matrix A

$$\begin{cases} \frac{1}{2\|A\|} A < I, \\ \|A\| \leq \frac{6|\bar{p}|^2}{\epsilon}, \\ \text{if } \delta = \frac{1}{2\|A\|}, \text{ then } (I - \delta A)^{-1} A \leq 2\|A\|I \leq \frac{12}{\epsilon} |\bar{p}|^2 I, \end{cases}$$

where $\|A\| = \sup_{\zeta \in \mathbb{R}^{2n}} \frac{|A\zeta \cdot \zeta|}{\zeta \cdot \zeta}$.

We finish the proof of the theorem before proving this lemma.

We can rewrite the matrix inequality in the following form:

$$\frac{-12|\bar{p}|^2}{\epsilon} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \frac{12|\bar{p}|^2}{\epsilon} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Thus X and Y are bounded independently of α (it is already the case for \bar{p} according to item 2. of Lemma 2.8). In particular, $\frac{-12|\bar{p}|^2}{\epsilon} I \leq X \leq \frac{12|\bar{p}|^2}{\epsilon} I$. So there exists $\alpha_n \rightarrow 0$ such that $\bar{t} \rightarrow t_\infty$, $\bar{p} \rightarrow p_\infty$ and $(X, Y) \rightarrow (X_\infty, Y_\infty)$. Moreover, since u is a subsolution and v is a supersolution, by using the condition (16), one finds:

$$\begin{aligned} \tau_1 + G_* \left(\bar{x}, \bar{t}, \frac{|\bar{p}|^2 \bar{p}}{\epsilon} + \alpha_n \bar{x}, X + \alpha_n I \right) &\leq \frac{-\gamma}{T^2}, \\ \tau_2 + G^* \left(\bar{y}, \bar{t}, \frac{|\bar{p}|^2 \bar{p}}{\epsilon} - \alpha_n \bar{y}, Y - \alpha_n I \right) &\geq 0. \end{aligned}$$

Then, by using the ellipticity of the equation and the matrix inequality $X \leq Y$, we obtain:

$$\tau_2 + G^* \left(\bar{y}, \bar{t}, \frac{|\bar{p}|^2 \bar{p}}{\epsilon} - \alpha_n \bar{y}, X - \alpha_n I \right) \geq 0.$$

Subtracting the two inequalities, yields:

$$\frac{\gamma}{T^2} + G_*(\bar{x}, \bar{t}, \frac{|\bar{p}|^2 \bar{p}}{\epsilon} + \alpha_n \bar{x}, X + \alpha_n I) \leq G^*(\bar{y}, \bar{t}, \frac{|\bar{p}|^2 \bar{p}}{\epsilon} - \alpha_n \bar{y}, X - \alpha_n I).$$

ie:

$$\begin{aligned} & c(\bar{x}, \bar{t}) \left| \frac{|\bar{p}|^2 \bar{p}}{\epsilon} + \alpha_n \bar{x} \right| - c(\bar{y}, \bar{t}) \left| \frac{|\bar{p}|^2 \bar{p}}{\epsilon} - \alpha_n \bar{y} \right| \\ & - F_* \left(\frac{|\bar{p}|^2 \bar{p}}{\epsilon} + \alpha_n \bar{x}, X + \alpha_n I \right) + F^* \left(\frac{|\bar{p}|^2 \bar{p}}{\epsilon} - \alpha_n \bar{y}, X - \alpha_n I \right) \geq \frac{\gamma}{T^2}. \end{aligned}$$

We let α go to 0. By using item 1 of Lemma 2.8 and the fact that c is bounded, we obtain:

$$(c(\bar{x}, \bar{t}) - c(\bar{y}, \bar{t})) \frac{|p_\infty|^3}{\epsilon} - F_* \left(\frac{|p_\infty|^2 p_\infty}{\epsilon}, X_\infty \right) + F^* \left(\frac{|p_\infty|^2 p_\infty}{\epsilon}, X_\infty \right) \geq \frac{\gamma}{T^2}.$$

Moreover,

$$(c(\bar{x}, \bar{t}) - c(\bar{y}, \bar{t})) \frac{|p_\infty|^3}{\epsilon} \leq \frac{L_c |p_\infty|^4}{\epsilon},$$

where L_c is the Lipschitz constant of c . This implies:

$$\frac{L_c |p_\infty|^4}{\epsilon} - F_* \left(\frac{|p_\infty|^2 p_\infty}{\epsilon}, X_\infty \right) + F^* \left(\frac{|p_\infty|^2 p_\infty}{\epsilon}, X_\infty \right) \geq \frac{\gamma}{T^2}.$$

By using (17), we obtain:

$$\frac{\gamma}{2T^2} - F_* \left(\frac{|p_\infty|^2 p_\infty}{\epsilon}, X_\infty \right) + F^* \left(\frac{|p_\infty|^2 p_\infty}{\epsilon}, X_\infty \right) \geq \frac{\gamma}{T^2}.$$

We then distinguish two cases:

1st case: $p_\infty = 0$. In this case, we have $X_\infty = 0$ and $F_*(0, 0) = F^*(0, 0) = 0$. So, we obtain:

$$0 \geq \frac{\gamma}{2T^2},$$

what is absurd.

2nd case: $p_\infty \neq 0$. In this case $F_* = F^* = F$. So, we have:

$$0 \geq \frac{\gamma}{2T^2},$$

what is absurd.

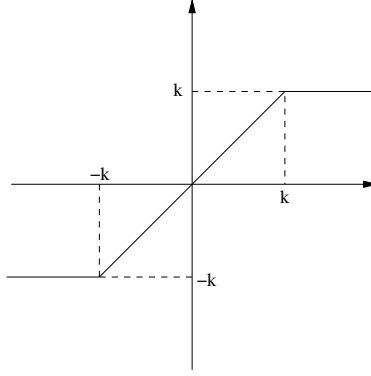


Figure 2: A truncature function T_k .

To achieve the proof in the case where the functions are not bounded, it suffices to use the fundamental property of geometric equations. We then consider the truncature functions T_k (see figure 2). For every k , we then have $T_k(u) \leq T_k(v)$ and by letting k go to infinity, we obtain the result. \square

We now prove Lemma 2.8.

Proof of Lemma 2.8

The function $(x, y) \rightarrow u(x, t) - v(y, t)$ is bounded (because u and v are bounded). Moreover, we have assumed that $\bar{M} > 0$. We then have:

$$\frac{|\bar{x} - \bar{y}|^4}{4\epsilon} + \frac{\alpha}{2}(|\bar{x}|^2 + |\bar{y}|^2) \leq C.$$

>From which

$$\frac{\alpha}{2}(|\bar{x}|^2 + |\bar{y}|^2) \leq C,$$

and

$$|\bar{x} - \bar{y}|^4 \leq C\epsilon,$$

and so $\lim_{\epsilon \rightarrow 0} |\bar{x} - \bar{y}| = 0$ and $\lim_{\alpha \rightarrow 0} \alpha \bar{x} = \lim_{\alpha \rightarrow 0} \alpha \bar{y} = 0$.

We set $M_h = \sup_{|x-y| \leq h, t \in [0, T]} (u(x, t) - v(y, t))$. Let (x_n^h, y_n^h, t_n^h) be such that $u(x_n^h, t_n^h) - v(y_n^h, t_n^h) \geq M_h - \frac{1}{n}$ with $|x_n^h - y_n^h| \leq h$ (x_n^h and y_n^h do not depend on α). We then have

$$\begin{aligned} & M_h - \frac{1}{n} - \frac{h^4}{4\epsilon} - \frac{\alpha}{2} (|x_n^h|^2 + |y_n^h|^2) \\ & \leq u(x_n^h, t_n^h) - v(y_n^h, t_n^h) - \frac{|x_n^h - y_n^h|^4}{4\epsilon} - \frac{\alpha}{2} (|x_n^h|^2 + |y_n^h|^2) \\ & \leq \bar{M} \\ & \leq u(\bar{x}, \bar{t}) - v(\bar{y}, \bar{t}). \end{aligned}$$

We let α go to 0:

$$\begin{aligned} M_h - \frac{1}{n} - \frac{h^4}{4\epsilon} &\leq \liminf_{\alpha \rightarrow 0} (u(\bar{x}, \bar{t}) - v(\bar{y}, \bar{t})) \\ &\leq \limsup_{\alpha \rightarrow 0} (u(\bar{x}, \bar{t}) - v(\bar{y}, \bar{t})). \end{aligned}$$

(We have used the fact that x_n^h and y_n^h do not depend on α). We let h go to 0 and we obtain:

$$\begin{aligned} M' - \frac{1}{n} &\leq \liminf_{\alpha \rightarrow 0} (u(\bar{x}, \bar{t}) - v(\bar{y}, \bar{t})) \\ &\leq \limsup_{\alpha \rightarrow 0} (u(\bar{x}, \bar{t}) - v(\bar{y}, \bar{t})). \end{aligned}$$

We let ϵ go to 0:

$$\begin{aligned} M' - \frac{1}{n} &\leq \liminf_{\epsilon \rightarrow 0} \liminf_{\alpha \rightarrow 0} (u(\bar{x}, \bar{t}) - v(\bar{y}, \bar{t})) \\ &\leq \limsup_{\epsilon \rightarrow 0} \limsup_{\alpha \rightarrow 0} (u(\bar{x}, \bar{t}) - v(\bar{y}, \bar{t})) \\ &\leq \limsup_{\epsilon \rightarrow 0} \limsup_{\alpha \rightarrow 0} \left(\sup_{|x-y| \leq c\epsilon^{\frac{1}{4}}} (u(x, t) - v(y, t)) \right) \\ &\leq \limsup_{h \rightarrow 0} \sup_{|x-y| \leq h} (u(x, t) - v(y, t)) \\ &= M'. \end{aligned}$$

so $\lim_{\epsilon \rightarrow 0} \lim_{\alpha \rightarrow 0} u(\bar{x}, \bar{y}) - v(\bar{y}, \bar{t}) = M'$. In the same way, we have $\lim_{\epsilon \rightarrow 0} \lim_{\alpha \rightarrow 0} \bar{M} = M'$. Therefore,

$$\lim_{\epsilon \rightarrow 0} \lim_{\alpha \rightarrow 0} \left(\frac{|\bar{x} - \bar{y}|^4}{4\epsilon} + \frac{\alpha}{2} (|\bar{x}|^2 + |\bar{y}|^2) \right) = \lim_{\epsilon \rightarrow 0} \lim_{\alpha \rightarrow 0} (u(\bar{x}, \bar{y}) - v(\bar{y}, \bar{t}) - \bar{M}) = M' - M' = 0.$$

So,

$$\begin{cases} \lim_{\epsilon \rightarrow 0} \lim_{\alpha \rightarrow 0} \frac{|\bar{x} - \bar{y}|^4}{4\epsilon} = 0, \\ \lim_{\epsilon \rightarrow 0} \lim_{\alpha \rightarrow 0} \frac{\alpha}{2} (|\bar{x}|^2 + |\bar{y}|^2) = 0. \end{cases}$$

What achieves the proof of Lemma 2.8. □

We now prove Lemma 2.9.

Proof of Lemma 2.9

By definition of the norm of A , we have: $\frac{A\zeta \cdot \zeta}{\zeta \cdot \zeta} \leq \|A\|$, so $\frac{A\zeta \cdot \zeta}{\|A\|} \leq I\zeta \cdot \zeta$, what gives the first result of the lemma.

Let $\zeta = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \in \mathbb{R}^{2n}$. We then have:

$$\begin{aligned} A\zeta \cdot \zeta &= \frac{2}{\epsilon} |\bar{p}|^2 Z(\zeta_1 - \zeta_2) \cdot (\zeta_1 - \zeta_2) \\ &\leq \frac{4}{\epsilon} |\bar{p}|^2 Z(\zeta_1 \cdot \zeta_1 + \zeta_2 \cdot \zeta_2) \\ &\leq \frac{4}{\epsilon} |\bar{p}|^2 \|Z\| |\zeta|^2. \end{aligned}$$

So, it suffices to show that $\|Z\| \leq \frac{3}{2}$:

$$\begin{aligned} Z\xi.\xi &= \frac{\xi^2}{2} + \frac{\bar{p}\xi \otimes \bar{p}\xi}{|\bar{p}|^2} \\ &\leq \frac{\xi^2}{2} + \xi^2 \\ &\leq \frac{3}{2}\xi^2. \end{aligned}$$

For the last item, it suffices to notice that if $B \geq 0$ and $C \geq 0$ with $B, C \in S^n$ such that $BC = CB$, then $CB \geq 0$. Indeed, $CB\xi.\xi = CB^{\frac{1}{2}}\xi.B^{\frac{1}{2}}\xi \geq 0$ (we have used the fact that B is positive, then symmetric and finally that C is positive). So, if $B \geq C$ and $D \geq 0$ with $D(B - C) = (B - C)D$, then $DB \geq DC$. So, it suffices to show that $A \leq 2\|A\|(I - \delta A) = 2\|A\|I - A$, ie $A \leq \|A\|I$ what is true by definition of the norm of A .

What achieves the proof of Lemma 2.9. \square

We now give an existence result by using the classic Perron's method (for the proof, we refer to Droniou, Imbert [19]):

THEOREM 2.10 (Existence and uniqueness)

Assume that there exists a subsolution $U^- \in USC(\mathbb{R}^n \times [0, T))$ and a supersolution $U^+ \in LSC(\mathbb{R}^n \times [0, T))$ of (15) such that $U^-(x, 0) = U^+(x, 0) = u_0(x)$, then, there exists a unique continuous solution of (15).

We now construct a sub and a supersolution which satisfy the initial condition. We begin by studying the problem $u_t + F(Du, D^2u) = 0$. We have the following Lemma:

LEMMA 2.11 (Existence and uniqueness, case $c = 0$)

There exists a unique solution u of the problem

$$\begin{cases} u_t + F(Du, D^2u) = 0, \\ u(x, 0) = u_0(x). \end{cases} \quad (18)$$

Moreover, u is uniformly continuous in time and its modulus of continuity, ω_F , depends only on the lipschitz constant of u_0 , B_0 . So, u satisfies:

$$\forall t, s \in [0, T), \forall x \in \mathbb{R}^n \quad |u(x, t) - u(x, s)| \leq \omega_F(|t - s|).$$

Proof of Lemma 2.11

We assume, in a first time, that $u_0 \in C_b^2 = \{u \in C^2, \exists C, \|Du\|_{L^\infty}, \|D^2u\|_{L^\infty} \leq C\}$. We set $u^\pm = u_0 \pm C_1 t$ with $C_1 = \inf_{x \in \mathbb{R}^n} \{-F^*(Du_0, D^2u_0), F_*(Du_0, D^2u_0)\}$ (C_1 depends only on the bounds of Du_0 and D^2u_0). It is then easy to check that u^+ is a supersolution and u^-

is a subsolution. Then, there exists a unique solution of (18) (Theorem 2.10) and, by the comparison principle, the following holds:

$$\forall t \in [0, T], \forall x \in \mathbb{R}^n, |u(x, t) - u_0(x)| \leq C_1 t. \quad (19)$$

We then set $v(x, t) = u(x, t + h)$. So v is solution of (18) and, by the comparison principle, we obtain:

$$u(x, t + h) - u(x, t) \leq \sup(u(x, h) - u_0) \leq C_1 h.$$

Similarly, we have:

$$u(x, t) - u(x, t + h) \leq \sup(u(x, h) - u_0) \leq C_1 h,$$

and so:

$$|u(x, t + h) - u(x, t)| \leq \sup(u(x, h) - u_0) \leq C_1 h.$$

We now assume that u_0 is only lipschitz. We set $u_\epsilon^0 = u_0 \star \rho_\epsilon$ where ρ_ϵ is a regularizing sequence, ie $\rho_\epsilon = \frac{1}{\epsilon^n} \rho(\frac{\cdot}{\epsilon})$ where $\rho \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$ and satisfies:

$$\rho \geq 0, \quad \text{supp}(\rho) \subset \bar{B}(0, 1), \quad \int_{\mathbb{R}^n} \rho(x) dx = 1.$$

Then $u_\epsilon^0 \in C_b^2$ and $\|Du_\epsilon^0\|_{L^\infty(\mathbb{R}^n)}, \|D^2u_\epsilon^0\|_{L^\infty(\mathbb{R}^n)} \leq \frac{B_0 C_2}{\epsilon}$. Indeed:

$$\begin{aligned} |Du_\epsilon^0(x)| &= |Du_0 \star \rho_\epsilon(x)| \\ &= \left| \int_{\mathbb{R}^n} Du_0(x - y) \rho_\epsilon(y) dy \right| \\ &\leq \int_{\mathbb{R}^n} |Du_0(x - y)| \rho_\epsilon(y) dy \\ &\leq B_0 \int_{\mathbb{R}^n} \rho_\epsilon(y) dy \\ &= B_0 \end{aligned}$$

and

$$\begin{aligned} |D^2u_\epsilon^0(x)| &= |Du_0 \star D\rho_\epsilon(x)| \\ &= \int_{\mathbb{R}^n} |Du_0(x - y)| |D\rho_\epsilon(y)| dy \\ &\leq B_0 \frac{1}{\epsilon} \int_{\mathbb{R}^n} \frac{1}{\epsilon^n} \left| D\rho\left(\frac{y}{\epsilon}\right) \right| dy \\ &= B_0 \frac{1}{\epsilon} \|D\rho\|_{L^1(\mathbb{R}^n)}. \end{aligned}$$

Moreover, $\|u_0 - u_\epsilon^0\|_{L^\infty(\mathbb{R}^n)} \leq B_0\epsilon$. Indeed, since $\int_{\mathbb{R}^n} \rho_\epsilon(x) dx = 1$

$$\begin{aligned} |u_0 - u_\epsilon^0(x)| &\leq \int_{\mathbb{R}^n} |u_0(x) - u_0(x-y)| \rho_\epsilon(y) dy \\ &\leq B_0 \int_{\bar{B}(0,\epsilon)} |y| \rho_\epsilon(y) dy \\ &\leq \epsilon B_0 \int_{\bar{B}(0,\epsilon)} \rho_\epsilon(y) dy = \epsilon B_0. \end{aligned}$$

We note u_ϵ the solution with initial condition u_ϵ^0 . Then, by the comparison principle, $\|u_{\epsilon'}(\cdot, t) - u_\epsilon(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq \|u_{\epsilon'}^0 - u_\epsilon^0\|_{L^\infty(\mathbb{R}^n)}$, and so u_ϵ converge uniformly (since u_ϵ^0 converge uniformly) to u which is, by stability (see Theorem 2.3 of Barles [6]), the solution of (18) with initial condition u_0 . We then have, by the comparison principle, $\|u_\epsilon(\cdot, t) - u(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq \|u_\epsilon^0 - u_0\|_{L^\infty(\mathbb{R}^n)}$. We then deduce:

$$\begin{aligned} \|u(\cdot, t+h) - u(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} &\leq 2\|u_0 - u_\epsilon^0\|_{L^\infty(\mathbb{R}^n)} + \|u_\epsilon(\cdot, t+h) - u_\epsilon(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \\ &\leq 2B_0\epsilon + C_1 \left(B_0, \frac{B_0 C_2}{\epsilon} \right) h. \end{aligned}$$

By taking the minimum on ϵ , we obtain the modulus of continuity of u , ω_F , which depends only on B_0 . □

REMARK 2.12 *In the case of dislocation dynamics, ie with F gived by (14), we have $C_1(B_0, \frac{B_0 C_2}{\epsilon}) \sim B_0 C_2 \frac{1}{\epsilon}$ and so $\omega_F(\delta) \sim \sqrt{\delta}$. In this case, an alternative proof can be founded in Chen, Giga, Goto [13], based on selfsimilar solutions (Wulff Shape) of the mean curvature motion.*

LEMMA 2.13 (Existence of sub and supersolutions, general case)

There exists U^+ (respectively U^-) supersolution (respectively subsolution) of (15) such that:

$$\begin{aligned} u_0(x) - \omega_F(t) - \|c\|_{L^\infty(\mathbb{R}^n \times [0, T])} B_0 t &\leq U^-(x, t) \\ &\leq U^+(x, t) \leq u_0(x) + \omega_F(t) + \|c\|_{L^\infty(\mathbb{R}^n \times [0, T])} B_0 t. \end{aligned}$$

for every $(x, t) \in \mathbb{R}^n \times (0, T)$.

Proof of Lemma 2.13

First, we study the problem (18). According to the Lemma 2.11, this problem has a unique continuous solution, u , which satisfies:

$$u_0(x) - \omega_F(t) \leq u(t, x) \leq u_0(x) + \omega_F(t).$$

We prove that $\|Du\|_{L^\infty(\mathbb{R}^n \times (0, T))} \leq \|Du_0\|_{L^\infty(\mathbb{R}^n)}$. Indeed, let us consider the function $u^h(x, t) = u(x+h, t) + \|Du_0\|_{L^\infty(\mathbb{R}^n)} |h|$. Then u^h is still solution of (18). Therefore, $u^h(x, 0) = u_0(x) +$

$h) + \|Du_0\|_{L^\infty(\mathbb{R}^n)} |h| \geq u_0(x)$. So, by the comparison principle, we have $u^h \geq u$. We deduce that, for every $h \in \mathbb{R}^n$,

$$u(x, t) - u(x + h, t) \leq \|Du_0\|_{L^\infty(\mathbb{R}^n)} |h|,$$

and so

$$\|Du\|_{L^\infty(\mathbb{R}^n \times (0, T))} \leq \|Du_0\|_{L^\infty(\mathbb{R}^n)}$$

We then set $U^+(x, t) = u(x, t) + \|c\|_{L^\infty(\mathbb{R}^n \times [0, T])} B_0 t$. Then, $\|DU^+\|_{L^\infty(\mathbb{R}^n \times (0, T))} \leq \|Du\|_{L^\infty(\mathbb{R}^n \times (0, T))} \leq B_0$ and U^+ is solution of:

$$\begin{cases} v_t - \|c\|_{L^\infty(\mathbb{R}^n \times [0, T])} B_0 + F(Dv, D^2v) = 0, \\ v(x, 0) = u_0, \end{cases}$$

and so U^+ is supersolution of (15) and satisfies:

$$\begin{aligned} U^+(x, t) &= u(x, t) + \|c\|_{L^\infty(\mathbb{R}^n \times [0, T])} B_0 t \\ &\leq u_0(x) + \omega_F(t) + \|c\|_{L^\infty(\mathbb{R}^n \times [0, T])} B_0 t. \end{aligned}$$

Similarly, we construct a subsolution U^- such that $U^-(x, t) \geq u_0(x) - \omega_F(t) - \|c\|_{L^\infty(\mathbb{R}^n \times [0, T])} B_0 t$ by setting $U^-(x, t) = u(x, t) - \|c\|_{L^\infty(\mathbb{R}^n \times [0, T])} B_0 t$. To achieve the proof, it suffices to apply the comparison principle to U^- and U^+ . \square

Finally, we proved the following Theorem:

THEOREM 2.14 (Existence and uniqueness for the local problem)

Let $T > 0$. Then, under the assumptions (H_0) , there exists a unique viscosity solution of the problem (15) in $\mathbb{R}^n \times [0, T)$.

2 Estimate on the solution of problem (15)

LEMMA 2.15 (A priori estimate)

Assume that $\|Du_0\|_{L^\infty(\mathbb{R}^n)} \leq B_0$ and $\frac{\partial u_0}{\partial x_n} \geq b_0$, with $B_0 > 0$ and $b_0 > 0$. Then, the solution of (15) given by the Theorem 2.10 satisfies

$$\|Du(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq B(t) \text{ and } \frac{\partial u}{\partial x_n} \geq b(t),$$

with $B(t) = B_0 e^{L_c t}$ and $b(t) = b_0 - B_0(e^{L_c t} - 1)$. Moreover, u is uniformly continuous in time and its modulus of continuity in time ω_u , defined by:

$$\forall x \in \mathbb{R}^n, \forall s, t \in [0, T), |u(x, t) - u(x, s)| \leq \omega_u(|t - s|),$$

satisfies:

$$\omega_u(\delta) \leq \omega_F(\delta) + \|c\|_{L^\infty} B_0 \delta + \omega_c(\delta) \int_0^T B(s) ds,$$

where ω_c is the modulus of continuity in time of c , and ω_F is the modulus of continuity in time of the solution of (18) (see Lemma 2.11).

Proof of Lemma 2.15 For the proof of the lipschitz estimate in space, we assume in a first time that u is bounded. We set $\phi^\epsilon(x, y, t) = B(t) (|x - y|^2 + \epsilon^2)^{1/2}$. We prove that $u(x, t) - u(y, t) \leq \phi^\epsilon$. We set:

$$M = \sup_{(x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times [0, T]} \{u(x, t) - u(y, t) - \phi^\epsilon(x, y, t)\},$$

Assume that $M > 0$. Then we set:

$$\bar{M} = \sup_{(x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times [0, T]} \left\{ u(x, t) - u(y, t) - \phi^\epsilon(x, y, t) - \frac{\alpha}{2}(|x|^2 + |y|^2) - \frac{\gamma}{T-t} \right\}.$$

For $\alpha > 0$, $\gamma > 0$ small enough, we have $\bar{M} > 0$. Moreover u is bounded, so the sup is reached in $(\bar{x}, \bar{y}, \bar{t})$ (with $\bar{x} \neq \bar{y}$) and

$$\frac{\alpha}{2}(|\bar{x}|^2 + |\bar{y}|^2) \leq C,$$

and so $\alpha\bar{x} \rightarrow 0$ and $\alpha\bar{y} \rightarrow 0$. We prove that $\bar{t} > 0$. Indeed, assume the opposite. Then, we have

$$u_0(\bar{x}) - u_0(\bar{y}) - \phi^\epsilon(\bar{x}, \bar{y}, 0) > 0,$$

ie

$$u_0(\bar{x}) - u_0(\bar{y}) > B_0 (|\bar{x} - \bar{y}|^2 + \epsilon^2)^{\frac{1}{2}} > B_0 |\bar{x} - \bar{y}|,$$

what is absurd since $\|Du_0\|_{L^\infty(\mathbb{R}^n)} \leq B_0$. We set

$$\bar{p} = D_x \phi^\epsilon = (|\bar{x} - \bar{y}|^2 + \epsilon^2)^{-1/2} (\bar{x} - \bar{y}) B(t) = -D_y \phi^\epsilon \neq 0 \text{ (because } \bar{x} \neq \bar{y}\text{),}$$

$$Z = D_x^2 \phi^\epsilon = \left((|\bar{x} - \bar{y}|^2 + \epsilon^2)^{-1/2} I - (|\bar{x} - \bar{y}|^2 + \epsilon^2)^{-3/2} (\bar{x} - \bar{y}) \otimes (\bar{x} - \bar{y}) \right) B(t) = D_y^2 \phi^\epsilon,$$

$$A = D^2 \phi^\epsilon = \begin{pmatrix} Z & -Z \\ -Z & Z \end{pmatrix}.$$

Then, by parabolic Ishii Lemma, applied to $\tilde{u} = u(x, t) - \frac{\alpha}{2}|x|^2$, $\tilde{v}(y, t) = v(y, t) + \frac{\alpha}{2}|y|^2$ and $\phi(x, y, t) = \phi^\epsilon(x, y, t) + \frac{\gamma}{T-t}$, for every β such that $\beta A < I$, there exists $\tau_1, \tau_2 \in \mathbb{R}$ and $X, Y \in S^n$ such that:

$$\tau_1 - \tau_2 = \frac{\gamma}{(T - \bar{t})^2} + L_c B(t) (|\bar{x} - \bar{y}|^2 + \epsilon^2)^{\frac{1}{2}},$$

$$(\tau_1, p + \alpha\bar{x}, X + \alpha I) \in \bar{\mathcal{P}}^+ u(\bar{x}, \bar{t}),$$

$$(\tau_2, p - \alpha\bar{y}, Y - \alpha I) \in \bar{\mathcal{P}}^- v(\bar{y}, \bar{t}),$$

$$\frac{-1}{\beta} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq (I - \beta A)^{-1} A.$$

So, the following holds

$$\tau_1 - c(\bar{x}, \bar{t})|\bar{p} + \alpha\bar{x}| + F_*(\bar{p} + \alpha\bar{x}, X + \alpha I) \leq 0,$$

$$\tau_2 - c(\bar{y}, \bar{t})|\bar{p} - \alpha\bar{y}| + F^*(\bar{p} - \alpha\bar{y}, Y - \alpha I) \geq 0.$$

The matrix inequality implies, in particular, that $X \leq Y$, so, by using the ellipticity of F , we deduce:

$$\tau_2 - c(\bar{y}, \bar{t})|\bar{p} - \alpha\bar{y}| + F^*(\bar{p} - \alpha\bar{y}, X - \alpha I) \geq 0.$$

>From that, by subtracting:

$$\begin{aligned} & \frac{\gamma}{(T - \bar{t})^2} + L_c B(t) (|\bar{x} - \bar{y}|^2 + \epsilon^2)^{\frac{1}{2}} - c(\bar{x}, \bar{t})|\bar{p} + \alpha\bar{x}| + c(\bar{y}, \bar{t})|\bar{p} - \alpha\bar{y}| \\ & + F_*(\bar{p} + \alpha\bar{x}, X + \alpha I) - F^*(\bar{p} - \alpha\bar{y}, X - \alpha I) \leq 0. \end{aligned}$$

We let α go to 0 (\bar{p} and X are bounded so they converge and we still note \bar{p} and X their limit):

$$\frac{\gamma}{(T - \bar{t})^2} + \lim_{\alpha \rightarrow 0} \left(L_c B(t) (|\bar{x} - \bar{y}|^2 + \epsilon^2)^{\frac{1}{2}} + (-c(\bar{x}, \bar{t}) + c(\bar{y}, \bar{t})) |\bar{p}| \right) + F_*(\bar{p}, X) - F^*(\bar{p}, X) \leq 0.$$

Now, $\bar{p} \neq 0$, therefore $F_*(\bar{p}, X) = F^*(\bar{p}, X)$. Moreover,

$$\begin{aligned} & L_c B(t) (|x - y|^2 + \epsilon^2)^{1/2} - c(x, t)|\bar{p}| + c(y, t)|\bar{p}| \\ & = (|x - y|^2 + \epsilon^2)^{1/2} \left(L_c B(t) - \frac{|x - y|B(t)}{|x - y|^2 + \epsilon^2} (c(x, t) - c(y, t)) \right) \\ & \geq (|x - y|^2 + \epsilon^2)^{1/2} \left(L_c B(t) - \frac{|x - y|^2 L_c B(t)}{|x - y|^2 + \epsilon^2} \right) \\ & \geq (|x - y|^2 + \epsilon^2)^{1/2} (L_c B(t) - L_c B(t)) \\ & \geq 0, \end{aligned}$$

so

$$\frac{\gamma}{(T - \bar{t})^2} \leq 0,$$

what is absurd. So $u(x, t) - u(y, t) \leq \phi^\epsilon$. By letting ϵ go to 0, we obtain:

$$u(x, t) - u(y, t) \leq B(t)|x - y|.$$

Exchanging x and y , yields

$$|u(x, t) - u(y, t)| \leq B(t)|x - y|,$$

what gives the first result in the case where u is bounded. If u is not bounded, we consider the truncature functions T_k (see figure 2). Then $T_k(u)$ is bounded and solution of the problem, and so:

$$|T_k(u(x, t)) - T_k(u(y, t))| \leq B(t)|x - y|.$$

Letting k go to infinity, yields:

$$|u(x, t) - u(y, t)| \leq B(t)|x - y|,$$

and we obtain the first estimate.

For the second estimate, we set, for $x = (x', x_n)$, $u^\lambda(x, t) = u(x', x_n + \lambda, t) - \lambda b(t)$. We have

$$\begin{aligned} u^\lambda(x', x_n, 0) &= u(x', x_n + \lambda) - \lambda b_0 \\ &\geq u(x', x_n, 0). \end{aligned}$$

Moreover,

$$\begin{aligned} &u_t^\lambda + G^*(x', x_n, t, Du^\lambda, D^2u^\lambda) \\ &= u_t - \lambda b'(t) - c(x', x_n, t)|Du| + F^*(Du, D^2u) \\ &= u_t + \lambda B_0 L_c e^{L_c t} - c(x', x_n, t)|Du| + F^*(Du, D^2u) \\ &\geq u_t + \lambda B_0 L_c e^{L_c t} - (c(x', x_n + \lambda, t) + \lambda L_c)|Du| + F^*(Du, D^2u) \\ &\geq \lambda B_0 L_c e^{L_c t} - \lambda B_0 L_c e^{L_c t} + u_t + G^*(x', x_n + \lambda, t, Du, D^2u) \\ &\geq 0, \end{aligned}$$

where u_t , Du , D^2u are taken at the point (x', x_n, t) . This is written in a formal way and that is justified by using a test function. So, we obtain that u^λ is a supersolution. By the comparison principle, we deduce $u^\lambda \geq u$, and so

$$u(x', x_n + \lambda, t) - u(x', x_n, t) \geq \lambda b(t).$$

what proves the second estimate. It thus remains to be shown that u is uniformly continuous in time. We set $\delta > 0$. For every $(x, t) \in \mathbb{R}^n \times (0, T)$ such that $t + \delta \leq T$, we set $v(x, t) = u(x, t + \delta)$. Then, v is a subsolution of

$$w_t - \omega_c(\delta)B(t + \delta) - c(x, t)|Dw| + F(Dw, D^2w) = 0$$

on $\mathbb{R}^n \times (0, T - \delta)$ in the sense of definition 2.1 (ii). Indeed, we have

$$v_t - c(x, t + \delta)|Dv| + F(Dv, D^2v) = 0,$$

and

$$-c(x, t + \delta)|Dv| \geq -\omega_c(\delta)B(t + \delta) - c(x, t)|Dv|,$$

what gives in a formal way:

$$v_t - \omega_c(\delta)B(t + \delta) - c(x, t)|Dv| + F(Dv, D^2v) \leq 0.$$

Moreover, $u + \omega_c(\delta) \int_0^{t+\delta} B(s)ds$ is solution of the same problem. So $\tilde{u} = u + \sup_{x \in \mathbb{R}^n} (u(x, \delta) - u_0(x))^+ + \omega_c(\delta) \int_0^{t+\delta} B(s)ds$ is a supersolution and $v(x, 0) \leq \tilde{u}(x, 0)$. By Lemma 2.13 and the comparison principle, we then have:

$$\begin{aligned} u(x, t + \delta) - u(x, t) &\leq \sup_{x \in \mathbb{R}^n} (u(x, \delta) - u_0(x))^+ + \omega_c(\delta) \int_0^{t+\delta} B(s)ds \\ &\leq \omega_F(\delta) + \|c\|_{L^\infty} B_0 \delta + \omega_c(\delta) \int_0^T B(s)ds. \end{aligned}$$

Similarly, v is a supersolution of

$$w_t + \omega_c(\delta)B(t + \delta) - c(x, t)|Dw| + F(Dw, D^2w) = 0$$

and $\tilde{u} = u - \sup_{x \in \mathbb{R}^n} (u(x, \delta) - u_0(x))^- - \omega_c(\delta) \int_0^{t+\delta} B(s)ds$ is subsolution. So, by the comparison principle, we have

$$\begin{aligned} u(x, t) - u(x, t + \delta) &\leq \omega_F(\delta) + \|c\|_{L^\infty} B\delta + \omega_c(\delta) \int_0^{t+\delta} B(s)ds \\ &\leq \omega_F(\delta) + \|c\|_{L^\infty} B_0\delta + \omega_c(\delta) \int_0^T B(s)ds, \end{aligned}$$

ie

$$|u(x, t) - u(x, t + \delta)| \leq \omega_F(\delta) + \|c\|_{L^\infty} B_0\delta + \omega_c(\delta) \int_0^T B(s)ds,$$

what achieves the proof of the lemma. \square

3 The non local problem : proof of Theorem 1.3

For the proof of Theorem 1.3, we will need the three following lemmas:

LEMMA 3.1 (Estimate on the characteristic functions)

Let $u^1 \in C(\mathbb{R}^n)$ satisfying

$$\frac{\partial u^1}{\partial x_n} \geq b$$

in the distributions' sense for some $b > 0$ and $u^2 \in L_{\text{loc}}^\infty(\mathbb{R}^n)$ satisfying the same condition. Then, we have the following estimate:

$$\|[u^2] - [u^1]\|_{L_{\text{unif}}^1} \leq \frac{2}{b} \|u^2 - u^1\|_{L^\infty}. \quad (20)$$

For the proof of this Lemma, we refer to the proof of Alvarez et al. [2] in the case $n = 2$, which adapts without difficulty to the case of any dimension.

LEMMA 3.2 (Convolution inequality)

For every $f \in L_{\text{unif}}^1(\mathbb{R}^n)$ and $g \in L_{\text{int}}^\infty(\mathbb{R}^n)$, the convolution product $f \star g$ is bounded and satisfies

$$\|f \star g\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L_{\text{unif}}^1(\mathbb{R}^n)} \|g\|_{L_{\text{int}}^\infty(\mathbb{R}^n)}.$$

For the proof, we refer to Alvarez et al.[3].

LEMMA 3.3 (Stability of the solution with respect to the velocity)

Let $T > 0$. We consider for $i = 1, 2$ two different equations:

$$\begin{cases} u_t^i = c^i(x, t)|Du^i| - F(Du^i, D^2u^i) & \text{in } \mathbb{R}^n \times (0, T), \\ u^i(x, 0) = u_0(x). \end{cases} \quad (21)$$

where c^i satisfy the assumption $(H_0)(ii)$, u_0 satisfies $(H_0)(iii)$ and F satisfies the assumptions (HF) . Then, for every $t \in [0, T)$, we have

$$\|u^1(\cdot, t) - u^2(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq \|c^1 - c^2\|_{L^\infty(\mathbb{R}^n \times (0, T))} \int_0^t B(s) ds,$$

where u^i are the solutions of (21) (see Theorem 2.14), $B(t) = B_0 e^{L_c t}$ with $L_c = \sup_i L_{c^i}$ (L_{c^i} is the Lipschitz constant of c^i).

Proof of Lemma 3.3

We set $K = \|c^1 - c^2\|_{L^\infty(\mathbb{R}^n \times (0, T))}$. We remark that u^1 is subsolution of

$$u_t - c^2(x, t)|Du| + F(Du, D^2u) - KB(t) = 0.$$

Indeed, we have:

$$\begin{aligned} u_t^1 - c^2(x, t)|Du^1| + F(Du^1, D^2u^1) &\leq c^1(x, t)|Du^1| - F(Du^1, D^2u^1) - c^2(x, t)|Du^1| + F(Du^1, D^2u^1) \\ &\leq \|c^1 - c^2\|_{L^\infty(\mathbb{R}^n \times (0, T))} B(t) \\ &\leq KB(t). \end{aligned}$$

It is a routine exercise to check that the differential inequality actually holds in the viscosity sense. Moreover, $u^2 + K \int_0^t B(s) ds$ is solution of the same problem. By the comparison principle (Theorem 2.7), we deduce

$$u^1 \leq u^2 + K \int_0^t B(s) ds.$$

>From what

$$\|u^1(\cdot, t) - u^2(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq \|c^1 - c^2\|_{L^\infty(\mathbb{R}^n \times (0, T))} \int_0^t B(s) ds.$$

□

We now prove Theorem 1.3.

Proof of Theorem 1.3

We set $\omega(\delta) = \omega_F(\delta) + \|c_0\|_{L^1} B_0 \delta$, where ω_F is defined in Lemma 2.11, and we define the set

$$E = \left\{ u \in L_{loc}^\infty(\mathbb{R}^n \times [0, T^*)), \left. \begin{array}{l} |Du(x, t)| \leq 2B_0, \\ \frac{\partial u}{\partial x_n}(x, t) \geq \frac{b_0}{2} \\ u \text{ is uniformly continuous in time and } \omega_u(\delta) \leq 2\omega(\delta) \end{array} \right\} \right.$$

where ω_u is the modulus of continuity in time of u , defined by

$$\forall x \in \mathbb{R}^n, \forall s, t \in [0, T^*), |u(x, t) - u(x, s)| \leq \omega_u(|t - s|).$$

For $u \in E$, we set $c(x, t) = (c_0 \star [u(\cdot, t)])(x)$. We see that c is bounded, lipschitz in space (with $L = |c_0|_{BV}$ as lipschitz constant) and uniformly continuous in time. Indeed,

$$\begin{aligned} \|c\|_{L^\infty(\mathbb{R}^n \times [0, T^*])} &\leq \sup_{t \in \mathbb{R}} \|c_0\|_{L^1} \| [u(\cdot, t)] \|_{L^\infty(\mathbb{R}^n)} \\ &\leq \|c_0\|_{L^1(\mathbb{R}^n)}. \end{aligned}$$

Moreover, for every t

$$\begin{aligned} \|Dc(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} &= \|Dc_0 \star [u(\cdot, t)]\|_{L^\infty(\mathbb{R}^n)} \\ &\leq \|Dc_0\|_{L^1(\mathbb{R}^n)} \| [u(\cdot, t)] \|_{L^\infty(\mathbb{R}^n)} \\ &\leq |c_0|_{BV}. \end{aligned}$$

Finally, for $0 < t, s < T^*$:

$$\begin{aligned} |c(x, t) - c(x, s)| &= |(c_0 \star [u(\cdot, t)])(x) - (c_0 \star [u(\cdot, s)])(x)| \\ &= |c_0 \star ([u(\cdot, t)] - [u(\cdot, s)])|(x)| \\ &\leq \|c_0\|_{L_{\text{int}}^\infty} \| [u(\cdot, t)] - [u(\cdot, s)] \|_{L_{\text{unif}}^1(\mathbb{R}^n)} \\ &\leq \frac{4\|c_0\|_{L_{\text{int}}^\infty}}{b_0} \|u(\cdot, t) - u(\cdot, s)\|_{L^\infty(\mathbb{R}^n)} \\ &\leq \frac{4\|c_0\|_{L_{\text{int}}^\infty}}{b_0} \omega_u(|t - s|) \\ &\leq \frac{8\|c_0\|_{L_{\text{int}}^\infty}}{b_0} \omega(|t - s|), \end{aligned}$$

so c is uniformly continuous in time and $\omega_c(\delta) \leq \frac{8\|c_0\|_{L_{\text{int}}^\infty}}{b_0} \omega(\delta)$.

For $u \in E$, we then define $v = \Phi(u)$ as the unique viscosity solution (see Theorem 2.14) of

$$\begin{cases} v_t = (c_0 \star [u])|Dv| - F(Dv, D^2v) \text{ in } \mathbb{R}^n \times (0, T^*), \\ v(x, t = 0) = u_0(x) \text{ in } \mathbb{R}^n, \end{cases} \quad (22)$$

with u_0 defined in Theorem 1.3. We show that $\Phi : E \rightarrow E$ is a contraction. First, we show that Φ is well defined. We have $\|Dv(\cdot, t)\| \leq B(t) \leq B_0 e^{Lt^*} \leq 2B_0$, by definition of T^* (see Lemma 2.15). Moreover, $\frac{\partial v}{\partial x_n} \geq b(t) = b_0 - B_0(e^{Lt} - 1)$ (see Lemma 2.15), and we want $\frac{\partial v}{\partial x_n} \geq \frac{b_0}{2}$, ie

$$\begin{aligned} B_0(e^{Lt} - 1) &\leq \frac{b_0}{2} \\ e^{Lt} &\leq \frac{b_0}{2B_0} + 1 \\ t &\leq \frac{\ln\left(\frac{b_0}{2B_0} + 1\right)}{L}, \end{aligned}$$

what is true according to the choice of T^* . It thus remains to be shown that v is uniformly continuous with $\omega_v(\delta) \leq 2\omega(\delta)$. Now, by the estimate of Lemma 2.15 on the modulus of continuity in time of the solution, we have:

$$\omega_v(\delta) \leq \omega_F(\delta) + \|c\|_{L^\infty} B_0 \delta + \omega_c(\delta) \int_0^{T^*} B(s) ds.$$

Since $\|c\|_{L^\infty(\mathbb{R}^n \times [0, T^*])} \leq \|c_0\|_{L^1}$, it suffices to show that $\omega_c(\delta) \int_0^{T^*} B(s) ds \leq \omega(\delta)$, ie

$$\begin{aligned} \frac{8\|c_0\|_{L_{\text{int}}^\infty}}{b_0} \omega(\delta) \int_0^{T^*} B(s) ds &\leq \omega(\delta) \\ \int_0^{T^*} B(s) ds &\leq \frac{b_0}{8\|c_0\|_{L_{\text{int}}^\infty}} \\ \frac{1}{L} (e^{LT^*} - 1) &\leq \frac{b_0}{8B_0\|c_0\|_{L_{\text{int}}^\infty}} \\ T^* &\leq \frac{\ln\left(\frac{Lb_0}{8B_0\|c_0\|_{L_{\text{int}}^\infty}} + 1\right)}{L}, \end{aligned}$$

what is true according to the choice of T^* and so $v \in E$.

It thus remains to be shown that Φ is a contraction. For $v^i = \Phi(u^i)$, according to the Lemmas 3.3 then 3.2 and 3.1, we have

$$\begin{aligned} \|v^2 - v^1\|_{L^\infty(\mathbb{R}^n \times (0, T^*))} &\leq 2B_0 T^* \|c_0 \star [u^2] - c_0 \star [u^1]\|_{L^\infty(\mathbb{R}^n \times (0, T^*))} \\ &\leq 2B_0 T^* \|c_0\|_{L_{\text{int}}^\infty(\mathbb{R}^n)} \sup_{t \in (0, T^*)} \|[u^2(\cdot, t)] - [u^1(\cdot, t)]\|_{L_{\text{unif}}^1(\mathbb{R}^n)} \\ &\leq \frac{8B_0 T^*}{b_0} \|c_0\|_{L_{\text{int}}^\infty(\mathbb{R}^n)} \|u^2 - u^1\|_{L^\infty(\mathbb{R}^n \times (0, T^*))} \\ &\leq \frac{1}{2} \|u^2 - u^1\|_{L^\infty(\mathbb{R}^n \times (0, T^*))}. \end{aligned}$$

And so Φ is a contraction on E which is a closed set for L^∞ topology. So, there exists a unique viscosity solution of (5) in E on $(0, T^*)$. \square

4 Appendix: proof of the Ishii Lemma

We are going to prove the parabolic Ishii Lemma (see Crandall Ishii [14]). The result is classic, but we give the proof for the reader's convenience. To do that, we will use an elliptic Ishii Lemma. First, we give some definitions:

DEFINITION 4.1 (Sub and superdifferential of order two)

If $u : \mathbb{R}^n \rightarrow \mathbb{R}$, then the superdifferential of order two of u , $\mathcal{D}^{2,+}u$, is defined by $(p, X) \in \mathbb{R}^n \times S^n$ belongs to $\mathcal{D}^{2,+}u(x, t)$ if $x \in \mathbb{R}$ and

$$u(y) \leq u(x) + \langle p, y - x \rangle + \frac{1}{2} \langle X(y - x), y - x \rangle + o(|y - x|^2)$$

as $\mathbb{R} \ni y \rightarrow x$. In a similar way, we defined the subdifferential of order two by $\mathcal{D}^{2,-}u = -\mathcal{D}^{2,+}(-u)$. We also defined the two following sets:

$$\bar{\mathcal{D}}^{2,+}u(x, t) = \left\{ (p, X) \in \mathbb{R}^n \times S^n, \exists (x_n, p_n, X_n) \in \mathbb{R}^n \times \mathbb{R}^n \times S^n \right. \\ \left. \text{such that } (p_n, X_n) \in \mathcal{D}^{2,+}u(x_n) \text{ and } (x_n, u(x_n), p_n, X_n) \rightarrow (x, u(x), p, X) \right\}.$$

The set $\bar{\mathcal{D}}^{2,-}u(x, t)$ is defined in a similar way. Lastly, we defined $\bar{\mathcal{D}}^2u(x) = \bar{\mathcal{D}}^{2,+}u(x) \cap \bar{\mathcal{D}}^{2,-}u(x)$.

LEMMA 4.2 (elliptic Ishii)

Let U and V be two open sets of \mathbb{R}^n , $u : U \rightarrow \mathbb{R}$ usc and $v : V \rightarrow \mathbb{R}$ lsc. Let $\phi : U \times V \rightarrow \mathbb{R}$ be of class C^2 . Assume that $(x, y) \rightarrow u(x) - v(y) - \phi(x, y)$ reaches a local maximum in $(\bar{x}, \bar{y}) \in U \times V$. We note $p_1 = D_x \phi(\bar{x}, \bar{y})$, $p_2 = -D_y \phi(\bar{x}, \bar{y})$ and $A = D^2 \phi(\bar{x}, \bar{y})$. Then, for every $\alpha > 0$ such that $\alpha A < I$, there exists $X, Y \in S^n$ such that:

$$(p_1, X) \in \bar{\mathcal{D}}^{2,+}u(\bar{x}), (p_2, Y) \in \bar{\mathcal{D}}^{2,-}v(\bar{y}), \\ \frac{-1}{\alpha} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq (I - \alpha A)^{-1} A.$$

For the proof, we refer to Droniou, Imbert [19]. We now prove the parabolic Ishii Lemma:

Proof of Lemma 2.4

The principle of this proof is to duplicate the variables in time and then pass to the limit using the compactness assumption (C). The key point is to regularize by sup-convolution using two different parameter for space and time.

We duplicate the variables by considering:

$$u(x, t) - v(y, s) - \phi(x, y, t) - \frac{|t - s|^2}{2\epsilon}.$$

This function admits a local maximum in $(x_\epsilon, y_\epsilon, t_\epsilon, s_\epsilon)$ with $(x_\epsilon, y_\epsilon, t_\epsilon, s_\epsilon) \rightarrow (\bar{x}, \bar{y}, \bar{t}, \bar{t})$ (because u and v are locally bounded). We note $\tilde{A}_\epsilon = D^2 \left(\phi(x, y, t) + \frac{|t-s|^2}{2\epsilon} \right)$. By elliptic Ishii Lemma, we have: for every α and α' such that:

$$\tilde{A}_\epsilon < \begin{pmatrix} \frac{I}{\alpha} & & & \\ & \frac{1}{\alpha'} & & \\ & & \frac{I}{\alpha} & \\ & & & \frac{1}{\alpha'} \end{pmatrix},$$

(This assumption holds for α such that $\alpha A < I$ and α' small enough) there exists $X_\epsilon, Y_\epsilon \in S^n$, $C, D \in \mathbb{R}^n$ and $\lambda, \sigma \in \mathbb{R}$ such that:

$$\left(\left(p_1^\epsilon, \phi_t + \frac{t-s}{\epsilon} \right), \begin{pmatrix} X_\epsilon & C \\ {}^t C & \lambda \end{pmatrix} \right) \in \bar{\mathcal{D}}^{2,+} u(x_\epsilon, t_\epsilon), \quad (23)$$

$$\left(\left(p_2^\epsilon, \frac{t-s}{\epsilon} \right), \begin{pmatrix} Y_\epsilon & D \\ {}^t D & \sigma \end{pmatrix} \right) \in \bar{\mathcal{D}}^{2,-} v(y_\epsilon, s_\epsilon), \quad (24)$$

and

$$\begin{pmatrix} \frac{-I}{\alpha} & & & \\ & \frac{-1}{\alpha'} & & \\ & & \frac{-I}{\alpha} & \\ & & & \frac{-1}{\alpha'} \end{pmatrix} \leq \left(\begin{array}{cc|cc} X_\epsilon & C & & \\ {}^t C & \lambda & & 0 \\ \hline & & -Y_\epsilon & -D \\ & & -{}^t D & -\sigma \end{array} \right) \leq B_\epsilon^{\alpha, \alpha'}, \quad (25)$$

where $p_1^\epsilon = D_x \phi(\cdot)$, $p_2^\epsilon = D_y \phi(\cdot)$ and $B_\epsilon^{\alpha, \alpha'}$ is the regularized by sup-convolution in the sense of quadratic form of parameters α and α' of:

$$\tilde{A}_\epsilon = \begin{pmatrix} D_x^2 \phi & v_1 & D_{xy} \phi & 0 \\ {}^t v_1 & \phi_{tt} + \frac{1}{\epsilon} & {}^t v_2 & -\frac{1}{\epsilon} \\ D_{xy} \phi & v_2 & D_y^2 \phi & 0 \\ 0 & -\frac{1}{\epsilon} & 0 & \frac{1}{\epsilon} \end{pmatrix},$$

where $v_1 = \phi_{xt}(\cdot)$ and $v_2 = \phi_{yt}(\cdot)$. So, for $\xi = (\xi', \xi_{n+1})$, $\eta = (\eta', \eta_{n+1}) \in \mathbb{R}^{n+1}$, we have:

$$B_\epsilon^{\alpha, \alpha'}(\xi, \eta) \cdot (\xi, \eta) = \sup_{(\zeta, \Gamma) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}} \left\{ \begin{array}{l} \tilde{A}_\epsilon(\zeta, \Gamma) \cdot (\zeta, \Gamma) - \frac{1}{\alpha} (|\xi' - \zeta'|^2 + |\Gamma' - \eta'|^2) \\ -\frac{1}{\alpha'} (|\xi_{n+1} - \zeta_{n+1}|^2 + |\Gamma_{n+1} - \eta_{n+1}|^2) \end{array} \right\}.$$

Moreover, we can see $B_\epsilon^{\alpha, \alpha'}$ as the regularized by supconvolution of the regularized by supconvolution of a quadratic form, so $B_\epsilon^{\alpha, \alpha'}$ is still a quadratic form (see the proof of Elliptic Ishii in Droniou, Imbert [19]). Equations (23) and (24) imply in particular:

$$\left(\phi_t(x_\epsilon, y_\epsilon, t_\epsilon) + \frac{t_\epsilon - s_\epsilon}{\epsilon}, p_1^\epsilon, X_\epsilon \right) \in \bar{\mathcal{P}}^+ u(x_\epsilon, t_\epsilon),$$

$$\left(\frac{t_\epsilon - s_\epsilon}{\epsilon}, p_2^\epsilon, Y_\epsilon \right) \in \bar{\mathcal{P}}^- v(y_\epsilon, s_\epsilon),$$

We note $\rho_\epsilon = \phi_t(x_\epsilon, y_\epsilon, t_\epsilon) + \frac{t_\epsilon - s_\epsilon}{\epsilon}$ and $\gamma_\epsilon = \frac{t_\epsilon - s_\epsilon}{\epsilon}$. We remark that $\phi_t(x_\epsilon, y_\epsilon, t_\epsilon) = \rho_\epsilon - \gamma_\epsilon$. Applying the vector $(\xi', 0, \eta', 0)$ to the matrix inequality (25), yields:

$$\frac{-1}{\alpha} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} (\xi', \eta') \cdot (\xi', \eta') \leq \begin{pmatrix} X_\epsilon & 0 \\ 0 & -Y_\epsilon \end{pmatrix} (\xi', \eta') \cdot (\xi', \eta') \leq B_\epsilon^{\alpha, \alpha'}(\xi', 0, \eta', 0) \cdot (\xi', 0, \eta', 0). \quad (26)$$

We show that the right hand go to $A^\alpha(\xi', \eta').(\xi', \eta')$ as $\alpha' \rightarrow 0$, where A^α is the regularized by sup-convolution of $A = D^2\phi$. Indeed, $B_\epsilon^{\alpha, \alpha'} = (B_\epsilon^\alpha)^{\alpha'}$. So

$$\begin{aligned} B_\epsilon^{\alpha, \alpha'}(\xi', 0, \eta', 0).(\xi', 0, \eta', 0) &\rightarrow B_\epsilon^\alpha(\xi', 0, \eta', 0).(\xi', 0, \eta', 0) \text{ when } \alpha' \rightarrow 0 \\ &= \sup_{(\zeta', \Gamma') \in \mathbb{R}^n \times \mathbb{R}^n} \left\{ \tilde{A}_\epsilon(\zeta', 0, \Gamma', 0).(\zeta', 0, \Gamma', 0) - \frac{1}{\alpha} (|\xi' - \zeta'|^2 + |\eta' - \Gamma'|^2) \right\} \\ &= \sup_{(\zeta', \Gamma') \in \mathbb{R}^n \times \mathbb{R}^n} \left\{ A(\zeta', \Gamma').(\zeta', \Gamma') - \frac{1}{\alpha} (|\xi' - \zeta'|^2 + |\eta' - \Gamma'|^2) \right\} \\ &= A^\alpha(\xi', \eta').(\xi', \eta'), \end{aligned}$$

what gives

$$\frac{-1}{\alpha} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X_\epsilon & 0 \\ 0 & -Y_\epsilon \end{pmatrix} \leq A^\alpha = (I - \alpha A)^{-1} A.$$

We then have, for a subsequence, $p_1^\epsilon \rightarrow p_1$, $p_2^\epsilon \rightarrow p_2$ and $(X_\epsilon, Y_\epsilon) \rightarrow (X, Y)$ (because p_1^ϵ , p_2^ϵ , X_ϵ and Y_ϵ are bounded). We choose ϵ small enough such that $|(\bar{x}, \bar{t}) - (x_\epsilon, t_\epsilon)| \leq r$ where r is defined in the assumption (C). Then $(\rho_\epsilon, p_1^\epsilon, X_\epsilon) \in \bar{\mathcal{P}}^{2,+} u(x_\epsilon, t_\epsilon)$. Moreover, $u(x_\epsilon, t_\epsilon)$, p_1^ϵ and X_ϵ are bounded (because u is locally bounded and by the last matrix inequality for X_ϵ), so, by the compactness assumption (C), ρ_ϵ is bounded from above. Similarly, by using the fact that $(-\gamma_\epsilon, -p_2^\epsilon, -Y_\epsilon) \in \bar{\mathcal{P}}^+(-v)(y_\epsilon, s_\epsilon)$, γ_ϵ is bounded from below. So, $\rho_\epsilon = \phi_t + \gamma_\epsilon$ and γ_ϵ are bounded. So, for a subsequence, we have: $\rho_\epsilon \rightarrow \rho$ and $\gamma_\epsilon \rightarrow \gamma$. Passing to the limit, yields:

$$\begin{aligned} \tau &= \rho - \gamma, \\ (\rho, p_1, X) &\in \bar{\mathcal{P}}^+ u(\bar{x}, \bar{t}), \\ (\gamma, p_2, Y) &\in \bar{\mathcal{P}}^- v(\bar{y}, \bar{t}), \\ \frac{-1}{\alpha} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} &\leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A^\alpha = (I - \alpha A)^{-1} A, \end{aligned}$$

what achieves the proof. \square

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