CONTINUOUS INTERIOR PENALTY hp-FINITE ELEMENT METHODS FOR TRANSPORT OPERATORS

ERIK BURMAN[1] AND ALEXANDRE ERN[2]

Abstract. A continuous interior penalty hp-finite element method that penalizes the jump of the discrete solution across mesh interfaces is introduced. Error estimates are obtained for first-order and advection-dominated transport operators. The analysis relies on three technical results that are of independent interest: an hp-inverse trace inequality, a local discontinuous to continuous hp-interpolation result, and hp-error estimates for continuous L^2-orthogonal projections.

1. Introduction

Continuous Interior Penalty (CIP) finite element methods have been introduced in the 1970s by Babuška and Zlámal [4] for the biharmonic operator and by Douglas and Dupont [11] for second-order elliptic and parabolic problems. The idea of such methods consists in penalizing the jump of the gradient of the discrete solution at mesh interfaces, thus weakly imposing C^1-continuity. Following this idea, Interior Penalty (IP) methods with discontinuous finite elements were designed for the biharmonic operator by Baker [5] and for second-order elliptic operators by Wheeler [21] and Arnold [1]. For second-order elliptic operators, only the jumps of the discrete solution at mesh interfaces are penalized. More recently, CIP methods experienced a further development. A priori error estimates that are uniform in the diffusion coefficient have been obtained for CIP linear finite element approximations to advection–diffusion equations by Burman and Hansbo [9]. A unified framework for the convergence analysis of both conforming and nonconforming linear finite elements with IP has been proposed by Burman [6]. Finally, a CIP linear finite element method with a nonlinear shock-capturing term that rigorously guarantees a discrete maximum principle for advection–diffusion–reaction problems has been investigated by Burman and Ern [7].

The goal of this paper is to present, for the first time, an hp-convergence analysis for a high-order CIP finite element method applied to first-order and advection-dominated second-order transport operators. The hp-version of the finite element method has been introduced in the 1980s following the analysis presented by Babuška and Dorr [2] and by Babuška and Suri [3] for second-order elliptic problems. For first-order and advection-dominated flow problems, hp-finite element approximations have been investigated by Houston, Schwab, and Süli for continuous finite elements with streamline diffusion stabilization and for IP discontinuous

Date: Draft version: 7th March 2005.

1991 *Mathematics Subject Classification.* 65N30, 65N12, 65N15, 65D05, 65N35.

Key words and phrases. Continuous interior penalty, hp-finite element method, convection–diffusion, hp-interpolation and projection, hp-inverse trace inequality.

finite elements [15, 17, 16], leading, respectively, to the so-called hp-Streamline Diffusion (SD) method and the hp-Discontinuous Galerkin (DG) method. These are, to date, the two established methods for the hp-finite element approximation of first-order and advection-dominated transport operators.

The CIP hp-finite element method investigated in this paper penalizes the jump of the gradient of the discrete solution at mesh interfaces. One advantage with respect to both the hp-SD method and the hp-DG method is that the stabilization parameter is independent of the diffusion coefficient. This can be important in nonlinear problems where this coefficient depends on the discrete solution. The other advantage with respect to the hp-DG method is that the CIP hp-finite element method requires less degrees of freedom (though only marginally less for high-order polynomials). The other advantage with respect to the hp-SD method is that the CIP hp-finite element method leads to a single, symmetric stabilization term. In the hp-SD method, the stabilization terms involve couplings with the second-order term, the source term and the time-derivative. This can cause severe problems when approximating stiff problems. The price to be paid for these advantages in the CIP hp-finite element method is on the one hand a slightly less compact discretization stencil and on the other hand a slight sub-optimality (proportional to $p^{\frac{1}{4}}$) in the error estimate. However, the present method has optimal convergence properties in the diffusion dominated regime, as opposed to the hp-DG method where a factor of $p^{\frac{1}{2}}$ is lost.

This paper is organized as follows. Section 2 introduces the discrete setting and the CIP hp-finite element method to approximate a first-order transport operator. Section 3 contains three technical results, that are of independent interest, for tensor-product finite elements. We prove an hp-inverse trace inequality, a local discontinuous to continuous hp-interpolation result, and hp-error estimates for continuous L^2-orthogonal projections. Section 4 presents the convergence analysis of the method in the spirit of the second Strang Lemma. Section 5 discusses the extension of the results derived in Sections 3 and 4 to simplicial finite elements. Section 6 investigates the extension of the results derived for tensor-product finite elements to advection–diffusion equations. Section 7 draws some conclusions.

2. Continuous interior penalty finite element methods

Let Ω be an open bounded and connected set in $\mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$ and outer normal n, let $\beta \in [W^{1,\infty}(\Omega)]^d$ be a vector field, and let $\sigma \in L^\infty(\Omega)$. Let $f \in L^2(\Omega)$, let $\partial\Omega^\pm = \{x \in \partial\Omega;\ \pm\beta(x)\cdot n(x) > 0\}$, and consider the problem

$$\begin{cases} \sigma u + \beta\cdot\nabla u = f, \\ \qquad u|_{\partial\Omega^-} = 0. \end{cases} \tag{1}$$

Define $W = \{w \in L^2(\Omega);\ \beta\cdot\nabla w \in L^2(\Omega)\}$ and observe that functions in W have traces in $L^2(\partial\Omega; \beta\cdot n)$. Consider the operator $A : W \ni w \mapsto \sigma w + \beta\cdot\nabla w \in L^2(\Omega)$. Henceforth, it is assumed that there is $\sigma_0 > 0$ such that

$$\sigma - \tfrac{1}{2}\nabla\cdot\beta \geqslant \sigma_0, \quad \text{a.e. in } \Omega. \tag{2}$$

Then, letting $V = \{w \in W;\ w|_{\partial\Omega^-} = 0\}$, $A : V \to L^2(\Omega)$ is an isomorphism, i.e., (1) is well-posed; see, e.g., [13].

Let $\mathcal{K}$ be a subdivision of Ω into non-overlapping rectangular cells $\{\kappa\}$. For $\kappa \in \mathcal{K}$, h_κ denotes its diameter. Set $h = \max_{\kappa\in\mathcal{K}} h_\kappa$. Assume that (i) $\mathcal{K}$ covers

$\overline{\Omega}$ exactly, (ii) $\mathcal{K}$ does not contain any hanging nodes, and (iii) $\mathcal{K}$ is quasi-uniform in the sense that there exists a constant $\rho > 0$, independent of h, such that $\rho h \leqslant \min_{\kappa\in\mathcal{K}} h_\kappa$. Each $\kappa \in \mathcal{K}$ is an affine image of the unit hypercube $\widehat{\kappa} = [-1, 1]^d$, i.e., $\kappa = F_\kappa(\widehat{\kappa})$. Let $\mathcal{F}$ denote the set of interior faces ($(d-1)$-manifolds) of the mesh, i.e., the set of faces that are not included in the boundary $\partial\Omega$. For $F \in \mathcal{F}$, h_F denotes its diameter.

Let $p \geqslant 1$ and let $\mathbb{Q}_{p,d}(\widehat{\kappa})$ be the space of polynomials of degree at most p in each variable. Introduce the continuous and discontinuous finite element spaces

$$V_h^p = \{\, v_h \in C^0(\overline{\Omega});\ \forall\kappa \in \mathcal{K},\ v_h|_\kappa \circ F_\kappa \in \mathbb{Q}_{p,d}(\widehat{\kappa}) \,\}, \tag{3}$$

$$W_h^p = \{\, w_h \in L^2(\Omega);\ \forall\kappa \in \mathcal{K},\ w_h|_\kappa \circ F_\kappa \in \mathbb{Q}_{p,d}(\widehat{\kappa}) \,\}. \tag{4}$$

For a subset $R \subset \Omega$, $(\cdot,\cdot)_R$ denotes the $L^2(R)$–scalar product, $\|\cdot\|_R = (\cdot,\cdot)_R^{1/2}$ the corresponding norm, and $\|\cdot\|_{s,R}$ the $H^s(R)$–norm. For $s \geqslant 1$, let $H^s(\mathcal{K})$ be the space of piecewise H^s functions. For $v \in H^2(\mathcal{K})$ and an interior face $F = \kappa_1 \cap \kappa_2$, where κ_1 and κ_2 are two distinct elements of $\mathcal{K}$ with respective outer normals n_1 and n_2, introduce the (scalar-valued) jump $[\nabla v{\cdot}n]_F = \nabla v|_{\kappa_1}{\cdot}n_1 + \nabla v|_{\kappa_2}{\cdot}n_2$ (the subscript F is dropped when there is no ambiguity). Similarly, for $v \in H^1(\mathcal{K})$, define the (scalar-valued) jump $[v]_F = v|_{\kappa_1} - v|_{\kappa_2}$ (the arbitrariness in the sign of $[v]_F$ can be avoided by considering the vector-valued jump $[v]_F = v|_{\kappa_1} n_1 + v|_{\kappa_2} n_2$; nothing that is stated hereafter depends on this arbitrariness).

On $H^1(\Omega) \times H^1(\Omega)$ define the standard Galerkin bilinear form

$$a(v,w) = \big((\sigma - \nabla{\cdot}\beta)v, w\big)_\Omega - (v, \beta{\cdot}\nabla w)_\Omega + (\beta{\cdot}nv, w)_{\partial\Omega^+}, \tag{5}$$

and on $H^q(\mathcal{K}) \times H^q(\mathcal{K})$, $q > \frac{3}{2}$, define the CIP bilinear form

$$j(v,w) = \sum_{F\in\mathcal{F}} \frac{h_F^2}{p^\alpha} |\beta{\cdot}n|_F ([\nabla v{\cdot}n], [\nabla w{\cdot}n])_F, \tag{6}$$

where $|\beta{\cdot}n|_F$ denotes the L^∞-norm of the normal component of β on F. Since $W^{1,\infty}(\Omega) \subset C^0(\overline{\Omega})$, the field β is continuous by assumption and, therefore, the quantity $\beta{\cdot}n$ is single-valued on all interior faces $F \in \mathcal{F}$. The exponent α will be determined by the convergence analysis in §4; see (39).

The finite element approximation to (1) consists of seeking $u_h \in V_h^p$ such that

$$a(u_h, v_h) + j(u_h, v_h) = (f, v_h)_\Omega, \quad \forall v_h \in V_h^p. \tag{7}$$

For $v \in H^q(\mathcal{K})$, $q > \frac{3}{2}$, consider the norm

$$\|v\|_{a,j}^2 = \|\sigma_0^{\frac{1}{2}} v\|_\Omega^2 + \tfrac{1}{2}\||\beta{\cdot}n|^{\frac{1}{2}} v\|_{\partial\Omega}^2 + j(v,v). \tag{8}$$

The well-posedness of the approximate problem (7) results from the following

Lemma 2.1 (Coerciveness)**.** *For all $v \in H^q(\mathcal{K})$, $q > \frac{3}{2}$, $a(v,v) + j(v,v) \geqslant \|v\|_{a,j}^2$.*

Proof. Straightforward verification using the divergence formula. □

Remark 2.1. Assume $p \geqslant 3$ and let $Z_h^p = V_h^p \cap C^1(\overline{\Omega})$ be the C^1-conforming subspace of V_h^p. Observe that $j(z_h, v_h) = 0$ for all $z_h \in Z_h^p$ and for all $v_h \in V_h^p$. Also observe that for a fixed mesh,

$$\lim_{p\to\infty} \frac{\dim Z_h^p}{\dim V_h^p} = 1. \tag{9}$$

Hence, in the discrete problem (7), the CIP bilinear form is not active on Z_h^p and only acts on a marginal fraction of the degrees of freedom.

3. Technical results

Henceforth, c denotes a generic constant, independent of p and h, but that can depend on the space dimension d and the quasi-uniformity parameter ρ. Its actual value can change at each occurrence.

3.1. *hp***-trace inequalities.** Let $\{g_j\}_{0\leqslant j\leqslant p}$ be the Gauß–Lobatto nodes in the unit interval $[-1,1]$. Set $I_{p,d}=\{0,\dots,p\}^d$ and $I^0_{p,d}=\{1,\dots,p{-}1\}^d$. For a multi-index $(i)=(i_1,\dots,i_d)\in I_{p,d}$, the tensor-product Gauß–Lobatto node $a_{\widehat{\kappa},(i)}$ in the unit hypercube $\widehat{\kappa}$ is the point with coordinates equal to $(g_{i_1},\dots,g_{i_d})$.

Let $\kappa\in\mathcal{K}$. Introduce the tensor-product Gauß–Lobatto nodes in K such that $a_{\kappa,(i)}=F_\kappa(a_{\widehat{\kappa},(i)})$ for all $(i)\in I_{p,d}$ and define the space

$$\mathbb{Q}^0_{p,d}(\kappa)=\{\,v\in\mathbb{Q}_{p,d}(\kappa);\ \forall(i)\in I^0_{p,d},\ v(a_{\kappa,(i)})=0\,\}. \tag{10}$$

In other words, $\mathbb{Q}^0_{p,d}(\kappa)$ is the subspace of $\mathbb{Q}_{p,d}(\kappa)$ spanned by those polynomials that vanish at all the interior tensor-product Gauß–Lobatto nodes in κ.

Lemma 3.1. *The following trace and inverse trace inequalities hold:*

$$\forall v\in\mathbb{Q}_{p,d}(\kappa),\quad \|v\|_{\partial\kappa}\leqslant d\left(\frac{p(p+1)}{2}(2+\tfrac{1}{p})^d\frac{\operatorname{meas}(\partial\kappa)}{\operatorname{meas}(\kappa)}\right)^{\frac12}\|v\|_\kappa, \tag{11}$$

$$\forall v\in\mathbb{Q}^0_{p,d}(\kappa),\quad \|v\|_{\kappa}\leqslant \left(\frac{2d}{p(p+1)}(2+\tfrac{1}{p})^{d-1}\frac{\operatorname{meas}(\kappa)}{\operatorname{meas}(\partial\kappa)}\right)^{\frac12}\|v\|_{\partial\kappa}, \tag{12}$$

with the convention that $\operatorname{meas}(\partial\kappa)=1$ *if* $d=1$.

Proof. Let $\{\varpi_j\}_{0\leqslant j\leqslant p}$ be the weights associated with the one-dimensional Gauß–Lobatto nodes. Recall that (see, e.g., [19])

$$\varpi_j=\frac{2}{p(p+1)}\frac{1}{L_p^2(g_j)},\quad j\in\{0,\dots,p\}, \tag{13}$$

where L_p is the Legendre polynomial of degree p. For a multi-index $(i)=(i_1,\dots,i_d)\in I_{p,d}$, set $\varpi_{(i)}=\prod_{l=1}^d\varpi_{i_l}$. Recall that (see, e.g., [19])

$$\forall v\in\mathbb{Q}_{p,d}(\widehat{\kappa}),\quad \|v\|_{\widehat{\kappa}}\leqslant\left(\sum_{(i)\in I_{p,d}}\varpi_{(i)}v(a_{\widehat{\kappa},(i)})^2\right)^{\frac12}\leqslant(2+\tfrac1p)^{\frac d2}\|v\|_{\widehat{\kappa}}. \tag{14}$$

(1) Proof of (11). Let $\widehat{v}\in\mathbb{Q}_{p,d}(\widehat{\kappa})$. Set $\widehat{F}_\pm=\{x_1=\pm1\}$. Using the first inequality in (14) in dimension $(d{-}1)$ on $\widehat{F}_\pm$ yields

$$\|\widehat{v}\|^2_{\widehat{F}_+\cup\widehat{F}_-}\leqslant\sum_{(i')\in I_{p,d-1}}\varpi_{(i')}\widehat{v}(a_{\widehat{F}_+,(i')})^2+\sum_{(i')\in I_{p,d-1}}\varpi_{(i')}\widehat{v}(a_{\widehat{F}_-,(i')})^2,$$

where $a_{\widehat{F}_\pm,(i')}$ are the tensor-product Gauß–Lobatto nodes on $\widehat{F}_\pm$. Using the fact that $\varpi_0=\varpi_p=\frac{2}{p(p+1)}$ leads to

$$\|\widehat{v}\|^2_{\widehat{F}_+\cup\widehat{F}_-}\leqslant\tfrac{p(p+1)}{2}\sum_{\substack{(i)\in I_{p,d}\\ i_1\in\{0,p\}}}\varpi_{(i)}\widehat{v}(a_{\widehat{\kappa},(i)})^2.$$

Hence, owing to the second inequality in (14) in dimension d, it is inferred that

$$\|\widehat{v}\|^2_{\widehat{F}_+\cup\widehat{F}_-} \leqslant \tfrac{p(p+1)}{2} \sum_{(i)\in I_{p,d}} \varpi_{(i)}\widehat{v}(a_{\widehat{\kappa},(i)})^2 \leqslant \tfrac{p(p+1)}{2}(2+\tfrac{1}{p})^d\|v\|^2_{\widehat{\kappa}}.$$

Summing over all pairs of opposite faces yields

$$\|v\|^2_{\partial\widehat{\kappa}} \leqslant d\tfrac{p(p+1)}{2}(2+\tfrac{1}{p})^d\|v\|^2_{\widehat{\kappa}}. \tag{15}$$

Let $v \in \mathbb{Q}_{p,d}(\kappa)$. Then, $v\circ F_\kappa \in \mathbb{Q}_{p,d}(\widehat{\kappa})$. Use (15) and the fact that $\mathrm{meas}(\widehat{\kappa}) = \frac{1}{d}\mathrm{meas}(\partial\widehat{\kappa})$ to infer (11).
(2) Proof of (12). Let $\widehat{v} \in \mathbb{Q}^0_{p,d}(\widehat{\kappa})$. Using the first inequality in (14) in dimension d and the second inequality in (14) in dimension $(d-1)$ yields

$$\begin{aligned}\|\widehat{v}\|^2_{\widehat{\kappa}} &\leqslant \sum_{(i)\in I_{p,d}} \varpi_{(i)}\widehat{v}(a_{\widehat{\kappa},(i)})^2 \leqslant \sum_{l=1}^{d}\sum_{\substack{(i)\in I_{p,d}\\ i_l\in\{0,p\}}} \varpi_{(i)}\widehat{v}(a_{\widehat{\kappa},(i)})^2\\ &\leqslant \sum_{l=1}^{d}\tfrac{2}{p(p+1)}\sum_{\substack{(i)\in I_{p,d}\\ i_l\in\{0,p\}}}(\varpi_{i_1}\dots\varpi_{i_{l-1}}\varpi_{i_{l+1}}\dots\varpi_{i_p})\widehat{v}(a_{\widehat{\kappa},(i)})^2\\ &\leqslant \sum_{l=1}^{d}\tfrac{2}{p(p+1)}(2+\tfrac{1}{p})^{d-1}\|\widehat{v}\|^2_{\{x_l=\pm1\}} = \tfrac{2}{p(p+1)}(2+\tfrac{1}{p})^{d-1}\|\widehat{v}\|^2_{\partial\widehat{\kappa}}.\end{aligned}$$

Let $v \in \mathbb{Q}^0_{p,d}(\kappa)$. Then, $v\circ F_\kappa \in \mathbb{Q}^0_{p,d}(\widehat{\kappa})$. Use the above inequality and the fact that $\mathrm{meas}(\widehat{\kappa}) = \frac{1}{d}\mathrm{meas}(\partial\widehat{\kappa})$ to infer (12). □

While the trace inequalities in Lemma 3.1 are valid for arbitrarily large p and d, the main focus in this paper is set on the case where d is not large with respect to p (for instance, $d \leqslant 3$ or, more generally, $d \leqslant cp$). This allows to simplify the constants in the trace inequalities as follows:

$$\forall v \in \mathbb{Q}_{p,d}(\kappa), \quad \|v\|_{\partial\kappa} \leqslant c\left(\frac{p^2}{h_\kappa}\right)^{\frac{1}{2}}\|v\|_\kappa, \tag{16}$$

$$\forall v \in \mathbb{Q}^0_{p,d}(\kappa), \quad \|v\|_{\kappa} \leqslant c\left(\frac{h_\kappa}{p^2}\right)^{\frac{1}{2}}\|v\|_{\partial\kappa}. \tag{17}$$

An important observation is that the inverse trace inequality (17) is optimal (asymptotically in p) with respect to the trace inequality (16).

3.2. **Continuous hp-interpolation.** The goal is to construct an operator $\mathcal{I}_{\mathrm{Os}} : W^p_h \to V^p_h$ endowed with a local interpolation property.

Let $\kappa \in \mathcal{K}$. For a node ν in κ, set $\mathcal{K}_\nu = \{\kappa' \in \mathcal{K};\ \nu \in \kappa'\}$; then, for $w_h \in W^p_h$, define $\mathcal{I}_{\mathrm{Os}}w_h$ locally in κ by the value it takes at all the tensor-product Gauß–Lobatto nodes by setting

$$\mathcal{I}_{\mathrm{Os}}w_h(\nu) = \frac{1}{\mathrm{card}(\mathcal{K}_\nu)}\sum_{\kappa\in\mathcal{K}_\nu} w_h|_\kappa(\nu). \tag{18}$$

Clearly, $\mathcal{I}_{\mathrm{Os}}w_h \in V^p_h$. The operator $\mathcal{I}_{\mathrm{Os}}$ is sometimes referred to as the Oswald interpolation operator; it has been considered in [6, 12, 14, 18].

Lemma 3.2. *There exists c, independent of p and h, such that, for all $\kappa \in \mathcal{K}$, the following estimate holds:*

$$\forall w_h \in W_h^p, \quad \|w_h - \mathcal{I}_{\mathrm{Os}} w_h\|_\kappa \leqslant c \left(\frac{h_\kappa}{p^2}\right)^{\frac{1}{2}} \sum_{F \in \mathcal{F}(\kappa)} \|[w_h]\|_F, \tag{19}$$

where $\mathcal{F}(\kappa) = \{F \in \mathcal{F};\ F \cap \kappa \neq \emptyset\}$.

Proof. Let $w_h \in W_h^p$ and let $\kappa \in \mathcal{K}$. Set $\delta_h = (w_h - \mathcal{I}_{\mathrm{Os}} w_h)|_\kappa$. For each tensor-product Gauß–Lobatto node $\nu \in \kappa$, let φ_ν denote the associated nodal basis function. By construction, δ_h vanishes at all the nodes located in the interior of κ. Hence,

$$\delta_h = \sum_{\nu \in \partial\kappa} \delta_h(\nu)\varphi_\nu.$$

(1) Define $\partial\kappa_{d-1}$ to be the set of $(d-1)$-manifolds in $\partial\kappa$ and for $m \in \{d-2, \dots, 0\}$, define $\partial\kappa_m$ to be the set obtained by taking the intersection of any two distinct elements in $\partial\kappa_{m+1}$. For instance, for $d = 3$, $\partial\kappa_2$, $\partial\kappa_1$, and $\partial\kappa_0$ are, respectively, the set of faces, edges, and vertices of $\partial\kappa$. Partition the boundary nodes of κ as follows. For $m \in \{0, \dots, d-1\}$, denote by $\mathcal{V}_m$ the set of nodes in the interior of $\partial\kappa_m$; then, $\bigcup_{m=0}^{d-1} \mathcal{V}_m$ forms a partition of the boundary nodes of κ. Hence,

$$\delta_h = \sum_{\nu \in \mathcal{V}_{d-1}} \delta_h(\nu)\varphi_\nu + \sum_{m=0}^{d-2} r_m,$$

with

$$r_m = \sum_{\nu \in \mathcal{V}_m} \delta_h(\nu)\varphi_\nu.$$

Observe that for all $m \in \{0, \dots, d-2\}$, $r_m \in \mathbb{Q}^0_{p,d}(\kappa)$ and

$$r_m \in \mathbb{Q}^0_{p,l}(\partial\kappa_l), \quad \forall l \in \{m+1, \dots, d-1\}.$$

(2) On each face $F \subset \partial\kappa$ and for each node ν located in the interior of F, equation (18) implies that $\delta_h(\nu) = \epsilon_F [w_h]_F(\nu)$ with $\epsilon_F = \pm 1$. Hence,

$$\sum_{\nu \in \mathcal{V}_{d-1}} \delta_h(\nu)\varphi_\nu = \sum_{F \subset \partial\kappa} \psi_F,$$

with

$$\psi_F = \epsilon_F \sum_{\nu \in \mathring{F}} [w_h]_F(\nu)\varphi_\nu.$$

Since $\psi_F \in \mathbb{Q}^0_{p,d}(\kappa)$ and ψ_F vanishes on $\partial\kappa \setminus F$, the inverse trace inequality (17) yields

$$\|\psi_F\|_\kappa \leqslant c \left(\frac{h_\kappa}{p^2}\right)^{\frac{1}{2}} \|\psi_F\|_{\partial\kappa} = c \left(\frac{h_\kappa}{p^2}\right)^{\frac{1}{2}} \|\psi_F\|_F.$$

Hence,

$$\|\delta_h\|_\kappa \leqslant c \sum_{F \subset \partial\kappa} \left(\frac{h_\kappa}{p^2}\right)^{\frac{1}{2}} \|\psi_F\|_F + \sum_{m=0}^{d-2} \|r_m\|_\kappa := \mathrm{(I)} + \mathrm{(II)}. \tag{20}$$

The rest of the proof consists of estimating both terms in the right-hand side of (20).
(3) If $d = 1$, term (II) vanishes and $\psi_F|_F = \epsilon_F [w_h]_F$; hence, (20) readily yields (19).
(4) Assume $d = 2$.

(4.a) Consider term (I). Let $F \subset \partial\kappa$. Since $\psi_F - \epsilon_F[w_h]_F$ is in $\mathbb{Q}^0_{p,1}(F)$, the inverse trace inequality (17) yields

$$\|\psi_F - \epsilon_F[w_h]_F\|_F \leqslant c\left(\frac{h_F}{p^2}\right)^{\frac{1}{2}} \|[w_h]_F\|_{\partial F}, \tag{21}$$

since ψ_F vanishes on ∂F. Owing to the trace inequality (16),

$$\|[w_h]_F\|_{\partial F} \leqslant c\left(\frac{p^2}{h_F}\right)^{\frac{1}{2}} \|[w_h]_F\|_F. \tag{22}$$

Inequalities (21) and (22) imply

$$\|\psi_F - \epsilon_F[w_h]_F\|_F \leqslant c\|[w_h]_F\|_F.$$

Collecting the above estimates and using the triangle inequality yields

$$\|\psi_F\|_\kappa \leqslant c\left(\frac{h_\kappa}{p^2}\right)^{\frac{1}{2}} (\|[w_h]_F\|_F + \|\psi_F - \epsilon_F[w_h]_F\|_F) \leqslant c\left(\frac{h_\kappa}{p^2}\right)^{\frac{1}{2}} \|[w_h]_F\|_F. \tag{23}$$

(4.b) Consider term (II). Since $r_0 \in \mathbb{Q}^0_{p,2}(\kappa)$ and $r_0 \in \mathbb{Q}^0_{p,1}(F)$ for all $F \subset \partial\kappa$, using the inverse trace inequality (17) twice yields

$$\|r_0\|_\kappa \leqslant c\left(\frac{h_\kappa}{p^2}\right)^{\frac{1}{2}} \sum_{F\subset\partial\kappa} \|r_0\|_F \leqslant c\left(\frac{h_\kappa}{p^2}\right)^{\frac{1}{2}} \sum_{F\subset\partial\kappa} \left(\frac{h_F}{p^2}\right)^{\frac{1}{2}} \|r_0\|_{\partial F}.$$

Observe that ∂F consists of two nodes and that for $\nu \in \partial F$,

$$r_0(\nu) = \sum_{F'\in\mathcal{F}_\nu} \eta_{F,F'}[w_h]_{F'}(\nu),$$

where $\mathcal{F}_\nu$ denotes the set of faces in the mesh containing the node ν and where the coefficients $\eta_{F,F'}$ can be bounded independently of p and h. Then, for all $F \subset \partial\kappa$, the following holds:

$$\begin{aligned}
\|r_0\|_{\partial F} &\leqslant c \sum_{\nu\in\partial F} \sum_{F'\in\mathcal{F}_\nu} |[w_h]_{F'}(\nu)| \\
&\leqslant c \sum_{\nu\in\partial F} \sum_{F'\in\mathcal{F}_\nu} \left(\frac{p^2}{h_{F'}}\right)^{\frac{1}{2}} \|[w_h]_{F'}\|_{F'} \\
&\leqslant c \sum_{F';F\cap F'\neq\emptyset} \left(\frac{p^2}{h_{F'}}\right)^{\frac{1}{2}} \|[w_h]_{F'}\|_{F'},
\end{aligned}$$

owing to the trace inequality (16). Hence,

$$\begin{aligned}
(\mathrm{II}) &\leqslant c\left(\frac{h_\kappa}{p^2}\right)^{\frac{1}{2}} \sum_{F\subset\partial\kappa} \left(\frac{h_F}{p^2}\right)^{\frac{1}{2}} \sum_{F';F\cap F'\neq\emptyset} \left(\frac{p^2}{h_{F'}}\right)^{\frac{1}{2}} \|[w_h]_{F'}\|_{F'} \\
&\leqslant c\left(\frac{h_\kappa}{p^2}\right)^{\frac{1}{2}} \sum_{F\in\mathcal{F}(\kappa)} \|[w_h]_F\|_F.
\end{aligned}$$

The proof is complete.

(5) For $d \geqslant 3$, the proof proceeds similarly. Term (I) is estimated as for $d = 2$. Term (II) is equal to $\|r_0\|_\kappa + \|r_1\|_\kappa$. The quantity $\|r_0\|_\kappa$ is estimated as for $d = 2$

by using the inverse trace inequality (17) three times and the trace inequality (16) twice. To estimate $\|r_1\|_\kappa$, use twice the inverse trace inequality (17) to infer

$$\|r_1\|_\kappa \leqslant c\left(\frac{h_\kappa}{p^2}\right)^{\frac{1}{2}} \sum_{F\subset\partial\kappa}\left(\frac{h_F}{p^2}\right)^{\frac{1}{2}} \sum_{E\subset\partial F}\|r_1\|_E,$$

where $E\subset\partial F$ means for all edges in ∂F. Straightforward algebra shows that

$$r_1|_E = \sum_{\nu\in\overset{\circ}{E}}\delta_h(\nu)\varphi_\nu = \sum_{\nu\in\overset{\circ}{E}}\sum_{F'\in\mathcal{F}_\nu}\eta_{F,F'}[w_h]_{F'}(\nu)\varphi_\nu.$$

Observe that the coefficient $\eta_{F,F'}$ is independent of ν. Let $\mathcal{F}_E$ be the set of faces in the mesh that contain E. Then,

$$r_1|_E = \sum_{F'\in\mathcal{F}_E}\eta_{F,F'}\sum_{\nu\in\overset{\circ}{E}}[w_h]_{F'}(\nu)\varphi_\nu.$$

Hence,

$$\|r_1\|_E \leqslant c\sum_{F'\in\mathcal{F}_E}\left\|\sum_{\nu\in\overset{\circ}{E}}[w_h]_{F'}(\nu)\varphi_\nu\right\|_E \leqslant c\sum_{F'\in\mathcal{F}_E}(\|[w_h]_{F'}\|_E + \|\zeta_{h,F'}\|_E),$$

with $\zeta_{h,F'} = \sum_{\nu\in\overset{\circ}{E}}[w_h]_{F'}(\nu)\varphi_\nu - [w_h]_{F'}$. Since $\zeta_{h,F'}\in\mathbb{Q}^0_{p,1}(E)$, the inverse trace inequality (17) yields

$$\|\zeta_{h,F'}\|_E \leqslant c\left(\frac{h_E}{p^2}\right)^{\frac{1}{2}}\sum_{\nu\in\partial E}|[w_h](\nu)|.$$

The conclusion is now straightforward. □

3.3. ***hp*-error estimate for continuous $L^2(\Omega)$-orthogonal projection.** Let $\Pi_h : L^2(\Omega)\to V_h^p$ be the $L^2(\Omega)$-orthogonal projector onto V_h^p. The purpose of this section is to investigate the approximation properties of Π_h in the L^2- and the H^1-norm.

First, we recall the following local hp-approximation property [10, 16]. Let $\Pi_h^* : L^2(\Omega)\to W_h^p$ be the $L^2(\Omega)$-orthogonal projector onto W_h^p. Then, there is c, independent of p and h, such that for all $\kappa\in\mathcal{K}$ and all $w\in H^q(\mathcal{K})$, $q\geqslant 1$,

$$\|w-\Pi_h^* w\|_\kappa \leqslant c\left(\frac{h}{p}\right)^s\|w\|_{s,\kappa}, \tag{24}$$

$$\|\nabla(w-\Pi_h^* w)\|_\kappa \leqslant cp^{\frac{1}{2}}\left(\frac{h}{p}\right)^{s-1}\|w\|_{s,\kappa}, \tag{25}$$

with $s=\min(p+1,q)$. The global counterpart of (24)–(25) for the continuous $L^2(\Omega)$-orthogonal projector Π_h is the following.

Lemma 3.3. *There exists c, independent of p and h, such that for all $w\in H^q(\Omega)$, $q\geqslant 1$,*

$$\|w-\Pi_h w\|_\Omega \leqslant c\left(\frac{h}{p}\right)^s\|w\|_{s,\Omega}, \tag{26}$$

$$\|\nabla(w-\Pi_h w)\|_\Omega \leqslant cp^{\frac{1}{2}}\left(\frac{h}{p}\right)^{s-1}\|w\|_{s,\Omega}, \tag{27}$$

with $s = \min(p+1, q)$.

Proof. Let $w \in H^q(\Omega)$, $q \geqslant 1$.
(1) Proof of (26). Since $V_h^p \subset W_h^p$, the definition of Π_h and Π_h^* leads to

$$(\Pi_h w - \Pi_h^* w, v_h) = 0, \quad \forall v_h \in V_h^p.$$

Hence, $\Pi_h \Pi_h^* w = \Pi_h w$. Using the fact that $\mathcal{I}_{\mathrm{Os}}(\Pi_h^* w) \in V_h^p$ yields

$$\|\Pi_h w - \Pi_h^* w\|_\Omega = \|\Pi_h(\Pi_h^* w) - \Pi_h^* w\|_\Omega \leqslant \|\mathcal{I}_{\mathrm{Os}}(\Pi_h^* w) - \Pi_h^* w\|_\Omega.$$

Using Lemma 3.2, it is inferred that for all $\kappa \in \mathcal{K}$,

$$\begin{aligned} \|\mathcal{I}_{\mathrm{Os}}(\Pi_h^* w) - \Pi_h^* w\|_\kappa &\leqslant c \left(\frac{h_\kappa}{p^2}\right)^{\frac{1}{2}} \sum_{F \in \mathcal{F}(\kappa)} \|[\Pi_h^* w]_F\|_F \\ &\leqslant c \left(\frac{h_\kappa}{p^2}\right)^{\frac{1}{2}} \sum_{F \in \mathcal{F}(\kappa)} \|[\Pi_h^* w - w]_F\|_F, \end{aligned}$$

since $[w]_F = 0$ by assumption. Recalling the fact [16] that there is c, independent of p and h, such that for all $\kappa \in \mathcal{K}$,

$$\|w - \Pi_h^* w\|_{\partial\kappa} \leqslant c \left(\frac{h_\kappa}{p}\right)^{s-\frac{1}{2}} \|w\|_{s,\kappa},$$

it is readily deduced that

$$\|\Pi_h w - \Pi_h^* w\|_\Omega \leqslant c p^{-\frac{1}{2}} \left(\frac{h}{p}\right)^s \|w\|_{s,\Omega}. \tag{28}$$

Conclude using the triangle inequality

$$\|w - \Pi_h w\|_\Omega \leqslant \|w - \Pi_h^* w\|_\Omega + \|\Pi_h^* w - \Pi_h w\|_\Omega,$$

and Equation (24).
(2) Proof of (27). Let $\kappa \in \mathcal{K}$. Recalling the inverse inequality

$$\|\nabla v_h\|_\kappa \leqslant c \frac{p^2}{h_\kappa} \|v_h\|_\kappa, \tag{29}$$

valid for all $v_h \in W_h^p$ (see, e.g., [10] for the proof of the p version of this inequality) and using (28), it is inferred that

$$\left(\sum_{\kappa \in \mathcal{K}} \|\nabla(\Pi_h^* w - \Pi_h w)\|_\kappa^2\right)^{\frac{1}{2}} \leqslant c p^{\frac{1}{2}} \left(\frac{h}{p}\right)^{s-1} \|w\|_{s,\Omega}. \tag{30}$$

Conclude using the triangle inequality and Equation (25). □

Remark 3.1. Equation (28) shows that $\Pi_h w$ superconverges to $\Pi_h^* w$ by a factor of $p^{-\frac{1}{2}}$. This remarkable property allows to compensate for the loss of one power of p in the inverse inequality (29) and thus to recover a sub-optimality factor of $p^{\frac{1}{2}}$ in (27), which is exactly the same as in (25).

4. Convergence analysis

The purpose of this section is to show how the results established in §3, namely the hp-inverse trace inequality, the local hp-interpolation result, and the hp-error estimate for the continuous $L^2(\Omega)$-orthogonal projection, can be used to analyze the convergence of the CIP hp-finite element method introduced in §2.

Let u solve (1) and let u_h solve (7). Henceforth, it is assumed that the exact solution u is smooth enough, i.e., $u \in H^q(\Omega)$, $q > \frac{3}{2}$. Bounds on the approximation error $u - u_h$ are obtained in the spirit of the Second Strang Lemma by establishing consistency and boundedness properties for the discrete setting. Recall that the discrete setting satisfies the stability property stated in Lemma 2.1.

Lemma 4.1 (Consistency)**.** *Let $u \in H^q(\Omega)$, $q > \frac{3}{2}$, solve* (1) *and let u_h solve* (7)*. Then, for all $v_h \in V_h^p$,*

$$a(u - u_h, v_h) + j(u - u_h, v_h) = 0. \tag{31}$$

Proof. Since $u \in H^q(\Omega)$, $q > \frac{3}{2}$, it is inferred that $[\nabla u{\cdot}n]_F = 0$ for all $F \in \mathcal{F}$. Hence, $j(u, v_h) = 0$ for all $v_h \in V_h^p$, whence (31) is readily deduced. $\square$

For all $\kappa \in \mathcal{K}$, set $\beta_{\kappa,\infty} := \|\beta\|_{L^\infty(\Delta_\kappa)}$ where $\Delta_\kappa = \{\kappa' \in \mathcal{K};\ \kappa \cap \kappa' \neq \emptyset\}$. Introduce the (semi-)norm

$$\|v\|_{h,\frac{1}{2}} = \left(\sum_{\kappa \in \mathcal{K}} h_\kappa^{-1} \beta_{\kappa,\infty} \|v\|_\kappa^2 \right)^{\frac{1}{2}}. \tag{32}$$

Lemma 4.2 (Boundedness)**.** *There is c, independent of p and h, such that for all $z \in H^q(\mathcal{K}) \cap (V_h^p)^\perp$, $q > \frac{3}{2}$,*

$$\sup_{v_h \in V_h^p} \frac{a(z, v_h) + j(z, v_h)}{\|v_h\|_{a,j}} \leqslant \|z\|_{a,j} + c(p^2 h^{\frac{1}{2}} + p^{\frac{\alpha}{2}-1}) \|z\|_{h,\frac{1}{2}}. \tag{33}$$

Proof. The only term to estimate is $(z, \beta{\cdot}\nabla v_h)_\Omega$. Let β_h the $L^2(\Omega)$-orthogonal projection of β onto W_h^0. Set $w_h = \beta_h{\cdot}\nabla v_h$. Observe that

$$(z, \beta{\cdot}\nabla v_h)_\Omega = (z, (\beta - \beta_h){\cdot}\nabla v_h)_\Omega + (z, w_h)_\Omega. \tag{34}$$

(1) Since $\beta \in [W^{1,\infty}(\Omega)]^d$, the first term in the right-hand side of (34) is estimated as follows:

$$\begin{aligned}(z, (\beta - \beta_h){\cdot}\nabla v_h)_\Omega &\leqslant \left(\sum_{\kappa \in \mathcal{K}} h_\kappa^{-1} \beta_{\kappa,\infty} \|z\|_\kappa^2 \right)^{\frac{1}{2}} \left(\sum_{\kappa \in \mathcal{K}} h_\kappa^3 \beta_{\kappa,\infty}^{-1} \|\beta\|_{W^{1,\infty}(\kappa)}^2 \|\nabla v_h\|_\kappa^2 \right)^{\frac{1}{2}} \\ &\leqslant c p^2 h^{\frac{1}{2}} \|z\|_{h,\frac{1}{2}} \|v_h\|_\Omega,\end{aligned}$$

owing to the inverse inequality (29).

(2) Let us estimate the second term in the right-hand side of (34). Using the fact

that $z \in (V_h^p)^\perp$ yields

$$
\begin{aligned}
(z, w_h)_\Omega &= (z, w_h - \mathcal{I}_{\mathrm{Os}} w_h)_\Omega \\
&\leqslant \left(\sum_{\kappa\in\mathcal{K}} h_\kappa^{-1} \beta_{\kappa,\infty} \|z\|_\kappa^2 \right)^{\frac12} \left(\sum_{\kappa\in\mathcal{K}} h_\kappa \beta_{\kappa,\infty}^{-1} \|w_h - \mathcal{I}_{\mathrm{Os}} w_h\|_\kappa^2 \right)^{\frac12} \\
&\leqslant c\|z\|_{h,\frac12} \left(\sum_{\kappa\in\mathcal{K}} \sum_{F\in\mathcal{F}(\kappa)} \beta_{\kappa,\infty}^{-1} \left(\frac{h_\kappa}{p}\right)^2 \|[w_h]_F\|_F^2 \right)^{\frac12},
\end{aligned}
$$

owing to Lemma 3.2. Observe that

$$
\|[w_h]\|_F = \|[\beta_h\cdot\nabla v_h]_F\|_F \leqslant \|[(\beta-\beta_h)\cdot\nabla v_h]_F\|_F + \|[\beta\cdot\nabla v_h]_F\|_F. \tag{35}
$$

(2.a) Using the fact that $\beta \in [W^{1,\infty}(\Omega)]^d$, together with inequalities (16) and (29), the first term in the right-hand side of (35) is estimated as follows:

$$
\begin{aligned}
\|[(\beta-\beta_h)\cdot\nabla v_h]_F\|_F &\leqslant c \sum_{\kappa\supset F} h_\kappa \|\beta\|_{W^{1,\infty}(\kappa)} \|\nabla v_h|_\kappa\|_F \\
&\leqslant c \sum_{\kappa\supset F} h_\kappa \|\beta\|_{W^{1,\infty}(\kappa)} \left(\frac{p^2}{h_\kappa}\right)^{\frac12} \|\nabla v_h\|_\kappa \\
&\leqslant c \sum_{\kappa\supset F} h_\kappa \|\beta\|_{W^{1,\infty}(\kappa)} \left(\frac{p^2}{h_\kappa}\right)^{\frac12} \frac{p^2}{h_\kappa} \|v_h\|_\kappa \\
&\leqslant c \sum_{\kappa\supset F} \|\beta\|_{W^{1,\infty}(\kappa)} \frac{p^3}{h_\kappa^{\frac12}} \|v_h\|_\kappa,
\end{aligned}
$$

where the notation $\kappa \supset F$ means for the two elements sharing F.
(2.b) To estimate the second term in the right-hand side of (35), notice that $[\beta\cdot\nabla v_h]_F = |\beta\cdot n|_F \|[\nabla v_h\cdot n]\|_F$ owing to the continuity of β and v_h. Hence,

$$
\begin{aligned}
\sum_{\kappa\in\mathcal{K}} \sum_{F\in\mathcal{F}(\kappa)} \beta_{\kappa,\infty}^{-1} \left(\frac{h_\kappa}{p}\right)^2 \|[\beta\cdot\nabla v_h]_F\|_F^2 &\leqslant c \sum_{F\in\mathcal{F}} \left(\frac{h_F}{p}\right)^2 |\beta\cdot n|_F \|[\nabla v_h\cdot n]_F\|_F^2 \\
&\leqslant c p^{\alpha-2} j(v_h, v_h).
\end{aligned}
$$

(2.c) Collecting the estimates obtained in steps (2.a) and (2.b) yields

$$
(z, w_h)_\Omega \leqslant c\|z\|_{h,\frac12} (p^2 h^{\frac12} + p^{\frac{\alpha}{2}-1}) \|v\|_{a,j}.
$$

The proof is complete. □

Lemma 4.3. *There is c, independent of p and h, such that*

$$
\|u - u_h\|_{a,j} \leqslant c\|u - \Pi_h u\|_{a,j} + c(p^2 h^{\frac12} + p^{\frac{\alpha}{2}-1}) \|u - \Pi_h u\|_{h,\frac12}. \tag{36}
$$

Proof. Owing to Lemma 2.1 and Lemma 4.1,

$$
\begin{aligned}
\|\Pi_h u - u_h\|_{a,j}^2 &\leqslant a(\Pi_h u - u_h, \Pi_h u - u_h) + j(\Pi_h u - u_h, \Pi_h u - u_h) \\
&= a(\Pi_h u - u, \Pi_h u - u_h) + j(\Pi_h u - u, \Pi_h u - u_h).
\end{aligned}
$$

Since $\Pi_h u - u \in (V_h^p)^\perp$, Lemma 4.2 leads to

$$
\|\Pi_h u - u_h\|_{a,j} \leqslant \|\Pi_h u - u\|_{a,j} + c(p^2 h^{\frac12} + p^{\frac{\alpha}{2}-1}) \|\Pi_h u - u\|_{h,\frac12}.
$$

Conclude using the triangle inequality. □

Lemma 4.4. *There is c, independent of p and h, such that for all $w \in H^q(\Omega)$, $q > \frac{3}{2}$,*

$$\|w - \Pi_h w\|_{h,\frac{1}{2}} \leqslant c p^{-\frac{1}{2}} \left(\frac{h}{p}\right)^{s-\frac{1}{2}} \|w\|_{s,\Omega}, \tag{37}$$

$$\|w - \Pi_h w\|_{a,j} \leqslant c(p^{\frac{1}{4}} + p^{2-\frac{\alpha}{2}}) \left(\frac{h}{p}\right)^{s-\frac{1}{2}} \|w\|_{s,\Omega}, \tag{38}$$

with $s = \min(p+1, q)$.

Proof. Estimate (37) directly results from (26). To prove (38), first notice that owing to the trace inequality

$$\|v\|_{\partial\Omega} \leqslant c\|v\|_{\Omega}^{\frac{1}{2}} \|\nabla v\|_{\Omega}^{\frac{1}{2}},$$

valid for all $v \in H^1(\Omega)$, it is inferred from Lemma 3.3 that

$$\|w - \Pi_h w\|_{\partial\Omega} \leqslant c p^{\frac{1}{4}} \left(\frac{h}{p}\right)^{s-\frac{1}{2}} \|w\|_{s,\Omega}.$$

To control $j(w - \Pi_h w, w - \Pi_h w)$, observe that for all $\kappa \in \mathcal{K}$,

$$\begin{aligned} \|\nabla(w - \Pi_h w)\|_{\partial\kappa} &\leqslant \|\nabla(w - \Pi_h^* w)\|_{\partial\kappa} + \|\nabla(\Pi_h w - \Pi_h^* w)\|_{\partial\kappa} \\ &\leqslant \|\nabla(w - \Pi_h^* w)\|_{\partial\kappa} + p h_\kappa^{-\frac{1}{2}} \|\nabla(\Pi_h w - \Pi_h^* w)\|_{\kappa}, \end{aligned}$$

owing to the trace inequality (16). Recall the fact [16] that there is c, independent of p and h, such that for all $\kappa \in \mathcal{K}$,

$$\|\nabla(w - \Pi_h^* w)\|_{\partial\kappa} \leqslant c p \left(\frac{h_\kappa}{p}\right)^{s-\frac{3}{2}} \|w\|_{s,\kappa}.$$

As a result,

$$\begin{aligned} j(w - \Pi_h w, w - \Pi_h w) &\leqslant \sum_{F \in \mathcal{F}} h_F^2 p^{-\alpha} |\beta \cdot n|_F \|[\nabla(w - \Pi_h w) \cdot n]_F\|_F^2 \\ &\leqslant c \sum_{\kappa \in \mathcal{K}} \left(h_\kappa^2 p^{-\alpha} \|\nabla(w - \Pi_h^* w)\|_{\partial\kappa}^2 + h_\kappa p^{2-\alpha} \|\nabla(\Pi_h w - \Pi_h^* w)\|_\kappa^2 \right) \\ &\leqslant c \left(p^{4-\alpha} (\tfrac{h}{p})^{2s-1} \|w\|_{s,\Omega}^2 + h p^{2-\alpha} \sum_{\kappa \in \mathcal{K}} \|\nabla(\Pi_h w - \Pi_h^* w)\|_\kappa^2 \right) \\ &\leqslant c p^{4-\alpha} (\tfrac{h}{p})^{2s-1} \|w\|_{s,\Omega}^2, \end{aligned}$$

owing to (30). The conclusion is straightforward. □

We are now in a position to state the main result of this section.

Theorem 4.1. *Let $u \in H^q(\Omega)$, $q > \frac{3}{2}$, solve* (1) *and let u_h solve* (7). *Take*

$$\alpha = \frac{7}{2}. \tag{39}$$

Then, there is c, independent of p and h, such that

$$\|u - u_h\|_{a,j} \leqslant c(p^{\frac{1}{4}} + p^{\frac{3}{2}} h^{\frac{1}{2}}) \left(\frac{h}{p}\right)^{s-\frac{1}{2}} \|u\|_{s,\Omega}, \tag{40}$$

with $s = \min(p+1, q)$*. Hence, if* $h \leqslant p^{-\frac{5}{2}}$*, the following holds:*

$$\|u - u_h\|_{a,j} \leqslant cp^{\frac{1}{4}} \left(\frac{h}{p}\right)^{s-\frac{1}{2}} \|u\|_{s,\Omega}. \tag{41}$$

Proof. Since $\frac{\alpha}{2} - 1 = \frac{3}{4}$, Lemma 4.3 yields

$$\begin{aligned}\|u - u_h\|_{a,j} &\leqslant c\|u - \Pi_h u\|_{a,j} + c(p^2 h^{\frac{1}{2}} + p^{\frac{3}{4}})\|u - \Pi_h u\|_{h,\frac{1}{2}} \\ &\leqslant c\|u - \Pi_h u\|_{a,j} + c(p^{\frac{3}{2}} h^{\frac{1}{2}} + p^{\frac{1}{4}}) \left(\frac{h}{p}\right)^{s-\frac{1}{2}} \|w\|_{s,\Omega},\end{aligned}$$

owing to (37). Since $2 - \frac{\alpha}{2} = \frac{1}{4}$, (38) leads to

$$\|u - \Pi_h u\|_{a,j} \leqslant cp^{\frac{1}{4}} \left(\frac{h}{p}\right)^{s-\frac{1}{2}} \|u\|_{s,\Omega},$$

whence (40) is readily deduced. Finally, under the assumption that $h \leqslant p^{-\frac{5}{2}}$, (41) directly results from (40). $\square$

5. Extension to simplices

The goal of this section is to discuss the extension of the analysis presented in §2–§4 to simplicial finite elements. Most of the proofs of the results stated hereafter are easily adapted from those presented above and are therefore only sketched.

Let $\mathcal{K}$ be a subdivision of Ω into affine simplices $\{\kappa\}$. Assume that (i) $\mathcal{K}$ covers $\overline{\Omega}$ exactly, (ii) $\mathcal{K}$ does not contain any hanging nodes, and (iii) $\mathcal{K}$ is locally quasi-uniform in the sense that there exists a constant $\rho > 0$, independent of h, such that $\rho \max_{E\subset\Delta_\kappa} h_E \leqslant \min_{E\subset\Delta_\kappa} h_E$ where $E \subset \Delta_\kappa$ means for all edges in the patch Δ_κ and h_E denotes the diameter of E.

Let $p \geqslant 1$ and let $\mathbb{P}_{p,d}(\widehat{\kappa})$ be the space of polynomials of total degree at most p. Introduce the continuous and discontinuous finite element spaces

$$V_h^p = \{\, v_h \in C^0(\overline{\Omega});\ \forall \kappa \in \mathcal{K},\ v_h|_\kappa \in \mathbb{P}_{p,d}(\kappa)\,\}, \tag{42}$$

$$W_h^p = \{\, w_h \in L^2(\Omega);\ \forall \kappa \in \mathcal{K},\ w_h|_\kappa \in \mathbb{P}_{p,d}(\kappa)\,\}. \tag{43}$$

5.1. **Trace and inverse trace inequalities on simplices.**

Recall the following trace inequality due to Warburton and Hesthaven [20].

Lemma 5.1. *Let* $\kappa \in \mathcal{K}$*. The following holds:*

$$\forall v \in \mathbb{P}_{p,d}(\kappa), \quad \|v\|_{\partial\kappa} \leqslant \left(\frac{(p+1)(p+d)}{d} \frac{\mathrm{meas}(\partial\kappa)}{\mathrm{meas}(\kappa)}\right)^{\frac{1}{2}} \|v\|_\kappa. \tag{44}$$

To investigate inverse trace inequalities in $\mathbb{P}_{p,d}(\kappa)$, set $n_{p,d} = \dim(\mathbb{P}_{p,d})$ and let $\{\phi_i\}_{1\leqslant i\leqslant n_{p,d}}$ be the canonical basis of $\mathbb{P}_{p,d}(\kappa)$. Let $\mathcal{A} = \{a_i\}_{1\leqslant i\leqslant n_{p,d}}$ be a set of nodes in κ. The set $\mathcal{A}$ is said to be *unisolvent* in $\mathbb{P}_{p,d}(\kappa)$ if the matrix with coefficients $(\phi_i(a_j))_{1\leqslant i,j\leqslant n_{p,d}}$ is invertible. It is said to be *admissible* if it is unisolvent in $\mathbb{P}_{p,d}(\kappa)$ and if for all dimension $d' < d$ and for any simplex $\kappa' \subset \partial\kappa$ of dimension d', the set of nodes in κ' is unisolvent in $\mathbb{P}_{p,d'}(\kappa')$. Given a unisolvent set of nodes $\mathcal{A}$, define

$$\mathbb{P}_{p,d}^{\mathcal{A}}(\kappa) = \{\, v \in \mathbb{P}_{p,d}(\kappa);\ \forall a_i \in \mathcal{A},\ v(a_i) = 0 \text{ if } a_i \notin \partial\kappa\,\}. \tag{45}$$

Lemma 5.2. *For all unisolvent set of nodes $\mathcal{A}$ in $\mathbb{P}_{p,d}(\kappa)$, the following holds:*

$$\inf_{v\in\mathbb{P}^{\mathcal{A}}_{p,d}(\kappa)\setminus\{0\}} \frac{\|v\|_{\partial\kappa}}{\|v\|_{\kappa}} \leqslant \left(\frac{2p+d}{d}\frac{\text{meas}(\partial\kappa)}{\text{meas}(\kappa)}\right)^{\frac{1}{2}}. \tag{46}$$

Proof. Estimate (46) is established on the reference simplex and then by mapping it to an arbitrary element in $\mathcal{K}$.
(1) Choose the $L^2(\widehat{\kappa})$-orthonormal basis of $\mathbb{P}_{p,d}(\widehat{\kappa})$ considered in [20]. Define the matrix $\widehat{F}$ of order $n_{p,d}$ such that $(\widehat{V},\widehat{F}\widehat{V})_{\mathbb{R}^{n_{p,d}}} = \|\widehat{v}\|^2_{\partial\widehat{\kappa}}$ for all $\widehat{v}\in\mathbb{P}_{p,d}(\widehat{\kappa})$, where $\widehat{V}$ is the coordinate vector of $\widehat{v}$ in the selected basis and where $(\cdot,\cdot)_{\mathbb{R}^{n_{p,d}}}$ denotes the Euclidean scalar product in $\mathbb{R}^{n_{p,d}}$. Clearly, $\widehat{F}$ is symmetric and positive semi-definite. The advantage of the basis selected in [20] is that $\widehat{F}$ admits a block-structure with blocks having rank 1.
(2) To specify this structure, define $J_{p,d}=\{(i)\in I_{p,d},\ \sum_{l=1}^{d} i_l\leqslant p\}$. The matrix $\widehat{F}$ is then conveniently indexed by multi-indices (i) and (j) in $J_{p,d}$, and its entries are given by

$$\widehat{F}_{(i)(j)}=\prod_{l=1}^{d-1}\delta_{i_l j_l}\,(-1)^{i_d}\left(\frac{2N_d(i)+d}{2}\right)^{\frac{1}{2}}(-1)^{j_d}\left(\frac{2N_d(j)+d}{2}\right)^{\frac{1}{2}},$$

where $N_d(i)=\sum_{l=1}^{d} i_l$. Let $r\in\{0,\dots,p\}$. With any $(i')\in I_{p,d-1}$ such that $\sum_{l=1}^{d-1} i'_l = r$, we can associate a block $\widehat{F}_{r,(i')}$ in $\widehat{F}$ corresponding to the multi-indices $(i)=(i'_1,\dots,i'_{d-1},i_d)$. In [20], it is proven that the block $\widehat{F}_{r,(i')}$ is of size $(p-r+1)$, of rank 1, and its trace is equal to

$$\text{tr}(\widehat{F}_{r,(i')})=\sum_{i_d=0}^{p-r}\tfrac{1}{2}(2r+2i_d+d)=\tfrac{1}{2}(p-r+1)(p+r+d).$$

(3) In [20], the largest eigenvalue of $\widehat{F}$ is estimated whereas here we are interested in the lowest non-zero eigenvalue. Since the blocks $\widehat{F}_{r,(i')}$ are of rank 1, the spectrum of $\widehat{F}$ is equal to

$$\sigma(\widehat{F})=\{\tfrac{1}{2}(p-r+1)(p+r+d)\}_{0\leqslant r\leqslant p}\cup\{0\}.$$

Since the product $(p-r+1)(p+r+d)$ is monotonically decreasing in r, it is inferred that the smallest non-zero eigenvalue of $\widehat{F}$ is $\frac{2p+d}{2}$. The conclusion is straightforward. $\square$

Lemma 5.2 shows that in any simplex of dimension $d\geqslant 2$, the inverse trace inequality cannot be optimal with respect to the trace inequality, i.e., at best a factor $p^{\frac{1}{2}}$ is lost.

The rest of the analysis presented in this section is restricted to triangles and relies on the following hypothesis.

Hypothesis 5.1. *There exists an admissible set of nodes $\mathcal{A}^*$ in $\mathbb{P}_{p,2}(\widehat{\kappa})$ such that*

$$\forall v\in\mathbb{P}^{\mathcal{A}^*}_{p,2}(\widehat{\kappa}),\quad \|v\|_{\widehat{\kappa}}\leqslant cp^{-\frac{1}{2}}\|v\|_{\partial\widehat{\kappa}}. \tag{47}$$

Henceforth, for all $\kappa\in\mathcal{K}$, $\mathbb{P}^0_{p,d}(\kappa)$ denotes the polynomial set $\mathbb{P}^{\mathcal{A}^*_\kappa}_{p,2}(\kappa)$ where $\mathcal{A}^*_\kappa$ is the image of the set $\mathcal{A}^*$ by the transformation mapping $\widehat{\kappa}$ onto κ. If d is not large with respect to p, estimate (44) and Hypothesis 5.1 lead to the following trace and inverse trace inequalities on triangles:

$$\forall v \in \mathbb{P}_{p,d}(\kappa), \quad \|v\|_{\partial\kappa} \leqslant c \left(\frac{p^2}{h_\kappa}\right)^{\frac{1}{2}} \|v\|_\kappa, \tag{48}$$

$$\forall v \in \mathbb{P}^0_{p,d}(\kappa), \quad \|v\|_\kappa \leqslant c \left(\frac{h_\kappa}{p}\right)^{\frac{1}{2}} \|v\|_{\partial\kappa}. \tag{49}$$

5.2. ***hp*-interpolation and projection on triangles.** Let $\kappa \in \mathcal{K}$. For a node ν in $\mathcal{A}^*_\kappa$, set $\mathcal{K}_\nu = \{\kappa' \in \mathcal{K};\ \nu \in \kappa'\}$; then, for $w_h \in W^p_h$, define $\mathcal{I}_{\mathrm{Os}} w_h$ locally in κ by the value it takes at all the nodes in $\mathcal{A}^*_\kappa$ by setting

$$\mathcal{I}_{\mathrm{Os}} w_h(\nu) = \frac{1}{\mathrm{card}(\mathcal{K}_\nu)} \sum_{\kappa \in \mathcal{K}_\nu} w_h|_\kappa(\nu). \tag{50}$$

Clearly, $\mathcal{I}_{\mathrm{Os}} w_h \in V^p_h$.

Lemma 5.3. *There exists c, independent of p and h, such that, for all $\kappa \in \mathcal{K}$, the following estimate holds:*

$$\forall w_h \in W^p_h, \quad \|w_h - \mathcal{I}_{\mathrm{Os}} w_h\|_\kappa \leqslant c \left(\frac{h_\kappa}{p}\right)^{\frac{1}{2}} \sum_{F \in \mathcal{F}(\kappa)} \|[w_h]\|_F. \tag{51}$$

Proof. The proof is the same as that of Lemma 3.2 except that the trace and inverse trace inequalities (48) and (49) are used instead of (16) and (17). In dimension 2, the inverse trace inequality (49) is used once so that a factor of $p^{-\frac{1}{2}}$ is lost in (51) with respect to (19). □

Lemma 5.4. *There exists c, independent of p and h, such that for all $w \in H^q(\Omega)$, $q \geqslant 1$,*

$$\|w - \Pi_h w\|_\Omega \leqslant c \left(\frac{h}{p}\right)^s \|w\|_{s,\Omega}, \tag{52}$$

$$\|\nabla(w - \Pi_h w)\|_\Omega \leqslant cp \left(\frac{h}{p}\right)^{s-1} \|w\|_{s,\Omega}, \tag{53}$$

with $s = \min(p+1, q)$.

Proof. The proof is similar to that of Lemma 3.3 except that the bound (51) is used instead of (19). □

Remark 5.1. The superconvergence of $\Pi_h w$ to $\Pi^*_h w$ is lost on simplices since

$$\|\Pi_h w - \Pi^*_h w\|_\Omega \leqslant c \left(\frac{h}{p}\right)^s \|w\|_{s,\Omega}.$$

As a result, the L^2-estimate (52) is still optimal, but the H^1-estimate (53) is now suboptimal by a factor of p.

5.3. **Convergence analysis on triangles.** Let $u \in H^q(\Omega)$, $q > \frac{3}{2}$, solve (1) and let u_h solve (7). The convergence analysis proceeds as in §4.

Lemma 5.5 (Boundedness)**.** *There is c, independent of p and h, such that for all $z \in H^q(\mathcal{K}) \cap (V_h^p)^\perp$, $q > \frac{3}{2}$,*

$$\sup_{v_h \in V_h^p} \frac{a(z, v_h) + j(z, v_h)}{\|v_h\|_{a,j}} \leqslant \|z\|_{a,j} + c(p^{\frac{\alpha}{2}-\frac{1}{2}} + p^2 h^{\frac{1}{2}})\|z\|_{h,\frac{1}{2}}. \tag{54}$$

Proof. The proof is similar to that of Lemma 4.2. However, on simplices, β_h can be chosen to be in V_h^1. As a result, there is no need to invoke the triangle inequality (35), and it is directly inferred that

$$\sum_{\kappa \in \mathcal{K}} \sum_{F \in \mathcal{F}(\kappa)} \beta_{\kappa,\infty}^{-1} \left(\frac{h_\kappa}{p}\right)^2 \|[\beta_h \cdot \nabla v_h]_F\|_F^2 \leqslant c p^{\alpha-2} j(v_h, v_h).$$

The conclusion is straightforward. □

Lemma 5.6. *There is c, independent of p and h, such that for all $w \in H^q(\Omega)$, $q > \frac{3}{2}$,*

$$\|w - \Pi_h w\|_{h,\frac{1}{2}} \leqslant c p^{-\frac{1}{2}} \left(\frac{h}{p}\right)^{s-\frac{1}{2}} \|w\|_{s,\Omega}, \tag{55}$$

$$\|w - \Pi_h w\|_{a,j} \leqslant c(p^{\frac{1}{2}} + p^{\frac{5}{2}-\frac{\alpha}{2}}) \left(\frac{h}{p}\right)^{s-\frac{1}{2}} \|w\|_{s,\Omega}, \tag{56}$$

with $s = \min(p+1, q)$.

Proof. Similar to that of Lemma 4.4. □

We are now in a position to state the main result of this section.

Theorem 5.1. *Let $u \in H^q(\Omega)$, $q > \frac{3}{2}$, solve* (1) *and let u_h solve* (7). *Take $\alpha = \frac{7}{2}$. Then, under Hypothesis* 5.1, *there is c, independent of p and h, such that*

$$\|u - u_h\|_{a,j} \leqslant c(p^{\frac{3}{4}} + p^{\frac{3}{2}} h^{\frac{1}{2}}) \left(\frac{h}{p}\right)^{s-\frac{1}{2}} \|u\|_{s,\Omega}, \tag{57}$$

with $s = \min(p+1, q)$. Hence, if $h \leqslant p^{-\frac{3}{2}}$, the following holds:

$$\|u - u_h\|_{a,j} \leqslant c p^{\frac{3}{4}} \left(\frac{h}{p}\right)^{s-\frac{1}{2}} \|u\|_{s,\Omega}. \tag{58}$$

Proof. Proceeding as in the proof of Theorem 4.1, it is inferred that

$$\|u - u_h\|_{a,j} \leqslant c \left(p^{\frac{1}{2}} + p^{\frac{5}{2}-\frac{\alpha}{2}} + p^{\frac{\alpha}{2}-1} + p^{\frac{3}{2}} h^{\frac{1}{2}}\right) \left(\frac{h}{p}\right)^{s-\frac{1}{2}} \|u\|_{s,\Omega}.$$

Since $\alpha = \frac{7}{2}$, it is clear that $\frac{5}{2} - \frac{\alpha}{2} = \frac{\alpha}{2} - 1 = \frac{3}{4} \geqslant \frac{1}{2}$. The conclusion is straightforward. □

6. Advection–diffusion problems

The purpose of this section is to show that the CIP bilinear form j defined by equation (6) is also suitable to approximate advection–diffusion problems in both the advection dominated and the diffusion dominated regimes. The discussion is restricted to finite element approximations on tensor-product meshes.

Consider the following PDE with mixed Neumann–Robin boundary conditions

$$\begin{cases} -\varepsilon\Delta u + \beta\cdot\nabla u + \sigma u = f, \\ (\beta u - \varepsilon\nabla u)\cdot n|_{\partial\Omega^-} = 0, \\ \varepsilon\nabla u\cdot n|_{\partial\Omega\setminus\partial\Omega^-} = 0, \end{cases} \tag{59}$$

where $\varepsilon > 0$ is a real parameter and where f, β, and σ satisfy the assumptions stated in §2. Without loss of generality, it is assumed that the problem (59) is non-dimensionalized so that β is of order unity. Note that Neumann and Robin boundary conditions are natural outflow and inflow conditions, respectively, for advection–diffusion problems.

On $H^1(\Omega)\times H^1(\Omega)$ define the standard Galerkin bilinear form

$$a_\varepsilon(u,v) = \varepsilon(\nabla u,\nabla v)_\Omega + \big((\sigma-\nabla\cdot\beta)u,v\big)_\Omega - (u,\beta\cdot\nabla v)_\Omega + (\beta\cdot nu,v)_{\partial\Omega^+}. \tag{60}$$

The finite element approximation to (59) consists of seeking $u_h \in V_h^p$ such that

$$a_\varepsilon(u_h,v_h) + j(u_h,v_h) = (f,v_h)_\Omega, \quad \forall v_h \in V_h^p. \tag{61}$$

For $v \in H^q(\mathcal{K})$, $q > \frac{3}{2}$, consider the norm

$$\|v\|_{\varepsilon,j}^2 = \|\varepsilon^{\frac12}\nabla v\|_\Omega^2 + \|\sigma_0^{\frac12} v\|_\Omega^2 + \tfrac12\||\beta\cdot n|^{\frac12} v\|_{\partial\Omega}^2 + j(v,v). \tag{62}$$

The discrete problem (61) is clearly endowed with stability and consistency properties.

Lemma 6.1 (Coerciveness)**.** *For all $v \in H^q(\mathcal{K})$, $q > \frac32$, $a_\varepsilon(v,v) + j(v,v) \geqslant \|v\|_{\varepsilon,j}^2$.*

Lemma 6.2 (Consistency)**.** *Let $u \in H^q(\Omega)$, $q > \frac32$, solve* (59) *and let u_h solve* (61)*. Then, for all $v_h \in V_h^p$,*

$$a_\varepsilon(u-u_h,v_h) + j(u-u_h,v_h) = 0. \tag{63}$$

Proposition 6.1 (Dominant advection)**.** *Let $u \in H^q(\Omega)$, $q > \frac32$, solve* (59) *and let u_h solve* (61)*. Take $\alpha = \frac72$. Assume $h \leqslant p^{-\frac32}$. Then, there is c, independent of p and h, such that*

$$\|u-u_h\|_{\varepsilon,j} \leqslant c(\varepsilon^{\frac12} p h^{-\frac12} + p^{\frac14}) \left(\frac{h}{p}\right)^{s-\frac12} \|u\|_{s,\Omega}, \tag{64}$$

with $s = \min(p+1,q)$. Hence, if $\varepsilon \leqslant hp^{-\frac32}$, the following holds:

$$\|u-u_h\|_{\varepsilon,j} \leqslant c p^{\frac14} \left(\frac{h}{p}\right)^{s-\frac12} \|u\|_{s,\Omega}. \tag{65}$$

Proof. Proceed as in the proof of Theorem 4.1. The only additional term to estimate is $\|\varepsilon^{\frac12}\nabla(u-\Pi_h u)\|_\Omega$. Use (27) to infer

$$\|\varepsilon^{\frac12}\nabla(u-\Pi_h u)\|_\Omega \leqslant c p^{\frac12}\varepsilon^{\frac12}\left(\frac{h}{p}\right)^{s-1}\|u\|_{s,\Omega}.$$

The conclusion is straightforward. □

Proposition 6.2 (Dominant diffusion)**.** *Let $u \in H^q(\Omega)$, $q \geqslant 2$, solve* (59) *and let u_h solve* (61)*. Take $\alpha = \frac{7}{2}$. Assume $h \leqslant p^{-\frac{3}{2}}$. Then, there is c, independent of p and h, such that*

$$\|u - u_h\|_{\varepsilon,j} \leqslant c(1 + \varepsilon^{-\frac{1}{2}})(\varepsilon^{\frac{1}{2}} + 1 + h^{\frac{1}{2}} p^{-\frac{3}{4}}) \left(\frac{h}{p}\right)^{s-1} \|u\|_{s,\Omega}, \tag{66}$$

with $s = \min(p+1, q)$. Hence, if $\varepsilon \geqslant \max(1, hp^{-\frac{3}{2}})$, the following holds:

$$\|u - u_h\|_{\varepsilon,j} \leqslant c\varepsilon^{\frac{1}{2}} \left(\frac{h}{p}\right)^{s-1} \|u\|_{s,\Omega}. \tag{67}$$

Proof. Recall the following interpolation result [3]: there exists $u_h^* \in V_h^p$ such that

$$\|u - u_h^*\|_{1,\Omega} \leqslant c \left(\frac{h}{p}\right)^{s-1} \|u\|_{s,\Omega}. \tag{68}$$

(1) Let us first prove that

$$j(u - u_h^*, u - u_h^*)^{\frac{1}{2}} \leqslant ch^{\frac{1}{2}} p^{-\frac{3}{4}} \left(\frac{h}{p}\right)^{s-1} \|u\|_{s,\Omega}. \tag{69}$$

Let $\Pi_{h,2}^* : H^2(\Omega) \to W_h^p$ be the H^2-orthogonal projection onto W_h^p. Recall that $\Pi_{h,2}^*$ is endowed with optimal hp-approximation properties in the L^2-, H^1- and H^2-norms; see [10]. Let $\kappa \in \mathcal{K}$ and let $F \subset \partial\kappa$. Owing to the trace inequality (16), it is inferred that

$$\begin{aligned}
\|\nabla(u - u_h^*)|_\kappa\|_F &\leqslant \|\nabla(u - \Pi_{h,2}^* u)|_\kappa\|_F + \|\nabla(\Pi_{h,2}^* u - u_h^*)|_\kappa\|_F \\
&\leqslant c \left(\tfrac{h_\kappa}{p}\right)^{s-\frac{3}{2}} \|u\|_{s,\kappa} + cph_\kappa^{-\frac{1}{2}} \|\nabla(\Pi_{h,2}^* u - u_h^*)\|_\kappa \\
&\leqslant c \left(\tfrac{h_\kappa}{p}\right)^{s-\frac{3}{2}} \|u\|_{s,\kappa} + cph_\kappa^{-\frac{1}{2}} (\|\nabla(u - \Pi_{h,2}^* u)\|_\kappa + \|\nabla(u - u_h^*)\|_\kappa) \\
&\leqslant cp^{\frac{1}{2}} \left(\tfrac{h_\kappa}{p}\right)^{s-\frac{3}{2}} \|u\|_{s,\kappa} + cph_\kappa^{-\frac{1}{2}} \|\nabla(u - u_h^*)\|_\kappa.
\end{aligned}$$

As a result,

$$\begin{aligned}
j(u - u_h^*, u - u_h^*) &\leqslant \sum_{F \in \mathcal{F}} \frac{h_F^2}{p^{\frac{7}{4}}} |\beta{\cdot}n|_F \|[\nabla(u - u_h^*){\cdot}n]\|_F^2 \\
&\leqslant c \sum_{\kappa \in \mathcal{K}} \frac{h_\kappa^2}{p^{\frac{7}{4}}} \left(p \left(\tfrac{h_\kappa}{p}\right)^{2s-3} \|u\|_{s,\kappa}^2 + p^2 h_\kappa^{-1} \|\nabla(u - u_h^*)\|_\kappa^2 \right),
\end{aligned}$$

yielding (69).

(2) Using (69), it is easily shown that

$$\|u - u_h^*\|_{\varepsilon,j} \leqslant c(\varepsilon^{\frac{1}{2}} + 1 + h^{\frac{1}{2}} p^{-\frac{3}{4}}) \left(\frac{h}{p}\right)^{s-1} \|u\|_{s,\Omega}.$$

Owing to Lemma 6.1 and Lemma 6.2,

$$\begin{aligned}
\|u - u_h\|_{\varepsilon,j}^2 &\leqslant a_\varepsilon(u - u_h, u - u_h^*) + j(u - u_h, u - u_h^*) \\
&\leqslant (1 + \varepsilon^{-\frac{1}{2}}) \|u - u_h\|_{\varepsilon,j} \|u - u_h^*\|_{\varepsilon,j}.
\end{aligned}$$

Hence,

$$\|u-u_h\|_{\varepsilon,j} \leqslant (1+\varepsilon^{-\frac{1}{2}})\|u-u_h^*\|_{\varepsilon,j}$$
$$\leqslant (1+\varepsilon^{-\frac{1}{2}})(\varepsilon^{\frac{1}{2}}+1+h^{\frac{1}{2}}p^{-\frac{3}{4}})\left(\frac{h}{p}\right)^{s-1}\|u\|_{s,\Omega},$$

yielding (66). The conclusion is straightforward. □

Proposition 6.2 shows that in the diffusion dominated regime, the CIP bilinear need not be modified to guarantee optimal convergence in h and p. In particular, the perturbation induced by the CIP bilinear form converges to zero at a rate that is faster by a factor of $h^{\frac{1}{2}}p^{-\frac{3}{4}}$ with respect to the error in the energy norm. Furthermore, the above results indicate that the Peclet number to detect the regime transition is $\frac{h}{\varepsilon}p^{-\frac{3}{2}}$.

7. Conclusions

The CIP hp-finite element method analyzed in this paper can be used efficiently to approximate first-order transport operators and advection–diffusion equations on tensor-product meshes. The error estimates are suboptimal by a factor of $p^{\frac{1}{4}}$ in the advection-dominated regime, while optimality is recovered in the diffusion-dominated regime. In contrast, several questions remain open for simplicial finite elements. In particular, the existence of an admissible set of nodes yielding the best possible inverse trace inequality (46) and the sharpness of the estimates derived in Lemma 5.3 and in Lemma 5.4 deserve further investigation. Finally, it is worthwhile to mention that the design of a CIP hp-finite element method for the Oseen equation is straightforward by combining the above results with those of [8].

References

[1] D.N. Arnold. An interior penalty finite element method with discontinuous elements. *SIAM J. Numer. Anal.*, 19(4):742–760, 1982.

[2] I. Babuška and M.R. Dorr. Error estimates for the combined h and p versions of the finite element method. *Numer. Math.*, 37(2):257–277, 1981.

[3] I. Babuška and M. Suri. The hp version of the finite element method with quasi-uniform meshes. *RAIRO, Math. Mod. Numer. Anal.*, 21:199–238, 1987.

[4] I. Babuška and M. Zlámal. Nonconforming elements in the finite element method with penalty. *SIAM J. Numer. Anal.*, 10:863–875, 1973.

[5] G.A. Baker. Finite element methods for elliptic equations using nonconforming elements. *Math. Comp.*, 31(137):45–59, 1977.

[6] E. Burman. A unified analysis for conforming and nonconforming stabilized finite element methods using interior penalty. *SIAM J. Numer. Anal.*, 2004. accepted.

[7] E. Burman and A. Ern. Stabilized galerkin approximation of convection-diffusion-reaction equations: discrete maximum principle and convergence. *Math. Comp.*, 2005. accepted.

[8] E. Burman, M. Fernández, and P. Hansbo. Edge stabilization for the incompressible Navier–Stokes equation: a continuous interior penalty method. Technical Report RR-5349, INRIA, 2004.

[9] E. Burman and P. Hansbo. Edge stabilization for galerkin approximations of convection–diffusion–reaction problems. *Comput. Methods Appl. Mech. Engrg.*, 193:1437–1453, 2004.

[10] C. Canuto and A. Quarteroni. Approximation results for orthogonal polynomials in Sobolev spaces. *Math. Comp.*, 38:67–86, 1982.

[11] J. Douglas Jr. and T. Dupont. *Interior Penalty Procedures for Elliptic and Parabolic Galerkin Methods*, volume 58 of *Lecture Notes in Physics*. Springer-Verlag, Berlin, 1976.

[12] L. El Alaoui and A. Ern. Residual and hierarchical a posteriori estimates for nonconforming mixed finite element methods. *Math. Model. Numer. Anal. (M2AN)*, 2004. accepted.

[13] A. Ern and J.-L. Guermond. *Theory and Practice of Finite Elements*, volume 159 of *Applied Mathematical Sciences*. Springer-Verlag, New York, NY, 2004.

[14] R.H.W. Hoppe and B. Wohlmuth. Element-oriented and edge-oriented local error estimators for non-conforming finite element methods. *Math. Model. Numer. Anal. (M2AN)*, 30:237–263, 1996.

[15] P. Houston, C. Schwab, and E. Süli. Stabilized *hp*-finite element methods for first-order hyperbolic problems. *SIAM J. Numer. Anal.*, 37(5):1618–1643 (electronic), 2000.

[16] P. Houston, C. Schwab, and E. Süli. Discontinuous *hp*-finite element methods for advection–diffusion–reaction problems. *SIAM J. Numer. Anal.*, 39(6):2133–2163, 2002.

[17] P. Houston and E. Süli. Stabilised *hp*-finite element approximation of partial differential equations with nonnegative characteristic form. *Computing*, 66(2):99–119, 2001.

[18] O. Karakashian and F. Pascal. A-posteriori error estimates for a discontinuous Galerkin approximation of second order elliptic problems. *SIAM J. Numer. Anal.*, 41(6):2374–2399, 2003.

[19] A. Quarteroni and A. Valli. *Numerical Approximation of Partial Differential Equations*, volume 23 of *Springer Series in Computational Mathematics*. Springer-Verlag, 1997.

[20] T. Warburton and J.S. Hesthaven. On the constants in *hp*-finite element trace inverse inequalities. *Comput. Methods Appl. Mech. Engrg.*, 192:2765–2773, 2003.

[21] M.F. Wheeler. An elliptic collocation-finite element method with interior penalties. *SIAM J. Numer. Anal.*, 15(1):152–161, 1978.

[1]Institut d'Analyse et Calcul Scientifique (CMCS/IACS), Ecole Polytechnique Fédérale de Lausanne, 1015 Lausanne, Switzerland

[2]CERMICS, Ecole nationale des ponts et chaussées, Champs sur Marne, 77455 Marne la Vallée Cedex 2, France

E-mail address: `Erik.Burman@epfl.ch`

E-mail address: `ern@cermics.enpc.fr`