EULER SCHEME AND TEMPERED DISTRIBUTIONS

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ABSTRACT. Given a smooth $\mathbb{R}^d$-valued diffusion $(X_t^x, t \in [0,1])$ starting at point x, we study how fast the Euler scheme $X_1^{n,x}$ with time step $1/n$ converges in law to the random variable X_1^x. Precisely, we look for which class of test functions f the approximate expectation $\mathbb{E}[f(X_1^{n,x})]$ converges with speed $1/n$ to $\mathbb{E}[f(X_1^x)]$.

When f is smooth with polynomially growing derivatives [14], or, under a uniform hypoellipticity condition for X, when f is only measurable and bounded [1], it is known that there exists a constant $C_1 f(x)$ such that

$$\mathbb{E}[f(X_1^{n,x})] - \mathbb{E}[f(X_1^x)] = C_1 f(x)/n + O\left(1/n^2\right). \tag{1}$$

If X is uniformly elliptic, we expand this result to the case when f is a tempered distribution. In such a case, $\mathbb{E}[f(X_1^x)]$ (resp. $\mathbb{E}[f(X_1^{n,x})]$) has to be understood as $\langle f, p(1,x,\cdot)\rangle$ (resp. $\langle f, p_n(1,x,\cdot)\rangle$) where $p(t,x,\cdot)$ (resp. $p_n(t,x,\cdot)$) is the density of X_t^x (resp. $X_t^{n,x}$). In particular, (1) is valid when f is a measurable function with polynomial growth, a Dirac mass or any derivative of a Dirac mass. We even show that (1) remains valid when f is a measurable function with exponential growth. Actually our results are symmetric in the two space variables x and y of the transition density and we prove that

$$\partial_x^\alpha \partial_y^\beta p_n(t,x,y) - \partial_x^\alpha \partial_y^\beta p(t,x,y) = \partial_x^\alpha \partial_y^\beta \pi(t,x,y)/n + r_n(t,x,y)$$

for a function π and a $O(1/n^2)$ remainder r_n which are shown to be of gaussian type. We give applications to option pricing and hedging, proving numerical convergence rates for prices, deltas and gammas.

1. Introduction and results

Let $d, r \geq 1$ be two integers. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which lives a r-dimensional Brownian motion B. We denote by $\mathcal{F}_t = \sigma(B_s, 0 \leq s \leq t)$ the filtration generated by B. Let us give two functions $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times r}$. We systematically use (column) vector and matrix notations, so that $b(x)$ should be thought of as a vector of size d and $\sigma(x)$ as a matrix of size $d\times r$. We denote transposition by a star and define a $d\times d$ matrix-valued function by putting $a = \sigma\sigma^*$. For a multiindex $\alpha \in \mathbb{N}^d$, $|\alpha| = \alpha_1 + \cdots + \alpha_d$ is its length and ∂^α is the differential operator $\partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}$. Equipping $\mathbb{R}^d$ with the euclidian norm $\|\cdot\|$, we denote by

- $C_{\text{pol}}^\infty(\mathbb{R}^d)$ the set of infinitely differentiable functions $f : \mathbb{R}^d \to \mathbb{R}$ with polynomially growing derivatives of any order, i.e. such that for all $\alpha \in \mathbb{N}^d$, there exists $c \geq 0$ and $q \in \mathbb{N}$ such that for all $x \in \mathbb{R}^d$,

$$|\partial^\alpha f(x)| \leq c\left(1 + \|x\|^q\right), \tag{2}$$

- $C_b^\infty(\mathbb{R}^d)$ the set of infinitely differentiable functions $f : \mathbb{R}^d \to \mathbb{R}$ with bounded derivatives of any order, i.e. such that $\partial^\alpha f \in L^\infty(\mathbb{R}^d)$ for all $\alpha \in \mathbb{N}^d$.

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We shall make use of the following assumptions:

(A) For all $i \in \{1, \dots, d\}$ and $j \in \{1, \dots, r\}$, b_i and $\sigma_{i,j}$ belong to $C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$ and have bounded first derivatives.

(B) For all $i \in \{1, \dots, d\}$ and $j \in \{1, \dots, r\}$, b_i and $\sigma_{i,j}$ belong to $C^\infty_b(\mathbb{R}^d)$.

(C) There exists $\eta > 0$ such that for all $x, \xi \in \mathbb{R}^d$, $\xi^* a(x) \xi \geq \eta \|\xi\|^2$.

(C) is known as the uniform ellipticity condition.

It is well known that, given $x \in \mathbb{R}$, the hypothesis (A) guarantees the existence and the $\mathbb{P}$-almost sure uniqueness of a solution $X^x = (X^x_t, t \geq 0)$ of the stochastic differential equation (SDE)

$$X^x_t = x + \int_0^t b(X^x_s)\, ds + \int_0^t \sigma(X^x_s)\, dB_s. \tag{3}$$

1.1. **Motivation.** Let us fix a time horizon $T > 0$. Without loss of generality, we can and do assume that $T = 1$. We try to estimate the law of X^x_1. To do so, the most natural idea is to approach X^x by its Euler scheme of order $n \geq 1$, say $X^{n,x} = (X^{n,x}_t, t \geq 0)$, defined as follows. We consider the regular subdivision $\mathfrak{S}_n = \{0 = t^n_0 < t^n_1 < \cdots < t^n_{n-1} < t^n_n = 1\}$ of the interval $[0, 1]$, i.e. $t^n_k = k/n$, and we put $X^{n,x}_0 = x$ and, for all $k \in \{0, \dots, n-1\}$ and $t \in [t^n_k, t^n_{k+1}]$,

$$X^{n,x}_t = X^{n,x}_{t^n_k} + b\left(X^{n,x}_{t^n_k}\right)(t - t^n_k) + \sigma\left(X^{n,x}_{t^n_k}\right)\left(B_t - B_{t^n_k}\right). \tag{4}$$

We measure the weak error between $X^{n,x}_1$ and X^x_1 by the quantities

$$\Delta^n_1 f(x) = \mathbb{E}\left[f\left(X^{n,x}_1\right)\right] - \mathbb{E}\left[f\left(X^x_1\right)\right]$$

and we try to find the largest space of test functions f for which for each x there exists a constant $C_1 f(x)$ such that

$$\Delta^n_1 f(x) = C_1 f(x)/n + O\left(1/n^2\right). \tag{5}$$

Practical interest of such an expansion has to be underlined (see, for instance, [7, 14]). When (5) holds, one can use the Euler scheme plus a Monte-Carlo method to estimate $\mathbb{E}\left[f\left(X^x_1\right)\right]$ and then, in a time of order nN, gets an error of order $1/\sqrt{N} + 1/n$, where N stands for the number of independants copies of $X^{n,x}_1$ generated by the Monte-Carlo procedure. Given a tolerance $\varepsilon \ll 1$, in order to minimize the time of calculus, one should then choose $N = O\left(n^2\right)$ and gets a result in a time of order $1/\varepsilon^3$.

One can even do better using Romberg's extrapolation technique: if one runs N independant copies $(X^{2n,x}_{i,1}, X^{n,x}_{i,1})$ of the couple $(X^{2n,x}_1, X^{n,x}_1)$, which still requires a time of order nN, then computing $\frac{1}{N}\sum_{i=1}^N (2f(X^{2n,x}_{i,1}) - f(X^{n,x}_{i,1}))$ one gets an estimate of $\mathbb{E}\left[f\left(X^x_1\right)\right]$ whose accuracy is of order $1/\sqrt{N} + 1/n^2$, since (5) implies that $\mathbb{E}[2f(X^{2n,x}_1) - f(X^{n,x}_1)] = \mathbb{E}[f(X^x_1)] + O(1/n^2)$. Given a tolerance $\varepsilon \ll 1$, one should now choose $N = O\left(n^4\right)$ and gets a result in a time of order $1/\varepsilon^{5/2}$.

1.2. **Previous results.** Using Itô expansions, D. Talay and L. Tubaro [14] have shown that (5) holds when $f \in C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$. Using Malliavin calculus, V. Bally and D. Talay [1] have extended this result to the case of measurable and bounded f's, when X is uniformly hypoelliptic. If (C) holds, $X^{n,x}_1$ and X^x_1 have densities, say $p_n(1, x, \cdot)$ and $p(1, x, \cdot)$ respectively (in this paper, densities are always taken with respect to the Lebesgue measure). For each pair (x, y), the authors [2] get an expansion of the form

$$p_n(1, x, y) - p(1, x, y) = \pi(1, x, y)/n + r_n(1, x, y)/n^2. \tag{6}$$

They also find constants $c_1 \geq 0$ and $c_2 > 0$ such that for all $n \geq 1$ and $x, y \in \mathbb{R}^d$, $|\pi(1,x,y)| + |r_n(1,x,y)| \leq c_1 \exp(-c_2 \|x-y\|^2)$.

Besides, V. Konakov and E. Mammen [9] have proposed an analytical approach for this problem based on the so-called parametrix method. If (C) holds, for each pair (x,y), they get an expansion of arbitrary order j of $p_n(1,x,y)$ but whose terms depend on n:

$$p_n(1,x,y) - p(1,x,y) = \sum_{i=1}^{j-1} \pi_{n,i}(1,x,y)/n^i + O\left(1/n^j\right). \tag{7}$$

For each i, they also find constants $c_1 \geq 0$ and $c_2 > 0$ such that for all $n \geq 1$ and $x, y \in \mathbb{R}^d$, $|\pi_{n,i}(1,x,y)| \leq c_1 \exp(-c_2 \|x-y\|^2)$. To do so, the authors use upper bounds on the partial derivatives of p - which they find in [4] - and prove analogous bounds on p_n's ones.

For a link with generalized Watanabe distributions on Wiener's space, see [12]. For the general case of Lévy driven stochastic differential equations, (4) holds under regularity assumptions on f and integrability conditions on the Lévy process (see [7, 13]). For the rate of convergence of the process $(X_t^{n,x} - X_t^x, t \in [0,1])$, see [5, 6]. As for the simulation of densities, see [8].

1.3. **Main results.** Our main result can be seen as an improvement of (6). It gives a first order *functional* expansion for p_n. In order to state it shortly, we introduce an increasing family of functional spaces $\left(\mathcal{G}_l(\mathbb{R}^d), l \in \mathbb{Z}\right)$. For $l \in \mathbb{Z}$, we define $\mathcal{G}_l(\mathbb{R}^d)$ as the set of all measurable functions $\pi : (0,1] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ such that

- for all $t \in (0,1]$, $\pi(t,\cdot,\cdot)$ is infinitely differentiable,
- for all $\alpha, \beta \in \mathbb{N}^d$, there exists two constants $c_1 \geq 0$ and $c_2 > 0$ such that for all $t \in (0,1]$ and $x, y \in \mathbb{R}^d$,

$$\left|\partial_x^\alpha \partial_y^\beta \pi(t,x,y)\right| \leq c_1 t^{-(|\alpha|+|\beta|+d+l)/2} \exp\left(-c_2 \|x-y\|^2 / t\right). \tag{8}$$

We say that a subset $\mathcal{B} \subset \mathcal{G}_l(\mathbb{R}^d)$ is bounded if, in (8), c_1 and c_2 can be chosen independently on $\pi \in \mathcal{B}$. We also introduce the space $\mathcal{G}(\mathbb{R}^d)$ defined in the same way as $\mathcal{G}_l(\mathbb{R}^d)$ with (8) replaced by the following two conditions:

$$\left|\partial_x^\alpha \partial_y^\beta \pi(t,x,y)\right| \leq c_1 t^{-(|\alpha|+|\beta|+d)/2} \exp\left(-c_2 \|x-y\|^2 / t\right), \tag{9}$$

$$\left|\partial_x^\alpha \left(\pi\left(t,x,x+y\sqrt{t}\right)\right)\right| \leq c_1 t^{-d/2} \exp\left(-c_2 \|y\|^2\right). \tag{10}$$

We say that a subset $\mathcal{B} \subset \mathcal{G}(\mathbb{R}^d)$ is bounded if, in (9) and (10), c_1 and c_2 can be chosen independently on $\pi \in \mathcal{B}$. We are now able to state our main result as follows.

Theorem 1. *Under (B) and (C),*

(i) *for all $t \in (0,1]$ and $x \in \mathbb{R}^d$, X_t^x has a density $p(t,x,\cdot)$ and $p \in \mathcal{G}(\mathbb{R}^d)$,*
(ii) *for all $t \in (0,1]$, $x \in \mathbb{R}^d$ and $n \geq 1$, $X_t^{n,x}$ has a density $p_n(t,x,\cdot)$ and $(p_n, n \geq 1)$ is a bounded sequence in $\mathcal{G}(\mathbb{R}^d)$,*
(iii) *there exists $\pi \in \mathcal{G}_1(\mathbb{R}^d)$ and a bounded sequence $(\pi_n, n \geq 1)$ in $\mathcal{G}_4(\mathbb{R}^d)$ such that for all $n \geq 1$,*

$$p_n - p = \pi/n + \pi_n/n^2. \tag{11}$$

Statement (i) is already known (see [4], theorem 7, page 260). Statement (ii), which has essentially been proved in [9], and statement (iii) are proved in Section 3.2. The function π can be expressed in terms of p by

$$\pi(t,x,y) = \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} p(s,x,z) L_2^*(p(t-s,\cdot,y))(z)\, dzds, \tag{12}$$

where the differential operator L_2^* is explicitely given in terms of the functions a and b by

$$\begin{aligned} -L_2^* = &\sum_{i=1}^d \left(b\cdot\nabla b_i + \frac{1}{2}\operatorname{tr}\left(a\nabla^2 b_i\right)\right)\partial_i \\ &+ \sum_{i,j=1}^d \left(\frac{1}{2} b\cdot\nabla a_{i,j} + a_j\cdot\nabla b_i + \frac{1}{4}\operatorname{tr}\left(a\nabla^2 a_{i,j}\right)\right)\partial_{ij} + \frac{1}{2}\sum_{i,j,k=1}^d a_k\cdot\nabla a_{i,j}\partial_{ijk}. \end{aligned} \tag{13}$$

Here, $\cdot$, a_k, tr, ∇ and ∇^2 respectively stand for the inner product in $\mathbb{R}^d$, the k-th column of a, the trace of a matrix, the gradient vector and the hessian matrix. In the case when $t=1$, (12) agrees with V. Bally and D. Talay's expression for π ([2], definition 2.2, page 100), but seems preferable because it does not involve differentiation with respect to t and makes clearly appear that the space differential operator L_2^* is of order less than 3, when V. Bally and D. Talay's operator $\mathcal{U}$ involves a fourth order differentiation in space.

As a consequence, we can state

Corollary 2. *Under (B) and (C), for all $\alpha,\beta\in\mathbb{N}^d$, there exists $c_1\geq 0$ and $c_2>0$ such that for all $n\geq 1$, $t\in(0,1]$ and $x,y\in\mathbb{R}^d$,*

$$\partial_x^\alpha\partial_y^\beta p_n(t,x,y) - \partial_x^\alpha\partial_y^\beta p(t,x,y) = \frac{1}{n}\,\partial_x^\alpha\partial_y^\beta \pi(t,x,y) + r_n(t,x,y)$$

and

$$|r_n(t,x,y)| \leq c_1 n^{-2} t^{-(|\alpha|+|\beta|+d+4)/2} \exp\left(-c_2\left\|x-y\right\|^2/t\right).$$

We shall now prove that if X is elliptic the expansion (5) is valid in the very general case when f is a tempered distribution. Let us denote by $\mathcal{S}(\mathbb{R}^d)$ Schwartz's space, i.e. the space of infinitely differentiable functions $\varphi:\mathbb{R}^d\to\mathbb{R}$ such that $x\mapsto x^\alpha\partial^\beta\varphi(x)\in L^\infty(\mathbb{R}^d)$ for all $\alpha,\beta\in\mathbb{N}^d$ (x^α stands for $x_1^{\alpha_1}\cdots x_d^{\alpha_d}$), and let us denote by $\mathcal{S}'(\mathbb{R}^d)$ the space of tempered distributions. The seminorms $(N_q, q\in\mathbb{N})$ are defined on $\mathcal{S}(\mathbb{R}^d)$ by

$$N_q(\varphi) = \sum_{|\alpha|\leq q, |\beta|\leq q} \sup_{x\in\mathbb{R}^d}\left|x^\alpha\partial^\beta\varphi(x)\right|,$$

and the order $\#S$ of $S\in\mathcal{S}'(\mathbb{R}^d)$ is the smallest integer q such that there is a $c\geq 0$ such that $|\langle S,\varphi\rangle|\leq cN_q(\varphi)$ for all $\varphi\in\mathcal{S}(\mathbb{R}^d)$. Note that whenever $\pi\in\mathcal{G}_l(\mathbb{R}^d)$, $\pi(t,x,\cdot)$ and $\pi(t,\cdot,y)$ belong to $\mathcal{S}(\mathbb{R}^d)$. More precisely, for $\mathcal{B}\subset\mathcal{G}_l(\mathbb{R}^d)$ bounded, there exists $c\geq 0$ such that for all $\pi\in\mathcal{B}$, $t\in(0,1]$ and $x,y\in\mathbb{R}^d$,

$$N_q(\pi(t,x,\cdot)) \leq ct^{-(d+l+q)/2}\left(1+\|x\|^q\right) \qquad \text{and} \qquad N_q(\pi(t,\cdot,y)) \leq ct^{-(d+l+q)/2}\left(1+\|y\|^q\right).$$

Applying a tempered distribution S to (11), t and x or t and y being fixed, we immediately deduce from Theorem 1

Theorem 3. *Under (B) and (C), for all $S \in \mathcal{S}'(\mathbb{R}^d)$, there exists $c \geq 0$ such that for all $n \geq 1$, $t \in (0,1]$ and $x, y \in \mathbb{R}^d$,*

$$\begin{aligned} \langle S, p_n(t,x,\cdot)\rangle - \langle S, p(t,x,\cdot)\rangle &= \frac{1}{n}\langle S, \pi(t,x,\cdot)\rangle + r'_n(t,x), \\ \langle S, p_n(t,\cdot,y)\rangle - \langle S, p(t,\cdot,y)\rangle &= \frac{1}{n}\langle S, \pi(t,\cdot,y)\rangle + r''_n(t,y), \end{aligned}$$

and

$$\left|r'_n(t,x)\right| + \left|r''_n(t,x)\right| \leq cn^{-2}t^{-(d+4+\#S)/2}\left(1 + \|x\|^{\#S}\right).$$

Let us define $\mathbb{E}\left[S(Y)\right]$ by $\langle S, p_Y\rangle$ when $S \in \mathcal{S}'(\mathbb{R}^d)$ and Y is a random variable with density $p_Y \in \mathcal{S}(\mathbb{R}^d)$. Note that, when S is a measurable and polynomially growing function, this definition coincides with the usual expectation. We then have proved that, under (B) and (C), (5) is valid for f's being only tempered distributions, and not only for $t = 1$, but also for any time $t \in (0,1]$, and we have even explicited the way the $O(1/n^2)$ remainder depends on t, f and x. Precisely, this remainder grows slower than $\|x\|^{\#f}$ as x tends to infinity, and explodes slower than $t^{-(\#f+d+4)/2}$ as t tends to 0.

As the particular case when S is a measurable and polynomially growing function, let us state

Corollary 4. *Assume (B) and (C). Let $f : \mathbb{R}^d \to \mathbb{R}$ be a measurable function such that there exists $c' \geq 0$ and $q \in \mathbb{N}$ such that for all $x \in \mathbb{R}^d$, $|f(x)| \leq c'(1 + \|x\|^q)$. Then there exists $c \geq 0$ such that for all $n \geq 1$, $t \in (0,1]$ and $x \in \mathbb{R}^d$,*

$$\mathbb{E}[f(X_t^{n,x})] - \mathbb{E}[f(X_t^x)] = \frac{1}{n}\int_{\mathbb{R}^d} f(y)\pi(t,x,y)\,dy + r_n(t,x) \tag{14}$$

and

$$|r_n(t,x)| \leq cn^{-2}t^{-2}\left(1 + \|x\|^q\right).$$

Proof. Multiplying (11) by $f(y)$ and integrating in y leads to (14) with the remainder $r_n(t,x) = n^{-2}\int_{\mathbb{R}^d} f(y)\pi_n(t,x,y)\,dy$. Since $|f(y)| \leq c'(1+\|y\|^q)$ and $(\pi_n, n \geq 1)$ is bounded in $\mathcal{G}_4(\mathbb{R}^d)$, we can find $c_1 \geq 0$ and $c_2 > 0$ such that for all $n \geq 1$, $t \in (0,1]$ and $x \in \mathbb{R}^d$, $|r_n(t,x)| \leq c_1 n^{-2}t^{-(d+4)/2}\int_{\mathbb{R}^d}(1 + \|y\|^q)\exp(-c_2\|x-y\|^2/t)\,dy$. Setting $\zeta = (y-x)/\sqrt{t}$ leads to $|r_n(t,x)| \leq c_1 n^{-2}t^{-2}\int_{\mathbb{R}^d}\exp(1 + \left\|x + \zeta\sqrt{t}\right\|^q)\exp(-c_2\|\zeta\|^2)\,d\zeta$. To complete the proof, it remains to observe that there exists $c \geq 0$ such that for all $t \in (0,1]$ and $x, \zeta \in \mathbb{R}^d$, $\left\|x + \zeta\sqrt{t}\right\|^q \leq c(\|x\|^q + \|\zeta\|^q)$. ☐

As far as extending the class of f's for which (5) holds is concerned, we can even do better. Indeed, if for $\mu \in (0,2)$ we denote by $\mathcal{E}_\mu$ the set of all measurable functions $f : \mathbb{R}^d \to \mathbb{R}$ such that there exists $c_1, c_2 \geq 0$ such that for all $y \in \mathbb{R}^d$,

$$|f(y)| \leq c_1\exp\left(c_2\|y\|^\mu\right),$$

it is easy to adapt the preceding proof to get

Corollary 5. *Under (B) and (C), for all $\mu \in (0,2)$ and $f \in \mathcal{E}_\mu$, there exists $c_1, c_2 \geq 0$ such that for all $n \geq 1$, $t \in (0,1]$ and $x \in \mathbb{R}^d$, $f(X_t^x)$ and $f(X_t^{n,x})$ are integrable and*

$$\mathbb{E}[f(X_t^{n,x})] - \mathbb{E}[f(X_t^x)] = \frac{1}{n}\int_{\mathbb{R}^d} f(y)\pi(t,x,y)\,dy + r_n(t,x) \tag{15}$$

with

$$|r_n(t,x)| \leq c_1 n^{-2}t^{-2}\exp\left(c_2\|x\|^\mu\right).$$

In particular, (5) remains true under (B) and (C) when $f \in \mathcal{E} = \cup_{\mu\in(0,2)}\mathcal{E}_\mu$. In the same way, differentiating (5) α times in x, multiplying by $f(y)$ and integrating in y leads to

Corollary 6. *Under (B) and (C), for all $\alpha \in \mathbb{N}^d$, $\mu \in (0,2)$ and $f \in \mathcal{E}_\mu$, there exists $c_1, c_2 \geq 0$ such that for all $n \geq 1$, $t \in (0,1]$ and $x \in \mathbb{R}^d$,*

$$\partial_x^\alpha \mathbb{E}[f(X_t^{n,x})] - \partial_x^\alpha \mathbb{E}[f(X_t^x)] = \frac{1}{n}\int_{\mathbb{R}^d} f(y)\partial_x^\alpha \pi(t,x,y)\,dy + r_n(t,x) \tag{16}$$

with

$$|r_n(t,x)| \leq c_1 n^{-2} t^{-(|\alpha|+4)/2} \exp\left(c_2 \|x\|^\mu\right).$$

This result can now be used in the context of financial markets.

1.4. **Application to option pricing and hedging.** Let $S^v = (S^{v,1},\ldots,S^{v,d})$ be a basket of assets satisfying

$$\frac{dS_t^{v,i}}{S_t^{v,i}} = \mu_i(S_t^v)\,dt + \sum_{j=1}^r \sigma_{i,j}(S_t^v)\,dB_t^j, \qquad S_0^{v,i} = v^i > 0,$$

with $\mu, \sigma \in C_b^\infty(\mathbb{R}^d)$ and σ satisfying (C). Given a measurable and polynomially growing function ϕ, we try to estimate the price $\text{Price} = \mathbb{E}[\phi(S_t^v)]$, the deltas $\text{Delta}_i = \partial_v^{e_i}\mathbb{E}[\phi(S_t^v)]$ and the gammas $\text{Gamma}_{i,j} = \partial_v^{e_i+e_j}\mathbb{E}[\phi(S_t^v)]$ of the european option of maturity t and payoff ϕ ($(e_1,\ldots,e_d)$ is the canonical base of $\mathbb{R}^d$). To do so, let us set $x = \ln v$ (i.e. $x^i = \ln v^i$) and $X_t^{x,i} = \ln(S_t^{v,i})$. Then X is the solution of (3) with $b = \mu - \|\sigma\|^2/2 \in C_b^\infty(\mathbb{R}^d)$, where $\|\sigma\|_i^2(x) = \sum_{j=1}^r \sigma_{i,j}^2(x)$. If we set $\exp(x) = (\exp(x^1),\ldots,\exp(x^d))$ and $f(x) = \phi(\exp(x))$, we define a function $f \in \mathcal{E}_1$ and, since $\text{Price} = \mathbb{E}[f(X_t^x)]$, (15) leads to

$$\text{Price}^n - \text{Price} = C_t^{\text{Price}}\phi(v)/n + O\left(n^{-2}t^{-2}\exp\left(c_2\|\ln v\|\right)\right),$$

where Price^n stands for the approximated price $\mathbb{E}[f(X_t^{n,x})]$ and

$$C_t^{\text{Price}}\phi(v) = \int_{(\mathbb{R}_+^*)^d} \phi(u)\frac{\pi(t,\ln v,\ln u)}{u_1\cdots u_d}\,du.$$

Besides, if we set $\text{Delta}_i^n = \partial_v^{e_i}\mathbb{E}[f(X_t^{n,\ln v})]$ and $\text{Gamma}_{i,j}^n = \partial_v^{e_i+e_j}\mathbb{E}[f(X_t^{n,\ln v})]$, (16) shows that

$$\begin{aligned}
\text{Delta}^n - \text{Delta} &= C_t^{\text{Delta}}\phi(v)/n + O\left(n^{-2}t^{-5/2}\exp\left(c_2\|\ln v\|\right)\right),\\
\text{Gamma}^n - \text{Gamma} &= C_t^{\text{Gamma}}\phi(v)/n + O\left(n^{-2}t^{-3}\exp\left(c_2\|\ln v\|\right)\right),
\end{aligned}$$

where

$$\begin{aligned}
C_t^{\text{Delta}}\phi(v)_i &= \int_{(\mathbb{R}_+^*)^d} \phi(u)\frac{\partial_2^{e_i}\pi(t,\ln v,\ln u)}{u_1\cdots u_d}\,du,\\
C_t^{\text{Gamma}}\phi(v)_{i,j} &= \int_{(\mathbb{R}_+^*)^d} \phi(u)\frac{\partial_2^{e_i+e_j}\pi(t,\ln v,\ln u)}{u_1\cdots u_d}\,du.
\end{aligned}$$

Eventually we have proved that applying the Euler scheme of order n to the logarithm of the underlying leads to approximations of the price, the deltas and the gammas which converge to the true price, deltas and gammas with speed $1/n$, at least when the drift and volatility of the underlying satisfy (B) and (C), which in the context of financial markets seems not to be a restricting hypothesis. Note that the principal part of the error explodes as t tends to 0 as $t^{-1/2}$ for the prices, t^{-1} for the deltas and $t^{-3/2}$ for the gammas.

1.5. **A preliminary result.** In order to prove Theorem 1, we first seek an expansion for the error operator

$$\Delta_t^n = P_t^n - P_t$$

where, for $f \in C_{\mathrm{pol}}^{\infty}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, we have set $P_t f(x) = \mathbb{E}[f(X_t^x)]$ and $P_t^n f(x) = \mathbb{E}\left[f(X_t^{n,x})\right]$. Precisely, we look for operators C_t and R_t^n such that $R_t^n = O(1/n^2)$ and $\Delta_t^n = C_t/n + R_t^n$. The following theorem, interesting in itself, is proved in Section 2. It can be seen as an improvement of [14]. It not only gives explicit formulas for $C_t f(x)$ and $R_t^n f(x)$ but also provides useful information about their dependencies on n, t, f and x. Note that it does not require neither (B) nor (C). In order to state it shortly, let us

- denote by $\mathcal{L}\left(C_{\mathrm{pol}}^{\infty}(\mathbb{R}^d)\right)$ the space of endomorphisms of $C_{\mathrm{pol}}^{\infty}(\mathbb{R}^d)$,
- say that a subset $\mathcal{B} \subset C_{\mathrm{pol}}^{\infty}(\mathbb{R}^d)$ is bounded if, in (2), c and q can be chosen independently on $f \in \mathcal{B}$,
- say that $T \in \mathcal{L}\left(C_{\mathrm{pol}}^{\infty}(\mathbb{R}^d)\right)$ is bounded if for all bounded $\mathcal{B} \subset C_{\mathrm{pol}}^{\infty}(\mathbb{R}^d)$, $\{Tf | f \in \mathcal{B}\}$ is a bounded subset of $C_{\mathrm{pol}}^{\infty}(\mathbb{R}^d)$,
- denote by $\mathcal{L}_b\left(C_{\mathrm{pol}}^{\infty}(\mathbb{R}^d)\right)$ the space of bounded endomorphisms of $C_{\mathrm{pol}}^{\infty}(\mathbb{R}^d)$,
- say that a $\mathcal{L}_b\left(C_{\mathrm{pol}}^{\infty}(\mathbb{R}^d)\right)$-valued family $(T_i, i \in I)$ is bounded if for all bounded $\mathcal{B} \subset C_{\mathrm{pol}}^{\infty}(\mathbb{R}^d)$, $\{T_i f | f \in \mathcal{B}, i \in I\}$ is a bounded subset of $C_{\mathrm{pol}}^{\infty}(\mathbb{R}^d)$,
- say that $(T_i, i \in I)$ is a $O(h(i))$ family in $\mathcal{L}_b\left(C_{\mathrm{pol}}^{\infty}(\mathbb{R}^d)\right)$ if the family $(h(i)^{-1}T_i, i \in I)$ is bounded.

It is already known that $(P_t, t \in [0,1])$ is a bounded family in $\mathcal{L}_b\left(C_{\mathrm{pol}}^{\infty}(\mathbb{R}^d)\right)$. A proof can de found in [11], lemma 3.9, page 15. Using Lemma 25, this proof straightforwardly adapts uniformly in n so that $(P_t^n, t \in [0,1], n \geq 1)$ is also bounded in $\mathcal{L}_b\left(C_{\mathrm{pol}}^{\infty}(\mathbb{R}^d)\right)$. We are now in the position to state the main result of the first step:

Theorem 7. *Under (A), $(\Delta_t^n, t \in [0,1], n \geq 1)$ is a $O(1/n)$ family in $\mathcal{L}_b\left(C_{\mathrm{pol}}^{\infty}(\mathbb{R}^d)\right)$, and there exists a $O(t)$ process $(C_t, t \in [0,1])$ and a $O(1/n^2)$ family $(R_t^n, t \in [0,1], n \geq 1)$ in $\mathcal{L}_b\left(C_{\mathrm{pol}}^{\infty}(\mathbb{R}^d)\right)$ such that*

$$\Delta_t^n = C_t/n + R_t^n.$$

Moreover, C_t is explicitely given in terms of $(P_t, t \in [0,1])$ and of L_2^* (see (13)) by

$$C_t = \frac{1}{2} \int_0^t P_s L_2^* P_{t-s} \, ds.$$

1.6. **Organization of the paper.** Section 2 is our first step on the way to prove Theorem 1. It is dedicated to the proof Theorem 7. We also derive an expansion for Δ_t^n of arbitrary order, but whose terms depend on n, see (26), and we explain how to recursively construct the differential operators which appear in it.

Section 3 is our second and final step. It is devoted to the proof of Theorem 1. To sum up, we use Theorem 7 and express C_t and R_t^n in terms of the densities of X_t^x and $X_t^{n,x}$, making appear kernels for C and R^n. The section begins with a study of the space convolution in $\mathcal{G}(\mathbb{R}^d)$ which allows to control these kernels. As in Section 2, we also give a *functional* expansion for $p_n - p$ of arbitrary order, but whose terms depend on n, see (43), thus improving (7).

Eventually, Section 4 is an appendix where we have gathered useful results on the Euler scheme and technical lemmas that are used in Sections 2 and 3.

2. First step: expansion for $\mathbb{E}\left[f\left(X_t^{n,x}\right)\right]$

2.1. **Operators associated with the Euler scheme.** Let us denote by L the infinitesimal generator of the diffusion X and by $(L^x, x \in \mathbb{R}^d)$ its tangent infinitesimal generator, i.e.

$$L = \sum_{i=1}^{d} b_i \partial^{e_i} + \frac{1}{2}\sum_{i,j=1}^{d} a_{i,j}\partial^{e_i+e_j} \qquad \text{and} \qquad L^x = \sum_{i=1}^{d} b_i(x)\partial^{e_i} + \frac{1}{2}\sum_{i,j=1}^{d} a_{i,j}(x)\partial^{e_i+e_j}.$$

We use the convention that L and L^x act on y, so that, for instance, $L\psi(t,x,y)$ and $L^x\psi(t,x,y)$ respectively stand for $L\left(\psi(t,x,\cdot)\right)(y)$ and $L^x\left(\psi(t,x,\cdot)\right)(y)$. L^x is the infinitesimal generator of $(X_t^{n,x}, t \in [0,1/n])$. Besides, for each $x \in \mathbb{R}^d$ we define a sequence of operators $(L_j^x, j \in \mathbb{N})$ by putting $L_0^x = I$ (the identity operator) and

$$L_{j+1}^x = L^x L_j^x - L_j^x L,$$

and we set $L_j^* f(x) = L_j^x f(x)$. Observe that $L_1^* = 0$. Besides, L_2^* is given by (13) so that, under (A), $L_2^* \in \mathcal{L}_b\left(C_{\mathrm{pol}}^\infty(\mathbb{R}^d)\right)$ and there exists a family $(g_{2,\alpha}^*, 1 \le |\alpha| \le 3)$ in $C_{\mathrm{pol}}^\infty(\mathbb{R}^d)$ such that

$$L_2^* = \sum_{1\le|\alpha|\le 3} g_{2,\alpha}^* \partial^\alpha. \tag{17}$$

Under (A), L and L^x belong to $\mathcal{L}_b\left(C_{\mathrm{pol}}^\infty(\mathbb{R}^d)\right)$ for each $x \in \mathbb{R}^d$, and, by induction, so does L_j^x. We can describe L_j^x more precisely. Indeed, defining the powers of an operator A by $A^0 = I$ and $A^{j+1} = AA^j$, inductions on j lead to $L_j^x = \sum_{i=0}^{j}(-1)^i\binom{j}{i}\left(L^x\right)^{j-i}L^i$ and to the existence of a family $(g_{j,\alpha}, h_{j,\alpha}, j \in \mathbb{N}^*, 1 \le |\alpha| \le 2j)$ in $C_{\mathrm{pol}}^\infty(\mathbb{R}^d)$ such that

$$\forall x \in \mathbb{R}^d, \qquad (L^x)^j = \sum_{1\le|\alpha|\le 2j} g_{j,\alpha}(x)\partial^\alpha \qquad \text{and} \qquad L^j = \sum_{1\le|\alpha|\le 2j} h_{j,\alpha}\partial^\alpha.$$

Hence, for each $j \in \mathbb{N}^*$ one can find a family $(m_{j,\alpha}, 1 \le |\alpha| \le 2j)$ of integers and a family $(g_{j,\alpha,l}, h_{j,\alpha,l}, 1 \le |\alpha| \le 2j, 1 \le l \le m_{j,\alpha})$ in $C_{\mathrm{pol}}^\infty(\mathbb{R}^d)$ such that for all $x \in \mathbb{R}^d$,

$$L_j^x = \sum_{1\le|\alpha|\le 2j}\left(\sum_{l=1}^{m_{j,\alpha}} g_{j,\alpha,l}(x)h_{j,\alpha,l}\right)\partial^\alpha. \tag{18}$$

Remark 8. Note that when (B) holds, the functions $g_{j,\alpha,l}$, $h_{j,\alpha,l}$ and $g_{2,\alpha}^*$ all belong to $C_b^\infty(\mathbb{R}^d)$ (in fact they are polynomial in b, σ and their derivatives).

We are now in the position to define the families of operators $\Phi^j = (\Phi_{t_k^n,s,t}^{n,j}, t \in [0,1], n \ge 1, k \in \{0,\ldots,\lfloor nt\rfloor\}, s \in [t_k^n, t_{k+1}^n \wedge t])$ and $\Psi^j = (\Psi_{t_k^n,t}^{n,j}, t \in [0,1], n \ge 1, k \in \{0,\ldots,\lfloor nt\rfloor\})$ as follows:

$$\forall f \in C_{\mathrm{pol}}^\infty(\mathbb{R}^d), \quad \Phi_{t_k^n,s,t}^{n,j} f(x) = \mathbb{E}\left[L_j^{X_{t_k^n}^{n,x}} P_{t-s} f\left(X_s^{n,x}\right)\right] \quad \text{and} \quad \Psi_{t_k^n,t}^{n,j} = \Phi_{t_k^n,t_k^n,t}^{n,j}. \tag{19}$$

Observe that since $s \in [t_k^n, t_{k+1}^n]$

$$\Phi_{t_k^n,s,t}^{n,j} = \sum_{1\le|\alpha|\le 2j}\sum_{l=1}^{m_{j,\alpha}} P_{t_k^n}^n\left(g_{j,\alpha,l}P_{s-t_k^n}^n\left(h_{j,\alpha,l}\partial^\alpha P_{t-s}\right)\right) \quad \text{and} \quad \Psi_{t_k^n,t}^{n,j} = P_{t_k^n}^n L_j^* P_{t-t_k^n}. \tag{20}$$

Then we have

Proposition 9. *Under (A), Φ^j and Ψ^j are bounded families in $\mathcal{L}_b\left(C^\infty_{\mathrm{pol}}(\mathbb{R}^d)\right)$.*

Proof. $(P_t, t \in [0,1])$ and $(P^n_t, t \in [0,1], n \geq 1)$ are bounded families in $\mathcal{L}_b\left(C^\infty_{\mathrm{pol}}(\mathbb{R}^d)\right)$, see the discussion preceding Theorem 7. Besides, multiplication by a function in $C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$ and differentiation are bounded operators on $C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$. As a sum of compositions of bounded families in $\mathcal{L}_b\left(C^\infty_{\mathrm{pol}}(\mathbb{R}^d)\right)$, Φ^j is a bounded family in $\mathcal{L}_b\left(C^\infty_{\mathrm{pol}}(\mathbb{R}^d)\right)$. Then obviously so is Ψ^j. □

2.2. **Itô expansions.** We recall (see [11], theorem 3.11, page 16) that for $f \in C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$, $(s,y) \mapsto P_{t-s}f(y)$ is infinitely differentiable on $[0,t] \times \mathbb{R}^d$ and

$$\forall (s,y) \in [0,t] \times \mathbb{R}^d, \qquad (\partial_s + L)P_{t-s}f(y) = 0. \tag{21}$$

Since ∂_s and L^x_j commute, (21) and the definition of L^x_j imply

$$(\partial_s + L^x)L^x_j P_{t-s} = L^x_{j+1} P_{t-s}. \tag{22}$$

For a measurable family (A_s) in $\mathcal{L}_b\left(C^\infty_{\mathrm{pol}}(\mathbb{R}^d)\right)$, we denote by $\int_{t_1}^{t_2} A_s\, ds$ the element of $\mathcal{L}\left(C^\infty_{\mathrm{pol}}(\mathbb{R}^d)\right)$ which maps f to $x \mapsto \int_{t_1}^{t_2} A_s f(x)\, ds$. The following lemma states that $\Phi^{n,j+1}_{t^n_k,\cdot,t}$ is the derivative of $\Phi^{n,j}_{t^n_k,\cdot,t}$ on the interval $[t^n_k, t^n_{k+1} \wedge t]$.

Lemma 10. *Under (A), for all $j \in \mathbb{N}$, $t \in [0,1]$, $n \geq 1$, $k \in \{0, \ldots, \lfloor nt \rfloor\}$ and $s \in [t^n_k, t^n_{k+1} \wedge t]$,*

$$\Phi^{n,j}_{t^n_k,s,t} = \Psi^{n,j}_{t^n_k,t} + \int_{t^n_k}^s \Phi^{n,j+1}_{t^n_k,s',t}\, ds'. \tag{23}$$

Proof. Conditionaly on $\mathcal{F}_{t^n_k}$, for $f \in C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$, $(s,y) \mapsto L^{X^{n,x}_{t^n_k}}_j P_{t-s}f(y)$ is infinitely differentiable on $[t^n_k, t^n_{k+1} \wedge t] \times \mathbb{R}^d$ so that we can apply Itô's formula to it and to $X^{n,x}$ between t^n_k and s. Using (22) for the second equality, we get

$$\begin{aligned} &L^{X^{n,x}_{t^n_k}}_j P_{t-s}f\left(X^{n,x}_s\right) - L^{X^{n,x}_{t^n_k}}_j P_{t-t^n_k}f\left(X^{n,x}_{t^n_k}\right) - M_s \\ &\qquad = \int_{t^n_k}^s \left(\frac{\partial}{\partial s} + L^{X^{n,x}_{t^n_k}}\right) L^{X^{n,x}_{t^n_k}}_j P_{t-s'}f\left(X^{n,x}_{s'}\right)\, ds' = \int_{t^n_k}^s L^{X^{n,x}_{t^n_k}}_{j+1} P_{t-s'}f\left(X^{n,x}_{s'}\right)\, ds' \end{aligned}$$

where $M_s = \sum_{i=1}^d \sum_{j=1}^r \sigma_{i,j}(X^{n,x}_{t^n_k}) \int_{t^n_k}^s \partial^{e_i}\left(L^{X^{n,x}_{t^n_k}}_j P_{t-s'}f\left(X^{n,x}_{s'}\right)\right) dB^j_{s'}$. Since $\{L^x_j P_{t-s'}f | s' \in [t^n_k, t^n_{k+1} \wedge t]\}$ is bounded in $C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$, (54) imply that $(M_s, s \in [t^n_k, t^n_{k+1} \wedge t])$ is a square-integrable martingale and thus has zero mean. Hence, taking expectations and using (19) and Fubini's theorem, we have

$$\Phi^{n,j}_{t^n_k,s,t}f(x) - \Psi^{n,j}_{t^n_k,t}f(x) = \int_{t^n_k}^s \mathbb{E}\left[L^{X^{n,x}_{t^n_k}}_{j+1} P_{t-s'}f\left(X^{n,x}_{s'}\right)\right] ds' = \int_{t^n_k}^s \Phi^{n,j+1}_{t^n_k,s',t}f(x)\, ds',$$

which concludes the proof. □

2.3. **Proof of Theorem 7.** For $n \geq 1$ and $0 \leq s' \leq s \leq t$, let us set $Q^n_{s,t} = P^n_s P_{t-s}$ and $\Delta Q^n_{s',s,t} = Q^n_{s,t} - Q^n_{s',t}$. Observe that $P^n_t = Q^n_{t,t}$ and $P_t = Q^n_{0,t}$ so that

$$\Delta^n_t = P^n_t - P_t = Q^n_{t,t} - Q^n_{0,t} = \sum_{k=0}^{\lfloor nt \rfloor - 1} \Delta Q^n_{t^n_k, t^n_{k+1}, t} + \Delta Q^n_{\lfloor nt \rfloor/n, t, t}. \tag{24}$$

Since $Q^n_{s,t} = \Phi^{n,0}_{t^n_k, s, t}$, by iterating (23), and using the convention that a sum over an empty set is zero, we then have for $k \in \{0, \dots, \lfloor nt \rfloor - 1\}$ and $j \geq 1$,

$$\Delta Q^n_{t^n_k, t^n_{k+1}, t} = \Phi^{n,0}_{t^n_k, t^n_{k+1}, t} - \Phi^{n,0}_{t^n_k, t^n_k, t} = \sum_{i=2}^{j} \frac{\Psi^{n,i}_{t^n_k, t}}{i! n^i} + R^{n,j+1}_{t^n_k, t}$$

(note that $\Psi^{n,1}_{s,t} = 0$ since $L^*_1 = 0$) and

$$\Delta Q^n_{\lfloor nt \rfloor/n, t, t} = \Phi^{n,0}_{\lfloor nt \rfloor/n, t, t} - \Phi^{n,0}_{\lfloor nt \rfloor/n, \lfloor nt \rfloor/n, t} = \sum_{i=2}^{j} \frac{(t - \lfloor nt \rfloor/n)^i}{i!} \Psi^{n,i}_{\lfloor nt \rfloor/n, t} + R^{n,j+1}_{\lfloor nt \rfloor/n, t}$$

where, for $k \in \{0, \dots, \lfloor nt \rfloor\}$,

$$R^{n,j+1}_{t^n_k, t} = \int_{t^n_k}^{t^n_{k+1} \wedge t} \int_{t^n_k}^{s_1} \cdots \int_{t^n_k}^{s_j} \Phi^{n,j+1}_{t^n_k, s_{j+1}, t} \, ds_{j+1} \cdots ds_2 ds_1. \tag{25}$$

From Proposition 9, $(R^{n,j+1}_{t^n_k, t}, t \in [0,1], n \geq 1, k \in \{0, \dots, \lfloor nt \rfloor\})$ is a $O(1/n^{j+1})$ family in $\mathcal{L}_b\left(C^\infty_{\text{pol}}(\mathbb{R}^d)\right)$. Using (24) we finally get for $j \geq 1$,

$$\Delta^n_t = \sum_{i=2}^{j} \frac{1}{i! n^i} \sum_{k=0}^{\lfloor nt \rfloor - 1} \Psi^{n,i}_{t^n_k, t} + R^{n,j}_t, \tag{26}$$

where

$$R^{n,j}_t = \sum_{k=0}^{\lfloor nt \rfloor} R^{n,j+1}_{t^n_k, t} + \sum_{i=2}^{j} \frac{(t - \lfloor nt \rfloor/n)^i}{i!} \Psi^{n,i}_{\lfloor nt \rfloor/n, t}. \tag{27}$$

Note that $(R^{n,j}_{\lfloor nt \rfloor/n}, t \in [0,1], n \geq 1)$ is a $O(t/n^j)$ family in $\mathcal{L}_b\left(C^\infty_{\text{pol}}(\mathbb{R}^d)\right)$ but that because of the second term of the r.h.s. of (27), $(R^{n,j}_t, t \in [0,1], n \geq 1)$ is only $O(1/n^{j \wedge 2})$.

In the particular case when $j = 1$, we have $\Delta^n_t = R^{n,1}_t$ so that we have proved that $(\Delta^n_t, t \in [0,1], n \geq 1)$ is $O(1/n)$ in $\mathcal{L}_b\left(C^\infty_{\text{pol}}(\mathbb{R}^d)\right)$, which was the first statement of Theorem 7.

In the particular case when $j = 2$, if we set

$$\Psi^{(2)}_{s,t} = P_s L^*_2 P_{t-s} \tag{28}$$

and

$$C_t = \frac{1}{2}\int_0^t \Psi^{(2)}_{s,t}\,ds, \tag{29}$$

$$A^n_{1,t} = \frac{1}{2n}\left(\frac{1}{n}\sum_{k=0}^{\lfloor nt\rfloor-1}\Psi^{(2)}_{t^n_k,t} - \int_0^t \Psi^{(2)}_{s,t}\,ds\right), \tag{30}$$

$$A^n_{2,t} = \frac{1}{2n^2}\sum_{k=0}^{\lfloor nt\rfloor-1}\left(\Psi^{n,2}_{t^n_k,t} - \Psi^{(2)}_{t^n_k,t}\right), \tag{31}$$

$$R^n_t = A^n_{1,t} + A^n_{2,t} + R^{n,2}_t, \tag{32}$$

we have

$$\Delta^n_t = C_t/n + R^n_t. \tag{33}$$

As a composition of bounded families, $(\Psi^{(2)}_{s,t}, 0 \le s \le t \le 1)$ is a bounded family in $\mathcal{L}_b\left(C^\infty_{\text{pol}}(\mathbb{R}^d)\right)$, so that $(C_t, t \in [0,1])$ is $O(t)$ in $\mathcal{L}_b\left(C^\infty_{\text{pol}}(\mathbb{R}^d)\right)$. It remains to prove that $(R^n_t, t \in [0,1], n \ge 1)$ is $O(1/n^2)$ in $\mathcal{L}_b\left(C^\infty_{\text{pol}}(\mathbb{R}^d)\right)$. We have already proved that it is true of $(R^{n,2}_t, t \in [0,1], n \ge 1)$.

For $(A^n_{1,t}, t \in [0,1], n \ge 1)$, observe that, if we set $L^\#_3 = LL^*_2 - L^*_2 L \in \mathcal{L}_b\left(C^\infty_{\text{pol}}(\mathbb{R}^d)\right)$, as $\partial_s P_s = LP_s = P_s L$, we have $\partial_s \Psi^{(2)}_{s,t} = P_s LL^*_2 P_{t-s} - P_s L^*_2 L P_{t-s} = P_s L^\#_3 P_{t-s}$. Hence the family $(\Psi^{(2)}_{t^n_k,t} - \Psi^{(2)}_{s,t}, t \in [0,1], n \ge 1, k \in \{0,\ldots,\lfloor nt\rfloor - 1\}, s \in [t^n_k, t^n_{k+1}])$ satisfies

$$\Psi^{(2)}_{t^n_k,t} - \Psi^{(2)}_{s,t} = -\int_{t^n_k}^s P_u L^\#_3 P_{t-u}\,du \tag{34}$$

and thus is $O(1/n)$ in $\mathcal{L}_b\left(C^\infty_{\text{pol}}(\mathbb{R}^d)\right)$. As a consequence,

$$A^n_{1,t} = \frac{1}{2n}\sum_{k=0}^{\lfloor nt\rfloor-1}\int_{t^n_k}^{t^n_{k+1}}\left(\Psi^{(2)}_{t^n_k,t} - \Psi^{(2)}_{s,t}\right)ds - \frac{1}{2n}\int_{\lfloor nt\rfloor/n}^t \Psi^{(2)}_{s,t}\,ds \tag{35}$$

is $O(1/n^2)$ in $\mathcal{L}_b\left(C^\infty_{\text{pol}}(\mathbb{R}^d)\right)$.

As for $(A^n_{2,t}, t \in [0,1], n \ge 1)$, note that from (28) and (20) applied with $j = 2$,

$$\Psi^{n,2}_{t^n_k,t} - \Psi^{(2)}_{t^n_k,t} = P^n_{t^n_k} L^*_2 P_{t-t^n_k} - P_{t^n_k} L^*_2 P_{t-t^n_k} = \Delta^n_{t^n_k} L^*_2 P_{t-t^n_k}. \tag{36}$$

Since $(\Delta^n_t, t \in [0,1], n \ge 1)$ is $O(1/n)$ in $\mathcal{L}_b\left(C^\infty_{\text{pol}}(\mathbb{R}^d)\right)$, so is the family $(\Psi^{n,2}_{t^n_k,t} - \Psi^{(2)}_{t^n_k,t}, t \in [0,1], n \ge 1, k \in \{0,\ldots,\lfloor nt\rfloor - 1\})$, as the composition of a bounded family by a $O(1/n)$ family in $\mathcal{L}_b\left(C^\infty_{\text{pol}}(\mathbb{R}^d)\right)$. This completes the proof of Theorem 7.

Remark 11. It is noteworthy that the family $(R'^n_t, t \in [0,1], n \ge 1)$ defined by

$$R'^n_t = R^n_t + \frac{1}{2n}\int_{\lfloor nt\rfloor/n}^t \Psi^{(2)}_{s,t}\,ds - \frac{1}{2}\left(t - \lfloor nt\rfloor/n\right)^2 \Psi^{n,2}_{\lfloor nt\rfloor/n,t}$$

is $O(t/n^2)$ in $\mathcal{L}_b\left(C^\infty_{\text{pol}}(\mathbb{R}^d)\right)$. In particular, $(R^n_{\lfloor nt\rfloor/n}, t \in [0,1], n \ge 1)$ is $O(t/n^2)$ in $\mathcal{L}_b\left(C^\infty_{\text{pol}}(\mathbb{R}^d)\right)$.

3. Second step: expansion for the density of $X^{n,x}_t$

This section is devoted to the proof of Theorem 1.

3.1. **Space convolutions.** Let us denote by $\mathcal{T}_1$ the unit triangle $\{(s,t) \in \mathbb{R}^2 | 0 < s < t \leq 1\}$. For $l \in \mathbb{Z}$, we define $\mathcal{H}_l(\mathbb{R}^d)$ as the space of measurable functions $\rho : \mathcal{T}_1 \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ such that

- for all $(s,t) \in \mathcal{T}_1$, $\rho(s,t,\cdot,\cdot)$ is infinitely differentiable,
- for all $\alpha, \beta \in \mathbb{N}^d$, there exists two constants $c_1 \geq 0$ and $c_2 > 0$ such that for all $(s,t) \in \mathcal{T}_1$ and $x, y \in \mathbb{R}^d$,

$$\left|\partial_x^\alpha \partial_y^\beta \rho(s,t,x,y)\right| \leq c_1 t^{-(|\alpha|+|\beta|+d+l)/2} \exp\left(-c_2 \|x-y\|^2/t\right). \tag{37}$$

We say that a subset $\mathcal{B} \subset \mathcal{H}_l(\mathbb{R}^d)$ is bounded if, in (37), c_1 and c_2 can be chosen independently on $\rho \in \mathcal{B}$. We also introduce the space $\mathcal{H}(\mathbb{R}^d)$ which is defined in the same way as $\mathcal{H}_l(\mathbb{R}^d)$ with (37) replaced by

$$\left|\partial_x^\alpha \partial_y^\beta \rho(s,t,x,y)\right| \leq c_1 t^{-(|\alpha|+|\beta|+d)/2} \exp\left(-c_2 \|x-y\|^2/t\right), \tag{38}$$

$$\left|\partial_x^\alpha \left(\rho\left(s,t,x,x+y\sqrt{t}\right)\right)\right| \leq c_1 t^{-d/2} \exp\left(-c_2 \|y\|^2\right), \tag{39}$$

and we say that a subset $\mathcal{B} \subset \mathcal{H}(\mathbb{R}^d)$ is bounded if, in (38) and (39), c_1 and c_2 can be chosen independently on $\rho \in \mathcal{B}$.

For $\pi_1, \pi_2 \in \mathcal{G}(\mathbb{R}^d)$ (see (9)-(10) for the definition of this space), $g \in C_b^\infty(\mathbb{R}^d)$ and $\gamma \in \mathbb{N}^d$, we define a function $\pi_1 *_{g,\gamma} \pi_2$ on $\mathcal{T}_1 \times \mathbb{R}^d \times \mathbb{R}^d$ by putting

$$(\pi_1 *_{g,\gamma} \pi_2)(s,t,x,y) = \int_{\mathbb{R}^d} g(z) \pi_1(s,x,z) \partial_2^\gamma \pi_2(t-s,z,y)\, dz.$$

Notation ∂_2 means differentiation with respect to the second argument, here z. We partition the unit triangle $\mathcal{T}_1$ into $\mathcal{T}_1^- = \{(s,t) \in \mathcal{T}_1 | 0 < s \leq t/2\}$ and $\mathcal{T}_1^+ = \{(s,t) \in \mathcal{T}_1 | t/2 < s < t\}$, and, for $\epsilon = \pm$, we define $(\pi_1 *_{g,\gamma} \pi_2)_\epsilon (s,t,x,y) = \mathbf{1}_{\mathcal{T}_1^\epsilon}(s,t) (\pi_1 *_{g,\gamma} \pi_2)(s,t,x,y)$, so that $\pi_1 *_{g,\gamma} \pi_2 = (\pi_1 *_{g,\gamma} \pi_2)_- + (\pi_1 *_{g,\gamma} \pi_2)_+$.

Proposition 12. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two bounded subsets of $\mathcal{G}(\mathbb{R}^d)$, $g \in C_b^\infty(\mathbb{R}^d)$ and $\gamma \in \mathbb{N}^d$. Then $\{\pi_1 *_{g,\gamma} \pi_2 | \pi_1 \in \mathcal{B}_1, \pi_2 \in \mathcal{B}_2\}$ is a bounded subset of $\mathcal{H}_{|\gamma|}(\mathbb{R}^d)$.*

Before proving Proposition 12 and for the sake of clarity, let us state apart the following technical lemma, whose proof is a straightforward application of Lebesgue's dominated convergence theorem:

Lemma 13. *Let $l \in \mathbb{Z}$, $(\chi_i, i \in I)$ be a family of measurable functions mapping $\mathcal{T}_1 \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$ into $\mathbb{R}$ such that*

- *for all $i \in I$, $(s,t) \in \mathcal{T}_1$ and $\zeta \in \mathbb{R}^d$, $\chi_i(s,t,\cdot,\cdot,\zeta)$ is infinitely differentiable,*
- *for all $\alpha, \beta \in \mathbb{N}^d$, there exists two constants $c_1 \geq 0$ and $c_2 > 0$ such that for all $i \in I$, $(s,t) \in \mathcal{T}_1$ and $x, y, \zeta \in \mathbb{R}^d$,*

$$\left|\partial_x^\alpha \partial_y^\beta \chi_i(s,t,x,y,\zeta)\right| \leq c_1 t^{-(|\alpha|+|\beta|+d+l)/2} \exp\left(-c_2 \|x-y\|^2/t - c_2 \|\zeta\|^2\right), \tag{40}$$

and let us define $\mathcal{I}(\chi_i)(s,t,x,y) = \int_{\mathbb{R}^d} \chi_i(s,t,x,y,\zeta)\, d\zeta$. Then $\{\mathcal{I}(\chi_i) | i \in I\}$ is a bounded subset of $\mathcal{H}_l(\mathbb{R}^d)$.

Proof of Proposition 12. It is enough to show that both $\mathcal{B}_\epsilon \equiv \{(\pi_1 *_{g,\gamma} \pi_2)_\epsilon | \pi_1 \in \mathcal{B}_1, \pi_2 \in \mathcal{B}_2\}$ are bounded.

Step 1. Let us first treat $\mathcal{B}_-$. After the change of variables $z = x + \zeta\sqrt{s}$, we get $(\pi_1 *_{g,\gamma} \pi_2)_- = \mathcal{I}(\chi^-_{\pi_1,\pi_2})$ with

$$\chi^-_{\pi_1,\pi_2}(s,t,x,y,\zeta) = \mathbf{1}_{\mathcal{T}_1^-}(s,t) s^{d/2} g(x+\zeta\sqrt{s}) \pi_1(s,x,x+\zeta\sqrt{s}) \partial_2^\gamma \pi_2(t-s,x+\zeta\sqrt{s},y).$$

It is enough to check that the family $\left(\chi^{-}_{\pi_1,\pi_2}, (\pi_1,\pi_2)\in\mathcal{B}_1\times\mathcal{B}_2\right)$ satisfies the assumptions of Lemma 13 with $l=|\gamma|$. The first point is obvious. In order to check the second one, let us fix $\alpha,\beta\in\mathbb{N}^d$. According to Leibniz's formula, $\partial_x^\alpha\partial_y^\beta\chi_{\pi_1,\pi_2}(s,t,x,y,\zeta)$ can be written as a weighted sum of terms of the form

$$\chi^{-,\alpha_1,\alpha_2,\alpha_3}_{\pi_1,\pi_2}(s,t,x,y,\zeta)=\mathbf{1}_{\mathcal{T}_1^-}(s,t)s^{d/2}\partial^{\alpha_1}g(x+\zeta\sqrt{s})$$
$$\partial_x^{\alpha_2}\left(\pi_1(s,x,x+\zeta\sqrt{s})\right)\partial_2^{\gamma+\alpha_3}\partial_3^\beta\pi_2(t-s,x+\zeta\sqrt{s},y),$$

with $|\alpha_1|+|\alpha_2|+|\alpha_3|=|\alpha|$, so that in order to check (40) it is enough to show that for each such $(\alpha_1,\alpha_2,\alpha_3)$ one can find $c_1\geq 0$ and $c_2>0$ such that for all $(\pi_1,\pi_2)\in\mathcal{B}_1\times\mathcal{B}_2$, $(s,t)\in\mathcal{T}_1$ and $x,y,\zeta\in\mathbb{R}^d$, $|\chi^{-,\alpha_1,\alpha_2,\alpha_3}_{\pi_1,\pi_2}(s,t,x,y,\zeta)|$ is less than the r.h.s. of (40), with $l=|\gamma|$. Now, $\mathcal{B}_1$ and $\mathcal{B}_2$ are bounded subsets of $\mathcal{G}(\mathbb{R}^d)$ so that one can find $c_3,c_5\geq 0$ and $c_4>0$ such that for all $(\pi_1,\pi_2)\in\mathcal{B}_1\times\mathcal{B}_2$, $(s,t)\in\mathcal{T}_1$ and $x,y,\zeta\in\mathbb{R}^d$,

$$\left|\partial_x^{\alpha_2}\left(\pi_1(s,x,x+\zeta\sqrt{s})\right)\right|\leq c_3 s^{-d/2}\exp\left(-c_4\left\|\zeta\right\|^2\right)$$

and

$$\mathbf{1}_{\mathcal{T}_1^-}(s,t)\left|\partial_2^{\gamma+\alpha_3}\partial_3^\beta\pi_2(t-s,x+\zeta\sqrt{s},y)\right|$$
$$\leq\mathbf{1}_{\mathcal{T}_1^-}(s,t)c_3(t-s)^{-(|\alpha_3|+|\beta|+|\gamma|+d)/2}\exp\left(-c_4\left\|x-y+\zeta\sqrt{s}\right\|^2/(t-s)\right)$$
$$\leq\mathbf{1}_{\mathcal{T}_1^-}(s,t)c_5 t^{-(|\alpha|+|\beta|+|\gamma|+d)/2}\exp\left(-c_4\left\|x-y+\zeta\sqrt{s}\right\|^2/t\right)$$

where, for the last inequality, we have used the fact that when $(s,t)\in\mathcal{T}_1^-$, $t/2\leq t-s\leq t\leq 1$. Now, using the fact that $\|x-z\|^2\geq\|x\|^2/2-\|z\|^2$ for all $x,z\in\mathbb{R}^d$, we see that for all $(s,t)\in\mathcal{T}_1^-$, $\|\zeta\|^2+\left\|x-y+\zeta\sqrt{s}\right\|^2/t\geq(\|x-y\|^2/t+\|\zeta\|^2)/2$. Since $g\in C_b^\infty(\mathbb{R}^d)$, we can eventually find $c_1\geq 0$ and $c_2>0$ such that for all $(\pi_1,\pi_2)\in\mathcal{B}_1\times\mathcal{B}_2$, $(s,t)\in\mathcal{T}_1$ and $x,y,\zeta\in\mathbb{R}^d$,

$$\left|\chi^{-,\alpha_1,\alpha_2,\alpha_3}_{\pi_1,\pi_2}(s,t,x,y,\zeta)\right|\leq c_1 t^{-(|\alpha|+|\beta|+d+|\gamma|)/2}\exp\left(-c_2\left\|x-y\right\|^2/t-c_2\left\|\zeta\right\|^2\right),$$

which completes Step 1.

Step 2. Let us now treat $\mathcal{B}_+$. After $|\gamma|$ integrations by parts, we have

$$(\pi_1 *_{g,\gamma}\pi_2)_+(s,t,x,y)=\mathbf{1}_{\mathcal{T}_1^+}(s,t)\int_{\mathbb{R}^d}\partial_z^\gamma(g(z)\pi_1(s,x,z))\pi_2(t-s,z,y)\,dz.$$

Using Leibniz's formula and making the change of variables $z=y-\zeta\sqrt{t-s}$, we get that $(\pi_1 *_{g,\gamma}\pi_2)_+$ is a weighted sum of terms of the form $\mathcal{I}(\chi^{+,\gamma_1,\gamma_2}_{\pi_1,\pi_2})$ with

$$\chi^{+,\gamma_1,\gamma_2}_{\pi_1,\pi_2}(s,t,x,y,\zeta)=\mathbf{1}_{\mathcal{T}_1^+}(s,t)(t-s)^{d/2}\partial^{\gamma_1}g(y-\zeta\sqrt{t-s})$$
$$\partial_3^{\gamma_2}\pi_1(s,x,y-\zeta\sqrt{t-s})\pi_2(t-s,y-\zeta\sqrt{t-s},y)$$

and $|\gamma_1|+|\gamma_2|=|\gamma|$, so that we are now in the position to apply the same arguments as in Step 1 and get that the family $(\chi^{+,\gamma_1,\gamma_2}_{\pi_1,\pi_2},(\pi_1,\pi_2)\in\mathcal{B}_1\times\mathcal{B}_2)$ satisfies the assumptions of Lemma 13 with $l=|\gamma|$, which completes the proof. □

We also have

Proposition 14. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two bounded subsets of $\mathcal{G}(\mathbb{R}^d)$ and $g\in C_b^\infty(\mathbb{R}^d)$. Then $\{\pi_1 *_{g,0}\pi_2|\pi_1\in\mathcal{B}_1,\pi_2\in\mathcal{B}_2\}$ is a bounded subset of $\mathcal{H}(\mathbb{R}^d)$.*

Proof. From Proposition 12, we know that $\{\pi_1 *_{g,0} \pi_2 | \pi_1 \in \mathcal{B}_1, \pi_2 \in \mathcal{B}_2\}$ is a bounded subset of $\mathcal{H}_0(\mathbb{R}^d)$. It remains to prove that (39) holds for $\rho = \pi_1 *_{g,0} \pi_2$ with constants c_1 and c_2 which do not depend on $(\pi_1, \pi_2) \in \mathcal{B}_1 \times \mathcal{B}_2$. As in the proof of Proposition 12, we treat $(\pi_1 *_{g,0} \pi_2)_-$ and $(\pi_1 *_{g,0} \pi_2)_+$ separately but analogously. For instance, let us treat the first term. We have $(\pi_1 *_{g,0} \pi_2)_- = \mathcal{I}(\chi^-_{\pi_1,\pi_2})$ with

$$\chi^-_{\pi_1,\pi_2}(s,t,x,y,\zeta) = \mathbf{1}_{\mathcal{T}_1^-}(s,t) s^{d/2} g(x + \zeta\sqrt{s}) \pi_1(s, x, x + \zeta\sqrt{s}) \pi_2(t - s, x + \zeta\sqrt{s}, y).$$

Then we write $\partial_x^\alpha \left(\chi^-_{\pi_1,\pi_2}\left(s,t,x,x + y\sqrt{t},\zeta\right)\right)$ as a weighted sum of terms of the form

$$\begin{aligned}\tilde{\chi}^{-,\alpha_1,\alpha_2,\alpha_3}_{\pi_1,\pi_2}(s,t,x,y,\zeta) = \mathbf{1}_{\mathcal{T}_1^-}(s,t) s^{d/2} \partial^{\alpha_1} g(x + \zeta\sqrt{s}) \\ \partial_x^{\alpha_2}\left(\pi_1(s,x,x+\zeta\sqrt{s})\right) \partial_x^{\alpha_3}\left(\pi_2(t-s, x+\zeta\sqrt{s}, x + y\sqrt{t})\right),\end{aligned}$$

with $|\alpha_1| + |\alpha_2| + |\alpha_3| = |\alpha|$. Then we use (10) twice and the same arguments as in the preceding proof to get $c_1 \geq 0$ and $c_2 > 0$ such that for all $(\pi_1, \pi_2) \in \mathcal{B}_1 \times \mathcal{B}_2$, $(s,t) \in \mathcal{T}_1$ and $x, y, \zeta \in \mathbb{R}^d$, $|\tilde{\chi}^{-,\alpha_1,\alpha_2,\alpha_3}_{\pi_1,\pi_2}(s,t,x,y,\zeta)| \leq c_1 t^{-d/2} \exp(-c_2 \|y\|^2 - c_2 \|\zeta\|^2)$, which completes the proof. $\square$

3.2. **Proof of Theorem 1.** In this section, we assume (B) and (C).

Lemma 15. *Under (B) and (C), for all $t \in (0,1]$, $n \geq 1$ and $x \in \mathbb{R}^d$, $X_t^{n,x}$ has a density $p_n(t,x,\cdot)$ and $(p_n, n \geq 1)$ is a bounded sequence in $\mathcal{G}(\mathbb{R}^d)$.*

Proof. It is known that for all $n \geq 1$, $k \in \{1, \ldots, n\}$ and $x \in \mathbb{R}^d$, $X^{n,x}_{t^n_k}$ has a density $p_{n,k}(x,\cdot)$ such that $p_{n,k}$ is infinitely differentiable and satisfies (9)-(10) with $t = t^n_k$ and two constants c_1 and c_2 which do not depend on n and k (see the proof of theorem 1.1, page 278, in [9]). Since $\lfloor nt \rfloor / n \geq t/2$ for all $t \geq 1/n$, this shows that the sequence $(\tilde{p}_n, n \geq 1)$ defined by $\tilde{p}_n(t,x,y) = \mathbf{1}_{\{nt \geq 1\}} p_{n,\lfloor nt \rfloor}(x,y)$ is bounded in $\mathcal{G}(\mathbb{R}^d)$. If we denote by $\Gamma(t,x,\cdot)$ the density of $x + b(x)t + \sigma(x)B_t$ ($t \in (0,1], x \in \mathbb{R}^d$), we observe that when $k \in \{1, \ldots, n-1\}$ and $t \in (t^n_k, t^n_{k+1})$, $X^{n,x}_t$ has the density $p_n(t,x,\cdot) = \int_{\mathbb{R}^d} p_{n,k}(x,z)\Gamma(t - t^n_k, z, \cdot)\, dz = (\tilde{p}_n *_{1,0} \Gamma)(t^n_k, t, x, \cdot)$. Hence, for all $t \in (0,1]$, $n \geq 1$ and $x \in \mathbb{R}^d$, $X^{n,x}_t$ has the density

$$p_n(t,x,\cdot) = \begin{cases} p_{n,k}(x,\cdot) & \text{if} \quad t = t^n_k, k \in \{1,\ldots,n\}, \\ \Gamma(t,x,\cdot) & \text{if} \quad t \in (0, t^n_1), \\ (\tilde{p}_n *_{1,0} \Gamma)(t^n_k, t, x, \cdot) & \text{if} \quad t \in (t^n_k, t^n_{k+1}), k \in \{1, \ldots, n-1\}. \end{cases}$$

Observing that $\Gamma \in \mathcal{G}(\mathbb{R}^d)$ and applying Proposition 14, we get that $(p_n, n \geq 1)$ is a bounded sequence in $\mathcal{G}(\mathbb{R}^d)$. $\square$

Recall (33). We shall now explicit C_t and R^n_t as integral operators on $\mathbb{R}^d$. To this end, note that, applying recursively Lebesgue's dominated convergence theorem, we have that for all $t \in (0,1]$, $f \in C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and $\alpha \in \mathbb{N}^d$,

$$\partial^\alpha P_t f(x) = \int_{\mathbb{R}^d} f(y) \partial_2^\alpha p(t,x,y)\, dy. \tag{41}$$

The next lemma explicits C_t as an integral operator. The function π which appears there should be thought of as the kernel of C.

Lemma 16. *Under (B) and (C), there exists $\pi \in \mathcal{G}_1(\mathbb{R}^d)$, given by (12), such that for all $t \in (0,1]$, $f \in C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,*

$$C_t f(x) = \int_{\mathbb{R}^d} f(y)\pi(t,x,y)\,dy.$$

Proof. Using (28)-(29) for the first equality, (17) for the third one and (41) for the fourth one, we have

$$\begin{aligned}
2C_t f(x) &= \int_0^t P_s L_2^* P_{t-s} f(x)\,ds \\
&= \int_0^t \int_{\mathbb{R}^d} p(s,x,z) L_2^* P_{t-s} f(z)\,dzds \\
&= \sum_{1\le|\alpha|\le 3} \int_0^t \int_{\mathbb{R}^d} g^*_{2,\alpha}(z) p(s,x,z) \partial^\alpha P_{t-s} f(z)\,dzds \\
&= \sum_{1\le|\alpha|\le 3} \int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(y) g^*_{2,\alpha}(z) p(s,x,z) \partial_2^\alpha p(t-s,z,y)\,dydzds.
\end{aligned}$$

Using Fubini's theorem, we see that to complete the proof it is enough to show that the function π defined by

$$\pi(t,x,y) = \frac{1}{2} \sum_{1\le|\alpha|\le 3} \int_0^t (p *_{g^*_{2,\alpha},\alpha} p)(s,t,x,y)\,ds \tag{42}$$

belongs to $\mathcal{G}_1(\mathbb{R}^d)$. Now, $p \in \mathcal{G}(\mathbb{R}^d)$ and, from Remark 8, $g^*_{2,\alpha} \in C^\infty_b(\mathbb{R}^d)$ so that we can apply Proposition 12: $p*_{g^*_{2,\alpha},\alpha} p \in \mathcal{H}_{|\alpha|}(\mathbb{R}^d)$. In particular, $\int_0^t (p*_{g^*_{2,\alpha},\alpha}p)(s,\cdot,\cdot,\cdot)ds \in \mathcal{G}_{|\alpha|-2}$. Since $|\alpha| \le 3$ and by monotonicity of $(\mathcal{G}_l(\mathbb{R}^d), l \in \mathbb{Z})$, we finally get that $\pi \in \mathcal{G}_1(\mathbb{R}^d)$. To complete the proof, note that (42) can be rewritten as (12). □

We have a similar representation for $A^n_{1,t}$, recall (30). We say that a sequence $(\pi^n, n \ge 1)$ is $O(1/n^j)$ in $\mathcal{G}_l(\mathbb{R}^d)$ if $(n^j \pi^n, n \ge 1)$ is bounded in $\mathcal{G}_l(\mathbb{R}^d)$.

Lemma 17. *Under (B) and (C), there exists a $O(1/n^2)$ sequence $(\pi^n_1, n \ge 1)$ in $\mathcal{G}_2(\mathbb{R}^d)$ such that for all $t \in (0,1]$, $f \in C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,*

$$A^n_{1,t} f(x) = \int_{\mathbb{R}^d} f(y)\pi^n_1(t,x,y)\,dy.$$

Proof. Recall (34). From Remark 8, there is a family $(g^\#_{3,\alpha}, 1 \le |\alpha| \le 4)$ in $C^\infty_b(\mathbb{R}^d)$ such that $L^\#_3 = \sum_{1\le|\alpha|\le 4} g^\#_{3,\alpha}\partial^\alpha$, so that, using (41) twice, we have

$$\Psi^{(2)}_{t^n_k,t} f(x) - \Psi^{(2)}_{s,t} f(x) = -\sum_{1\le|\alpha|\le 4} \int_{t^n_k}^s \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(y) g^\#_{3,\alpha}(z) p(u,x,z) \partial_2^\alpha p(t-u,z,y)\,dydzdu.$$

Using (35), we get $A^n_{1,t} f(x) = \int_{\mathbb{R}^d} f(y)\pi^n_1(t,x,y)\,dy$ with $\pi^n_1 = \pi^n_{1,1} + \pi^n_{1,2}$ and

$$\begin{aligned}
\pi^n_{1,1}(t,x,y) &= -\frac{1}{2n} \sum_{1\le|\alpha|\le 4} \sum_{k=0}^{\lfloor nt\rfloor -1} \int_{t^n_k}^{t^n_{k+1}} \int_{t^n_k}^s \left(p *_{g^\#_{3,\alpha},\alpha} p\right)(u,t,x,y)\,duds, \\
\pi^n_{1,2}(t,x,y) &= -\frac{1}{2n} \sum_{1\le|\alpha|\le 3} \int_{\lfloor nt\rfloor/n}^t (p *_{g^*_{2,\alpha},\alpha} p)(s,t,x,y)\,ds.
\end{aligned}$$

Now Proposition 12 states that $p *_{g^{\#}_{3,\alpha},\alpha} p$ and $p *_{g^{*}_{2,\alpha},\alpha} p$ belong to $\mathcal{H}_{|\alpha|}(\mathbb{R}^d)$. Hence $(\int_{t_k^n}^{t_{k+1}^n}\int_{t_k^n}^{s}(p *_{g^{\#}_{3,\alpha},\alpha} p)(u,\cdot,\cdot,\cdot)\,duds, n\geq 1, k\in\{0,\dots,\lfloor nt\rfloor-1\})$ is $O(1/n^2)$ in $\mathcal{G}_{|\alpha|}$ and $(\int_{\lfloor nt\rfloor/n}^{\cdot}(p *_{g^{*}_{2,\alpha},\alpha} p)(s,\cdot,\cdot,\cdot)\,ds, n\geq 1)$ is $O(1/n)$ in $\mathcal{G}_{|\alpha|}$. As a consequence, $(\pi^n_{1,1}, n\geq 1)$ (resp. $(\pi^n_{1,2}, n\geq 1)$) is $O(1/n^2)$ in $\mathcal{G}_2(\mathbb{R}^d)$ (resp. $\mathcal{G}_1(\mathbb{R}^d)$). Eventually, $(\pi^n_1, n\geq 1)$ is $O(1/n^2)$ in $\mathcal{G}_2(\mathbb{R}^d)$. □

We shall now prove analogous lemmas for $A^n_{2,t}$ and $R^{n,2}_t$.

Lemma 18. *Under (B) and (C), there exists a $O(1/n^2)$ sequence $(\pi^n_2, n\geq 1)$ in $\mathcal{G}_3(\mathbb{R}^d)$ such that for all $t\in(0,1]$, $f\in C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$ and $x\in\mathbb{R}^d$,*

$$A^n_{2,t}f(x)=\int_{\mathbb{R}^d} f(y)\pi^n_2(t,x,y)\,dy.$$

Proof. From (20) and (28), $\Psi^{n,2}_{0,t}=\Psi^{(2)}_{0,t}$ so that (31) reads

$$\begin{aligned}
2n^2A^n_{2,t}f(x) &= \sum_{k=1}^{\lfloor nt\rfloor-1}\left(\Psi^{n,2}_{t_k^n,t}-\Psi^{(2)}_{t_k^n,t}\right)\\
&= \sum_{k=1}^{\lfloor nt\rfloor-1}(P^n_{t_k^n}-P_{t_k^n})L_2^*P_{t-t_k^n}f(x)\\
&= \sum_{k=1}^{\lfloor nt\rfloor-1}\int_{\mathbb{R}^d}(p_n-p)(t_k^n,x,z)L_2^*P_{t-t_k^n}f(z)\,dz\\
&= \sum_{1\leq|\alpha|\leq 3}\sum_{k=1}^{\lfloor nt\rfloor-1}\int_{\mathbb{R}^d}(p_n-p)(t_k^n,x,z)g^*_{2,\alpha}(z)\partial^\alpha P_{t-t_k^n}f(z)\,dz\\
&= \sum_{1\leq|\alpha|\leq 3}\sum_{k=1}^{\lfloor nt\rfloor-1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(p_n-p)(t_k^n,x,z)g^*_{2,\alpha}(z)f(y)\partial_2^\alpha p(t-t_k^n,z,y)\,dydz
\end{aligned}$$

where we have used (36) for the second equality, (17) for the fourth one and (41) for the fifth one. From Remark 8, $g^*_{2,\alpha}\in C^\infty_b(\mathbb{R}^d)$ so that to complete the proof it is enough to show that whenever $g\in C^\infty_b(\mathbb{R}^d)$ and $\alpha\in\mathbb{N}^d$, the sequence $(\pi^n, n\geq 1)$ defined by

$$\pi^n(t,x,y)=\sum_{k=1}^{\lfloor nt\rfloor-1}\int_{\mathbb{R}^d}(p_n-p)(t_k^n,x,z)g(z)\partial_2^\alpha p(t-t_k^n,z,y)dz=\sum_{k=1}^{\lfloor nt\rfloor-1}((p_n-p)*_{g,\alpha}p)(t_k^n,t,x,y)$$

is bounded in $\mathcal{G}_{|\alpha|}(\mathbb{R}^d)$. And to do so, it is enough to show that the sequence $(\rho^n_{t_k^n}, n\geq 1, k\in\{1,\dots,n-1\})$ defined by

$$\rho^n_{t_k^n}(t,x,y)=\mathbf{1}_{\mathcal{T}_1}(t_k^n,t)\left((p_n-p)*_{g,\alpha}p\right)(t_k^n,t,x,y)$$

is $O(1/n)$ in $\mathcal{G}_{|\alpha|+2}(\mathbb{R}^d)$. Let us write $\rho^{n,-}_{t_k^n}(t,x,y)=\mathbf{1}_{\mathcal{T}_1^-}(t_k^n,t)\rho^n_{t_k^n}(t,x,y)$ and $\rho^{n,+}_{t_k^n}(t,x,y)=\mathbf{1}_{\mathcal{T}_1^+}(t_k^n,t)\rho^n_{t_k^n}(t,x,y)$ so that $\rho^n_{t_k^n}=\rho^{n,-}_{t_k^n}+\rho^{n,+}_{t_k^n}$.

Let us first prove that $(\rho^{n,-}_{t_k^n}, n\geq 1, k\in\{1,\dots,n-1\})$ is $O(1/n)$ in $\mathcal{G}_{|\alpha|+2}(\mathbb{R}^d)$. Note that $\rho^{n,-}_{t_k^n}=P^n_{t_k^n}\pi_{t_k^n}-P_{t_k^n}\pi_{t_k^n}\equiv\Delta^n_{t_k^n}\pi_{t_k^n}$ (see (46) in the appendix for the definition of $P^n_s\pi$

and $P_s\pi$ when $\pi \in \mathcal{G}_l(\mathbb{R}^d)$) where the sequence $(\pi_{t_k^n}, n \geq 1, k \in \{1, \dots, n-1\})$ defined by $\pi_{t_k^n}(t,x,y) = \mathbf{1}_{\mathcal{T}_1^-}(t_k^n, t) g(x) \partial_2^\alpha p(t - t_k^n, x, y)$ is bounded in $\mathcal{G}_{|\alpha|}(\mathbb{R}^d)$. Thus, from (26)-(27) and (25) applied with $j = 1$,

$$\rho_{t_k^n}^{n,-} = \sum_{m=0}^{k-1} \int_{t_m^n}^{t_{m+1}^n} \int_{t_m^n}^{s_1} \Phi_{t_m^n, s_2, t_k^n}^{n,2} \pi_{t_k^n} \, ds_2 ds_1,$$

and, since $k \leq \lfloor nt \rfloor$ when $(t_k^n, t) \in \mathcal{T}_1$, Proposition 24 in the appendix gives the result.

Let us now prove the same for $\rho^{n,+}$. After $|\alpha|$ integrations by parts and after setting $z = y - \zeta\sqrt{t-s}$, we get that $((p_n - p) *_{g,\alpha} p)_+$ is a weighted sum of terms of the form $\mathcal{I}(\chi_{\alpha_1,\alpha_2}^{n,+})$ - see Lemma 13 - with

$$\chi_{\alpha_1,\alpha_2}^{n,+}(s,t,x,y,\zeta) = \mathbf{1}_{\mathcal{T}_1^+}(s,t)(t-s)^{d/2} \partial^{\alpha_1} g(y - \zeta\sqrt{t-s})$$
$$\partial_3^{\alpha_2}(p_n - p)(s, x, y - \zeta\sqrt{t-s}) p(t-s, y - \zeta\sqrt{t-s}, y)$$

and $|\alpha_1| + |\alpha_2| = |\alpha|$. Now, from Corollary 22 in the appendix, $(p_n - p, n \geq 1)$ is $O(1/n)$ in $\mathcal{G}_2(\mathbb{R}^d)$ so that, using the same arguments as in Step 2 of the proof of Proposition 12, we get that $((p_n - p) *_{g,\alpha} p)_+$ is $O(1/n)$ in $\mathcal{H}_{|\alpha|+2}$. Since $\rho_{t_k^n}^{n,+}(t,x,y) = \mathbf{1}_{\mathcal{T}_1^+}(t_k^n, t)((p_n - p) *_{g,\alpha} p)_+(t_k^n, t, x, y)$, we conclude that $(\rho_{t_k^n}^{n,+}, n \geq 1, k \in \{1, \dots, n-1\})$ is $O(1/n)$ in $\mathcal{G}_{|\alpha|+2}(\mathbb{R}^d)$. $\square$

Lemma 19. *Under (B) and (C), there exists a $O(1/n^2)$ sequence $(\pi_3^n, n \geq 1)$ in $\mathcal{G}_4(\mathbb{R}^d)$ such that for all $t \in (0,1]$, $f \in C_{\text{pol}}^\infty(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,*

$$R_t^{n,2} f(x) = \int_{\mathbb{R}^d} f(y) \pi_3^n(t,x,y) \, dy.$$

Proof. From (27) and (25) applied with $j = 2$,

$$R_t^{n,2} f(x) = \sum_{k=0}^{\lfloor nt \rfloor} \int_{t_k^n}^{t_{k+1}^n \wedge t} \int_{t_k^n}^{s_1} \int_{t_k^n}^{s_2} \Phi_{t_k^n, s_3, t}^{n,3} f(x) \, ds_3 ds_2 ds_1 + \frac{1}{2} \Psi_{\lfloor nt \rfloor/n, t}^{n,2} (t - \lfloor nt \rfloor/n)^2.$$

It is enough to apply Lemmas 21 and 23 to conclude. $\square$

Theorem 1 follows from Lemmas 15, 16, 17, 18 and 19.

Remark 20. Note that (26)-(27) and (25) combined with Lemmas 21 and 23 imply that we have an expansion of arbitrary order j for $p_n - p$:

$$(p_n - p)(t, \cdot, \cdot) = \sum_{i=2}^{j} \frac{1}{i! n^i} \sum_{k=0}^{\lfloor nt \rfloor - 1} \psi_{t_k^n}^{n,i}(t, \cdot, \cdot) + \sum_{i=2}^{j} \frac{(t - \lfloor nt \rfloor/n)^i}{i!} \psi_{\lfloor nt \rfloor/n}^{n,i}(t, \cdot, \cdot)$$
$$+ \sum_{k=0}^{\lfloor nt \rfloor} \int_{t_k^n}^{t_{k+1}^n \wedge t} \int_{t_k^n}^{s_1} \cdots \int_{t_k^n}^{s_j} \varphi_{t_k^n}^{n,j+1}(s_{j+1}, t, \cdot, \cdot) \, ds_{j+1} \cdots ds_2 ds_1.$$

Since $(\varphi_{t_k^n}^{n,j}, n \geq 1, k \in \{0, \dots, n\})$ and $(\psi_{t_k^n}^{n,j}, n \geq 1, k \in \{0, \dots, n\})$ are respectively bounded in $\mathcal{H}_{2j}(\mathbb{R}^d)$ and $\mathcal{G}_{2j}(\mathbb{R}^d)$, this gives

$$p_n - p = \sum_{i=1}^{j-1} \frac{\pi_{n,i}}{n^i} + \sum_{i=2}^{j} (t - \lfloor nt \rfloor/n)^i \pi'_{n,i} + r_j^n \tag{43}$$

where $(\pi_{n,i}, n \geq 1)$ and $(\pi'_{n,i}, n \geq 1)$ are respectively bounded in $\mathcal{G}_{2i-2}(\mathbb{R}^d)$ and $\mathcal{G}_{2i}(\mathbb{R}^d)$ and $(r^n_j, n \geq 1)$ is $O(1/n^j)$ in $\mathcal{G}_{2j}(\mathbb{R}^d)$. In particular, when $t = 1$ and no differentiation is applied neither in x nor in y, this boils down to the result of V. Konakov and E. Mammen [9]. Expansion (43) is much richer in the sense that it allows for infinite differentiation in x and y and also precises the way the coefficients explode when t tends to 0.

4. Appendix

4.1. Kernels of Φ and Ψ.

Here we explicit the kernels of $\Phi^{n,j}_{t^n_k,s,t}$ and $\Psi^{n,j}_{t^n_k,t}$.

Lemma 21. *Under (B) and (C), for each $j \in \mathbb{N}^*$, there exists a bounded sequence $(\varphi^{n,j}_{t^n_k}, n \geq 1, k \in \{0, \ldots, n\})$ in $\mathcal{H}_{2j}(\mathbb{R}^d)$ such that for all $t \in (0, 1]$, $n \geq 1$, $k \in \{0, \ldots, \lfloor nt \rfloor\}$, $s \in (t^n_k, t^n_{k+1} \wedge t)$, $f \in C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,*

$$\Phi^{n,j}_{t^n_k,s,t} f(x) = \int_{\mathbb{R}^d} f(y) \varphi^{n,j}_{t^n_k}(s, t, x, y)\, dy. \tag{44}$$

Proof. From (19), (18) and (41) and using Fubini's theorem, we have (44) with

$$\varphi^{n,j}_{t^n_k}(s, t, x, y) = \mathbf{1}_{]t^n_k,t^n_{k+1}[}(s) \sum_{1 \leq |\alpha| \leq 2j} \sum_{l=1}^{m_{j,\alpha}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_n(t^n_k, x, z_1) g_{j,\alpha,l}(z_1) p_n(s - t^n_k, z_1, z_2) h_{j,\alpha,l}(z_2) \partial^\alpha_2 p(t - s, z_2, y)\, dz_1 dz_2,$$

if $k \geq 1$ and

$$\varphi^{n,j}_0(s, t, x, y) = \mathbf{1}_{]0,1/n[}(s) \sum_{1 \leq |\alpha| \leq 2j} \sum_{l=1}^{m_{j,\alpha}} \int_{\mathbb{R}^d} g_{j,\alpha,l}(x) p_n(s, x, z_2) h_{j,\alpha,l}(z_2) \partial^\alpha_2 p(t - s, z_2, y)\, dz_2.$$

As $(p_n, n \geq 1)$ is bounded in $\mathcal{G}(\mathbb{R}^d)$, Proposition 12 shows that $(\varphi^{n,j}_0, n \geq 1)$ is bounded in $\mathcal{H}_{|\alpha|}(\mathbb{R}^d)$, so that to prove the lemma it is enough to show that whenever $g, h \in C^\infty_b(\mathbb{R}^d)$ and $\alpha \in \mathbb{N}^d$, the sequence $(\phi^{n,j}_{t^n_k}, n \geq 1, k \in \{1, \ldots, n\})$ of functions defined on $\mathcal{T}_1 \times \mathbb{R}^d \times \mathbb{R}^d$ by

$$\phi^{n,j}_{t^n_k}(s, t, x, y) = \mathbf{1}_{]t^n_k,t^n_{k+1}[}(s) \int_{\mathbb{R}^d} (p_n *_{g,0} p_n)(t^n_k, s, x, z_2) h(z_2) \partial^\alpha_2 p(t - s, z_2, y)\, dz_2$$

is bounded in $\mathcal{H}_{|\alpha|}(\mathbb{R}^d)$. Now, setting $q^n_{t^n_k}(s, x, z) = \mathbf{1}_{]t^n_k,t^n_{k+1}[}(s)(p_n *_{g,0} p_n)(t^n_k, s, x, z)$, it follows from Proposition 14 that $(q^n_{t^n_k}, n \geq 1, k \in \{1, \ldots, n\})$ is a bounded sequence in $\mathcal{G}(\mathbb{R}^d)$. Then Proposition 12 shows that $\phi^{n,j}_{t^n_k} = q^n_{t^n_k} *_{h,\alpha} p$ is bounded in $\mathcal{H}_{|\alpha|}(\mathbb{R}^d)$. □

Corollary 22. *Under (B) and (C), $(p_n - p, n \geq 1)$ is $O(1/n)$ in $\mathcal{G}_2(\mathbb{R}^d)$.*

Proof. From (26) applied with $j = 1$ and (25), we have for all $f \in C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} f(y)(p_n - p)(t, x, y)\, dy = \sum_{k=0}^{\lfloor nt \rfloor} \int_{t^n_k}^{t^n_{k+1} \wedge t} \int_{t^n_k}^{s_1} \Phi^{n,2}_{t^n_k,s_2,t} f(x)\, ds_2 ds_1,$$

so that Lemma 21 implies that

$$(p_n - p)(t, x, y) = \sum_{k=0}^{\lfloor nt \rfloor} \int_{t^n_k}^{t^n_{k+1} \wedge t} \int_{t^n_k}^{s_1} \varphi^{n,2}_{t^n_k}(s_2, t, x, y)\, ds_2 ds_1$$

is $O(1/n)$ in $\mathcal{G}_2(\mathbb{R}^d)$. □

Setting $\psi^{n,j}_{t^n_k}(t,x,y) = \phi^{n,j}_{t^n_k}(t^n_k, t, x, y)$, we deduce from Lemma 21

Lemma 23. *Under (B) and (C), for each $j \in \mathbb{N}^*$, there exists a bounded sequence $(\psi^{n,j}_{t^n_k}, n \geq 1, k \in \{0, \ldots, n\})$ in $\mathcal{G}_{2j}(\mathbb{R}^d)$ such that for all $t \in (0,1]$, $n \geq 1$, $k \in \{0, \ldots, \lfloor nt \rfloor\}$, $f \in C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,*

$$\Psi^{n,j}_{t^n_k,t} f(x) = \int_{\mathbb{R}^d} f(y) \psi^{n,j}_{t^n_k}(t,x,y)\, dy. \tag{45}$$

4.2. **Operators on $\mathcal{G}_l(\mathbb{R}^d)$.** When $\pi \in \mathcal{G}_l(\mathbb{R}^d)$, $\pi(t,\cdot,y) \in L^\infty(\mathbb{R}^d)$ so that for $s \in [0,1]$ and $n \geq 1$ we can define two functions $P_s\pi$ and $P^n_s\pi$ on $(0,1] \times \mathbb{R}^d \times \mathbb{R}^d$ by $P_s\pi(t,\cdot,y) = \mathbf{1}_{\{s \leq t\}} P_s(\pi(t,\cdot,y))$ and $P^n_s\pi(t,\cdot,y) = \mathbf{1}_{\{s\leq t\}} P^n_s(\pi(t,\cdot,y))$, i.e.

$$P_s\pi(t,x,y) = \mathbf{1}_{\{s\leq t\}} \mathbb{E}\left[\pi\left(t, X^x_s, y\right)\right] \quad \text{and} \quad P^n_s\pi(t,x,y) = \mathbf{1}_{\{s\leq t\}}\mathbb{E}\left[\pi\left(t, X^{n,x}_s, y\right)\right]. \tag{46}$$

We also write $\Delta^n_s \pi = P^n_s\pi - P_s\pi$. For $j \in \mathbb{N}^*$ we denote by Φ^j the family $(\Phi^{n,j}_{t^n_m,s',s}, s \in [0,1], n \geq 1, m \in \{0,\ldots,\lfloor ns\rfloor\}, s' \in [t^n_m, t^n_{m+1} \wedge s])$ of operators on $\mathcal{G}_l(\mathbb{R}^d)$ defined as in (19) by

$$\Phi^{n,j}_{t^n_m,s',s}\pi(t,x,y) = \mathbb{E}\left[L_j^{X^{n,x}_{t^n_m}} P_{s-s'}\pi\left(t, X^{n,x}_{s'}, y\right)\right],$$

i.e., using (18),

$$\Phi^{n,j}_{t^n_m,s',s} = \sum_{1\leq|\alpha|\leq 2j} \sum_{l=1}^{m_{j,\alpha}} P^n_{t^n_m}\left(g_{j,\alpha,l} P^n_{s'-t^n_m}\left(h_{j,\alpha,l}\partial^\alpha_x P_{s-s'}\right)\right). \tag{47}$$

Denoting by $\mathcal{L}_b(\mathcal{G}_l(\mathbb{R}^d), \mathcal{G}_{l'}(\mathbb{R}^d))$ the space of all morphisms mapping any bounded subset of $\mathcal{G}_l(\mathbb{R}^d)$ into a bounded subset of $\mathcal{G}_{l'}(\mathbb{R}^d)$, we then have

Proposition 24. *Under (B) and (C), $(P_s, s \in [0,1])$ and $(P^n_s, s \in [0,1], n \geq 1)$ are bounded families in $\mathcal{L}_b(\mathcal{G}_l(\mathbb{R}^d))$, and Φ^j is a bounded family in $\mathcal{L}_b(\mathcal{G}_l(\mathbb{R}^d), \mathcal{G}_{l+2j}(\mathbb{R}^d))$.*

Proof. Let us first deal with (P_s). Let $\pi \in \mathcal{G}_l(\mathbb{R}^d)$. P_s is measurable. Moreover, Lebesgue's dominated convergence theorem shows that $P_s\pi(t,x,\cdot)$ is infinitely differentiable and that for all $\beta \in \mathbb{N}^d$

$$\partial^\beta_y P_s\pi(t,x,y) = \mathbf{1}_{\{s\leq t\}}\mathbb{E}\left[\partial^\beta_3 \pi\left(t, X^x_s, y\right)\right].$$

Hypothesis (A) ensures that a version of X^x can be chosen such that for each $t \geq 0$, the map $x \mapsto X^x_t$ is infinitely differentiable (see, for example, [10]). Since $\partial^\beta_3\pi(t,\cdot,y) \in C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$, it follows from Theorem 3.14 page 16 in [11] that $\partial^\beta_y P_s\pi(t,\cdot,y)$ is infinitely differentiable and that for all $\alpha \in \mathbb{N}^d$ there exists universal polynomials $(\Pi_{\alpha,\mu}, |\mu| \leq |\alpha|)$ such that

$$\partial^\alpha_x \partial^\beta_y P_s\pi(t,x,y) = \mathbf{1}_{\{s\leq t\}} \sum_{|\mu|\leq|\alpha|} \mathbb{E}\left[\partial^\mu_2\partial^\beta_3\pi\left(t, X^x_s, y\right) \Pi_{\alpha,\mu}\left(\partial^\nu_x X^x_s, |\nu| \leq |\alpha|\right)\right] \tag{48}$$

with

$$\sup_{s\in[0,1],x\in\mathbb{R}^d} \mathbb{E}[\Pi_{\alpha,\mu}\left(\partial^\nu_x X^x_s, |\nu|\leq|\alpha|\right)^2] < \infty \tag{49}$$

for all $|\mu| \leq |\alpha|$. As a consequence, $P_s\pi(t,\cdot,\cdot)$ is infinitely differentiable and using Cauchy-Schwarz's inequality, (8) and (49), we see that for all bounded $\mathcal{B} \subset \mathcal{G}_l(R^d)$ and $\alpha, \beta \in \mathbb{N}^d$, there exists two constants $c_1 \geq 0$ and $c_2 > 0$ such that for all $\pi \in \mathcal{B}$, $s \in [0,1]$, $t \in (0,1]$ and $x, y \in \mathbb{R}^d$,

$$\left|\partial_x^\alpha \partial_y^\beta P_s\pi(t,x,y)\right| \leq c_1 \mathbf{1}_{\{s\leq t\}} t^{-(|\alpha|+|\beta|+d+l)/2} \mathbb{E}\left[\exp\left(-c_2 \left\|X_x^s - y\right\|^2/t\right)\right]^{1/2}. \tag{50}$$

Now, partitioning Ω into $\{\|X_x^s - y\| \leq \|x-y\|/2\}$ and $\{\|X_x^s - y\| > \|x-y\|/2\}$, we have

$$\mathbb{E}\left[\exp\left(-c_2 \left\|X_x^s - y\right\|^2/t\right)\right] \leq \mathbb{P}\left(\|X_x^s - y\| \leq \|x-y\|/2\right) + \exp\left(-c_2 \|x-y\|^2/4t\right). \tag{51}$$

Using (10) for $p \in \mathcal{G}(\mathbb{R}^d)$ for the fourth inequality, we can find $c_3, c_5 \geq 0$ and $c_4, c_6 > 0$ such that for all $s \in (0,1]$ and $x, y \in \mathbb{R}^d$,

$$\begin{aligned}
\mathbb{P}\left(\|X_s^x - y\| \leq \|x-y\|/2\right) &\leq \mathbb{P}\left(\|X_s^x - x\| \geq \|x-y\|/2\right) \\
&= \int_{\mathbb{R}^d} \mathbf{1}_{\{\|z-x\|\geq\|x-y\|/2\}} p(s,x,z)\, dz \\
&= \int_{\mathbb{R}^d} \mathbf{1}_{\{\|\xi\|\geq\|x-y\|/2\sqrt{s}\}} p(s,x,x+\xi\sqrt{s}) s^{d/2}\, d\xi \\
&\leq c_3 \int_{\mathbb{R}^d} \mathbf{1}_{\{\|\xi\|\geq\|x-y\|/2\sqrt{s}\}} \exp(-c_4\|\xi\|^2)\, d\xi \\
&\leq c_5 \exp\left(-c_6 \|x-y\|^2/s\right). \qquad (52)
\end{aligned}$$

Eventually, from (51) and (52), we can find $c_7 \geq 0$ and $c_8 > 0$ such that for all $s \in [0,1]$, $t \in (0,1]$ and $x, y \in \mathbb{R}^d$,

$$\begin{aligned}
\mathbf{1}_{\{s\leq t\}} \mathbb{E}\left[\exp\left(-c_2 \left\|X_x^s - y\right\|^2/t\right)\right] &\leq c_5 \exp\left(-c_6 \|x-y\|^2/t\right) + \exp\left(-c_2 \|x-y\|^2/4t\right) \\
&\leq c_7 \exp\left(-c_8 \|x-y\|^2/t\right). \qquad (53)
\end{aligned}$$

It is enough to inject (53) into (50) to complete the proof for (P_s).

This proof naturally extends to the case of (P_s^n). Indeed, (48) holds with (X^n, P^n) instead of (X, P). Moreover, from Lemma 26, (49) holds uniformly in n with X^n instead of X. Eventually, (52) holds with X^n instead of X, uniformly in n because $(p_n, n \geq 1)$ is bounded in $\mathcal{G}(\mathbb{R}^d)$.

As for Φ^j, it is enough to use (47), the boundedness of (P_s) and (P_s^n), Remark 8 and the facts that multiplication by a function in $\mathcal{B}$ belongs to $\mathcal{L}_b(\mathcal{G}_l(\mathbb{R}^d), \mathcal{G}_l(\mathbb{R}^d))$ and that $\partial_2^\alpha \in \mathcal{L}_b(\mathcal{G}_l(\mathbb{R}^d), \mathcal{G}_{l+|\alpha|}(\mathbb{R}^d))$. □

4.3. **Moments for the Euler scheme and its derivatives.** Let us assume (A). Then it is known that $X_t^{n,x}$ has bounded moments of any order and that for all $q \in \mathbb{N}$, one can find $c \geq 0$ such that for all $x \in \mathbb{R}^d$,

$$\sup_{t\in[0,1],n\geq 1} \mathbb{E}\left[\|X_t^{n,x}\|^q\right] \leq c\left(1 + \|x\|^q\right) \tag{54}$$

(see [15]). From (4), $x \mapsto X_t^{n,x}$ is infinitely differentiable and we shall see that analogous upper bounds hold for its derivatives. Following [11], for $m \geq 1$, we denote by $X_t^{(m),n,x}$ the m-th derivative of $x \mapsto X_t^{n,x}$ at point x. It should be thought of as a $d \times d^m$ matrix. For instance, $X_t^{(1),n,x}$ is the jacobian matrix of $x \mapsto X_t^{n,x}$. Differentiating (4), we have

$$X_t^{(1),n,x} = I + \int_0^t b^{(1)}(X_{\lfloor ns\rfloor/n}^{n,x}) X_{\lfloor ns\rfloor/n}^{(1),n,x}\, ds + \sum_{j=1}^r \int_0^t \sigma_j^{(1)}(X_{\lfloor ns\rfloor/n}^{n,x}) X_{\lfloor ns\rfloor/n}^{(1),n,x}\, dB_s^j, \tag{55}$$

where I stands for the identity matrix and σ_j is the j-th column of σ. Besides, by induction, there are for each $m \geq 2$ universal polynomials $P_{m,j}$, $j \in \{0, \dots, r\}$, such that

$$(56)\quad X_t^{(m),n,x} = \int_0^t b^{(1)}(X_{\lfloor ns \rfloor/n}^{n,x}) X_{\lfloor ns \rfloor/n}^{(m),n,x}\, ds + \sum_{j=1}^r \int_0^t \sigma_j^{(1)}(X_{\lfloor ns \rfloor/n}^{n,x}) X_{\lfloor ns \rfloor/n}^{(m),n,x}\, dB_s^j + \int_0^t Q_{m,0,\lfloor ns \rfloor/n}^{n,x}\, ds + \sum_{j=1}^r \int_0^t Q_{m,j,\lfloor ns \rfloor/n}^{n,x}\, dB_s^j,$$

where

$$(57)\quad \begin{cases} Q_{m,0,t}^{n,x} = P_{m,0}(b^{(2)}(X_t^{n,x}), \dots, b^{(m)}(X_t^{n,x}), X_t^{(1),n,x}, \dots, X_t^{(m-1),n,x}), \\ Q_{m,j,t}^{n,x} = P_{m,j}(\sigma_j^{(2)}(X_t^{n,x}), \dots, \sigma_j^{(m)}(X_t^{n,x}), X_t^{(1),n,x}, \dots, X_t^{(m-1),n,x}). \end{cases}$$

This is analogous to (1.8) page 4 in [11]. Then we have

Lemma 25. *Under (A), for all $m \geq 1$ and $q \in \mathbb{N}$, there exists $c \geq 0$ and $q' \in \mathbb{N}$ such that for all $x \in \mathbb{R}^d$,*

$$(58)\quad \sup_{t \in [0,1], n \geq 1} \mathbb{E}\left[\left\| X_t^{(m),n,x} \right\|^q\right] \leq c\left(1 + \|x\|^{q'}\right).$$

Proof. We give a proof by induction on m. Let us first assume that $m = 1$. Let $q \in \mathbb{N}$. From (55), and observing that (A) states that $b^{(1)}$ and all the $\sigma_j^{(1)}$ are bounded, Jensen's and Burkholder-Davis-Gundy's inequalities lead to the existence of $c \geq 0$ such that for all $t \in [0,1]$, $n \geq 1$ and $x \in \mathbb{R}^d$,

$$\mathbb{E}\left[\left\| X_t^{(1),n,x} \right\|^q\right] \leq c\left(1 + \int_0^t \mathbb{E}\left[\left\| X_{\lfloor ns \rfloor/n}^{(1),n,x} \right\|^q\right] ds\right).$$

Taking this inequality at time $\lfloor nt \rfloor/n$ and applying Gronwall's lemma, we get that

$$\sup_{t \in [0,1], n \geq 1, x \in \mathbb{R}^d} \mathbb{E}\left[\left\| X_{\lfloor nt \rfloor/n}^{(1),n,x} \right\|^q\right] < \infty.$$

From (4), one easily checks that the same holds at time t instead of $\lfloor nt \rfloor/n$, so that (58) holds for $m = 1$ with $q' = 0$.

Let us now assume that (58) holds for the $m-1$ first derivatives. Let $q \in \mathbb{N}$. From (56), and observing again that (A) states that $b^{(1)}$ and all the $\sigma_j^{(1)}$ are bounded, Jensen's and Burkholder-Davis-Gundy's inequalities lead to the existence of $c_1 \geq 0$ such that for all $t \in [0,1]$, $n \geq 1$ and $x \in \mathbb{R}^d$,

$$(59)\quad \mathbb{E}\left[\left\| X_t^{(m),n,x} \right\|^q\right] \leq c_1\left(\int_0^t \mathbb{E}\left[\left\| X_{\lfloor ns \rfloor/n}^{(m),n,x} \right\|^q\right] ds + \int_0^t \sum_{j=0}^r \mathbb{E}\left[\left\| Q_{m,j,\lfloor ns \rfloor/n}^{n,x} \right\|^q\right] ds\right).$$

Using (57), the induction hypothesis, (A) and (54), we find $c_2 \geq 0$ and $q' \in \mathbb{N}$ such that for all $s \in [0,1]$, $n \geq 1$ and $x \in \mathbb{R}^d$,

$$\sum_{j=0}^r \mathbb{E}\left[\left\| Q_{m,j,\lfloor ns \rfloor/n}^{n,x} \right\|^q\right] \leq c_2\left(1 + \|x\|^{q'}\right).$$

Thus, taking (59) at time $\lfloor nt \rfloor/n$ and applying Gronwall's lemma, we find $c \geq 0$ such that for all $x \in \mathbb{R}^d$,

$$\sup_{t \in [0,1], n \geq 1} \mathbb{E}\left[\left\| X_{\lfloor nt \rfloor/n}^{(m),n,x} \right\|^q\right] \leq c\left(1 + \|x\|^{q'}\right).$$

From (4), one easily checks that the same holds at time t instead of $\lfloor nt \rfloor / n$, which completes the proof. $\square$

Observe that, under (B), the above proof holds with $q' = 0$ so that we have

Lemma 26. *Under (B), for all $m \geq 1$ and $q \in \mathbb{N}$,*

$$\sup_{t \in [0,1], n \geq 1, x \in \mathbb{R}^d} \mathbb{E}\left[\left\| X_t^{(m),n,x} \right\|^q\right] < \infty.$$

References

[1] V. Bally and D. Talay. The Law of the Euler Scheme for Stochastic Differential Equations : Convergence Rate of the Distribution Function. *Probab. Th. Rel. Fields*, 104:43–60, 1996.

[2] V. Bally and D. Talay. The Law of the Euler Scheme for Stochastic Differential Equations : Convergence Rate of the Density. *Monte Carlo Meth. Appl.*, 2(2):93-128, 1996.

[3] A. Friedman. Partial Differential Equations of Parabolic Type. *Prentice-Hall, Englewood Cliffs, New Jersey*, 1964.

[4] A. Friedman. Stochastic Differential Equations and Applications. *Probability and Mathematical Statistics, Academic Press*, 1975.

[5] J. Jacod and P. Protter. Asymptotic Error Distributions for the Euler Method for Stochastic Differential Equations. *Ann. Probab.*, 26(1):267–307, 1998.

[6] J. Jacod. The Euler Scheme for Lévy Driven Stochastic Differential Equations: Limit Theorems. *Ann. Probab.*, 32(3A):1830–1872, 2004.

[7] J. Jacod, T. Kurtz, S. Méléard and P. Protter. The Approximate Euler Method for Lévy Driven Stochastic Differential Equations. To appear in the *Ann. de l'I.H.P.*, special issue devoted to the memory of P.-A. Meyer.

[8] A. Kohatsu-Higa and R. Pettersson. Variance Reduction Methods for Simulation of Densities on Wiener Space. *SIAM J. Numer. Anal.*, 40(2):431–450, 2002.

[9] V. Konakov and E. Mammen. Edgeworth Type Expansions for Euler Schemes for Stochastic Differential Equations. *Monte Carlo Meth. Appl.*, 8(3):271–285, 2002.

[10] H. Kunita. On the Decomposition of Solutions of Stochastic Differential Equations. *Lecture Notes in Math., Springer-Verlag*, 851:213–255, 1981.

[11] S. Kusuoka and D. Stroock. Applications of the Malliavin Calculus, Part II. *J. Fac. Sci., Univ. of Tokyo*, Sec. IA, 32(1):1–76, March 1985.

[12] P. Malliavin and A. Thalmaier. Numerical Error for SDE: Asymptotic Expansion and Hyperdistributions. *C. R. Acad. Sci. Paris*, Ser. I 336:851–856, 2003.

[13] P. Protter and D. Talay. The Euler Scheme for Lévy Driven Stochastic Differential Equations. *Ann. Prob.*, 25(1):393–423, 1997.

[14] D. Talay and L. Tubaro. Expansion of the Global Error for Numerical Schemes Solving Stochastic Differential Equations. *Stoch. Analysis and Appl.*, 8(4):94–120, 1990.

[15] D. Talay. Probabilistic Numerical Methods for Partial Differential Equations: Elements of Analysis. In Graham, C. (ed.) et al., Probabilistic Models for Nonlinear Partial Differential Equations. Lectures given at the 1st session of the Centro Internazionale Matematico Estivo, Montecatini Terme, Italy, May 22-30, 1995. Berlin: Springer. Lect. Notes Math. 1627:148–196, 1996.

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