ON THREE BINARY RELATIONS BETWEEN AGENTS AND CLASSIFICATION OF INFORMATION STRUCTURES IN WITSENHAUSEN'S INTRINSIC MODEL FOR DISCRETE STOCHASTIC CONTROL

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Abstract. Within the framework of Witsenhausen's intrinsic model for discrete stochastic control, we provide a unified framework to extend and study three binary relations between agents: the so called *precedence*, *subsystem* and *memory-communication* relations. These are tools to analyze *nonsequential systems*, those for which, in contrast to *sequential* ones, any *a priori* ordering of control actions is impossible independently of the set of control laws. We give localized versions of these relations are given; localizing the precedence relation provides an evocative characterization of Witsenhausen's *causality* property (C). Connections between these three binary relations are exhibited. We show in particular that the subsystem relation is the reflexive and transitive closure of the precedence relation. We give new characterizations of sequentiality, and introduce systems closed under precedence, as well as *partially nested systems*. We prove that partially nested systems without self information (causal ones in particular) are sequential. We end up with a summary table providing a classification of information structures in terms of binary relations.

Key words. Information structure, nonsequential stochastic control, binary relations

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1. Introduction.

In two papers [10, 11], Witsenhausen introduced and developed the so called *intrinsic model for discrete stochastic control*. It provides a transparent and elegant framework to deal with interactions between a finite number of agents (decision makers), without presupposing any ordering of actions. Such a framework is adapted to general stochastic control systems, also called information structures, when one deals with *nonsequential systems*, those for which, in contrast to *sequential* ones, any *a priori* ordering of control actions is impossible independently of the set of control laws. Witsenhausen's model has been used by a limited number of authors (see [5, 6, 1, 3] for some references) interested by nonclassical information structures. We share such an interest since our study of dual free stochastic controls in [2]. In this paper, we try to provide tools, based upon three binary relations between agents, in order to analyze general information structures.

In Section 2, we recall Witsenhausen's intrinsic model for discrete stochastic control and definition of causality. Examples are provided. In Section 3, we provide a unified framework to define and study three binary relations between agents, scattered in the litterature [5, 6, 11, 2]: the so called precedence, subsystem and memory-communication relations. We provide an extension by localizing such relations to any event. As an illustration, localizing the precedence relation provides an evocative characterization of Witsenhausen's causality property (C). Connections between these three binary relations are exhibited. By using binary relations tools, we show in particular that the subsystem relation is the reflexive and transitive closure of the precedence relation. In Section 4, we recall the typology of systems presented in [11] and relate it to the three binary relations. By using binary relations tools, we are able to give new characterizations of sequentiality. We introduce systems closed under precedence, those for which the precedence relation is transitive, and give their

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properties. The most interesting one is the equivalence between self information and sequentiality. We also introduce partially nested systems which are the extension of quasiclassical systems to non necessarily sequential systems. As pointed out in [12], strictly partially nested systems are adapted to a general dynamic programming formulation, in the case they are sequential. We prove that partially nested systems are closed under precedence. As a consequence, we show that partially nested systems without self information (causal ones in particular) are sequential.

2. Witsenhausen intrinsic model for discrete stochastic control. We present here the so called *intrinsic model for discrete stochastic control* introduced by Witsenhausen in [10] (see also [11]).

2.1. The model. Let A be a finite set representing agents. Each agent $\alpha \in A$ is supposed to take one decision $u_\alpha \in \mathbb{U}_\alpha$, where $\mathbb{U}_\alpha$ is the control set for agent α, equipped with σ-algebra $\mathcal{U}_\alpha$. This includes the (discrete time) dynamics case by considering that an individual taking one decision at each period is in fact made up of several different agents, one for each period. We put

$$\mathbb{U}_A \stackrel{\text{def}}{=} \prod_{\beta \in A} \mathbb{U}_\beta \,, \tag{2.1}$$

the product set, equipped with the σ-algebra generated by rectangles:

$$\mathcal{U}_A \stackrel{\text{def}}{=} \bigotimes_{\beta \in A} \mathcal{U}_\beta \,. \tag{2.2}$$

Let Ω be a measurable set, with σ-algebra $\mathcal{F}$. This is the sample space of random issues, but we do not equip it with a probability measure. We put

$$\mathbb{H} \stackrel{\text{def}}{=} \mathbb{U}_A \times \Omega \quad \text{and} \quad \mathcal{H} \stackrel{\text{def}}{=} \mathcal{U}_A \otimes \mathcal{F} \,. \tag{2.3}$$

The *information field* of agent α is a subfield $\mathcal{I}_\alpha \subset \mathcal{U}_A \otimes \mathcal{F}$. By this, the information of agent α may depend upon other agents decisions and upon realizations of the sample space Ω.

The collection consisting of A, $(\Omega, \mathcal{F})$, $(\mathbb{U}_\alpha, \mathcal{U}_\alpha, \mathcal{I}_\alpha)_{\alpha \in A}$ is called an *information structure* [10] or a *(stochastic control) system* [11].

For any subset B of A, we define the cylindric extension of $\bigotimes_{\beta \in B} \mathcal{U}_\beta$ to $\mathcal{U}_A$ by:

$$\mathcal{U}(B) \stackrel{\text{def}}{=} \bigotimes_{\beta \in B} \mathcal{U}_\beta \otimes \bigotimes_{\beta \notin B} \{\emptyset, \mathbb{U}_\beta\} \subset \mathcal{U}_A \,. \tag{2.4}$$

Any element in $\mathcal{U}(B)$ is of the form $K \times \prod_{\beta \notin B} \mathbb{U}_\beta$, where $K \in \bigotimes_{\beta \in B} \mathcal{U}_\beta$. We have that, for all $B \subset A$ and $C \subset A$

$$\mathcal{U}(B \cap C) = \mathcal{U}(B) \cap \mathcal{U}(C) \quad \text{and} \quad \mathcal{U}(B \cup C) = \mathcal{U}(B) \vee \mathcal{U}(C) \,. \tag{2.5}$$

For any $B \subset A$, let $\mathbb{U}_B \stackrel{\text{def}}{=} \prod_{\alpha \in B} \mathbb{U}_\alpha$ and ϖ_B denote the projection from $\mathbb{U}_A \times \Omega$ to $\mathbb{U}_B \times \Omega$:

$$\forall (u, \omega) \in \mathbb{U}_A \times \Omega \,, \quad \varpi_B(u, \omega) \stackrel{\text{def}}{=} ((u_\alpha)_{\alpha \in B}, \omega) \in \mathbb{U}_B \times \Omega \,. \tag{2.6}$$

We have that $\mathcal{U}(B) \otimes \mathcal{F} = \varpi_B^{-1}(\bigotimes_{\beta \in B} \mathcal{U}_\beta) \otimes \mathcal{F}$. Notice that $\varpi_\emptyset(u, \omega) = \omega$, while $\varpi_A(u, \omega) = (u, \omega)$.

2.2. The decision process. Quoting Witsenhausen in [10], "The decision process is considered as a feedback loop and the game is characterized by its interaction with the policies of the agents, without prejudging questions of chronological order. "We denote

$$\Lambda_\alpha \stackrel{\text{def}}{=} \{\lambda_\alpha : \mathbb{U}_A \times \Omega \to \mathbb{U}_\alpha \mid \lambda_\alpha^{-1}(\mathcal{U}_\alpha) \subset \mathcal{I}_\alpha\} \,. \tag{2.7}$$

Any $\lambda_\alpha \in \Lambda_\alpha$ represents a *possible policy* of agent α, in the sense that it depends, in a measurable way (with respect to $\mathcal{I}_\alpha$, i.e. taking information into account), upon all agents decisions and upon the random issue. We put

$$\Lambda_A \stackrel{\text{def}}{=} \prod_{\alpha \in A} \Lambda_\alpha \,. \tag{2.8}$$

Open loop policies are constant ones, that is are elements of

$$\Lambda_A^\perp \stackrel{\text{def}}{=} \prod_{\alpha \in A} \Lambda_\alpha^\perp \quad \text{where} \quad \Lambda_\alpha^\perp \stackrel{\text{def}}{=} \{\lambda_\alpha : \mathbb{U}_A \times \Omega \to \mathbb{U}_\alpha \mid \lambda_\alpha^{-1}(\mathcal{U}_\alpha) \subset \{\emptyset, \mathbb{U}_A \times \Omega\}\} \,. \tag{2.9}$$

For any $\lambda = (\lambda_\alpha)_{\alpha \in A} \in \Lambda_A$, the problem is to find, for any $\omega \in \Omega$, solutions $u \in \mathbb{U}_A$ (dependent upon ω) satisfying the *closed-loop equations* $\varpi_\alpha(u) = \lambda_\alpha(u, \omega)$, where ϖ_α is defined in (2.6), that is:

$$u_\alpha = \lambda_\alpha((u_\beta, \beta \in A), \omega) \,, \quad \forall \alpha \in A \,. \tag{2.10}$$

For an open loop policy $\lambda \in \Lambda_A^\perp$, equation (2.10) has a unique solution $u_\alpha = \lambda_\alpha$, by identifying the function λ_α with its constant value. However, in general, all cases are possible: no solution, multiple solutions, unique solution.

Witsenhausen says that *solvability property (S)* holds when, for any $\lambda \in \Lambda_A$ and any $\omega \in \Omega$, there exists one and only one $u \in \mathbb{U}_A$ satisfying (2.10).

Definition 2.1 ([11]). Absence of self information *is the property that* $\mathcal{I}_\alpha \subset \mathcal{U}(A \backslash \{\alpha\}) \otimes \mathcal{F}$ *for all agent* $\alpha \in A$. Witsenhausen proves in [11] that solvability property (S) implies absence of self information.

2.3. Causality. The subtle notion of causality is treated with care in [10] (see also [11, 6]). A weaker notion of *causal implementability property* (CI) is defined and studied by Andersland and Teneketzis in [1].

Let $n = \text{card}(A)$. For $k \in \{1, \ldots, n\}$, let S_A^k denote the set of injective mappings from $\{1, \ldots, k\}$ to A. Thus $S_A \stackrel{\text{def}}{=} S_A^n$ is the set of *total orderings* of A. We also put $S_A^0 = \emptyset$. For $0 \leq i \leq j \leq n$, let $T_i^j : S_A^j \to S_A^i$ denote the *truncation map* which restricts any $\sigma \in S_A^j$ to the domain $\{1, \ldots, i\}$, or to $\emptyset$ if $i = 0$.

An information structure is said to possess *causality property (C)* (or a system is said to be *causal*) if there exists (at least) one mapping φ from $\mathbb{U}_A \times \Omega$ towards S_A, with the property that

$$\forall k \in \{1, \ldots, n\} \,, \quad \forall s \in S_A^k \,, \quad \mathcal{I}_{s(k)} \cap (T_k^n \circ \varphi)^{-1}(s) \subset \mathcal{U}(\{s(1), \ldots, s(k-1)\}) \otimes \mathcal{F} \,. \tag{2.11}$$

Definition 2.2. *A* sequential system *is one for which the causality condition holds with a constant mapping* φ*, that is there exists* $\varsigma \in S_A$ *such that*

$$\forall k \in \{1, \ldots, n\} \,, \quad \mathcal{I}_{\varsigma(k)} \subset \mathcal{U}(\{\varsigma(1), \ldots, \varsigma(k-1)\}) \otimes \mathcal{F} \,. \tag{2.12}$$

Witsenhausen proves in [10] that causality property (C) implies (recursive) solvability (S). Andersland and Teneketzis prove in [1] that causality property (C) implies causal implementability property (CI), and that this latter implies (recursive) solvability (S).

2.4. Examples. We present examples, either with finite decision and sample sets (for which subfields are equivalently described by partitions as illustrated in Figures 2.1–2.2–2.3), or continuous decision and sample sets. The notation $\sigma()$ means the σ-field generated either by subsets or by a mapping.

Single agent with two decisions, two random issues. Consider the case of a single agent $A = \{a\}$, $\mathbb{U}_a = \{a_1, a_2\}$, together with $\mathcal{U}_a = 2^{\mathbb{U}_a}$, and with sample space $\Omega = \{\omega_-, \omega_+\}$ and $\mathcal{F} = 2^{\Omega}$. We have $\mathbb{H} = \mathbb{U}_a \times \Omega = \{a_1, a_2\} \times \{\omega_-, \omega_+\}$ and $\mathcal{H} = \mathcal{U}_a \otimes \mathcal{F} = 2^{\mathbb{U}_a} \otimes 2^{\Omega} = 2^{\mathbb{H}}$. The set of orderings S_A is reduced to the mapping ς such that $\varsigma(a) = 1$.

- $\mathcal{I}_a = \{\emptyset, \mathbb{H}\}$. The agent knows nothing. Possible policies coincide with open-loop ones, so that the decision process is elementary. Notice that the system is sequential, with $\varphi : \mathbb{H} \to S_A$ equal to $\varphi \equiv \varsigma$, where $\varsigma(a) = 1$.
- $\mathcal{I}_a = 2^{\mathbb{U}_a} \otimes \{\emptyset, \Omega\} = \sigma(\{a_1\} \times \{\omega_-, \omega_+\}, \{a_2\} \times \{\omega_-, \omega_+\})$. The agent takes his decision in function of his own decison: this is an example of self information, which precludes solvability. Indeed, Λ_A in (2.8) identifies with one of the four mappings from $\mathbb{U}_a$ to $\mathbb{U}_a$. Thus, equation (2.10) becomes $u_a = \lambda_a(u_a)$. Existence (and unicity) holds if and only if λ_a is the identity mapping. The system is non sequential.
- $\mathcal{I}_a = \{\emptyset, \mathbb{U}_a\} \otimes \mathcal{F} = \sigma(\{a_1, a_2\} \times \{\omega_-\}, \{a_1, a_2\} \times \{\omega_+\})$: the agent knows only the random issue. The system is sequential, with $\varphi : \mathbb{H} \to S_A$ equal to $\varphi \equiv \varsigma$, where $\varsigma(a) = 1$. Solvability holds and the decision process $u_a = \lambda_a(a, \omega)$ is recursive since $\lambda_a(a, \omega) = \lambda_a(\omega)$ because λ_a is measurable with respect to $\mathcal{I}_a = \{\emptyset, \mathbb{U}_a\} \otimes \mathcal{F}$, hence does not depend upon the variable a: the random issue is observed, and the decision depends upon it.

Two agents with two decisions, two random issues. Consider the case $A = \{a, b\}$, $\mathbb{U}_a = \{a_1, a_2\}$, $\mathbb{U}_b = \{b_1, b_2\}$, together with $\mathcal{U}_a = 2^{\mathbb{U}_a}$ and $\mathcal{U}_b = 2^{\mathbb{U}_b}$, with sample space $\Omega = \{\omega_-, \omega_+\}$ and $\mathcal{F} = 2^{\Omega}$. We have $\mathbb{H} = \mathbb{U}_a \times \mathbb{U}_b \times \Omega = \{a_1, a_2\} \times \{b_1, b_2\} \times \{\omega_-, \omega_+\}$ and $\mathcal{H} = \mathcal{U}_a \otimes \mathcal{U}_b \otimes \mathcal{F} = 2^{\mathbb{H}}$.

The set of orderings $S_A = \{ab, ba\}$ is reduced to two mappings: $ab = \varsigma_{ab}$ such that $\varsigma_{ab}(a) = 1$ and $\varsigma_{ab}(b) = 2$; the reverse for $ba = \varsigma_{ba}$. With the same type of notation, we also have $S_A^1 = \{a, b\}$.

- $\mathcal{I}_a = \{\emptyset, \mathbb{H}\}$, $\mathcal{I}_b = \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F}$. Agent a knows nothing, while agent b knows the random issue. There are no interactions between agents, just a dependence upon random issues: this is an example of static team (see subsection 4.1).
- $\mathcal{I}_a = \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F}$, $\mathcal{I}_b = \mathcal{U}_a \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F}$. This corresponds to Figure 2.1.
  The system is sequential with $\varphi \equiv ab$: agent a observes the random issue and takes his decision in function; agent b observes both agent a's decision and the random issue and takes his decision in function. Notice that $\mathcal{I}_a \subset \mathcal{I}_b$, which may be interpreted in different ways. One may say that agent a communicates his own information to agent b. If agent a is an individual at time $t = 0$, while agent b is the same individual at time $t = 1$, one may say that the information is not forgotten with time (memory of past knowledge).
- $\mathcal{I}_a = \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F}$, $\mathcal{I}_b = \mathcal{U}_a \otimes \{\emptyset, \mathbb{U}_b\} \otimes \{\emptyset, \Omega\}$. The system is sequential with $\varphi \equiv ab$. Here, $\mathcal{I}_a$ and $\mathcal{I}_b$ are not comparable: indeed, agent a observes only the random issue, while agent b observes only agent a's decision.
- $\mathcal{I}_a = \{\emptyset, \mathbb{U}_a\} \otimes \mathcal{U}_b \otimes \{\emptyset, \Omega\}$, $\mathcal{I}_b = \mathcal{U}_a \otimes \{\emptyset, \mathbb{U}_b\} \otimes \{\emptyset, \Omega\}$. This corresponds to Figure 2.2. Agent a observes only agent b's decision, while agent b observes

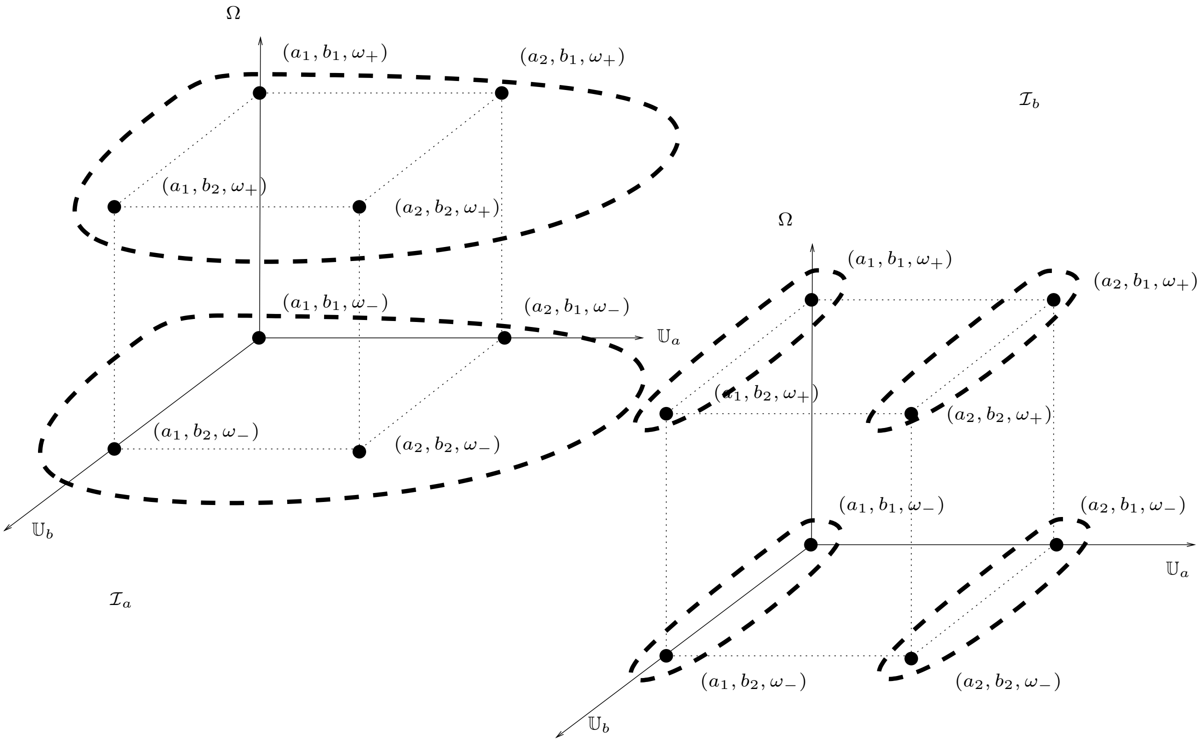

FIG. 2.1. A sequential information structure described by partitions

only agent a's decision: this corresponds to a *deadlock* situation [1] where the decision process may have no solution or multiple solutions. The system is not sequential.

- $\mathcal{I}_a = \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F}$, $\mathcal{I}_b = \sigma(\{a_1\} \times \{b_1, b_2\} \times \{\omega_-\})$. The system is sequential with $\varphi \equiv ab$. $\mathcal{I}_a$ and $\mathcal{I}_b$ are not comparable.
- $\mathcal{I}_a = \sigma(\{a_1, a_2\} \times \{b_1, b_2\} \times \{\omega_+\}, \{a_1, a_2\} \times \{b_1\} \times \{\omega_-\})$, $\mathcal{I}_b = \sigma(\{a_1, a_2\} \times \{b_1, b_2\} \times \{\omega_-\}, \{a_1\} \times \{b_1, b_2\} \times \{\omega_+\})$. This corresponds to Figure 2.3.
Define $\varphi_{|\mathbb{U}_A \times \mathbb{U}_b \times \{\omega_+\}} \equiv ab$ and $\varphi_{|\mathbb{U}_A \times \mathbb{U}_b \times \{\omega_-\}} \equiv ba$. We shall show that causality holds. However, it may easily be seen that the system is not sequential.
For $k = 1$, recall that $S_A^1 = \{a, b\}$. For $s = a \in S_A^1$, we have $(T_1^2 \circ \varphi)^{-1}(a) = \varphi^{-1}(ab) = \mathbb{U}_A \times \mathbb{U}_b \times \{\omega_+\}$ and $\mathcal{I}_{s(1)} \cap (T_1^2 \circ \varphi)^{-1}(s) = \mathcal{I}_a \cap (T_1^2 \circ \varphi)^{-1}(a) = \mathcal{I}_a \cap \mathbb{U}_A \times \mathbb{U}_b \times \{\omega_+\} = \{\emptyset, \{a_1, a_2\} \times \{b_1, b_2\} \times \{\omega_+\}\} \subset \mathcal{U}(\emptyset) \otimes \mathcal{F}$. When $s = b \in S_A^1$, we prove in the same way that $\mathcal{I}_{s(1)} \cap (T_1^2 \circ \varphi)^{-1}(s) = \{\emptyset, \{a_1, a_2\} \times \{b_1, b_2\} \times \{\omega_-\}\} \subset \mathcal{U}(\emptyset) \otimes \mathcal{F}$.
For $k = 2$, recall that $S_A^2 = S_A = \{ab, ba\}$. For $s = ab \in S_A^2$, we have $(T_2^2 \circ \varphi)^{-1}(ab) = \varphi^{-1}(ab) = \mathbb{U}_A \times \mathbb{U}_b \times \{\omega_+\}$ and $\mathcal{I}_{s(2)} \cap (T_2^2 \circ \varphi)^{-1}(s) = \mathcal{I}_b \cap \mathbb{U}_A \times \mathbb{U}_b \times \{\omega_+\} = \{\emptyset, \{a_1, a_2\} \times \{b_1, b_2\} \times \{\omega_+\}, \{a_1\} \times \{b_1, b_2\} \times \{\omega_+\}, \{a_2\} \times \{b_1, b_2\} \times \{\omega_+\}\} \subset \mathcal{U}(\{a\}) \otimes \mathcal{F} = \mathcal{U}(\{s(1)\}) \otimes \mathcal{F}$. When $s = ba \in S_A^2$, we prove in the same way that $\mathcal{I}_{s(2)} \cap (T_2^2 \circ \varphi)^{-1}(s) \subset \mathcal{U}(\{s(1)\}) \otimes \mathcal{F}$.
Thus (2.11) is satisfied, and causality holds true.

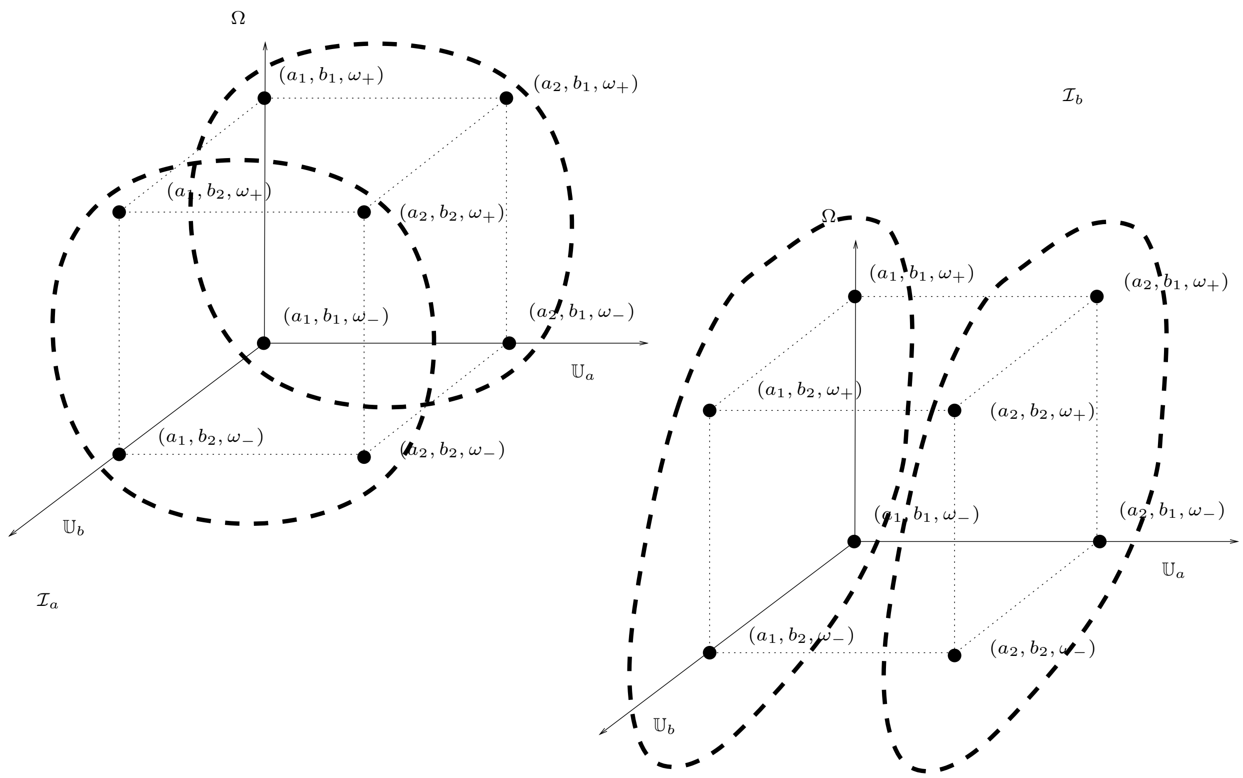

FIG. 2.2. An information structure with deadlock

State space observed and controlled systems. Witsenhausen's model includes the case where information is given by *signals*: agent α learns about the location of $h \in \mathbb{H}$ by a mapping $H_\alpha : \mathbb{H} \to \mathbb{Y}_\alpha$. Assuming that H_α is measurable from $(\mathbb{H}, \mathcal{H})$ to $(\mathbb{Y}_\alpha, \mathcal{Y}_\alpha)$, the connection between both approaches is given by $\mathcal{I}_\alpha = H_\alpha^{-1}(\mathcal{Y}_\alpha) = \sigma(H_\alpha)$.

The traditional framework where controls and random issue affect a state which, in turn, delivers information is also included in Witsenhausen's model. We shall illustrate it on an example drawn from [9].

Consider the following state equations

$$\begin{cases} x_1 &= x_0 + u_1 \\ x_2 &= x_1 - u_2 \end{cases}$$

together with output equations

$$\begin{cases} y_0 &= x_0 \\ y_1 &= x_1 + v \end{cases}$$

with controls $u_1 \in \mathbb{U}_1 = \mathbb{R}$ and $u_2 \in \mathbb{U}_2 = \mathbb{R}$, and random issue $\omega = (x_0, v) \in \Omega = \mathbb{R} \times \mathbb{R}$. All sets are equipped with their Borelian σ-algebras: $\mathcal{U}_1 = \mathcal{U}_2 = \mathcal{B}(\mathbb{R})$ and $\mathcal{F} = \mathcal{B}(\mathbb{R}^2)$. Putting $A = \{1, 2\}$, $\mathbb{H} = \mathbb{U}_1 \times \mathbb{U}_2 \times \Omega = \mathbb{R}^4$, with $\mathcal{H} = \mathcal{B}(\mathbb{R}^4)$, u_1, u_2, x_0 and v are seen either as "primitive variables" or as coordinate mappings with domain $\mathbb{H}$. The "variables" x_1, x_2, y_0 and y_1 are mappings with domain $\mathbb{H}$.

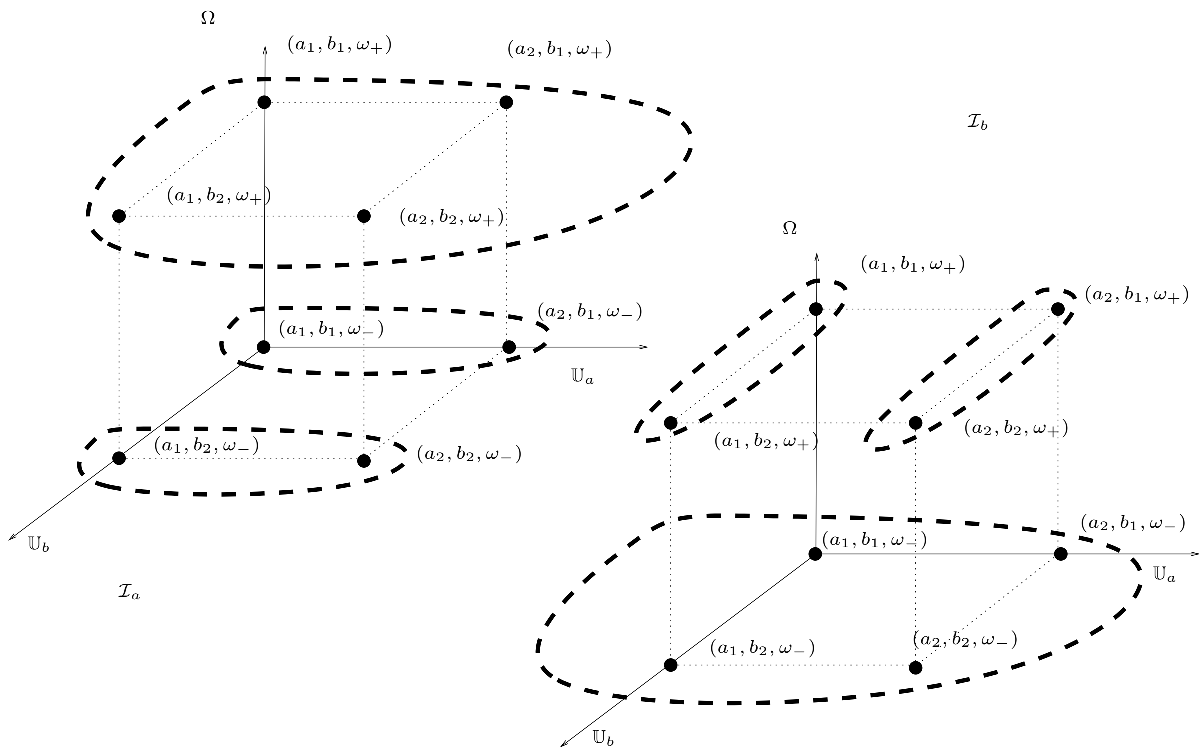

FIG. 2.3. A nonsequential information structure

- $\mathcal{I}_1 = \sigma(y_0) = \sigma(x_0) = \{\emptyset, \mathbb{U}_1\} \otimes \{\emptyset, \mathbb{U}_2\} \otimes \mathcal{B}(\mathbb{R}) \otimes \{\emptyset, \mathbb{R}\}$, $\mathcal{I}_2 = \sigma(y_0, y_1) = \sigma(x_0, x_0 + u_1 + v) = \sigma(x_0, u_1 + v)$. This is the *classical information pattern* (sequential and memory of past knowledge).
- $\mathcal{I}_1 = \sigma(y_0) = \sigma(x_0) = \{\emptyset, \mathbb{U}_1\} \otimes \{\emptyset, \mathbb{U}_2\} \otimes \mathcal{B}(\mathbb{R}) \otimes \{\emptyset, \mathbb{R}\}$, $\mathcal{I}_2 = \sigma(y_1) = \sigma(x_0 + u_1 + v)$. This is an example of *nonclassical information pattern* (Witsenhausen counterexample in stochastic optimal control).

3. Precedence, subsystem and memory-communication binary relations between agents. From now on, we shall make use of the terminology, notations and properties of binary relations as recalled in the Appendix. In brief, $\mathbf{1}_A$ is the *equality or diagonal relation*, $\mathfrak{R}^\infty$ is the *transitive closure* of a binary relation $\mathfrak{R}$, while $\mathfrak{R}^* = \mathbf{1}_A \cup \mathfrak{R}^\infty$ is the *reflexive and transitive closure*. The *complementary* relation $\neg\mathfrak{R}$ verifies $\alpha \neg\mathfrak{R} \beta \iff \neg(\alpha \mathfrak{R} \beta)$.

3.1. The precedence binary relation between agents. A precedence binary relation between agents was introduced by Ho and Chu in [5, 6] for the multi-agent LQG problem. An extension of precedence[1] to non necessarily LQG problems was given in [2], but not within the framework of Witsenhausen's intrinsic model.

We now proceed to define the precedence relation: it identifies the agents whose decisions indeed affect the observations of a given agent.

[1]We took inspiration from the definition in [6], while that in [5] rather relates to the subsystem relation to be seen in the sequel.

Definition 3.1. *Let $[\alpha] \subset A$ be the intersection of subsets $B \subset A$ such that $\mathcal{I}_\alpha \subset \mathcal{U}(B) \otimes \mathcal{F}$. Since A belongs to this intersection and by (2.5), $[\alpha]$ exists and is the smallest subset $B \subset A$ such that $\mathcal{I}_\alpha \subset \mathcal{U}(B) \otimes \mathcal{F}$. We define a* precedence *binary relation $\mathfrak{P}$ on A by*

$$\beta \,\mathfrak{P}\, \alpha \iff \beta \in [\alpha] \,, \tag{3.1}$$

and we say that β is a precedent *of α.* Any β in $[\alpha]$ affects the observations available to agent α. In other words, if β is a *precedent* of α, then $\mathcal{I}_\alpha$ indeed "depends upon u_β".

Examples. Consider the case $A = \{a, b\}$, $\mathbb{U}_a = \{a_1, a_2\}$, $\mathbb{U}_b = \{b_1, b_2\}$, $\Omega = \{\omega_-, \omega_+\}$.

- $\mathcal{I}_a = \{\emptyset, \mathbb{H}\}$, $\mathcal{I}_b = \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F}$: $[a] = [b] = \emptyset$, $\mathfrak{P} = \emptyset$.
- $\mathcal{I}_a = \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F}$, $\mathcal{I}_b = \mathcal{U}_a \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F}$ (see Figure 2.1): $[a] = \emptyset$, $[b] = \{a\}$, $\mathfrak{P} = \{(a, b)\}$.
- $\mathcal{I}_a = \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F}$, $\mathcal{I}_b = \mathcal{U}_a \otimes \{\emptyset, \mathbb{U}_b\} \otimes \{\emptyset, \Omega\}$: $[a] = \emptyset$, $[b] = \{a\}$, $\mathfrak{P} = \{(a, b)\}$.
- $\mathcal{I}_a = \{\emptyset, \mathbb{U}_a\} \otimes \mathcal{U}_b \otimes \{\emptyset, \Omega\}$, $\mathcal{I}_b = \mathcal{U}_a \otimes \{\emptyset, \mathbb{U}_b\} \otimes \{\emptyset, \Omega\}$ (see Figure 2.2): $[a] = \{b\}$, $[b] = \{a\}$, $\mathfrak{P} = \{(a, b), (b, a)\}$.
- $\mathcal{I}_a = \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F}$, $\mathcal{I}_b = \sigma(\{a_1\} \times \{b_1, b_2\} \times \{\omega_-\})$: $[a] = \emptyset$, $[b] = \{a\}$, $\mathfrak{P} = \{(a, b)\}$.
- $\mathcal{I}_a = \sigma(\{a_1, a_2\} \times \{b_1, b_2\} \times \{\omega_+\}, \{a_1, a_2\} \times \{b_1\} \times \{\omega_-\})$, $\mathcal{I}_b = \sigma(\{a_1, a_2\} \times \{b_1, b_2\} \times \{\omega_-\}, \{a_1\} \times \{b_1, b_2\} \times \{\omega_+\})$ (see Figure 2.3): $[a] = \{b\}$, $[b] = \{a\}$, $\mathfrak{P} = \{(a, b), (b, a)\}$.

For any $B \subset A$, we also introduce

$$[B] \stackrel{\text{def}}{=} \bigcup_{\beta \in B} [\beta] \,, \quad [B]^0 \stackrel{\text{def}}{=} B \quad \text{and} \quad \forall n \in \mathbb{N} \,, \quad [B]^{n+1} \stackrel{\text{def}}{=} [[B]^n] \,. \tag{3.2}$$

When B is a singleton $\{\alpha\}$, we denote $[\alpha]^n$ for $[\{\alpha\}]^n$.

The precedence relation is generally not reflexive: $\alpha \in [\alpha]$ means that agent α decisions affect its own observation[2]. Witsenhausen's Definition 2.1 of absence of self information precludes such a possibility.

Absence of self information is a property which translates straightforwardly with the precedence relation, as shown in the following Proposition which will be quite useful in the sequel. Its proof is a simple rewriting of Definition 2.1.

Proposition 3.2. *A system is without self information if and only if $\alpha \notin [\alpha]$ for all agent $\alpha \in A$ if and only if the complementary relation $\neg\mathfrak{P}$ of the precedence relation $\mathfrak{P}$ is reflexive if and only if $\mathbf{1}_A \cap \mathfrak{P} = \emptyset$.*

It is a straightforward consequence of Definition 3.1 that, for all $\alpha \in A$, $\beta \in A$ and $B \subset A$, $C \subset A$, we have:

$$\mathcal{I}_\beta \subset \mathcal{I}_\alpha \Rightarrow [\beta] \subset [\alpha] \,, \tag{3.3}$$

$$[\beta] \subset B \iff \mathcal{I}_\beta \subset \mathcal{U}(B) \otimes \mathcal{F} \,. \tag{3.4}$$

$$[B] \subset C \iff \mathcal{I}_B \subset \mathcal{U}(C) \otimes \mathcal{F} \,. \tag{3.5}$$

[2] Recall that, in a temporal framework, an agent is a decision maker at a given time.

3.2. The subsystem binary relation between agents. For any $B \subset A$, let $\mathcal{I}_B \subset \mathcal{H}$ be the *information of the subset B of agents*:

$$\mathcal{I}_B \stackrel{\text{def}}{=} \bigvee_{\beta \in B} \mathcal{I}_\beta \,. \tag{3.6}$$

Witsenhausen in [11] defines a subsystem as a subset B of A such that $\mathcal{I}_B \subset \mathcal{U}(B) \otimes \mathcal{F}$. Using the precedence binary relation, this is equivalent to the following definition.

Definition 3.3. *A subset B of A is a* subsystem *if $[B] \subset B$.* Thus, the information received by agents in B depend upon actions of nature and actions of members of B only.

As a consequence of (2.5), Witsenhausen notices in [11] that subsystems are closed under the intersection and union operations, and thus they form the closed sets of a topology τ on A. Connected components of (A, τ) are dynamically decoupled subsystems; a static coupling remains through the common dependence upon the random issue.

Definition 3.4 ([11]). *The closure $\langle B \rangle$, for the topology τ, of a subset $B \subset A$ is the smallest subsystem containing B; it is called the* subsystem generated by B. *The* subsystem generated by agent α *is the closure $\langle \alpha \rangle$ of the singleton $\{\alpha\}$. The corresponding* subsystem *binary relation $\mathfrak{S}$ between agents is as follows:*

$$\forall (\alpha, \beta) \in A^2 \,, \quad \beta \mathfrak{S} \alpha \iff \beta \in \langle \alpha \rangle \,. \tag{3.7}$$

In other words, $\beta \mathfrak{S} \alpha$ means that agent β belongs to the subsystem generated by agent α or, equivalently, that the subsystem generated by agent α contains the one generated by agent β ($\langle \beta \rangle \subset \langle \alpha \rangle$).

It is a consequence of the definition of a subsystem that, for $B \subset A$:

$$B \text{ is a subsystem } \iff \langle B \rangle = B \iff [B] \subset B \,. \tag{3.8}$$

Proposition 3.5 ([11]). *The subsystem relation $\mathfrak{S}$ is a pre-order (or a quasi order), namely it is reflexive and transitive.*

Proof. The subsystem relation $\mathfrak{S}$ is reflexive since $\alpha \in \langle \alpha \rangle$ for any agent $\alpha \in A$. It is also transitive. Indeed, let agents α, β and δ be such that $\alpha \mathfrak{S} \beta$ and $\beta \mathfrak{S} \delta$, that is $\alpha \in \langle \beta \rangle$ and $\beta \in \langle \delta \rangle$. From $\beta \in \langle \delta \rangle$, we deduce that $\langle \beta \rangle \subset \langle \delta \rangle$ and thus $\alpha \in \langle \delta \rangle$, that is $\alpha \mathfrak{S} \delta$. □

The relation $\mathfrak{S}$ is generally not anti-symmetric since $\langle \beta \rangle = \langle \alpha \rangle$ may occur with $\alpha \neq \beta$.

Examples. Consider the case $A = \{a, b\}$, $\mathbb{U}_a = \{a_1, a_2\}$, $\mathbb{U}_b = \{b_1, b_2\}$, $\Omega = \{\omega_-, \omega_+\}$.

- $\mathcal{I}_a = \{\emptyset, \mathbb{H}\}$, $\mathcal{I}_b = \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F}$: $\langle a \rangle = \{a\}$, $\langle b \rangle = \{b\}$, $\mathfrak{S} = \mathbf{1}_A = \{(a,a), (b,b)\}$.
- $\mathcal{I}_a = \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F}$, $\mathcal{I}_b = \mathcal{U}_a \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F}$ (see Figure 2.1): $\langle a \rangle = \{a\}$, $\langle b \rangle = \{a, b\}$, $\mathfrak{S} = \{(a,b), (a,a), (b,b)\}$.
- $\mathcal{I}_a = \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F}$, $\mathcal{I}_b = \mathcal{U}_a \otimes \{\emptyset, \mathbb{U}_b\} \otimes \{\emptyset, \Omega\}$: $\langle a \rangle = \{a\}$, $\langle b \rangle = \{a, b\}$, $\mathfrak{S} = \{(a,b), (a,a), (b,b)\}$.
- $\mathcal{I}_a = \{\emptyset, \mathbb{U}_a\} \otimes \mathcal{U}_b \otimes \{\emptyset, \Omega\}$, $\mathcal{I}_b = \mathcal{U}_a \otimes \{\emptyset, \mathbb{U}_b\} \otimes \{\emptyset, \Omega\}$ (see Figure 2.2): $\langle a \rangle = \{a, b\}$, $\langle b \rangle = \{a, b\}$, $\mathfrak{S} = \{(a,b), (b,a), (a,a), (b,b)\} = A^2$.
- $\mathcal{I}_a = \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F}$, $\mathcal{I}_b = \sigma(\{a_1\} \times \{b_1, b_2\} \times \{\omega_-\})$: $\langle a \rangle = \{a\}$, $\langle b \rangle = \{a, b\}$, $\mathfrak{S} = \{(a,b), (a,a), (b,b)\}$.

- $\mathcal{I}_a = \sigma(\{a_1,a_2\} \times \{b_1,b_2\} \times \{\omega_+\}, \{a_1,a_2\} \times \{b_1\} \times \{\omega_-\})$, $\mathcal{I}_b = \sigma(\{a_1,a_2\} \times \{b_1,b_2\} \times \{\omega_-\}, \{a_1\} \times \{b_1,b_2\} \times \{\omega_+\})$ (see Figure 2.3): $\langle a \rangle = \{a,b\}$, $\langle b \rangle = \{a,b\}$, $\mathfrak{S} = \{(a,b),(b,a),(a,a),(b,b)\} = A^2$.

3.3. The memory-communication binary relation between agents. The following definition of memory-communication is inspired by [5] and by [2].

Definition 3.6. *Let $\|\alpha\|$ be the union of subsets $B \subset A$ such that $\mathcal{I}_B \subset \mathcal{I}_\alpha$. Since $\{\alpha\}$ belongs to this union and by (3.6), $\|\alpha\|$ is the largest subset $B \subset A$ such that $\mathcal{I}_B \subset \mathcal{I}_\alpha$:*

$$B \subset \|\alpha\| \iff \mathcal{I}_B \subset \mathcal{I}_\alpha \,. \tag{3.9}$$

We define a memory-communication *binary relation $\mathfrak{M}$ on A by*

$$\forall (\alpha,\beta) \in A^2 \,, \quad \beta \,\mathfrak{M}\, \alpha \iff \beta \in \|\alpha\| \iff \mathcal{I}_\beta \subset \mathcal{I}_\alpha \,. \tag{3.10}$$

For any $B \subset A$, we also introduce

$$\|B\| \stackrel{\text{def}}{=} \bigcup_{\beta \in B} \|\beta\| \,. \tag{3.11}$$

When $\beta \in \|\alpha\|$, the observations made by agent β are part of those available to agent α. Note that $\alpha \in \|\alpha\|$, so that $\mathfrak{M}$ is a reflexive relation. $\mathfrak{M}$ is clearly transitive.

Proposition 3.7 ([2]). *The* memory-communication *binary relation $\mathfrak{M}$ is a pre-order (or a quasi order), namely it is reflexive and transitive.*

Examples. Consider the case $A = \{a,b\}$, $\mathbb{U}_a = \{a_1,a_2\}$, $\mathbb{U}_b = \{b_1,b_2\}$, $\Omega = \{\omega_-,\omega_+\}$.

- $\mathcal{I}_a = \{\emptyset, \mathbb{H}\}$, $\mathcal{I}_b = \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F}$: $\|a\| = \{a\}$, $\|b\| = \{a,b\}$, $\mathfrak{M} = \{(a,b),(a,a),(b,b)\}$.
- $\mathcal{I}_a = \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F}$, $\mathcal{I}_b = \mathcal{U}_a \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F}$ (see Figure 2.1): $\|a\| = \{a\}$, $\|b\| = \{a,b\}$, $\mathfrak{M} = \{(a,b),(a,a),(b,b)\}$.
- $\mathcal{I}_a = \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F}$, $\mathcal{I}_b = \mathcal{U}_a \otimes \{\emptyset, \mathbb{U}_b\} \otimes \{\emptyset, \Omega\}$: $\|a\| = \{a\}$, $\|b\| = \{b\}$, $\mathfrak{M} = \{(a,a),(b,b)\}$.
- $\mathcal{I}_a = \{\emptyset, \mathbb{U}_a\} \otimes \mathcal{U}_b \otimes \{\emptyset, \Omega\}$, $\mathcal{I}_b = \mathcal{U}_a \otimes \{\emptyset, \mathbb{U}_b\} \otimes \{\emptyset, \Omega\}$ (see Figure 2.2): $\|a\| = \{a\}$, $\|b\| = \{b\}$, $\mathfrak{M} = \{(a,a),(b,b)\}$.
- $\mathcal{I}_a = \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F}$, $\mathcal{I}_b = \sigma(\{a_1\} \times \{b_1,b_2\} \times \{\omega_-\})$: $\|a\| = \{a\}$, $\|b\| = \{b\}$, $\mathfrak{M} = \{(a,a),(b,b)\}$.
- $\mathcal{I}_a = \sigma(\{a_1,a_2\} \times \{b_1,b_2\} \times \{\omega_+\}, \{a_1,a_2\} \times \{b_1\} \times \{\omega_-\})$, $\mathcal{I}_b = \sigma(\{a_1,a_2\} \times \{b_1,b_2\} \times \{\omega_-\}, \{a_1\} \times \{b_1,b_2\} \times \{\omega_+\})$ (see Figure 2.3): $\|a\| = \{a\}$, $\|b\| = \{b\}$, $\mathfrak{M} = \{(a,a),(b,b)\}$.

3.4. Localization of systems and relations. Let any $h \in \mathbb{H}$ be called an *issue* and any $F \subset \mathbb{H}$ an *event*[3].

Suppose that an event $F \subset \mathbb{H}$ is given. We define a *localized information structure at event F* or a *localized (stochastic control) system at event F* as the collection consisting of A, $(\Omega, \mathcal{F})$, $(\mathbb{U}_\alpha, \mathcal{U}_\alpha, \mathcal{I}_\alpha)_{\alpha \in A}$ together with F. We also speak of an *information structure restricted to F* or a *(stochastic control) system restricted to F*.

Definition 3.8. *Let $[\alpha]_F \subset A$ be the intersection of subsets $B \subset A$ such that $\mathcal{I}_\alpha \cap F \subset \mathcal{U}(B) \otimes \mathcal{F}$: $[\alpha]$ is the smallest subset $B \subset A$ such that $\mathcal{I}_\alpha \cap F \subset \mathcal{U}(B) \otimes \mathcal{F}$. We define a* localized precedence binary relation $\mathfrak{P}_F$ at event F *on A by*

$$\beta \,\mathfrak{P}_F\, \alpha \iff \beta \in [\alpha]_F \,, \tag{3.12}$$

[3] We depart here from the tradition according to which an event is an element of $\mathcal{H}$.

and we say that β is a precedent *of α on the event F.* The *precedence* binary relation $\mathfrak{P}$ on A is $\mathfrak{P}_{\mathbb{H}}$.

DEFINITION 3.9. *A subset B of A is a* subsystem on F *if $[B]_F \subset B$. The closure $\langle B\rangle_F$ of B is the smallest subsystem on F containing B. The* subsystem generated by agent α on F *is the closure $\langle\alpha\rangle_F$ of the singleton $\{\alpha\}$. The corresponding* localized subsystem binary relation $\mathfrak{S}_F$ at event F *between agents is as follows:*

$$\forall(\alpha,\beta)\in A^2\,,\quad \beta\mathfrak{S}_F\alpha \iff \beta\in\langle\alpha\rangle_F\,. \tag{3.13}$$

DEFINITION 3.10 ([2]). *Let $\|\alpha\|_F$ be the union of subsets $B\subset A$ such that $\mathcal{I}_B\cap F\subset\mathcal{I}_\alpha$. $\|\alpha\|_F$ is the largest subset $B\subset A$ such that $\mathcal{I}_B\cap F\subset\mathcal{I}_\alpha$:*

$$B\subset\|\alpha\|_F \iff \mathcal{I}_B\cap F\subset\mathcal{I}_\alpha\,. \tag{3.14}$$

We define a localized memory-communication binary relation $\mathfrak{M}_F$ at event F *on A by*

$$\forall(\alpha,\beta)\in A^2\,,\quad \beta\,\mathfrak{M}_F\,\alpha \iff \beta\in\|\alpha\|_F \iff \mathcal{I}_\beta\cap F\subset\mathcal{I}_\alpha\,. \tag{3.15}$$

Notions of solvability, causality and self information may also be localized. All the results concerning systems remain true for localized systems, except the property that solvability on an event implies absence of self information on this event.

Consider the case $A=\{a,b\}$, $\mathbb{U}_a=\{a_1,a_2\}$, $\mathbb{U}_b=\{b_1,b_2\}$, $\Omega=\{\omega_-,\omega_+\}$.

- $\mathcal{I}_a=\sigma(\{a_1,a_2\}\times\{b_1,b_2\}\times\{\omega_+\},\{a_1,a_2\}\times\{b_1\}\times\{\omega_-\})$, $\mathcal{I}_b=\sigma(\{a_1,a_2\}\times\{b_1,b_2\}\times\{\omega_-\},\{a_1\}\times\{b_1,b_2\}\times\{\omega_+\})$ (see Figure 2.3).
  Let $F_+=\{a_1,a_2\}\times\{b_1,b_2\}\times\{\omega_+\}$ and $F_-=\{a_1,a_2\}\times\{b_1,b_2\}\times\{\omega_-\}$. We have $[a]_{F_+}=\emptyset$, $[b]_{F_+}=\{a\}$, $\mathfrak{P}_{F_+}=\{(b,a)\}$ and $[a]_{F_-}=\{b\}$, $[b]_{F_-}=\emptyset$, $\mathfrak{P}_{F_-}=\{(a,b)\}$. We have $\|a\|_{F_+}=\{a\}$, $\|b\|_{F_+}=\{a,b\}$, $\mathfrak{M}_{F_+}=\{(a,b),(a,a),(b,b)\}$ and $\|a\|_{F_-}=\{a,b\}$, $\|b\|_{F_-}=\{b\}$, $\mathfrak{M}_{F_-}=\{(b,a),(a,a),(b,b)\}$.

The following Proposition is a straightforward consequence of the hereabove Definitions.

PROPOSITION 3.11. *Let $(F_i)_{i\in I}$ be a finite or enumerable family of subsets of $\mathbb{H}$ such that $[B]_{F_i}\subset C$. Then both $F_-=\bigcap_{i\in I}$ and $F_+=\bigcup_{i\in I}$ are such that $[B]_{F_-}\subset C$ and $[B]_{F_+}\subset C$. The same holds true with $\langle B\rangle$.*

Localizing the precedence relation provides an evocative characterization of Witsenhausen's causality property (C), recalled in subsection 2.3, as follows.

A family $(K_i)_{i\in I}$ is said to *cover* $\mathbb{H}$ if $K_i\subset\mathbb{H}$ and $\bigcup_{i\in I}K_i=\mathbb{H}$. It is said to *disjointly cover* $\mathbb{H}$ if, in addition, the K_i are two by two disjoints. By these definitions, we allow for the K_i to be empty sets.

PROPOSITION 3.12. *An information structure possesses causality property (C) if and only if there exists a family $(K_\varsigma)_{\varsigma\in S_A}$ which disjointly covers $\mathbb{H}$ and such that*

$$\forall k\in\{1,\dots,n\}\,,\quad \forall s\in S_A^k\,,\quad [s(k)]_{K_s}\subset\{s(1),\dots,s(k-1)\}\,, \tag{3.16}$$

where $K_s \overset{\text{def}}{=} \bigcup_{\varsigma\in S_A,T_k^n\varsigma=s} K_\varsigma$.

Proof. If causality property (C) holds with the mapping φ, we put $K_\varsigma\overset{\text{def}}{=}\varphi^{-1}(\varsigma)$. Then (3.16) is a rewriting of (2.12).

Now, if there is a family $(K_\varsigma)_{\varsigma\in S_A}$ which disjointly covers $\mathbb{H}$, we can define $\varphi:\mathbb{H}\to S_A$ by $\varphi_{|K_\varsigma}\equiv\varsigma$. Then (2.12) is a rewriting of (3.16). ∎

3.5. Relationships between $\mathfrak{P}$ and $\mathfrak{S}$. Here are relationships between the precedence relation $\mathfrak{P}$ and the subsystem relation $\mathfrak{S}$.

PROPOSITION 3.13.

1. *For any $\alpha \in A$ and $\beta \in A$, we have:*

$$\beta \in \langle\alpha\rangle \Rightarrow [\beta] \subset \langle\alpha\rangle \,. \tag{3.17}$$

2. *For any $\alpha \in A$, we have:*

$$\langle\alpha\rangle = \bigcup_{n\in\mathbb{N}} [\alpha]^n \,. \tag{3.18}$$

Proof.

1. By definition of $\langle\alpha\rangle$, we have $\mathcal{I}_{\langle\alpha\rangle} = \bigvee_{\beta\in\langle\alpha\rangle} \mathcal{I}_\beta \subset \mathcal{U}(\langle\alpha\rangle)\otimes\mathcal{F}$. Thus $\beta \in \langle\alpha\rangle \Rightarrow \mathcal{I}_\beta \subset \mathcal{U}(\langle\alpha\rangle)\otimes\mathcal{F}$, that is $[\beta] \subset \langle\alpha\rangle$ by (3.4).
2. First, we prove by induction that $[\alpha]^n \subset \langle\alpha\rangle$ for $n \in \mathbb{N}$. By definition (3.2), this holds true for $n = 0$ since $\alpha \in \langle\alpha\rangle$. Assuming $[\alpha]^n \subset \langle\alpha\rangle$, we deduce that $[\alpha]^{n+1} = [[\alpha]^n] \subset [\langle\alpha\rangle]$ by (3.2). Now, since $\langle\alpha\rangle$ is a subsystem, we have $[\langle\alpha\rangle] \subset \langle\alpha\rangle$ by (3.8). Thus $[\alpha]^{n+1} \subset \langle\alpha\rangle$ and the induction is proven.
   Second, from $[\alpha]^n \subset \langle\alpha\rangle$, we deduce that $\bigcup_{n\in\mathbb{N}}[\alpha]^n \subset \langle\alpha\rangle$. On the other hand, $\bigcup_{n\in\mathbb{N}}[\alpha]^n$ is a subsystem by (3.8) since $[\bigcup_{n\in\mathbb{N}}[\alpha]^n] = \bigcup_{n\in\mathbb{N}}[\alpha]^{n+1} \subset \bigcup_{n\in\mathbb{N}}[\alpha]^n$ (notice that the enumerable union $\bigcup_{n\in\mathbb{N}}[\alpha]^n$ is in fact finite since A is finite). To sum up, $\bigcup_{n\in\mathbb{N}}[\alpha]^n$ is a subsystem containing α and contained in $\langle\alpha\rangle$: it is thus equal to this latter.

□

Proposition 3.13 provides the following links between the subsystem and the precedence binary relations.

THEOREM 3.14. *The relation $\mathfrak{S}$ is the reflexive and transitive closure $\mathfrak{P}^*$ of the precedence relation $\mathfrak{P}$. We thus have the following inclusions and equalities:*

$$\mathfrak{P} \subset \mathfrak{P}^\infty \subset \mathfrak{S} = \mathfrak{P}^* = \mathfrak{P}^\infty \cup \mathbf{1}_A \,. \tag{3.19}$$

Proof. By (3.18), we have that $\mathfrak{P} \subset \mathfrak{S}$, since $\beta \in [\alpha] \Rightarrow \beta \in \langle\alpha\rangle$. By Proposition 3.5, $\mathfrak{S}$ is reflexive and transitive. Thus, $\mathfrak{P} \subset \mathfrak{S} \Rightarrow \mathfrak{P}^* \subset \mathfrak{S}^* = \mathfrak{S}$. Now, by definition (A.9) of $\mathfrak{P}^\infty$, the identity (3.18) means that $\mathfrak{P}^\infty \cup \mathbf{1}_A = \mathfrak{S}$. We conclude with the equality (A.11), which is here $\mathfrak{P}^* = \mathfrak{P}^\infty \cup \mathbf{1}_A$. □

3.6. Relationships between $\mathfrak{P}$ and $\mathfrak{M}$. Here are relationships between the precedence relation $\mathfrak{P}$ and the memory-communication relation $\mathfrak{M}$.

PROPOSITION 3.15 ([2]). *For any agents $\alpha \in A$ and $\beta \in A$, we have*

$$\beta \in \|\alpha\| \Rightarrow [\beta] \subset [\alpha] \,. \tag{3.20}$$

For any $B \subset A$, we have:

$$[\|B\|] \subset [B] \,. \tag{3.21}$$

Proof. Let agents α and β be such that $\beta \in \|\alpha\|$. Thus, $\mathcal{I}_\beta \subset \mathcal{I}_\alpha$ by (3.10) and we conclude that $[\beta] \subset [\alpha]$ by (3.3), *i.e.* $[\|\alpha\|] \subset [\alpha]$. We deduce that, for any $B \subset A$, we have:

$$[\|B\|] = \bigcup_{\beta\in\|B\|} [\beta] = \bigcup_{\alpha\in B} \bigcup_{\beta\in\|\alpha\|} [\beta] \subset \bigcup_{\alpha\in B} [\alpha] = [B] \,.$$

□

4. A classification of systems.

4.1. A typology of systems. The following definitions are extended from [11].

A *static team* is a subset B of A such that $[B] = \emptyset$. Equivalently, by Theorem 3.14, the precedence relation $\mathfrak{P}$ is empty or the subsystem relation $\mathfrak{S}$ is reduced to the equality relation $\mathbf{1}_A$.

It follows from Definition 2.2 that a system is *sequential* if and only if there exists an ordering $(\alpha_0, \ldots, \alpha_{n-1})$ of A such that

$$\forall k = 0, \ldots, n-1 , \quad [\alpha_k] \subset \{\alpha_0, \ldots, \alpha_{k-1}\}^4 . \tag{4.1}$$

Equivalently, there exists an ordering $(\alpha_0, \ldots, \alpha_{n-1})$ of A such that $\alpha_i \mathfrak{P} \alpha_j \Rightarrow i < j$ (*i.e.* strictly compatible with $\mathfrak{P}$).

A system is *classical* if it is sequential and, in addition, $\alpha_k \in \|\alpha_{k+1}\|$ for $k = 0, \ldots, n-2$. Equivalently, by using the transitivity of $\mathfrak{M}$ (Proposition 3.7), there exists an ordering $(\alpha_0, \ldots, \alpha_{n-1})$ of A such that

$$\forall k = 0, \ldots, n-1 , \quad [\alpha_k] \subset \{\alpha_0, \ldots, \alpha_{k-1}\} \subset \{\alpha_0, \ldots, \alpha_{k-1}, \alpha_k\} \subset \|\alpha_k\| . \tag{4.2}$$

Thus, any agent knows what all "previous" agents know. Notice that $[\alpha_k] \subset [\alpha_{k+1}]$. Indeed, (4.2) together with the property that $[\|\alpha_k\|] \subset [\alpha_k]$ (see Proposition 3.15) imply that $[[\alpha_k]] \subset [\alpha_0] \cup \ldots \cup [\alpha_{k-1}] \subset [\|\alpha_k\|] \subset [\alpha_k]$, hence $[\alpha_{k-1}] \subset [\alpha_k]$.

A system is *strictly classical* if it is classical with $\mathcal{U}(\{\alpha_k\}) \otimes \mathcal{F} \subset \mathcal{I}_{\alpha_{k+1}}$. In other words, agent α_{k+1} knows also the actions of the "previous" agent α_k, for $k = 0, \ldots, n-2$.

A system is *quasiclassical* if it is sequential and that $\mathfrak{S} \subset \mathfrak{M}$.

A system is *strictly quasiclassical* if it is sequential and that $\beta \mathfrak{S} \alpha$, $\beta \neq \alpha$ implies $(\mathcal{I}_\beta \bigvee (\mathcal{U}(\{\beta\}) \otimes \mathcal{F})) \subset \mathcal{I}_\alpha$. In other words, agent α knows what know and do all the other agents which form the subsystem that he generates.

4.2. Characterizations of sequential systems. The equivalence of the two first assertions in the following Theorem is due to Witsenhausen in [11]. The others are new. They will prove useful in the sequel for characterizing partially nested systems without self information.

THEOREM 4.1. *The following assertions are equivalent:*
1. *the system is sequential;*
2. *the system is without self information and the subsystem relation $\mathfrak{S}$ is an order;*
3. *the precedence relation $\mathfrak{P}$ is* acyclic *(that is, $\forall \alpha \in A$, $\forall n \geq 1$, $\alpha \notin [\alpha]^n$);*
4. *the relation $\neg(\mathfrak{P}^\infty)$ is reflexive (*i.e. $\mathbf{1}_A \cap \mathfrak{P}^\infty = \emptyset$*);*
5. *the* graph of precedence $G(\mathfrak{P})$ *is a forest.*

[4] For $k = 0$, this means that $[\alpha_0] = \emptyset$.

Proof. The proof is a straightforward transcription of Proposition A.1 in the Appendix with $\mathfrak{R} = \mathfrak{P}$.

For this, recall that a system is sequential if and only if there exists an ordering of A strictly compatible with $\mathfrak{P}$.

Also, a system is without self information and $\mathfrak{S}$ is an order if and only if $\neg\mathfrak{P}$ is reflexive and $\mathfrak{S} = \mathfrak{P}^*$ is an order. □

4.3. Definition and properties of systems closed under precedence. The following definition covers a class of systems with original properties.

DEFINITION 4.2. *A system is closed under precedence if the precedence binary relation $\mathfrak{P}$ is transitive.* Equivalently, for any $B \subset A$, $[B]$ is a subsystem, by (3.8).

Here is a Proposition which will prove important for partially nested systems.

PROPOSITION 4.3. *A system closed under precedence is sequential if and only if it is without self information.*

A system closed under precedence and without self information is sequential.

Proof. On the one hand, by Theorem 4.1, a system is sequential if and only if the relation $\neg(\mathfrak{P}^\infty)$ is reflexive.

On the other hand, by Proposition 3.2, a system is without self information if and only if the relation $\neg\mathfrak{P}$ is reflexive.

We conclude with the fact that $\mathfrak{P}^\infty = \mathfrak{P}$ for a system closed under precedence. □

4.4. Definition and properties of partially nested systems. We here generalize the quasiclassical systems to non necessarily sequential systems. The terminology is taken from [5, 6]. A partially nested system is one in which any agent has the information available to those agents which are its precedents.

PROPOSITION 4.4. *The following conditions are equivalent.*

1. $\mathfrak{S} \subset \mathfrak{M}$*, that is*

$$\forall \alpha \in A\,, \quad \langle \alpha \rangle \subset \|\alpha\|\,. \tag{4.3}$$

2. $\mathfrak{P} \subset \mathfrak{M}$*, that is*

$$\forall \alpha \in A\,, \quad [\alpha] \subset \|\alpha\|\,. \tag{4.4}$$

3. $\mathfrak{P}^\infty \subset \mathfrak{M}$*, that is*

$$\forall \alpha \in A\,, \quad \forall n \geq 1\,, \quad [\alpha]^n \subset \|\alpha\|\,. \tag{4.5}$$

Proof. $\mathfrak{S} \subset \mathfrak{M}$ implies $\mathfrak{P} \subset \mathfrak{M}$, since $\mathfrak{P} \subset \mathfrak{S}$ by Theorem 3.14. On the other hand, $\mathfrak{P} \subset \mathfrak{M}$ implies $\mathfrak{P}^\infty \subset \mathfrak{M}$ since $\mathfrak{M}$ is transitive, by Proposition 3.7. At last, $\mathfrak{P}^\infty \subset \mathfrak{M}$ implies $\mathfrak{S} \subset \mathfrak{M}$ since $\mathfrak{S}$ is the reflexive closure of $\mathfrak{P}^\infty$ by Theorem 3.14 and since $\mathfrak{M}$ is reflexive by Proposition 3.7. □

DEFINITION 4.5. *A system which satisfies any of the three equivalent assertions of Proposition 4.4 is said to be a* partially nested *system.*

The following Proposition, proved in [2], provides a curious property of the precedence binary relation. As a consequence, the assertions of Proposition 4.4 hold true for a partially nested system.

PROPOSITION 4.6 ([2]). *A partially nested system is closed under precedence.*

Proof. The proof is from [2].

		Binary relations between agents				
		subsystem $\mathfrak{S}$		precedence $\mathfrak{P}$		memory $\mathfrak{M}$
Properties		pre-order		$\mathfrak{S} = \mathfrak{P}^\infty \cup \mathbf{1}_A$		pre-order
no self information	$\iff$			$\neg\mathfrak{P}$ reflexive		
static team	$\iff$	$\mathfrak{S} = \mathbf{1}_A$	or	$\mathfrak{P} = \emptyset$		
sequential	$\iff$	order	or	$\neg\mathfrak{P}^\infty$ reflexive		
sequential	$\iff$	order	or	acyclic		
classical	$\Rightarrow$			acyclic	and	$\mathfrak{M} \supset \mathfrak{P}$
quasiclassical	$\iff$			acyclic	and	$\mathfrak{M} \supset \mathfrak{P}$
closed under precedence	$\iff$			$\mathfrak{P} = \mathfrak{P}^\infty$		
partially nested	$\iff$					$\mathfrak{M} \supset \mathfrak{P}$
partially nested	$\Rightarrow$			$\mathfrak{P} = \mathfrak{P}^\infty$		

Table 4.1
Binary relations characterization of a typology of systems

For a partially nested system, we have $\mathfrak{P} \subset \mathfrak{M}$ by Proposition 4.4. Let $(\alpha, \beta, \delta) \in A^3$ be such that $\alpha\mathfrak{P}\beta$ and $\beta\mathfrak{P}\delta$. We have

$$\begin{aligned} \beta\mathfrak{P}\delta \Rightarrow\ & \beta\mathfrak{M}\delta \quad \text{by}\, \mathfrak{P} \subset \mathfrak{M} \\ \Rightarrow & \beta \in \|\delta\| \text{ by definition (3.10)} \\ \Rightarrow & [\beta] \subset [\delta] \text{ by (3.20).} \end{aligned}$$

On the other hand, since $\alpha\mathfrak{P}\beta$, we have $\alpha \in [\beta]$ by equation (3.1). Combining $\alpha \in [\beta]$ and $[\beta] \subset [\delta]$, we obtain $\alpha \in [\delta]$, that is $\alpha\mathfrak{P}\delta$ by definition of $\mathfrak{P}$. Thus, $\mathfrak{P}$ is transitive. $\square$

The following Theorem is a direct corollary of Propositions 4.3 and 4.6.

Theorem 4.7. *A partially nested system without self information is sequential. As a consequence, a causal partially nested system is sequential.*

4.5. A summary table of results. In the following table, for a given system, you find in the corresponding line either equivalent characterizations or implications. For instance, a system is quasiclassical if and only if $\mathfrak{P}$ is acyclic and $\mathfrak{M} \supset \mathfrak{P}$.

5. Conclusion. We have provided a unified framework to define and study three binary relations between agents scattered in the litterature. The terminology and properties of binary relations has allowed us to obtain new results and the typology of systems is expressed in a compact form (see Table 4.1).

Define a strictly partially nested system as a partially nested one with the additional property that $\beta\mathfrak{S}\alpha$, $\beta \neq \alpha$ implies $\mathcal{U}(\{\beta\}) \otimes \mathcal{F} \subset \mathcal{I}_\alpha$. In other words, agent α knows what know and do all the other agents which form the subsystem that he generates. Witsenhausen shows in [12] that, under appropriate assumptions on control and/or random sets, strictly partially nested systems exhibit policy independence of conditional expectations. He suggests in the conclusion of [12] a mechanism of stochastic dynamic programming to solve team problems. This mechanism supposes sequentiality. With our results, we show that strictly partially nested systems without self information are sequential, allowing thus for an extension of stochastic dynamic programming.

Appendix A. Recalls on binary relations.

We follow here [7, 8].

A *binary relation* on A is a subset $\mathfrak{R}$ of $A^2 = A \times A$. As is traditional, we shall from now on denote

$$\forall(\alpha,\beta) \in A^2 \,, \quad \alpha\mathfrak{R}\beta \iff (\alpha,\beta) \in \mathfrak{R} \,. \tag{A.1}$$

When $\alpha\mathfrak{R}\beta$, we say that β is *related* to α. Well known binary relations are the *empty relation* $\emptyset$, the *universal relation* A^2, and the *equality or diagonal relation*

$$\forall(\alpha,\beta) \in A^2 \,, \quad \alpha\mathbf{1}_A\beta \iff \alpha = \beta \quad \text{or equivalently} \quad \mathbf{1}_A \stackrel{\text{def}}{=} \{(\alpha,\alpha) \mid \alpha \in A\} \,. \tag{A.2}$$

For each $\alpha \in A$, we define a subset $\alpha\mathfrak{R}$ of A by

$$\alpha\mathfrak{R} \stackrel{\text{def}}{=} \{\beta \in A \mid \alpha\mathfrak{R}\beta\} \,. \tag{A.3}$$

We thus have

$$\forall(\alpha,\beta) \in A^2 \,, \quad \alpha\mathfrak{R}\beta \iff \beta \in \alpha\mathfrak{R} \,. \tag{A.4}$$

If B is a subset of A, we define

$$B\mathfrak{R} \stackrel{\text{def}}{=} \bigcup_{\beta \in B} \beta\mathfrak{R} \,. \tag{A.5}$$

The set $\mathfrak{B}_A$ of all binary relations on A is equipped with the inclusion $\subset$:

$$\mathfrak{R}_- \subset \mathfrak{R}_+ \iff \big(\forall(\alpha,\beta) \in A^2 \,,\, \alpha\mathfrak{R}_-\beta \Rightarrow \alpha\mathfrak{R}_+\beta\big) \iff (\forall\alpha \in A \,,\, \alpha\mathfrak{R}_- \subset \alpha\mathfrak{R}_+) \,. \tag{A.6}$$

The *converse* $\mathfrak{R}^{-1}$ of a binary relation is

$$\forall(\alpha,\beta) \in A^2 \,, \quad \alpha\mathfrak{R}^{-1}\beta \iff \beta\mathfrak{R}\alpha \,. \tag{A.7}$$

We have $\mathfrak{R}_- \subset \mathfrak{R}_+ \Rightarrow \mathfrak{R}_-^{-1} \subset \mathfrak{R}_+^{-1}$.

The *directed graph* $G(\mathfrak{R})$ built from $\mathfrak{R}$ is $(A, \mathfrak{R})$, where elements of A are called *vertices* and those of $\mathfrak{R}$ *edges*. Thus notions attached to graphs are easily transfered to relations.

A *chain* in a binary relation $\mathfrak{R}$ is a sequence $(\alpha_0, \ldots, \alpha_n)$ for some $n \geq 1$ such that $\alpha_i\mathfrak{R}\alpha_{i+1}$ for $i = 0, \ldots, n-1$; this chain is said to be *from* α_1 *to* α_n, and its *length* is n. We also say that α_1 and α_n are *joined* by a chain of length n. A chain in a relation is the equivalent of a *path* in a graph. A chain is *simple* if the α_i are all distinct.

The chain $(\alpha_1, \ldots, \alpha_n)$ is a *cycle* if $\alpha_n\mathfrak{R}\alpha_1$. A cycle is *trivial* if $n = 1$, otherwise it is *nontrivial*. A binary relation $\mathfrak{R}$ is said to be *acyclic* if there is no cycle in $\mathfrak{R}$. This corresponds to acyclicity of the directed graph $G(\mathfrak{R})$ built from $\mathfrak{R}$.

The *composition* $\mathfrak{R} \circ \mathfrak{R}'$ of two binary relations is defined by

$$\forall(\alpha,\beta) \in A^2 \,, \quad \alpha(\mathfrak{R} \circ \mathfrak{R}')\beta \iff \exists\delta \in A \,, \quad \alpha\mathfrak{R}\delta \quad \text{and} \quad \delta\mathfrak{R}'\beta \,. \tag{A.8}$$

By simplicity, we shall abbreviate $\mathfrak{R} \circ \mathfrak{R}' \stackrel{\text{def}}{=} \mathfrak{R}\mathfrak{R}'$. In the composition $\mathfrak{R}^2 \stackrel{\text{def}}{=} \mathfrak{R}\mathfrak{R}$, β is related to α is there is a chain of length 2 from α to β. We define as well $\mathfrak{R}^n \stackrel{\text{def}}{=} \underbrace{\mathfrak{R}\cdots\mathfrak{R}}_{n\,times}$, etc. and

$$\mathfrak{R}^\infty \stackrel{\text{def}}{=} \bigcup_{n\geq 1} \mathfrak{R}^n \tag{A.9}$$

where two elements are related if and only if they may be joined by a chain of any length:

$$\forall(\alpha,\beta) \in A^2\,, \quad \alpha\mathfrak{R}^\infty\beta \iff \text{there exists a chain in } \mathfrak{R} \text{ from } \alpha \text{ to } \beta\,. \tag{A.10}$$

A binary relation $\mathfrak{R}$ is said to be *reflexive* if $\mathbf{1}_A \subset \mathfrak{R}$, *symmetric* if $\mathfrak{R}^{-1} \subset \mathfrak{R}$, *anti-symmetric* if $\mathfrak{R} \cap \mathfrak{R}^{-1} \subset \mathbf{1}_A$, *transitive* if $\mathfrak{R}^2 \subset \mathfrak{R}$.

The *transitive closure* of a binary relation is the smallest transitive binary relation which contains $\mathfrak{R}$: it coincides with $\mathfrak{R}^\infty$.

The *reflexive and transitive closure* $\mathfrak{R}^*$ of a binary relation $\mathfrak{R}$ is the smallest reflexive and transitive binary relation which contains $\mathfrak{R}$. We have

$$\mathfrak{R}^* = \mathbf{1}_A \cup \mathfrak{R}^\infty\,. \tag{A.11}$$

An *equivalence relation* is a reflexive, symmetric and transitive binary relation. A *pre-order* or *quasi order* is a reflexive and transitive binary relation. An *order* is a reflexive, anti-symmetric and transitive binary relation.

The *complementary* relation $\neg\mathfrak{R}$ of a binary relation $\mathfrak{R}$ is $\neg\mathfrak{R} \stackrel{\text{def}}{=} A^2\backslash\mathfrak{R}$, that is

$$\alpha\neg\mathfrak{R}\beta \iff \neg\,(\alpha\mathfrak{R}\beta) \iff (\alpha,\beta) \notin \mathfrak{R}\,. \tag{A.12}$$

An *ordering* $(\alpha_0,\ldots,\alpha_{n-1})$ of A is a bijection from $\{0,\ldots,n-1\}$ to A. Such an ordering is said to be *strictly compatible* (resp. *compatible*) with a binary relation $\mathfrak{R}$ if $\alpha_i\mathfrak{R}\alpha_j \Rightarrow i < j$ (resp. $i \leq j$). Notice that a strictly compatible ordering corresponds to a *topological sort* of the directed graph $G(\mathfrak{R})$ built from $\mathfrak{R}$.

PROPOSITION A.1. *The following assertions are equivalent for a binary relation* $\mathfrak{R}$.

1. *There exists an ordering of A strictly compatible with* $\mathfrak{R}$.
2. $\neg\mathfrak{R}$ *is reflexive (*i.e. $\mathbf{1}_A \cap \mathfrak{R} = \emptyset$*) and* $\mathfrak{R}^*$ *is an order.*
3. $\mathfrak{R}$ *is acyclic.*
4. $\neg(\mathfrak{R}^\infty)$ *is reflexive (*i.e. $\mathbf{1}_A \cap \mathfrak{R}^\infty = \emptyset$*).*
5. *the graph* $G(\mathfrak{R})$ *is a forest.*

Proof.

(1) $\Rightarrow$ **(2)** Let an ordering $(\alpha_0,\ldots,\alpha_{n-1})$ of A supposed to be strictly compatible with $\mathfrak{R}$. Let $\mathfrak{O}$ be the binary relation on A defined by $\alpha_i\mathfrak{O}\alpha_j \iff i \leq j$. Clearly, $\mathfrak{O}$ is an order relation such that $\mathfrak{R} \subset \mathfrak{O}\backslash\mathbf{1}_A$.
Thus, on the one hand, $\mathfrak{R} \subset \mathfrak{O}\backslash\mathbf{1}_A \Rightarrow \mathbf{1}_A \subset \neg\mathfrak{R}$, so that $\neg\mathfrak{R}$ is reflexive, *i.e.* $\mathbf{1}_A \cap \mathfrak{R} = \emptyset$. On the other hand, $\mathfrak{R} \subset \mathfrak{O} \Rightarrow \mathfrak{R}^* \subset \mathfrak{O}$ since $\mathfrak{O}$ is reflexive and transitive. We deduce that $\mathfrak{R}^* \cap (\mathfrak{R}^*)^{-1} \subset \mathfrak{O} \cap \mathfrak{O}^{-1}$, where $\mathfrak{O} \cap \mathfrak{O}^{-1} \subset \mathbf{1}_A$ since $\mathfrak{O}$ is antisymmetric. Thus, $\mathfrak{R}^* \cap (\mathfrak{R}^*)^{-1} \subset \mathbf{1}_A$ and $\mathfrak{R}^*$ is antisymmetric.

(2) $\Rightarrow$ **(3)** Assume that $\mathfrak{R}$ is not acyclic, and let $(\alpha_0,\ldots,\alpha_k)$ denote a cycle. Notice that for any $i \in \{0,\ldots,k\}$ and $j \in \{0,\ldots,k\}$, we have $\alpha_i\mathfrak{R}^\infty\alpha_j$, and thus $\alpha_i\mathfrak{R}^*\alpha_j$ because $\mathfrak{R}^\infty \subset \mathfrak{R}^*$. Since $\mathfrak{R}^*$ is an order, this implies $\alpha_i = \alpha_j$. Thus the cycle may be reduced to a single element α_0 which satisfies $\alpha_0\mathfrak{R}\alpha_0$: this contradicts the assumption that $\neg\mathfrak{R}$ is reflexive. Thus, $\mathfrak{R}$ is acyclic.

(3) $\Rightarrow$ **(1)** We know from graph theory that, when the directed graph $G(\mathfrak{R})$ is acyclic, it is possible to perform a topological sort [4, p.485], in other words a strictly compatible ordering.

(3) $\iff$ **(4)** $\mathfrak{R}$ is not acyclic if and only if there exists an $\alpha \in A$ and a chain from α to α, if and only if there exists an $\alpha \in A$ such that $\alpha \mathfrak{R}^{\infty} \alpha$, if and only if $\neg(\mathfrak{R}^{\infty})$ is not reflexive, if and only if $\mathbf{1}_A \cap \mathfrak{R}^{\infty} = \emptyset$.

(3) $\iff$ **(5)** This is the definition of a forest.

$\square$

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