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A priori and a posteriori analysis of nonconforming finite elements with face penalty for advection–diffusion equations

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We analyze a nonconforming finite element method to approximate advection–diffusion–reaction equations. The method is stabilized by penalizing the jumps of the solution and those of its advective derivative across mesh interfaces. The a priori error analysis leads to (quasi-)optimal estimates in the mesh-size (sub–optimal of order $\frac{1}{2}$ in the L^2–norm optimal and optimal in the broken graph norm for quasi–uniform meshes) keeping the Péclet number fixed. Then, we investigate a residual a posteriori error estimator for the method. The estimator is semi-robust in the sense that it yields lower and upper bounds of the error which differ by a factor equal at most to the square root of the Péclet number. Finally, to illustrate the theory we present numerical results including adaptively generated meshes.

Keywords: nonconforming finite elements – face penalty – advection–diffusion – a posteriori error estimator – adaptive meshes

1. Introduction

Advection–diffusion equations in the dominant advection regime are encountered in many applications, including pollutant transport and the Navier–Stokes equations. It is well-known that the standard Galerkin approximation to these equations leads to oscillations when layers are not properly resolved. To stabilize this phenomenon, several well–established techniques have been proposed and analyzed in a conforming setting (e.g., streamline–diffusion [8, 21], subgrid viscosity [17, 18], and residual free bubbles [7]) as well as in a discontinuous setting (e.g., the Discontinuous Galerkin method analyzed in [22]).

An interesting compromise between conforming and discontinuous Galerkin methods consists of using nonconforming finite elements. In this paper, we are interested in low-order nonconforming finite elements such as the Crouzeix–Raviart finite element. This finite element presents various interesting features. First, the degrees of freedom are localized at the mesh faces, thereby leading to efficient communication and parallelization. Second, Crouzeix–Raviart finite elements have close links with finite volume box schemes; see, e.g., [4, 12] for Darcy's equations and [16] for advection–diffusion equations. This property is useful to reconstruct locally the diffusive flux in problems where conservativity properties are important, e.g., pollutant transport. Finally, keeping the mesh fixed, the Crouzeix–Raviart finite element space has approximately two times less degrees of freedom than the first-order Discontinuous Galerkin finite element space.

The topic of approximating advection–diffusion equations by Crouzeix–Raviart finite elements is not new; see [20, 25] where the streamline–diffusion paradigm is extended to the nonconforming setting. The difficulties with streamline–diffusion, in both conforming and nonconforming settings, is that the method involves a parameter depending on the diffusion coefficient and that the extension to time-dependent problems is not straightforward. This can be unpractical in nonlinear problems, e.g., the Navier–Stokes equations where the regions with dominant convection may not be known a priori. In this paper we consider a different technique to stabilize the nonconforming finite element approximation, namely that of penalizing the jumps of the solution and those of its advective derivative across mesh interfaces. Drawing on earlier ideas by Douglas and Dupont [14], the analysis of face penalty finite element methods has been recently extended to advection–diffusion equations with dominant advection [9, 10] and to the Stokes equations [11]; see also [26] for an application to incompressible flow problems. The advantage of using the face penalty technique rather than streamline–diffusion is that the former involves a single user-dependent parameter which is independent of the diffusion coefficient. Moreover, the face penalty technique is readily extendable to time-dependent problems.

The a posteriori error analysis of nonconforming finite element approximations to advection–diffusion equations is a much less explored topic. Even in a conforming setting, the analysis is harder than it seems at first sight. The main issue at stake is to derive a so-called robust error estimator for which the upper and lower bounds for the error differ by a factor that is independent of the Péclet number. The first main progress in this direction was achieved by Verfürth [30] in a conforming setting, the proposed error estimator yielding a factor between lower and upper error bounds which scales as the square root of the Péclet number. Such error estimators are henceforth called semi-robust. Further results in this direction include [2, 3]. Recently, robust error estimators, still in a conforming setting, have been proposed by Verfürth [31] and by Sangalli [28]. To this purpose, the norm with which the error is measured has to be modified; in particular, it includes the advective derivative of the error. In [31], the advective derivative is measured in a dual (non-local) norm. In [28], the advective derivative is measured in a non-standard interpolated norm of order $\frac{1}{2}$ introduced in [27] and that can be evaluated by solving a generalized eigenvalue problem on a fine mesh. The purpose of the present work is to propose and analyze a semi-robust error estimator for nonconforming finite element approximations to advection–diffusion equations. To our knowledge, it is the first semi-robust error estimator in this setting. The present analysis can be viewed as a first step towards establishing robust error estimators in the nonconforming setting.

This paper is organized as follows. Section 2 presents the model problem and the nonconforming finite element approximation with face penalty. Section 3 deals with the a priori error analysis and Section 4 with the residual a posteriori error analysis. Section 5 contains numerical results and Section 6 draws some conclusions.

2. The setting

2.1 *The model problem*

Let Ω be a polygonal domain of $\mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$ and outward normal n. Let $\varepsilon > 0$, $\beta \in [\mathscr{C}^{0,\frac{1}{2}}(\Omega)]^d$, and $\nu \in L^\infty(\Omega)$ be respectively the diffusion coefficient, the velocity field, and the reaction coefficient. Set $\partial\Omega_{\mathrm{in}} = \{x \in \partial\Omega : \beta\cdot n < 0\}$ and $\partial\Omega_{\mathrm{out}} = \{x \in \partial\Omega : \beta\cdot n \geqslant 0\}$. Let $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega_{\mathrm{in}})$ be the data. We are interested in the following advection–diffusion–reaction problem with mixed Robin–Neumann boundary conditions:

$$\begin{cases} -\varepsilon\Delta u + \beta\cdot\nabla u + \nu u = f & \text{in } \Omega\,, \\ -\varepsilon\nabla u\cdot n + \beta\cdot n u = g & \text{on } \partial\Omega_{\mathrm{in}}\,, \\ \nabla u\cdot n = 0 & \text{on } \partial\Omega_{\mathrm{out}}\,. \end{cases} \tag{2.1}$$

Without loss of generality, we assume that (2.1) is non-dimensionalized so that $\|\beta\|_{[L^\infty(\Omega)]^d}$ and the length scale of Ω are of order unity; hence, the parameter ε is the reciprocal of the Péclet number.

Under the assumption that there is $\sigma_0 > 0$ such that $\sigma = \nu - \frac{1}{2}\nabla\cdot\beta \geqslant \sigma_0$ in Ω and that $\nabla\cdot\beta \in L^\infty(\Omega)$, it is straightforward to verify using the Lax–Milgram Lemma that the following weak formulation of (2.1) is well posed:

$$\begin{cases} \text{Seek } u \in H^1(\Omega) \text{ such that} \\ a(u,v) = \int_\Omega fv - \int_{\partial\Omega_{\text{in}}} gv \quad \forall v \in H^1(\Omega)\,, \end{cases} \tag{2.2}$$

where

$$a(u,v) = \int_\Omega \varepsilon\nabla u\cdot\nabla v + \int_\Omega (\nu - \nabla\cdot\beta)uv - \int_\Omega u(\beta\cdot\nabla v) + \int_{\partial\Omega_{\text{out}}} (\beta\cdot n)uv\,. \tag{2.3}$$

2.2 *The discrete setting*

Let $(\mathscr{T}_h)_h$ be a shape–regular family of simplicial affine meshes of Ω. For an element $T \in \mathscr{T}_h$, let ∂T denote its boundary, h_T its diameter and set $h = \max_{T\in\mathscr{T}_h} h_T$. Let $\mathscr{F}_h$, $\mathscr{F}_h^i$, and $\mathscr{F}_h^\partial$ denote respectively the set of faces, internal, and external faces in $\mathscr{T}_h$. Let $\mathscr{F}_h^{\text{in}}$ and $\mathscr{F}_h^{\text{out}}$ be the set of faces belonging respectively to $\partial\Omega_{\text{in}}$ and to $\partial\Omega_{\text{out}}$ such that $\mathscr{F}_h^\partial = \mathscr{F}_h^{\text{in}} \cup \mathscr{F}_h^{\text{out}}$. For a face $F \in \mathscr{F}_h$, let h_F denote its diameter and $\mathscr{T}_F$ the set of elements in $\mathscr{T}_h$ containing F. For an element $T \in \mathscr{T}_h$, let $\mathscr{F}_T$ denote the set of faces belonging to T. Let $\mathscr{S}_h$ be the set of mesh vertices. For a vertex $s \in \mathscr{S}_h$, let $\mathscr{T}_s$ denote the set of elements in $\mathscr{T}_h$ containing s.

For an integer $k \geqslant 1$, let $H^k(\mathscr{T}_h) = \{v \in L^2(\Omega);\ \forall T \in \mathscr{T}_h, v|_T \in H^k(T)\}$. We introduce the discrete gradient operator $\nabla_h : H^1(\mathscr{T}_h) \to [L^2(\Omega)]^d$ such that for all $v \in H^1(\mathscr{T}_h)$ and for all $T \in \mathscr{T}_h, (\nabla_h v)|_T = \nabla(v|_T)$. Let $F \in \mathscr{F}_h^i$; then, there are $T_1(F)$ and $T_2(F) \in \mathscr{T}_h$ such that $F = T_1(F) \cap T_2(F)$. Conventionally, choose n_F to be the unit normal vector to F pointing from $T_1(F)$ towards $T_2(F)$. For $v \in H^1(\mathscr{T}_h)$, define its jump across F as

$$[\![v]\!]_F = v|_{T_1(F)} - v|_{T_2(F)} \quad \text{a.e. on } F\,. \tag{2.4}$$

For $F \in \mathscr{F}_h^\partial$, define n_F to be the unit normal to F pointing towards the exterior of Ω and for $v \in H^1(\mathscr{T}_h)$, set $[\![v]\!]_F = v|_{T(F)}$ where $T(F)$ is the mesh element of which F is a face. A similar notation is used for the jumps of vector-valued functions, the jump being taken componentwise.

For a subset $R \subset \Omega$, $(\cdot,\cdot)_{0,R}$ denotes the $L^2(\Omega)$–scalar product, $\|\cdot\|_{0,R}$ the associated norm, $\|\cdot\|_{k,R}$ the $H^k(R)$–norm for $k \geqslant 1$, and $v_{\infty,R}$ the $L^\infty(R)$–norm or $[L^\infty(R)]^d$–norm of the function v.

Consider the Crouzeix-Raviart finite element space $P^1_{\text{nc}}(\mathscr{T}_h)$ defined as [13]

$$P^1_{\text{nc}}(\mathscr{T}_h) = \{\, v_h \in L^2(\Omega);\ \forall T \in \mathscr{T}_h, v_h|_T \in P^1(T);\ \forall F \in \mathscr{F}_h^i, \textstyle\int_F [\![v_h]\!]_F = 0\}\,,$$

where $P^1(T)$ denotes the vector space of polynomials on T with degree less than or equal to 1. For further purposes, we restate some well–known results. There exists a constant c such that for all h, for all $v_h \in P^1_{\text{nc}}(\mathscr{T}_h)$, for all $T \in \mathscr{T}_h$, and for all $F \subset \partial T$,

$$\|v_h\|_{1,T} \leqslant c h_T^{-1} \|v_h\|_{0,T}\,, \tag{2.5}$$

$$\|v_h\|_{0,F} \leqslant c h_F^{-\frac{1}{2}} \|v_h\|_{0,T}\,, \tag{2.6}$$

$$\|[\![v_h]\!]_F\|_{0,F} \leqslant c h_F \|[\![\nabla_h v_h]\!]_F\|_{0,F}\,. \tag{2.7}$$

Let $P^1_{\mathrm{c}}(\mathscr{T}_h) = P^1_{\mathrm{nc}}(\mathscr{T}_h) \cap H^1(\Omega)$ be the usual first-order conforming finite element space. Let $\mathscr{I}_{\mathrm{Os}} : P^1_{\mathrm{nc}}(\mathscr{T}_h) \to P^1_{\mathrm{c}}(\mathscr{T}_h)$ be the so-called Oswald interpolation operator [15, 19] defined such that

$$\forall v_h \in P^1_{\mathrm{nc}}(\mathscr{T}_h)\,,\ \forall s \in \mathscr{S}_h\,,\quad \mathscr{I}_{\mathrm{Os}} v_h(s) = \frac{1}{\sharp(\mathscr{T}_s)} \sum_{T \in \mathscr{T}_s} v_h|_T(s)\,, \tag{2.8}$$

where $\sharp(\mathscr{T}_s)$ denotes the cardinal number of $\mathscr{T}_s$. This operator is endowed with the following approximation property [1, 9, 23]: There exists a constant c, independent of h, such that for all $v_h \in P^1_{\mathrm{nc}}(\mathscr{T}_h)$ and for all $T \in \mathscr{T}_h$,

$$\|v_h - \mathscr{I}_{\mathrm{Os}} v_h\|_{0,T} + h_T \|\nabla(v_h - \mathscr{I}_{\mathrm{Os}} v_h)\|_{0,T} \leqslant c \sum_{F \in \mathscr{F}_T^{\mathrm{Os}}} h_F^{\frac{1}{2}} \|[\![v_h]\!]_F\|_{0,F}\,, \tag{2.9}$$

where $\mathscr{F}_T^{\mathrm{Os}}$ denotes all the interior faces in the mesh containing a vertex of T. Using (2.7) and (2.9) yields

$$\|v_h - \mathscr{I}_{\mathrm{Os}} v_h\|_{0,T} + h_T \|\nabla(v_h - \mathscr{I}_{\mathrm{Os}} v_h)\|_{0,T} \leqslant c \sum_{F \in \mathscr{F}_T^{\mathrm{Os}}} h_F^{\frac{3}{2}} \|[\![\nabla_h v_h]\!]_F\|_{0,F}\,. \tag{2.10}$$

2.3 *The discrete bilinear forms*

Set $V = H^2(\mathscr{T}_h) \cap H^1(\Omega)$ and $V(h) = V + P^1_{\mathrm{nc}}(\mathscr{T}_h)$. Introduce the bilinear form a_h defined on $V(h) \times V(h)$ by

$$\begin{aligned} a_h(v,w) = &\int_\Omega \varepsilon \nabla_h v \cdot \nabla_h w + \int_\Omega (\nu - \nabla\cdot\beta) v w - \int_\Omega v(\beta\cdot\nabla_h w) \\ &+ \sum_{F \in \mathscr{F}_h^i} \int_F \beta\cdot n_F [\![vw]\!]_F + \int_{\partial\Omega_{\mathrm{out}}} (\beta\cdot n) v w\,, \end{aligned} \tag{2.11}$$

and equip $V(h)$ with the norm

$$\|v\|_{\varepsilon\beta\sigma,\Omega} = \|\varepsilon^{\frac{1}{2}} \nabla_h v\|_{0,\Omega} + \|\sigma^{\frac{1}{2}} v\|_{0,\Omega} + \||\beta\cdot n|^{\frac{1}{2}} v\|_{0,\partial\Omega}\,. \tag{2.12}$$

The bilinear form a_h is not $\|\cdot\|_{\varepsilon\beta\sigma,\Omega}$-coercive on $V(h)$ owing to the presence of the jump terms in (2.11). To control these terms, consider the bilinear form j_h defined on $V(h) \times V(h)$ by

$$j_h(v,w) = \sum_{F \in \mathscr{F}_h^i} \int_F (\beta\cdot n_F)[\![v]\!]_F w^{\downarrow}\,, \tag{2.13}$$

where $w^{\downarrow}$ is the so-called downwind value of w defined as $w^{\downarrow} = w|_{T_2(F)}$ if $\beta\cdot n_F \geqslant 0$ and $w^{\downarrow} = w|_{T_1(F)}$ otherwise.

LEMMA 2.1 There exists a constant $c > 0$ such that for all $v \in V(h)$,

$$a_h(v,v) + j_h(v,v) \geqslant c \left(\|v\|^2_{\varepsilon\beta\sigma,\Omega} + \sum_{F \in \mathscr{F}_h^i} \||\beta\cdot n_F|^{\frac{1}{2}} [\![v]\!]_F\|^2_{0,F} \right). \tag{2.14}$$

Proof. Straightforward verification using integration by parts. □

Working with the bilinear form $a_h + j_h$ alone is not sufficient to control the advective derivative of the discrete solution. To this purpose, we introduce the bilinear form s_h on $V(h) \times V(h)$ such that

$$s_h(v,w) = \sum_{F\in\mathscr{F}_h^i} \int_F \gamma \frac{h_F^2}{\beta_{\infty,F}} [\![\beta\cdot\nabla_h v]\!]_F [\![\beta\cdot\nabla_h w]\!]_F \,, \tag{2.15}$$

where $\gamma > 0$ is independent of ε (the contribution of a face $F \in \mathscr{F}_h^i$ is conventionally set to zero if $\beta_{\infty,F} = 0$). This leads to the following discrete problem:

$$\begin{cases} \text{Seek } u_h \in P^1_{\mathrm{nc}}(\mathscr{T}_h) \text{ such that for all } v_h \in P^1_{\mathrm{nc}}(\mathscr{T}_h), \\ a_h(u_h,v_h) + j_h(u_h,v_h) + s_h(u_h,v_h) = (f,v_h)_{0,\Omega} - (g,v_h)_{0,\partial\Omega_{\mathrm{in}}} \,. \end{cases} \tag{2.16}$$

Lemma 2.1 implies that the bilinear form $(a_h + j_h + s_h)$ is $\|\cdot\|_{\varepsilon\beta\sigma,\Omega}$–coercive; hence, (2.16) is well–posed owing to the Lax–Milgram Lemma.

REMARK 2.1 A term similar to the bilinear form j_h is also added in [20] to control the jumps across mesh interfaces. As in the discrete problem (2.16) where the bilinear form j_h is introduced in addition to the bilinear form s_h, this term is introduced in addition to the streamline diffusion term stabilizing the nonconforming finite element approximation. To avoid this additional term, it is possible to work with the Q^1_{rot} finite element on rectangular meshes [29] or to consider a nonconforming finite element space satisfying the patch-test of order three [24]; however, the dimension of this space is twice as larger as the dimension of the Crouzeix–Raviart finite element space. Alternatively, one can penalize the jumps of all the gradient components instead of just those of the advective components and take $j_h = 0$; we refer to [9] for more details.

3. A priori error analysis

In this section we present the convergence analysis for the discrete problem (2.16). The main result is Theorem 3.1. Henceforth, c denotes a generic positive constant, independent of h and ε, whose value can change at each occurrence. Since the advection–diffusion problem has been non-dimensionalized so that the field β is of order unity, the dependency on β can be hidden in the constants c in the error estimates. The same is done for the function ν since we are not interested in the asymptotic of strong reaction regimes. Finally, without loss of generality, we assume that $h \leqslant 1$ and $\varepsilon \leqslant 1$.

The error analysis is performed in the spirit of the Second Strang Lemma by considering two norms on $V(h)$, namely,

$$\|w\|_{A,\Omega} = \|w\|_{\varepsilon\beta\sigma,\Omega} + \left(\sum_{F\in\mathscr{F}_h^i} \| |\beta\cdot n_F|^{\frac12} [\![w]\!]_F \|^2_{0,F} \right)^{\frac12} + s_h(w,w)^{\frac12} \,, \tag{3.1}$$

$$\|w\|_{h,\frac12} = \|w\|_{A,\Omega} + \left(\sum_{T\in\mathscr{T}_h} h_T^{-1} \|w\|^2_{0,T} + \|w\|^2_{0,\partial T} \right)^{\frac12} . \tag{3.2}$$

Let u be the unique solution to (2.2) and let u_h be the unique solution to (2.16).

LEMMA 3.1 (STABILITY) The bilinear form $(a_h + j_h + s_h)$ is $\|\cdot\|_{A,\Omega}$–coercive.

Proof. Direct consequence of Lemma 2.1. □

LEMMA 3.2 (CONTINUITY) Let Π_h be the L^2-orthogonal projection onto $P^1_{\mathrm{c}}(\mathscr{T}_h)$. Then, there is a constant c such that for all $w \in V$ and for all $w_h \in P^1_{\mathrm{nc}}(\mathscr{T}_h)$,

$$a_h(\Pi_h w - w, w_h) \leqslant c \, \|\Pi_h w - w\|_{h,\frac{1}{2}} \|w_h\|_{A,\Omega} . \tag{3.3}$$

Proof. Let $w \in V$ and set $v = \Pi_h w - w$. Let $w_h \in P^1_{\mathrm{nc}}(\mathscr{T}_h)$ and let us estimate each term in $a_h(v, w_h)$.
(1) It is clear that

$$\int_\Omega \varepsilon \nabla_h v \cdot \nabla_h w_h + \int_\Omega (v - \nabla\cdot\beta) v w_h \leqslant c\|v\|_{\varepsilon\beta\sigma,\Omega}\|w_h\|_{\varepsilon\beta\sigma,\Omega} \leqslant c\|v\|_{h,\frac{1}{2}}\|w_h\|_{A,\Omega} .$$

(2) Let us write $\int_\Omega v\beta\cdot\nabla_h w_h = \int_\Omega v(\beta - \beta^1_h)\cdot\nabla_h w_h + \int_\Omega v\beta^1_h\cdot\nabla_h w_h$ where β^1_h is the L^2–orthogonal projection of β onto $[P^1_{\mathrm{c}}(\mathscr{T}_h)]^d$. Since $\beta \in [\mathscr{C}^{0,\frac{1}{2}}(\Omega)]^d$ and owing to the inverse inequality (2.5),

$$\int_\Omega v(\beta - \beta^1_h)\cdot\nabla_h w_h \leqslant c \sum_{T\in\mathscr{T}_h} h_T^{-\frac{1}{2}} \|v\|_{0,T}\|w_h\|_{0,T} \leqslant c\|v\|_{h,\frac{1}{2}}\|w_h\|_{A,\Omega} .$$

Furthermore, by construction $(v, \mathscr{I}_{\mathrm{Os}}(\beta^1_h\cdot\nabla_h w_h))_{0,\Omega} = 0$; hence, using (2.5), (2.6), (2.10), the regularity of β, and the shape–regularity of the mesh family yields

$$\begin{aligned}
\int_\Omega v\beta^1_h\cdot\nabla_h w_h &= \int_\Omega v(\beta^1_h\cdot\nabla_h w_h - \mathscr{I}_{\mathrm{Os}}(\beta^1_h\cdot\nabla_h w_h)) \\
&\leqslant c \sum_{T\in\mathscr{T}_h} \|v\|_{0,T} \left(\sum_{F\in\mathscr{F}_T^{\mathrm{Os}}} h_F^{\frac{1}{2}} \|[\![\beta^1_h\cdot\nabla_h w_h]\!]_F\|_{0,F} \right) \\
&\leqslant c \sum_{T\in\mathscr{T}_h} \|v\|_{0,T} \left(\sum_{F\in\mathscr{F}_T^{\mathrm{Os}}} h_F^{\frac{1}{2}} \|[\![\beta\cdot\nabla_h w_h]\!]_F\|_{0,F} \right) \\
&\quad + c \sum_{T\in\mathscr{T}_h} \|v\|_{0,T} \left(\sum_{F\in\mathscr{F}_T^{\mathrm{Os}}} h_F^{\frac{1}{2}} \|[\![(\beta^1_h - \beta)\cdot\nabla_h w_h]\!]_F\|_{0,F} \right) \\
&\leqslant c\|v\|_{h,\frac{1}{2}} s_h(w_h, w_h)^{\frac{1}{2}} + c\|v\|_{h,\frac{1}{2}}\|w_h\|_{0,\Omega} \leqslant c\|v\|_{h,\frac{1}{2}}\|w_h\|_{A,\Omega} .
\end{aligned}$$

(3) Since $v \in H^1(\Omega)$, $\beta\cdot n_F [\![v w_h]\!]_F = \beta\cdot n_F v [\![w_h]\!]_F$. Hence,

$$\sum_{F\in\mathscr{F}_h^i} \int_F \beta\cdot n_F [\![v w_h]\!]_F \leqslant c \left(\sum_{T\in\mathscr{T}_h} \|v\|^2_{0,\partial T} \right)^{\frac{1}{2}} \left(\sum_{F\in\mathscr{F}_h^i} \||\beta\cdot n_F|^{\frac{1}{2}} [\![w_h]\!]_F\|^2_{0,F} \right)^{\frac{1}{2}} \leqslant c\|v\|_{h,\frac{1}{2}}\|w_h\|_{A,\Omega} .$$

Similarly,

$$\sum_{F\in\mathscr{F}_h^{\mathrm{out}}} \int_F (\beta\cdot n) v w_h \leqslant c\|v\|_{h,\frac{1}{2}}\|w_h\|_{A,\Omega} .$$

Collecting the above inequalities yields (3.3). □

LEMMA 3.3 (ERROR ESTIMATION) Assume that $u \in H^2(\Omega)$. Set

$$R_h(u) = \sup_{w_h \in P^1_{\mathrm{nc}}(\mathscr{T}_h)} \frac{a_h(u,w_h) - (f,w_h)_{0,\Omega} + (g,w_h)_{0,\partial\Omega_{\mathrm{in}}}}{\|w_h\|_{A,\Omega}}. \tag{3.4}$$

Then, there exists a constant $c > 0$ such that

$$c\,\|u-u_h\|_{A,\Omega} \leqslant \|u - \Pi_h u\|_{h,\frac{1}{2}} + R_h(u). \tag{3.5}$$

Proof. Since $\|u-\Pi_h u\|_{A,\Omega} \leqslant \|u-\Pi_h u\|_{h,\frac{1}{2}}$, the triangle inequality yields

$$\|u-u_h\|_{A,\Omega} \leqslant \|u-\Pi_h u\|_{h,\frac{1}{2}} + \|\Pi_h u - u_h\|_{A,\Omega}.$$

Set $w_h = \Pi_h u - u_h$ and observe that $w_h \in V(h)$. Then, the $\|\cdot\|_{A,\Omega}$-coercivity of $(a_h + j_h + s_h)$ on $V(h) \times V(h)$ yields

$$c\|\Pi_h u - u_h\|^2_{A,\Omega} \leqslant a_h(\Pi_h u - u_h, w_h) + j_h(\Pi_h u - u_h, w_h) + s_h(\Pi_h u - u_h, w_h).$$

Moreover, using the fact that $s_h(u,w_h) = j_h(\Pi_h u, w_h) = 0$ since $u \in H^2(\Omega)$ and $\Pi_h u \in H^1(\Omega)$ leads to

$$\begin{aligned}
&a_h(\Pi_h u - u_h, w_h) + j_h(\Pi_h u - u_h, w_h) + s_h(\Pi_h u - u_h, w_h)\\
&= a_h(\Pi_h u - u, w_h) + s_h(\Pi_h u - u, w_h) + a_h(u,w_h) - (f,w_h)_{0,\Omega} + (g,w_h)_{0,\partial\Omega_{\mathrm{in}}}.
\end{aligned}$$

Owing to Lemma 3.2,

$$a_h(\Pi_h u - u, w_h) \leqslant c\|\Pi_h u - u\|_{h,\frac{1}{2}} \|w_h\|_{A,\Omega}.$$

Furthermore,

$$s_h(\Pi_h u - u, w_h) \leqslant s_h(\Pi_h u - u, \Pi_h u - u)^{\frac{1}{2}} s_h(w_h,w_h)^{\frac{1}{2}} \leqslant \|\Pi_h u - u\|_{h,\frac{1}{2}} \|w_h\|_{A,\Omega}.$$

The conclusion is straightforward. $\square$

LEMMA 3.4 (CONSISTENCY) Assume that $u \in H^2(\Omega)$. Then, there exists a constant c such that

$$|R_h(u)| \leqslant c\varepsilon^{\frac{1}{2}} h \|u\|_{2,\Omega}. \tag{3.6}$$

Proof. Let $w_h \in P^1_{\mathrm{nc}}(\mathscr{T}_h)$. Observe that

$$a_h(u,w_h) - (f,w_h)_{0,\Omega} + (g,w_h)_{0,\partial\Omega_{\mathrm{in}}} = \sum_{F \in \mathscr{F}^i_h} \int_F \varepsilon \nabla u \cdot n_F [\![w_h]\!]_F.$$

Since $P^1_{\mathrm{nc}}(\mathscr{T}_h)$ satisfies the patch–test of order zero,

$$a_h(u,w_h) - (f,w_h)_{\Omega} + (g,w_h)_{0,\partial\Omega_{\mathrm{in}}} = \sum_{F \in \mathscr{F}^i_h} \int_F \varepsilon(\nabla u - \Pi^0_F(\nabla u)) \cdot n_F [\![w_h]\!]_F,$$

where $\Pi^0_F : [L^2(F)]^d \to [P^0(F)]^d$ denotes the L^2-orthogonal projection on $[P^0(F)]^d$. Using the standard Crouzeix–Raviart face interpolation inequality [13] leads to

$$|a_h(u,w_h) - (f,w_h)_{\Omega} + (g,w_h)_{\partial\Omega_{\mathrm{in}}}| \leqslant c\,\varepsilon h \|u\|_{2,\Omega} \|\nabla_h w_h\|_{0,\Omega}.$$

whence (3.6) is readily inferred. $\square$

THEOREM 3.1 (CONVERGENCE) Assume that $u \in H^2(\Omega)$. Then, there exists a constant c such that

$$\|u-u_h\|_{A,\Omega} \leqslant ch(\varepsilon^{\frac{1}{2}}+h^{\frac{1}{2}})\|u\|_{2,\Omega}\,. \tag{3.7}$$

Proof. Observe that Π_h satisfies the following approximation property (see, e.g., [6] for local approximation properties of Π_h): There exists a constant c such that for all $v \in H^2(\Omega)$,

$$\|v-\Pi_h v\|_{h,\frac{1}{2}} \leqslant ch(\varepsilon^{\frac{1}{2}}+h^{\frac{1}{2}})\|v\|_{2,\Omega}\,, \tag{3.8}$$

and use Lemmas 3.3 and 3.4. □

REMARK 3.1 The a priori error estimate (3.7) shows that when keeping the Péclet number ε fixed, the convergence order in the mesh-size for the error $\|u-u_h\|_{A,\Omega}$ is 1 in the diffusion-dominated regime and $\frac{3}{2}$ in the advection-dominated regime. This estimate is similar to those derived for stabilized schemes in the conforming setting; see, e.g., [7, 8, 11, 17].

REMARK 3.2 The above analysis shares some common features with that presented in [9]. The main differences is that we consider mixed Robin–Neumann boundary conditions instead of Dirichlet boundary conditions, that the advective field is in $[\mathscr{C}^{0,\frac{1}{2}}(\Omega)]^d$ instead of being piecewise affine, and that the stabilization is achieved by using the bilinear form (j_h+s_h) instead of penalizing the jumps of all the gradient components across interfaces.

4. A posteriori error analysis

In this section we present the residual a posteriori error analysis for the discrete problem (2.16). The main results are Theorem 4.1 which yields a global upper bound for the error and Theorem 4.2 which yields a local lower bound for the error.

Let f_h, β_h and ν_h be the L^2–orthogonal projection of f, β and ν onto the space of piecewise constant functions on $\mathscr{T}_h$ respectively, and let g_h be the L^2–orthogonal projection of g onto the space of piecewise constant functions on $\mathscr{F}_h$. Let u be the unique solution to (2.2) and let u_h be the unique solution to (2.16). As in the previous section, we assume without loss of generality that $h \leqslant 1$ and $\varepsilon \leqslant 1$. Furthermore, define

$$\alpha_S = \min(\varepsilon^{-\frac{1}{2}}h_S, 1)\,, \tag{4.1}$$

where S belongs to $\mathscr{T}_h$ or to $\mathscr{F}_h$, and observe that

$$h_S^{\frac{1}{2}} \leqslant \max(\varepsilon^{\frac{1}{2}}, \alpha_S)\,, \tag{4.2}$$

$$h_S \leqslant \alpha_S\,. \tag{4.3}$$

Indeed, (4.2) trivially holds if $h_S \leqslant \varepsilon$ whereas if $\varepsilon \leqslant h_S$, then $h_S^{\frac{1}{2}} \leqslant \alpha_S$. Furthermore, (4.3) directly results from the fact that $h \leqslant 1$ and $\varepsilon \leqslant 1$.

THEOREM 4.1 (GLOBAL UPPER BOUND) There is a constant $c > 0$ such that

$$c\|u-u_h\|_{\varepsilon\beta\sigma,\Omega} \leqslant \left(\sum_{T\in\mathscr{T}_h}[\eta_T(u_h)^2+\delta_T(u_h)^2]+\sum_{F\in\mathscr{F}_h^i}\eta_F(u_h)^2\right)^{\frac{1}{2}}\,, \tag{4.4}$$

where we have introduced for all $T \in \mathscr{T}_h$ the local data error indicators

$$\delta_T(u_h) = \alpha_T(\|f - f_h\|_{0,T} + \|(\beta - \beta_h)\cdot\nabla u_h\|_{0,T} + \|(\nu - \nu_h)u_h\|_{0,T}) + \sum_{F\in\mathscr{F}_T^{(2)}} \varepsilon^{-\frac{1}{4}}\alpha_F^{\frac{1}{2}}\|g - g_h + (\beta - \beta_h)\cdot n u_h\|_{0,F}\,, \tag{4.5}$$

as well as the local residual error indicators

$$\eta_T(u_h) = \alpha_T\|f_h - \beta_h\cdot\nabla u_h - \nu_h u_h\|_{0,T} + \sum_{F\in\mathscr{F}_T^{(1)}} \varepsilon^{-\frac{1}{4}}\alpha_F^{\frac{1}{2}}\|\varepsilon[\![\nabla_h u_h]\!]_F\|_{0,F} + \sum_{F\in\mathscr{F}_T^{(2)}} \varepsilon^{-\frac{1}{4}}\alpha_F^{\frac{1}{2}}\|g_h + \varepsilon\nabla u_h\cdot n - \beta_h\cdot n_F u_h\|_{0,F}\,, \tag{4.6}$$

$$\eta_F(u_h) = h_F^{\frac{1}{2}}\max(\alpha_F, \varepsilon^{\frac{1}{2}})\|[\![\nabla_h u_h]\!]_F\|_{0,F}\,, \tag{4.7}$$

where $\mathscr{F}_T^{(1)} = \mathscr{F}_T \cap \{\mathscr{F}_h^i \cup \mathscr{F}_h^{\text{out}}\}$ and $\mathscr{F}_T^{(2)} = \mathscr{F}_T \cap \mathscr{F}_h^{\text{in}}$.

Proof. Let $v_h = \mathscr{I}_{\text{Os}}u_h \in P_c^1(\mathscr{T}_h)$ and set $w = u - v_h \in H^1(\Omega)$. Then,

$$c\|u - v_h\|_{\varepsilon\beta\sigma,\Omega}^2 \leqslant a_h(u - u_h, w) + a_h(u_h - v_h, w)\,.$$

Furthermore, for all $w_h \in P_c^1(\mathscr{T}_h)$ the following equality holds

$$a_h(u - u_h, w) = a_h(u - u_h, w - w_h) + j_h(u_h, w_h) + s_h(u_h, w_h)\,.$$

Hence,

$$c\|u - v_h\|_{\varepsilon\beta\sigma,\Omega}^2 \leqslant a_h(u - u_h, w - w_h) + j_h(u_h, w_h) + s_h(u_h, w_h) + a_h(u_h - v_h, w)\,.$$

Let us estimate the four terms in the right-hand side of the above equation. Set $w_h = \mathscr{C}_h w \in P_c^1(\mathscr{T}_h)$ where $\mathscr{C}_h$ denotes the Clément interpolant of w.
(1) Estimate of $a_h(u - u_h, w - w_h)$. Using the techniques presented in [30] yields

$$a_h(u - u_h, w - w_h) \leqslant c\left(\sum_{T\in\mathscr{T}_h}[\eta_T(u_h)^2 + \delta_T(u_h)^2]\right)^{\frac{1}{2}}\|w\|_{\varepsilon\beta\sigma,\Omega}\,.$$

(2) Estimate of $j_h(u_h, w_h)$. Let $F \in \mathscr{F}_h^i$.
(2.a) Assume $\alpha_F = 1$. Owing to (2.6) and (2.7),

$$\int_F \beta\cdot n_F[\![u_h]\!]_F w_h^{\downarrow} \leqslant ch_F^{\frac{1}{2}}\|[\![\nabla_h u_h]\!]_F\|_{0,F}\|w_h\|_{0,\mathscr{T}_F} = ch_F^{\frac{1}{2}}\alpha_F\|[\![\nabla_h u_h]\!]_F\|_{0,F}\|w_h\|_{0,\mathscr{T}_F}\,.$$

(2.b) Assume $\alpha_F = \varepsilon^{-\frac{1}{2}}h_F$. Since $\int_F[\![u_h]\!]_F = 0$, it follows that

$$\int_F \beta\cdot n_F[\![u_h]\!]_F w_h^{\downarrow} = \int_F(\beta - \Pi_F^0\beta)\cdot n_F[\![u_h]\!]_F w_h^{\downarrow} + \int_F \Pi_F^0\beta\cdot n_F[\![u_h]\!]_F(w_h^{\downarrow} - \Pi_F^0 w_h^{\downarrow})\,,$$

where Π_F^0 is defined in the proof of Lemma 3.4. Since $\beta \in [\mathscr{C}^{0,\frac{1}{2}}(\Omega)]^d$, using (2.6), (2.7), and (4.2) yields

$$\begin{aligned}\int_F (\beta - \Pi_F^0 \beta)\cdot n_F [\![u_h]\!]_F w_h^{\downarrow} &\leqslant c h_F^{\frac{1}{2}} \|[\![u_h]\!]_F\|_{0,F} \|w_h^{\downarrow}\|_{0,F} \\ &\leqslant c h_F \|[\![\nabla_h u_h]\!]_F\|_{0,F} \|w_h\|_{0,\mathscr{T}_F} \\ &\leqslant c h_F^{\frac{1}{2}} \max(\varepsilon^{\frac{1}{2}}, \alpha_F) \|[\![\nabla_h u_h]\!]_F\|_{0,F} \|w_h\|_{0,\mathscr{T}_F}\,.\end{aligned}$$

Moreover,

$$\begin{aligned}\int_F \Pi_F^0 \beta\cdot n_F [\![u_h]\!]_F (w_h^{\downarrow} - \Pi_F^0 w_h^{\downarrow}) &\leqslant c \|[\![u_h]\!]_F\|_{0,F} \|w_h^{\downarrow} - \Pi_F^0 w_h^{\downarrow}\|_{0,F} \\ &\leqslant c \varepsilon^{-\frac{1}{2}} h_F^{\frac{3}{2}} \|[\![\nabla_h u_h]\!]_F\|_{0,F} \|\varepsilon^{\frac{1}{2}} \nabla w_h\|_{0,\mathscr{T}_F}\,.\end{aligned}$$

Collecting the above estimates yields

$$j_h(u_h, w_h) \leqslant c \left(\sum_{F\in\mathscr{F}_h^i} \eta_F(u_h)^2 \right)^{\frac{1}{2}} (\|\varepsilon^{\frac{1}{2}} \nabla w_h\|_{0,\Omega} + \|w_h\|_{0,\Omega})\,.$$

Finally, owing to the L^2- and H^1-stability of the Clément interpolation operator [5], it is inferred that

$$j_h(u_h, w_h) \leqslant c \left(\sum_{F\in\mathscr{F}_h^i} \eta_F(u_h)^2 \right)^{\frac{1}{2}} \|w\|_{\varepsilon\beta\sigma,\Omega}\,.$$

(3) Estimate of $s_h(u_h, w_h)$. Let $F \in \mathscr{F}_h^i$.
(3.a) Assume $\alpha_F = 1$. Owing to (2.5) and (2.6),

$$\int_F \frac{h_F^2}{\beta_{\infty,F}} [\![\beta\cdot\nabla_h u_h]\!]_F [\![\beta\cdot\nabla w_h]\!]_F \leqslant c h_F^{\frac{1}{2}} \|[\![\nabla_h u_h]\!]_F\|_{0,F} \|w_h\|_{0,\mathscr{T}_F}\,.$$

(3.b) Assume $\alpha_F = \varepsilon^{-\frac{1}{2}} h_F$. Then,

$$\begin{aligned}\int_F \frac{h_F^2}{\beta_{\infty,F}} [\![\beta\cdot\nabla_h u_h]\!]_F [\![\beta\cdot\nabla w_h]\!]_F &\leqslant c h_F^2 \|[\![\nabla_h u_h]\!]_F\|_{0,F} h_F^{-\frac{1}{2}} \|\nabla w_h\|_{0,\mathscr{T}_F} \\ &\leqslant c \varepsilon^{-\frac{1}{2}} h_F^{\frac{3}{2}} \|[\![\nabla_h u_h]\!]_F\|_{0,F} \|\varepsilon^{\frac{1}{2}} \nabla w_h\|_{0,\mathscr{T}_F}\,.\end{aligned}$$

Collecting the above estimates and using the L^2- and H^1-stability of the Clément interpolation operator yields

$$s_h(u_h, w_h) \leqslant c \left(\sum_{F\in\mathscr{F}_h^i} \eta_F(u_h)^2 \right)^{\frac{1}{2}} \|w\|_{\varepsilon\beta\sigma,\Omega}\,.$$

(4) Estimate of $a_h(u_h - v_h, w)$.
(4.a) Estimate of the diffusive term. Let $T \in \mathscr{T}_h$. Use (2.10) to infer

$$\int_T \varepsilon \nabla_h (u_h - v_h) \cdot \nabla w \leqslant c \left(\sum_{F \in \mathscr{F}_T^{\mathrm{Os}}} \varepsilon^{\frac{1}{2}} h_F^{\frac{1}{2}} \| [\![\nabla_h u_h]\!]_F \|_{0,F} \right) \| \varepsilon^{\frac{1}{2}} \nabla w \|_{0,T} .$$

(4.b) Estimate of the reactive term. Let $T \in \mathscr{T}_h$. Use (2.10), and (4.3) to infer

$$\begin{aligned} \int_T \nu (u_h - v_h) w &\leqslant c \left(\sum_{F \in \mathscr{F}_T^{\mathrm{Os}}} h_F^{\frac{3}{2}} \| [\![\nabla_h u_h]\!]_F \|_{0,F} \right) \| w \|_{0,T} \\ &\leqslant c \left(\sum_{F \in \mathscr{F}_T^{\mathrm{Os}}} h_F^{\frac{1}{2}} \alpha_F \| [\![\nabla_h u_h]\!]_F \|_{0,F} \right) \| w \|_{0,T} . \end{aligned}$$

(4.c) Estimate of the advective and face terms. Observe that these terms can be written in the form $\sum_{T \in \mathscr{T}_h} \Xi_T$ with

$$\Xi_T = \int_T \beta \cdot \nabla_h (u_h - v_h) w - \int_{\partial T \cap \partial \Omega_{\mathrm{in}}} (\beta \cdot n_T)(u_h - v_h) w \tag{4.8}$$

$$= - \int_T (u_h - v_h) \beta \cdot \nabla w - \int_T (\nabla \cdot \beta)(u_h - v_h) w + \int_{\partial T^*} (\beta \cdot n_T)(u_h - v_h) w , \tag{4.9}$$

where n_T denotes the outward normal to T and $\partial T^* = \partial T \setminus (\partial T \cap \partial \Omega_{\mathrm{in}})$. If $\alpha_T = 1$, consider (4.8) and use (2.10) to infer

$$\begin{aligned} \int_T \beta \cdot \nabla_h (u_h - v_h) w &\leqslant c \left(\sum_{F \in \mathscr{F}_T^{\mathrm{Os}}} h_F^{\frac{1}{2}} \| [\![\nabla_h u_h]\!]_F \|_{0,F} \right) \| w \|_{0,T} \\ &\leqslant c \left(\sum_{F \in \mathscr{F}_T^{\mathrm{Os}}} h_F^{\frac{1}{2}} \alpha_F \| [\![\nabla_h u_h]\!]_F \|_{0,F} \right) \| w \|_{0,T} , \end{aligned}$$

owing to the shape–regularity of the mesh family. Moreover, if T has a face on $\partial \Omega_{\mathrm{in}}$, say F_T, using (2.6) and (2.10) leads to

$$\int_{\partial T \cap \partial \Omega_{\mathrm{in}}} (\beta \cdot n_T)(u_h - v_h) w \leqslant c \left(\sum_{F \in \mathscr{F}_T^{\mathrm{Os}}} h_F \| [\![\nabla_h u_h]\!]_F \|_{0,F} \right) \| |\beta \cdot n|^{\frac{1}{2}} w \|_{0,F_T} .$$

Hence,

$$|\Xi_T| \leqslant c \left(\sum_{F \in \mathscr{F}_T^{\mathrm{Os}}} h_F^{\frac{1}{2}} \alpha_F \| [\![\nabla_h u_h]\!]_F \|_{0,F} \right) \left(\| w \|_{0,T} + \| |\beta \cdot n|^{\frac{1}{2}} w \|_{0,F_T} \right) .$$

If $\alpha_T = \varepsilon^{-\frac{1}{2}} h_T$, consider (4.9). Owing to (2.10),

$$\int_T (u_h - v_h) \beta \cdot \nabla w \leqslant c \left(\sum_{F \in \mathscr{F}_T^{\mathrm{Os}}} \varepsilon^{-\frac{1}{2}} h_F^{\frac{3}{2}} \| [\![\nabla_h u_h]\!]_F \|_{0,F} \right) \| \varepsilon^{\frac{1}{2}} \nabla w \|_{0,T} .$$

Furthermore, the term $\int_T (\nabla\cdot\beta)(u_h - v_h)w$ is estimated as in step (4.b). Let $F \subset \partial T^*$. Assume first that $F \in \mathscr{F}_h^i$. Observe that

$$\int_F \beta\cdot n_F [\![u_h - v_h]\!]_F w = \int_F (\beta - \Pi_F^0\beta)\cdot n_F [\![u_h]\!]_F w + \int_F \Pi_h^0\beta\cdot n_F [\![u_h]\!]_F (w - \Pi_F^0 w),$$

since $v_h \in P_c^1(\mathscr{T}_h)$. Proceeding as above yields

$$\begin{aligned}\int_F \beta\cdot n_F [\![u_h - v_h]\!]_F w &\leqslant c h_F^{\frac{3}{2}} \|[\![\nabla_h u_h]\!]_F\|_{0,F}\|w\|_{0,F} + c\varepsilon^{-\frac{1}{2}} h_F^{\frac{3}{2}} \|[\![\nabla_h u_h]\!]_F\|_{0,F}\|\varepsilon^{\frac{1}{2}}\nabla w\|_{0,\mathscr{T}_F} \\ &\leqslant c\varepsilon^{-\frac{1}{2}} h_F^{\frac{3}{2}} \|[\![\nabla_h u_h]\!]_F\|_{0,F}(\|\varepsilon^{\frac{1}{2}}\nabla w\|_{0,\mathscr{T}_F} + \|w\|_{0,\mathscr{T}_F}),\end{aligned}$$

where we have used the trace inequality $\|w\|_{0,F} \leqslant c\|w\|_{0,\mathscr{T}_F}^{\frac{1}{2}} \|w\|_{1,\mathscr{T}_F}^{\frac{1}{2}}$ valid for all $w \in H^1(\Omega)$. Furthermore, if $F \subset \partial\Omega_{\text{out}}$, using (2.6), (2.10), and (4.2) yields

$$\begin{aligned}\int_F \beta\cdot n(u_h - v_h)w &\leqslant c\|u_h - v_h\|_{0,F}\||\beta\cdot n|^{\frac{1}{2}} w\|_{0,F} \\ &\leqslant c\left(\sum_{F' \in \mathscr{F}_{T(F)}^{\text{Os}}} h_{F'}\|[\![\nabla_h u_h]\!]_{F'}\|_{0,F'}\right)\||\beta\cdot n|^{\frac{1}{2}} w\|_{0,F} \\ &\leqslant c\left(\sum_{F' \in \mathscr{F}_{T(F)}^{\text{Os}}} h_{F'}^{\frac{1}{2}} \max(\varepsilon^{\frac{1}{2}}, \alpha_{F'})\|[\![\nabla_h u_h]\!]_{F'}\|_{0,F'}\right)\||\beta\cdot n|^{\frac{1}{2}} w\|_{0,F}.\end{aligned}$$

Collecting the above inequalities yields

$$a_h(u_h - v_h, w) \leqslant c\left(\sum_{F \in \mathscr{F}_h^i} \eta_F(u_h)^2\right)^{\frac{1}{2}} \|w\|_{\varepsilon\beta\sigma,\Omega}.$$

(5) Owing to steps (1)–(4) above, it is inferred that

$$c\|u - v_h\|_{\varepsilon\beta\sigma,\Omega} \leqslant \left(\sum_{T \in \mathscr{T}_h} [\eta_T(u_h)^2 + \delta_T(u_h)^2] + \sum_{F \in \mathscr{F}_h^i} \eta_F(u_h)^2\right)^{\frac{1}{2}}.$$

Using (2.10), (4.3), and the shape–regularity of the mesh family yields

$$\begin{aligned}\|\varepsilon^{\frac{1}{2}}\nabla_h(u_h - v_h)\|_{0,\Omega} + \|u_h - v_h\|_{0,\Omega} &\leqslant c\left(\sum_{F \in \mathscr{F}_h^i} (\varepsilon h_F + h_F^3)\|[\![\nabla_h u_h]\!]_F\|_{0,F}^2\right)^{\frac{1}{2}} \\ &\leqslant c\left(\sum_{F \in \mathscr{F}_h^i} (\varepsilon h_F + h_F\alpha_F^2)\|[\![\nabla_h u_h]\!]_F\|_{0,F}^2\right)^{\frac{1}{2}}.\end{aligned}$$

Moreover, for all $F \in \mathscr{F}_h^\partial$, owing to (2.6), (2.10), and (4.2),

$$\||\beta\cdot n|^{\frac{1}{2}}(u_h - v_h)\|_{0,F} \leqslant c \sum_{F' \in \mathscr{F}^{\mathrm{Os}}_{T(F)}} h_F \|[\![\nabla_h u_h]\!]_{F'}\|_{0,F'}$$
$$\leqslant c \sum_{F' \in \mathscr{F}^{\mathrm{Os}}_{T(F)}} h_{F'}^{\frac{1}{2}} \max(\varepsilon^{\frac{1}{2}}, \alpha_{F'}) \|[\![\nabla_h u_h]\!]_{F'}\|_{0,F'}\,.$$

Collecting the above estimates yields

$$\|u_h - v_h\|_{\varepsilon\beta\sigma,\Omega} \leqslant c \left(\sum_{F \in \mathscr{F}_h^i} \eta_F(u_h)^2\right)^{\frac{1}{2}}.$$

Use the triangle inequality to conclude. □

Let $T \in \mathscr{T}_h$ and let Δ_T denote the union of elements of $\mathscr{T}_h$ sharing at least a vertex with T. For all $w \in V(h)$, localize $\|w\|_{\varepsilon\beta\sigma,\Omega}$ as follows:

$$\|w\|_{\varepsilon\beta\sigma,\Delta_T} = \|\varepsilon^{\frac{1}{2}}\nabla_h v\|_{0,\Delta_T} + \|\sigma^{\frac{1}{2}} w\|_{0,\Delta_T} + \left(\sum_{F \in \mathscr{F}_{\Delta_T} \cap \mathscr{F}_h^\partial} \||\beta\cdot n|^{\frac{1}{2}} w\|_{0,F}^2\right)^{\frac{1}{2}},$$

where $\mathscr{F}_{\Delta_T}$ denotes the set of faces of the elements in Δ_T.

THEOREM 4.2 (LOCAL LOWER BOUND) There is a constant c such that for all $T \in \mathscr{T}_h$,

$$\eta_T(u_h) \leqslant c\Big((1+\varepsilon^{-\frac{1}{2}}\alpha_T)\|u - u_h\|_{\varepsilon\beta\sigma,\Delta_T} + \delta_{\Delta_T}(u_h)\Big), \tag{4.10}$$

where $\delta_{\Delta_T}(u_h) = \sum_{T' \in \Delta_T} \delta_{T'}(u_h)$, and for all $F \in \mathscr{F}_h^i$,

$$\eta_F(u_h) \leqslant c\varepsilon^{-\frac{1}{2}}\alpha_F(\|u - u_h\|_{\varepsilon\beta\sigma,\mathscr{T}_F} + \inf_{z_h \in [P_c^1(\mathscr{T}_h)]^d} \|\varepsilon^{\frac{1}{2}}(\nabla u - z_h)\|_{0,\mathscr{T}_F}). \tag{4.11}$$

Proof. The upper bound (4.10) is obtained by using the techniques presented in [30]. To prove (4.11), let $z_h \in [P_c^1(\mathscr{T}_h)]^d$ and let $F \in \mathscr{F}_h^i$. Observe that $[\![\nabla_h u_h]\!]_F = [\![\nabla_h u_h - z_h]\!]_F$. Then, using (2.6) and the triangle inequality yields

$$\|[\![\nabla_h u_h]\!]_F\|_{0,F} \leqslant c h_F^{-\frac{1}{2}} \|\nabla_h u_h - z_h\|_{0,\mathscr{T}_F} \leqslant c h_F^{-\frac{1}{2}} (\|\nabla u - \nabla_h u_h\|_{0,\mathscr{T}_F} + \|\nabla u - z_h\|_{0,\mathscr{T}_F}).$$

The conclusion is straightforward. □

REMARK 4.1 Keeping ε fixed, $\delta_{\Delta_T}(u_h)$ and $\inf_{z_h \in [P_c^1(\mathscr{T}_h)]^d} \|\varepsilon^{\frac{1}{2}}(\nabla u - z_h)\|_{0,\mathscr{T}_F}$ converge at least with the same order as $\|u - u_h\|_{\varepsilon\beta\sigma,\Delta_T}$ and $\|u - u_h\|_{\varepsilon\beta\sigma,\mathscr{T}_F}$, respectively.

5. Numerical results

In this section two test cases are presented to illustrate the above theoretical results. In both cases, $\Omega = (0,1) \times (0,1)$ and we consider a shape–regular family of unstructured triangulations of Ω with mesh-size $h_i = h_0 \times 2^{-i}$ with $h_0 = 0.1$ and $i \in \{0, \cdots, 4\}$. The diffusion coefficient ε takes the values $\{10^{-2}, 10^{-4}, 10^{-6}\}$, the reaction coefficient ν is set to 1, and the parameter γ in (2.15) is set to 0.005.

5.1 *Test case 1*

Let $\beta = (1,0)^T$ and choose the data f and g such that the exact solution of (2.1) is

$$u(x,y) = \frac{1}{2}\left(1-\tanh(\frac{0.5-x}{a_w})\right), \tag{5.1}$$

with internal layer width $a_w = 0.05$.

Table 5.1 presents the convergence results for the error $\|u-u_h\|_{A,\Omega}$; $N_{\rm fa}$ denotes the number of degrees of freedom (i.e., the number of mesh faces) and ω denotes the convergence order with respect to the mesh–size. In the advection–dominated regime ($\varepsilon = 10^{-4}$ and $\varepsilon = 10^{-6}$), the error decreases as $h^{\frac{3}{2}}$. In the intermediate regime ($\varepsilon = 10^{-2}$), the convergence order changes from $\frac{3}{2}$ to 1 as the mesh is refined. These results are in agreement with the estimate derived in Theorem 3.1.

Mesh		$\varepsilon = 10^{-2}$		$\varepsilon = 10^{-4}$		$\varepsilon = 10^{-6}$	
i	$N_{\rm fa}$	$\|u-u_h\|_{A,\Omega}$	ω	$\|u-u_h\|_{A,\Omega}$	ω	$\|u-u_h\|_{A,\Omega}$	ω
0	374	1.04	-	1.01	-	$9.99\,10^{-1}$	-
1	1441	$4.05\,10^{-1}$	1.40	$3.76\,10^{-1}$	1.46	$3.71\,10^{-1}$	1.47
2	5621	$1.53\,10^{-1}$	1.43	$1.29\,10^{-1}$	1.57	$1.26\,10^{-1}$	1.59
3	22330	$6.02\,10^{-2}$	1.35	$4.52\,10^{-2}$	1.52	$4.40\,10^{-2}$	1.52
4	88961	$2.45\,10^{-2}$	1.30	$1.61\,10^{-2}$	1.49	$1.55\,10^{-2}$	1.51

Table 1. Numerical errors and convergence orders for the different values of ε

Let $\eta_1(u_h)$, $\eta_2(u_h)$, and $\delta(u_h)$ be the global error estimators defined as

$$\eta_1(u_h) = \left(\sum_{T\in\mathscr{T}_h}\eta_T(u_h)^2\right)^{\frac{1}{2}}, \quad \eta_2(u_h) = \left(\sum_{F\in\mathscr{F}_h^i}\eta_F(u_h)^2\right)^{\frac{1}{2}}, \quad \delta(u_h) = \left(\sum_{T\in\mathscr{T}_h}\delta_T(u_h)^2\right)^{\frac{1}{2}}, \tag{5.2}$$

where the local error indicators $\eta_T(u_h)$, $\eta_F(u_h)$, and $\delta_T(u_h)$ are defined in (4.6), (4.7), and (4.5), respectively. The asymptotic behavior of the global error estimators with respect to the number of degrees of freedom is presented in Figure 1. The error $u-u_h$ measured in the norm $\|\cdot\|_{\varepsilon\beta\sigma,\Omega}$ is also presented in Figure 1. For the three values of the diffusion coefficient, the error estimator $\eta_1(u_h)$ has approximately the same convergence order as the error. In the diffusion–dominated regime, the error estimators $\eta_2(u_h)$ and $\delta(u_h)$ exhibit a super-convergent behavior. In the advection–dominated regime, the convergence order of $\|u-u_h\|_{\varepsilon\beta\sigma,\Omega}$ and $\eta_1(u_h)$, is close to $\frac{3}{2}$ while the convergence order of $\delta(u_h)$ and $\eta_2(u_h)$ is close to 1. The efficiency index evaluated as

$$I = \frac{\eta_1(u_h)+\eta_2(u_h)+\delta(u_h)}{\|u-u_h\|_{\varepsilon\beta\sigma,\Omega}}, \tag{5.3}$$

is in the range 9.7 to 33.7 for $\varepsilon = 10^{-2}$, 81.1 to 347.4 for $\varepsilon = 10^{-4}$, and 95.8 to 1571.4 for $\varepsilon = 10^{-6}$. The increase of the efficiency index is roughly proportional to $\varepsilon^{-\frac{1}{2}}$, in agreement with the theoretical results of Section 4.

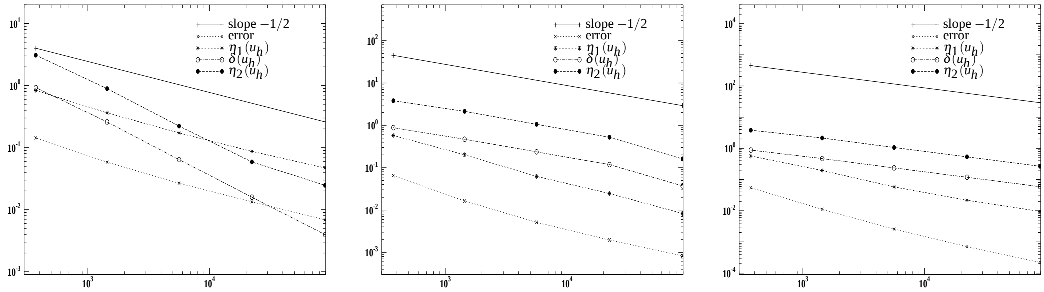

FIG. 1. Exact error and global error estimators against degrees of freedom. Left: $\varepsilon = 10^{-2}$; center: $\varepsilon = 10^{-4}$; right: $\varepsilon = 10^{-6}$

5.2 *Test case 2*

The goal of this section is to present a test case for which the mesh is adaptively refined based on the a posteriori error analysis. Let Γ_1 denote the lower horizontal edge of Ω and let Γ_2 denote its left vertical edge. Set $\beta = (2,1)^T$, $f = 0$, and g such that $g = 1$ on Γ_1 and $g = 0$ on Γ_2. Owing to the discontinuity of the Robin boundary condition, the solution exhibits an inner layer located along the line $\{x = 2y\}$. Similar results are obtained if the data g ensures a sharp but continuous transition from 0 to 1 at the origin. Figure 2 presents the contour lines of the computed solution for the different values of ε.

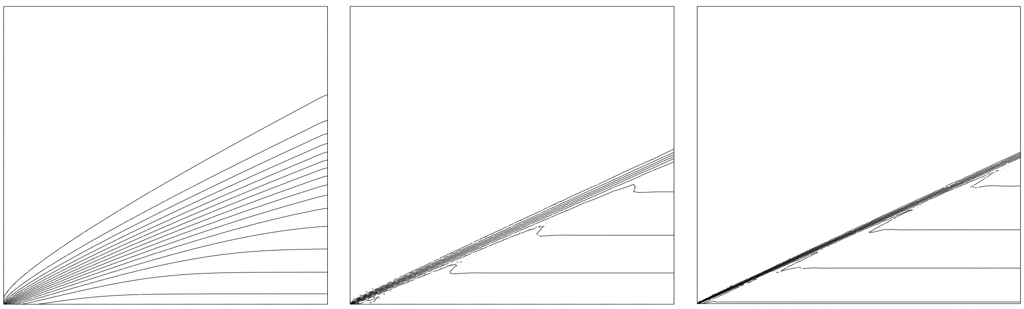

FIG. 2. Contour lines of the solution for test case 2. Left: $\varepsilon = 10^{-2}$; center: $\varepsilon = 10^{-4}$; right: $\varepsilon = 10^{-6}$

To refine the mesh adaptively using the local error indicator $\eta_T(u_h)$ (evaluated by setting $\alpha_T = \varepsilon^{-\frac{1}{2}} h_T$), the following algorithm is considered:

(i) Construct an initial mesh $\mathscr{T}_h^0$. Set $i := 0$.

(ii) Compute the approximate solution u_h^i on $\mathscr{T}_h^i$ and compute the local error indicators $\eta_{T_i}(u_h^i)$ for all $T_i \in \mathscr{T}_h^i$.

(iii) If the global error is sufficiently small, stop; otherwise, compute the quantities

$$\hat{h}_{T_i} = l(\eta_{T_i}(u_h^i))h_{T_i},$$

where $l(\eta_{T_i}(u_h^i)) = \frac{1}{2}$ if $\eta_{T_i}(u_h^i) \leqslant S_i$ and $l(\eta_{T_i}(u_h^i)) = 1$ otherwise. The threshold S_i is evaluated as $S_i = \frac{1}{2nt_i} \sum_{T_i \in \mathscr{T}_h^i} \eta_{T_i}(u_h^i)$ where nt_i denotes the number of triangles in the mesh $\mathscr{T}_h^i$.

(iv) Using the quantities $\hat{h}_{T_i}$ to construct a new mesh $\mathscr{T}_h^{i+1}$. Go to step (ii).

Figure 3 presents the adaptively refined meshes after five iterations of the above algorithm. For the three values of the diffusion coefficient, the mesh is refined at the origin. In the diffusion–dominated regime the mesh is refined around the inner layer and at the outflow layer. In the advection–dominated regime the meshes are refined along the inner layer. The refined zone becomes smaller as the diffusion coefficient ε takes smaller values, indicating that the local error indicator $\eta_T(u_h)$ alone can detect the inner layer.

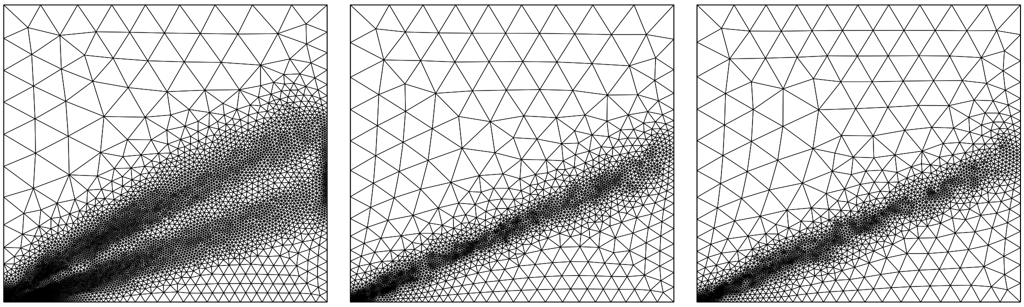

FIG. 3. Adaptive meshes after five iterations. Left: $\varepsilon = 10^{-2}$ and $N_{\mathrm{fa}} = 18157$; center: $\varepsilon = 10^{-4}$ and $N_{\mathrm{fa}} = 7145$; right: $\varepsilon = 10^{-6}$ and $N_{\mathrm{fa}} = 6934$

Figure 4 presents the asymptotic behavior of the three global error estimators as a function of the number of degrees of freedom in the adaptively refined meshes. The local error indicator $\eta_S(u_h)$, where S denotes either a triangle T or a face F, is evaluated by setting $\alpha_S = \varepsilon^{-\frac{1}{2}} h_S$. In the diffusion–dominated regime the convergence order of $\eta_1(u_h)$ and $\eta_2(u_h)$ is greater than 1, and $\eta_2(u_h)$ converges faster than $\eta_1(u_h)$. In the advection–dominated regime $\eta_1(u_h)$ and $\eta_2(u_h)$ exhibit the same convergence order except on the coarser meshes where $\eta_1(u_h)$ super-converges.

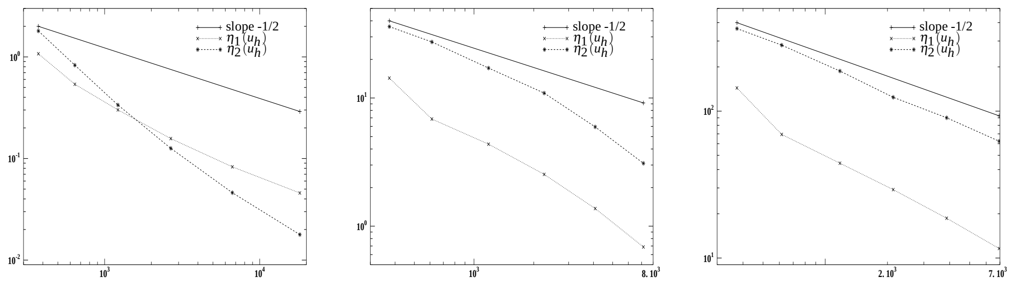

FIG. 4. Global error estimators against degrees of freedom. Left: $\varepsilon = 10^{-2}$; center: $\varepsilon = 10^{-4}$; right: $\varepsilon = 10^{-6}$

6. Conclusions

In this paper we have presented an a priori and an a posteriori error analysis for a nonconforming finite element method to approximate advection–diffusion equations. The method is stabilized by penalizing the jumps of the solution and those of its advective derivative across mesh interfaces. The a priori error analysis leads to (quasi-)optimal error estimates in the mesh-size in the sense that keeping the Péclet number fixed the estimates are sub–optimal of order $\frac{1}{2}$ in the L^2–norm and optimal in the broken graph norm for quasi–uniform meshes. These estimates are similar to those obtained with other methods. A drawback of the present scheme is the presence of face–oriented bilinear forms leading to a stencil larger than that resulting from the use of the Crouzeix–Raviart finite element. When solving nonlinear problems, e.g., the Navier–Stokes equations, these terms can be treated in the framework of a nonlinear iterative solver thus avoiding the widening of the stencil; see, e.g., [26]. Finally, the a posteriori error analysis of the present scheme leads to semi-robust error indicators, meaning that the factor between the lower and upper bounds scales as the square root of the Péclet number. The present analysis provides the first semi-robust a posteriori error estimator in a nonconforming setting and can be viewed as a first step towards establishing robust a posteriori error estimators in this setting.

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