DISCONTINUOUS GALERKIN METHODS FOR FRIEDRICHS' SYSTEMS. PART II. SECOND-ORDER ELLIPTIC PDE'S

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Abstract. This paper is the second part of a work attempting to give a unified analysis of Discontinuous Galerkin methods. The setting under scrutiny is that of Friedrichs' systems endowed with a particular 2×2 structure in which some of the unknowns can be eliminated to yield a system of second-order elliptic-like PDE's for the remaining unknowns. For such systems, a general Discontinuous Galerkin method is proposed and analyzed. The key feature is that the unknowns that can be eliminated at the continuous level can also be eliminated at the discrete level by solving local problems. All the design constraints on the boundary operators that weakly enforce boundary conditions and on the interface operators that penalize interface jumps are fully stated. Examples are given for advection–diffusion–reaction, linear elasticity, and a simplified version of the magneto-hydrodynamics equations. Comparisons with well-known Discontinuous Galerkin approximations for the Poisson equation are presented.

Key words. Friedrichs' systems, Finite Elements, Partial Differential Equations, Discontinuous Galerkin Method

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1. Introduction. Friedrichs's systems [8] are systems of first-order PDE's endowed with a symmetry and a positivity property. Such systems embrace both elliptic and hyperbolic PDE's, i.e., they include advection–reaction, advection–diffusion–reaction, linear elasticity, and Maxwell's equations in the elliptic regime, to cite a few examples. The analysis of this class of problems and its approximation by means of Discontinuous Galerkin (DG) methods has been initiated by Lesaint and Raviart [10, 11] and Johnson et al. [9]. A thorough systematic analysis generalizing [10, 11, 9] has been undertaken in the first part of this work [6].

In this second part, we specialize the setting to Friedrichs' systems that can be set into a 2×2 block structure such that (i) the dependent variable z can be partitioned into the form $z = (z^\sigma, z^u)$, and (ii) the σ-component, z^σ, can be eliminated to yield a system of second-order PDE's for the u-component, z^u, which is of elliptic type. To efficiently approximate the above Friedrichs' systems using DG methods, it is desirable to reproduce at the discrete level the possibility to eliminate the σ-component of the discrete unknown *locally* on each mesh element. This feature induces a non-trivial modification of the analysis presented in [6] that constitutes the scope of the present work. In particular, the design of boundary and interface operators has to be revised. The analysis presented herein shows that to recover stability while allowing for the local elimination in question requires an enhanced penalty on the boundary conditions and the interface jumps of the discrete u-component.

This paper is organized as follows. Section 2 briefly restates the main theoretical results of [6] on the well–posedness of Friedrichs' systems and introduces the above 2×2 block structure. Section 3 presents three important examples of Friedrichs' systems with 2×2 block structure, namely advection-diffusion-reaction equations written in mixed form, linear elasticity equations written in the stress–pressure–displacement form, and a simplified form of the magneto-hydrodynamics (MHD) equations. Sec-

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tion 4 formulates a general DG method for Friedrichs' systems with 2×2 structure and describes the technique to locally eliminate the σ-component of the discrete solution. The convergence analysis is reported in Section 5. All the design assumptions on the boundary operators which weakly enforce boundary conditions and on the interface operators which penalize interface jumps are stated. The key results are Theorem 5.8 which contains the main estimate for the σ- and u-component of the approximation error, and Theorem 5.14 which contains an improved estimate for the u-component of the error in the L^2-norm obtained using a duality argument. Finally, Section 6 applies the DG method to the PDE systems presented in §3; in particular, the link with the unified analysis of Arnold et al. [1] for the Poisson equation is explicated to illustrate the fact that various DG methods presented in the literature, e.g., the Local Discontinuous Galerkin (LDG) method of Cockburn and Shu [4], the Interior Penalty (IP) method of Douglas and Dupont [5], the method of Brezzi *et al.* [3], and the method of Bassi and Rebay [2], fit into the present framework.

2. Friedrichs' systems in block structure.

2.1. Main results on Friedrich's systems in general form. Let Ω be a bounded, open, and connected Lipschitz domain in $\mathbb{R}^d$. Let m be a positive integer. Let $\mathcal{K}$ and $\{\mathcal{A}^k\}_{1\le k\le d}$ be $(d+1)$ functions on Ω with values in $\mathbb{R}^{m,m}$ such that

$$\mathcal{K} \in [L^\infty(\Omega)]^{m,m}, \tag{A1}$$

$$\forall k\in\{1,\dots,d\},\ \mathcal{A}^k \in [L^\infty(\Omega)]^{m,m} \quad \text{and} \quad \sum_{k=1}^d \partial_k\mathcal{A}^k \in [L^\infty(\Omega)]^{m,m}, \tag{A2}$$

$$\forall k\in\{1,\dots,d\},\ \mathcal{A}^k = (\mathcal{A}^k)^t \ \text{ a.e. in } \Omega, \tag{A3}$$

$$\exists \mu_0>0, \quad \mathcal{K}+\mathcal{K}^t - \sum_{k=1}^d \partial_k\mathcal{A}^k \ge 2\mu_0\mathcal{I}_m \ \text{ a.e. on } \Omega, \tag{A4}$$

where $\mathcal{I}_m$ is the identity matrix in $\mathbb{R}^{m,m}$. Set $L=[L^2(\Omega)]^m$ and let $\mathfrak{D}(\Omega)$ be the space of $\mathfrak{C}^\infty$ functions that are compactly supported in Ω. A function z in L is said to have an A-weak derivative in L if the linear form

$$[\mathfrak{D}(\Omega)]^m \ni \phi \longmapsto -\int_\Omega \sum_{k=1}^d z^t\partial_k(\mathcal{A}^k\phi) \in \mathbb{R}, \tag{2.1}$$

is bounded on L. In this case, the function in L that can be associated with the above linear form by means of the Riesz representation theorem is denoted by Az. Clearly, if z is smooth, e.g., $z\in[\mathfrak{C}^1(\overline{\Omega})]^m$, $Az=\sum_{k=1}^d \mathcal{A}^k\partial_k z$. Define the so-called graph space $W=\{z\in L;\ Az\in L\}$ equipped with the graph norm $\|z\|_W = \|Az\|_L + \|z\|_L$. The space W is endowed with a Hilbert structure when equipped with the scalar product $(z,y)_L + (Az,Ay)_L$.

Define the operator $K\in\mathcal{L}(L;L)$ by $K: L\ni z\longmapsto \mathcal{K}z\in L$ and the operator $T\in\mathcal{L}(W;L)$ by $T=A+K$, i.e.,

$$Tz = \mathcal{K}z + \sum_{k=1}^d \mathcal{A}^k\partial_k z. \tag{2.2}$$

Let $K^*\in\mathcal{L}(L;L)$ be the adjoint operator of K, i.e., $K^*: L\ni z\longmapsto \mathcal{K}^t z\in L$. Let

$\tilde{T} \in \mathcal{L}(W; L)$ be the formal adjoint of T,

$$\tilde{T}z = \mathcal{K}^t z - \sum_{k=1}^{d} \partial_k(\mathcal{A}^k z). \tag{2.3}$$

In this definition $\sum_{k=1}^{d} \partial_k(\mathcal{A}^k z)$ is understood in the weak sense. One verifies that this weak derivative exists in L whenever z is in W. Moreover, the usual rule for differentiating products applies. Observe that assumption (A4) implies that

$$\forall z \in W, \quad (Tz, z)_L + (z, \tilde{T}z)_L \geq 2\mu_0 \|z\|_L^2. \tag{2.4}$$

Let $D \in \mathcal{L}(W; W')$ be the operator such that

$$\forall (z, y) \in W \times W, \quad \langle Dz, y \rangle_{W',W} = (Tz, y)_L - (z, \tilde{T}y)_L. \tag{2.5}$$

The operator D is self-adjoint and is a boundary operator in the sense that $\mathrm{Ker}(D)$ is the closure of $[\mathfrak{D}(\Omega)]^m$ in W; see [7] for the proof and further results.

Consider the following problem: For $f \in L$, seek $z \in W$ such that $Tz = f$. In general, boundary conditions must be enforced for this problem to be well–posed. In other words, one must find a closed subspace V of W such that the restricted operator $T : V \to L$ is an isomorphism. To achieve this goal, a simple approach inspired from Friedrichs' work [6, 8] consists of introducing an operator $M \in \mathcal{L}(W; W')$ such that

$$M \text{ is positive, i.e., } \langle Mz, z \rangle_{W',W} \geq 0 \text{ for all } z \text{ in } W, \tag{M1}$$

$$W = \mathrm{Ker}(D - M) + \mathrm{Ker}(D + M). \tag{M2}$$

Then by setting

$$V = \mathrm{Ker}(D - M) \qquad \text{and} \qquad V^* = \mathrm{Ker}(D + M^*), \tag{2.6}$$

where $M^* \in \mathcal{L}(W; W')$ is the adjoint of M and V and V^* are equipped with the graph norm, the following Theorem can be proved (see [7, 6] for a proof):

THEOREM 2.1. *Assume* (A1)–(A4) *and* (M1)–(M2). *Then, the restricted operators* $T : V \to L$ *and* $\tilde{T} : V^* \to L$ *are isomorphisms.*

As a result, for f in L, the following two problems are well-posed

$$\text{Seek } z \in V \text{ such that } Tz = f, \tag{2.7}$$

$$\text{Seek } z^* \in V^* \text{ such that } \tilde{T}z^* = f. \tag{2.8}$$

A key observation at this point is that the boundary conditions enforced in (2.7) and (2.8) are essential, i.e., they are enforced strongly by seeking the solutions in V and V^*, respectively. The key reason that lead us to focus on the theory of Friedrichs's systems is that it yields a way to enforce boundary conditions naturally, thus leading to a suitable framework for developing a DG theory. To see this, introduce the following bilinear forms on $W \times W$,

$$a(z, y) = (Tz, y)_L + \tfrac{1}{2}\langle (M - D)z, y \rangle_{W',W}, \tag{2.9}$$

$$a^*(z, y) = (\tilde{T}z, y)_L + \tfrac{1}{2}\langle (M^* + D)z, y \rangle_{W',W}. \tag{2.10}$$

It is clear that a and a^* are in $\mathcal{L}(W \times W; \mathbb{R})$. Consider the following problems: For $f \in L$,

$$\text{Seek } z \in W \text{ such that } a(z, y) = (f, y)_L,\ \forall y \in W, \tag{2.11}$$

$$\text{Seek } z^* \in W \text{ such that } a^*(z^*, y) = (f, y)_L,\ \forall y \in W. \tag{2.12}$$

The key result of this section is the following

THEOREM 2.2. *Assume* (A1)–(A4) *and* (M1)–(M2). *Then,*

(i) *There is a unique solution to* (2.11) *and this solution solves* (2.7)*;*

(ii) *There is a unique solution to* (2.12) *and this solution solves* (2.8).

Theorem 2.2 is proven in [6]. Contrary to (2.7) and (2.8), the boundary conditions in (2.11) and (2.8) are natural, i.e., they are weakly enforced. For this reason, problem (2.11) will constitute our working basis for designing DG methods; see §4.

2.2. The $2{\times}2$ block structure. We now assume that the $(d+1)$ $\mathbb{R}^{m,m}$-valued fields $\mathcal{K}$ and $\{\mathcal{A}^k\}_{1\le k\le d}$ have a $2{\times}2$ block structure, i.e., there are two positive integers m_σ and m_u such that $m = m_\sigma + m_u$ and

$$\mathcal{K} = \left[\begin{array}{c:c} \mathcal{K}^{\sigma\sigma} & \mathcal{K}^{\sigma u} \\ \hdashline \mathcal{K}^{u\sigma} & \mathcal{K}^{uu} \end{array}\right], \qquad \mathcal{A}^k = \left[\begin{array}{c:c} 0 & \mathcal{B}^k \\ \hdashline [\mathcal{B}^k]^t & \mathcal{C}^k \end{array}\right], \tag{2.13}$$

with obvious notation for the blocks of $\mathcal{K}$ and where for all $k \in \{1,\dots,d\}$, $\mathcal{B}^k$ is an $m_\sigma{\times}m_u$ matrix field and $\mathcal{C}^k$ is a symmetric $m_u{\times}m_u$ matrix field. To simplify the notation, define the operators $B = \sum_{k=1}^d \mathcal{B}^k\partial_k$, $\tilde{B} = \sum_{k=1}^d [\mathcal{B}^k]^t\partial_k$, and $C = \sum_{k=1}^d \mathcal{C}^k\partial_k$. Set $L_\sigma = [L^2(\Omega)]^{m_\sigma}$ and $L_u = [L^2(\Omega)]^{m_u}$.

The two key hypotheses on which the present work is based are the following:

$$\exists k_0 > 0,\ \forall \xi \in \mathbb{R}^{m_\sigma},\ \xi^t\mathcal{K}^{\sigma\sigma}\xi \ge k_0\|\xi\|^2_{\mathbb{R}^{m_\sigma}} \quad \text{a.e. on } \Omega, \tag{A5}$$

$$\forall k \in \{1,\dots,d\},\ \text{the } m_\sigma{\times}m_\sigma \text{ upper-left block of } \mathcal{A}^k \text{ is zero.} \tag{A6}$$

Assumption (A5), which means that $\mathcal{K}^{\sigma\sigma}$ is uniformly positive definite, implies that the matrix $\mathcal{K}^{\sigma\sigma}$ is invertible.

Assumptions (A5) and (A6) allow for the elimination of z^σ from the PDE system $Tz = f$. With obvious notation, partition z and f into (z^σ, z^u) and (f^σ, f^u), respectively. Then, z^σ is given by

$$z^\sigma = [\mathcal{K}^{\sigma\sigma}]^{-1}\Big(f^\sigma - \mathcal{K}^{\sigma u}z^u - Bz^u\Big), \tag{2.14}$$

and z^u solves the following second-order PDE:

$$\begin{aligned} &-\tilde{B}[\mathcal{K}^{\sigma\sigma}]^{-1}Bz^u + (C - \tilde{B}[\mathcal{K}^{\sigma\sigma}]^{-1}\mathcal{K}^{\sigma u} - \mathcal{K}^{u\sigma}[\mathcal{K}^{\sigma\sigma}]^{-1}B)z^u \\ &\qquad + (\mathcal{K}^{uu} - \mathcal{K}^{u\sigma}[\mathcal{K}^{\sigma\sigma}]^{-1}\mathcal{K}^{\sigma u})z^u = f^u - (\mathcal{K}^{u\sigma} + \tilde{B})[\mathcal{K}^{\sigma\sigma}]^{-1}f^\sigma. \end{aligned} \tag{2.15}$$

The objective of the present work is to design DG methods for approximating (2.15). The strategy consists of constructing a DG approximation to (2.11), but at variance with what has been done in [6], the construction is now specialized to the above $2{\times}2$ block structure so that the approximate unknown corresponding to z^σ can be eliminated locally on each mesh element by solving simple local problems.

Remark 2.1. The present study does not cover the DG approximation of the whole realm of second-order PDE's. Indeed, it is clear from (2.15) that the leading order term in the PDE, namely $\tilde{B}[\mathcal{K}^{\sigma\sigma}]^{-1}Bz^u$ (up to first-order terms), has a very particular structure since the matrices $(\mathcal{B}^k)^t[\mathcal{K}^{\sigma\sigma}]^{-1}\mathcal{B}^k$ are positive semi-definite. Hence, the PDE's covered by this work are elliptic-like; see §3 for various examples.

3. Examples. This section presents three examples of Friedrichs' systems endowed with the $2{\times}2$ block structure considered in §2.2.

3.1. Advection-diffusion-reaction. Consider the PDE

$$-\Delta u + \beta\cdot\nabla u + \mu u = f, \tag{3.1}$$

with $\beta \in [L^\infty(\Omega)]^d$, $\nabla\cdot\beta \in L^\infty(\Omega)$, $\mu \in L^\infty(\Omega)$, and $f \in L^2(\Omega)$. Assume that

$$\mu - \tfrac{1}{2}\nabla\cdot\beta \geq \mu_0 > 0 \quad \text{a.e. in } \Omega. \tag{3.2}$$

The PDE (3.1) can be written as a system of first-order PDE's in the form

$$\begin{cases} \sigma + \nabla u = 0, \\ \mu u + \nabla\cdot\sigma + \beta\cdot\nabla u = f. \end{cases} \tag{3.3}$$

Set $m = d + 1$, $m_\sigma = d$, and $m_u = 1$. Then, the mixed formulation (3.3) can be cast into the form of a Friedrichs' system with 2×2 block structure by introducing $(d+1)$ functions with values in $\mathbb{R}^{m,m}$, namely $\mathcal{K}$ and $\{\mathcal{A}^k\}_{1\leq k\leq d}$ such that

$$\mathcal{K} = \left[\begin{array}{c:c} \mathcal{I}_d & 0 \\ \hdashline 0 & \mu \end{array}\right], \quad \mathcal{A}^k = \left[\begin{array}{c:c} 0 & e^k \\ \hdashline (e^k)^t & \beta^k \end{array}\right], \tag{3.4}$$

where $\mathcal{I}_d$ is the identity matrix in $\mathbb{R}^{d,d}$, e^k is the k-th vector in the canonical basis of $\mathbb{R}^d$, and β^k is the k-th component of β in this basis. It is clear that hypotheses (A1)–(A6) hold. The graph space is $W = H(\mathrm{div};\Omega) \times H^1(\Omega)$ and for all $(\sigma,u),(\tau,v) \in W$,

$$\langle D(\sigma,u),(\tau,v)\rangle_{W',W} = \langle \sigma\cdot n, v\rangle_{-\frac{1}{2},\frac{1}{2}} + \langle \tau\cdot n, u\rangle_{-\frac{1}{2},\frac{1}{2}} + \int_{\partial\Omega} (\beta\cdot n)uv, \tag{3.5}$$

where $\langle,\rangle_{-\frac{1}{2},\frac{1}{2}}$ denotes the duality pairing between $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$. Note that (3.5) makes sense since functions in $H^1(\Omega)$ have traces in $H^{\frac{1}{2}}(\partial\Omega)$ and vector fields in $H(\mathrm{div};\Omega)$ have normal traces in $H^{-\frac{1}{2}}(\partial\Omega)$.

Homogeneous Dirichlet boundary conditions can be enforced by setting

$$\langle M(\sigma,u),(\tau,v)\rangle_{W',W} = \langle \sigma\cdot n, v\rangle_{-\frac{1}{2},\frac{1}{2}} - \langle \tau\cdot n, u\rangle_{-\frac{1}{2},\frac{1}{2}}. \tag{3.6}$$

With this choice $V = V^* = H(\mathrm{div};\Omega)\times H^1_0(\Omega)$. Let $\varrho \in L^\infty(\partial\Omega)$ be such that $2\varrho + \beta\cdot n \geq 0$ a.e. in $\partial\Omega$. Then, setting

$$\langle M(\sigma,u),(\tau,v)\rangle_{W',W} = -\langle \sigma\cdot n, v\rangle_{-\frac{1}{2},\frac{1}{2}} + \langle \tau\cdot n, u\rangle_{-\frac{1}{2},\frac{1}{2}} + \int_{\partial\Omega} (2\varrho + \beta\cdot n)uv, \tag{3.7}$$

yields $V = \{(\sigma,u) \in W;\, (-\sigma\cdot n + \varrho u)|_{\partial\Omega} = 0\}$ and $V^* = \{(\sigma,u) \in W;\, (\sigma\cdot n + \varrho u)|_{\partial\Omega} = 0\}$, i.e., a Robin boundary condition is enforced. A Neumann condition corresponds to $\varrho = 0$. We refer to [6] for more details.

Remark 3.1. Advection–diffusion–reaction equations with smooth tensor-valued diffusion can be handled by Friedrichs' formalism as follows. Let $\kappa = (\kappa_{kl})_{1\leq k,l\leq d}$ be a positive definite matrix-valued field defined on Ω whose lowest eigenvalue is uniformly bounded away from zero. Consider the PDE

$$-\nabla\cdot(\kappa^t\kappa\nabla u) + \beta\cdot\nabla u + \mu = f. \tag{3.8}$$

Here, κ is the square root of the diffusion tensor. The natural way to write this PDE in mixed form consists of setting

$$\begin{cases} \sigma + \kappa\nabla u = 0, \\ \mu u + \nabla\cdot(\kappa^t\sigma) + \beta\cdot\nabla u = f. \end{cases} \tag{3.9}$$

If the field κ is smooth, Friedrichs' formalism can be recovered by using the identities $\nabla\cdot(\kappa^t\sigma) = \kappa{:}\nabla\sigma + (\nabla\cdot\kappa)\cdot\sigma = \kappa{:}\nabla\sigma - (\nabla\cdot\kappa)\cdot(\kappa\nabla u)$ and setting

$$\mathcal{A}^k = \left[\begin{array}{c:c} 0 & \kappa_k \\ \hdashline (\kappa_k)^t & \beta^k - \kappa_k^t \nabla\cdot\kappa \end{array}\right], \tag{3.10}$$

where κ_k denotes the k-th column of κ. Hence, for property (A4) to hold, derivatives up to second-order of κ must be controlled.

3.2. Linear elasticity. Let α and γ be two positive functions in $L^\infty(\Omega)$ uniformly bounded away from zero by α_0 and γ_0, respectively. Consider the following set of PDE's

$$\begin{cases} \sigma + p\mathcal{I}_d - \frac{1}{2}(\nabla u + (\nabla u)^t) = 0, \\ \mathrm{tr}(\sigma) + (d+\gamma)p = 0, \\ -\frac{1}{2}\nabla\cdot(\sigma + \sigma^t) + \alpha u = f, \end{cases} \tag{3.11}$$

where σ is $\mathbb{R}^{d,d}$-valued, p is scalar-valued, u is $\mathbb{R}^d$-valued, and $f \in [L^2(\Omega)]^d$. The first and second equations in (3.11) imply $p = -\gamma^{-1}\nabla\cdot u$ and $\sigma = \frac{1}{2}(\nabla u + (\nabla u)^t) + \gamma^{-1}(\nabla\cdot u)\mathcal{I}_d$; γ is a compressibility coefficient, σ is the stress tensor, $\frac{1}{2}(\nabla u + (\nabla u)^t)$ is the strain tensor, and u represents the displacement field.

Set $m = d^2 + 1 + d$. The tensor σ in $\mathbb{R}^{d,d}$ is identified with the vector $\overline{\sigma} \in \mathbb{R}^{d^2}$ by setting $\overline{\sigma}_{[ij]} = \sigma_{ij}$ with $1 \le i,j \le d$ and $[ij] = d(j-1)+i$. Then, the mixed formulation (3.11) can be cast into the form of a Friedrichs' system by introducing the $(d+1)$ $\mathbb{R}^{m,m}$-valued fields with the following $3{\times}3$ block structure

$$\mathcal{K} = \left[\begin{array}{c:c:c} \mathcal{I}_{d^2} & \mathcal{Z} & 0 \\ \hdashline (\mathcal{Z})^t & (d{+}\gamma) & 0 \\ \hdashline 0 & 0 & \alpha\mathcal{I}_d \end{array}\right], \qquad \mathcal{A}^k = \left[\begin{array}{c:c:c} 0 & 0 & \mathcal{E}^k \\ \hdashline 0 & 0 & 0 \\ \hdashline (\mathcal{E}^k)^t & 0 & 0 \end{array}\right], \tag{3.12}$$

where $\mathcal{Z} \in \mathbb{R}^{d^2}$ has components given by $\mathcal{Z}_{[ij]} = \delta_{ij}$ with $1 \le i,j \le d$, and for all $k \in \{1,\ldots,d\}$, $\mathcal{E}^k \in \mathbb{R}^{d^2,d}$ has components given by $\mathcal{E}^k_{[ij],l} = -\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ with $1 \le i,j,l \le d$; here, δ_{ij} denotes the Kronecker symbol.

To recover the $2{\times}2$ structure introduced in §2.2, set $m_\sigma = d^2+1$ and $m_u = d$, i.e., the σ-component corresponds to the pair $(\overline{\sigma}, p)$. Then, hypotheses (A1)–(A6) hold. In particular, (A4)–(A5) result from the fact that for all $z = (\overline{\sigma}, p, u) \in \mathbb{R}^m$,

$$z^t\mathcal{K}z \ge (1 - \tfrac{d}{d+\frac{\gamma_0}{2}})\overline{\sigma}^2 + \tfrac{\gamma_0}{2}p^2 + \tfrac{d}{d+\frac{\gamma_0}{2}}(\overline{\sigma} + \tfrac{d+\frac{\gamma_0}{2}}{d}p\mathcal{Z})^2 + \alpha_0 u^2 \ge c(\overline{\sigma}^2 + p^2 + u^2), \tag{3.13}$$

where c only depends on d, α_0, and γ_0. Using the second Korn inequality for the variable u, it is readily seen that the graph space is $W = H_{\overline{\sigma}}{\times}L^2(\Omega){\times}[H^1(\Omega)]^d$ with $H_{\overline{\sigma}} = \{\overline{\sigma} \in [L^2(\Omega)]^{d^2};\ \nabla\cdot(\sigma + \sigma^t) \in [L^2(\Omega)]^d\}$. The boundary operator D takes the following form: For all $(\overline{\sigma}, p, u), (\overline{\tau}, q, v) \in W$,

$$\langle D(\overline{\sigma}, p, u), (\overline{\tau}, q, v)\rangle_{W',W} = -\langle \tfrac{1}{2}(\tau + \tau^t)\cdot n, u\rangle_{-\frac{1}{2},\frac{1}{2}} - \langle \tfrac{1}{2}(\sigma + \sigma^t)\cdot n, v\rangle_{-\frac{1}{2},\frac{1}{2}}, \tag{3.14}$$

where $\langle , \rangle_{-\frac{1}{2},\frac{1}{2}}$ denotes the duality pairing between $[H^{-\frac{1}{2}}(\partial\Omega)]^d$ and $[H^{\frac{1}{2}}(\partial\Omega)]^d$.

To enforce boundary conditions for (3.11), one possibility consists of setting for all $(\overline{\sigma}, p, u), (\overline{\tau}, q, v) \in W$,

$$\langle M(\overline{\sigma}, p, u), (\overline{\tau}, q, v)\rangle_{W',W} = \langle \tfrac{1}{2}(\tau + \tau^t)\cdot n, u\rangle_{-\frac{1}{2},\frac{1}{2}} - \langle \tfrac{1}{2}(\sigma + \sigma^t)\cdot n, v\rangle_{-\frac{1}{2},\frac{1}{2}}. \tag{3.15}$$

With this choice, the displacement is set to zero at $\partial\Omega$ as shown in the following

LEMMA 3.1. *Let M be given by* (3.15). *Then,* $V = V^* = H_{\overline{\sigma}}\times L^2(\Omega)\times[H^1_0(\Omega)]^d$.

Proof. It is clear that $V = V^*$ since $M + M^* = 0$. Observe that

$$\langle (D - M)(\overline{\sigma}, p, u), (\overline{\tau}, q, v)\rangle_{W',W} = -\langle (\tau + \tau^t)\cdot n, u\rangle_{-\frac{1}{2},\frac{1}{2}}. \tag{3.16}$$

Hence, it is clear that $H_{\overline{\sigma}}\times L^2(\Omega)\times[H^1_0(\Omega)]^d \subset \mathrm{Ker}(D - M) = V$. Conversely, let $(\overline{\sigma}, p, u) \in \mathrm{Ker}(D - M)$. Let $\theta \in [H^{-\frac{1}{2}}(\partial\Omega)]^d$. Consider the following problem: Seek $v_\theta \in [H^1(\Omega)]^d$ such that for all $w \in [H^1(\Omega)]^d$,

$$(v_\theta, w)_{[L^2(\Omega)]^d} + (\nabla v_\theta + (\nabla v_\theta)^t, \nabla w + (\nabla w)^t)_{[L^2(\Omega)]^{d,d}} = \langle \theta, w\rangle_{-\frac{1}{2},\frac{1}{2}}.$$

This problem is well–posed owing to the second Korn inequality and the Lax–Milgram Lemma. Set $\tau_\theta = \nabla v_\theta + (\nabla v_\theta)^t$. Since $\overline{\tau}_\theta \in H_{\overline{\sigma}}$, one can take $(\overline{\tau}, p, v) = (\overline{\tau}_\theta, 0, 0)$ in (3.16) yielding $\langle \theta, u\rangle_{-\frac{1}{2},\frac{1}{2}} = 0$. Since θ is arbitrary in $[H^{-\frac{1}{2}}(\partial\Omega)]^d$, $u \in [H^1_0(\Omega)]^d$. ☐

3.3. Simplified MHD. For the sake of simplicity we assume that the space dimension is three, i.e., $d = 3$. Let ν, μ, and σ be three functions in $L^\infty(\Omega)$, and let $\beta \in [L^\infty(\Omega)]^3$ be a vector field. A simplified (time-discretized) version of the MHD equations consists of seeking the electric field E and the magnetic field H such that

$$\begin{cases} \nu H + \nabla\times E = 0, \\ \sigma(E + \beta\times(\mu H)) - \nabla\times H = j, \end{cases} \tag{3.17}$$

where $j \in [L^2(\Omega)]^3$ is a given source term. The separation of the electromagnetic field (H, E) into magnetic and electric fields induces a natural partitioning of $[L^2(\Omega)]^6$ into $[L^2(\Omega)]^3 \times [L^2(\Omega)]^3$. Set $m = 6$. The PDE's (3.17) are recast into the form of a Friedrichs' system by introducing the following block structured matrices in $\mathbb{R}^{6,6}$,

$$\mathcal{K} = \left[\begin{array}{c|c} \nu\mathcal{I}_3 & 0 \\ \hline \sigma\mu\mathcal{V} & \sigma\mathcal{I}_3 \end{array}\right], \qquad \mathcal{A}^k = \left[\begin{array}{c|c} 0 & \mathcal{R}^k \\ \hline (\mathcal{R}^k)^t & 0 \end{array}\right], \tag{3.18}$$

where $\mathcal{R}^k_{ij} = \epsilon_{ikj}$ is the Levi-Civita permutation tensor, $1 \le i, j, k \le 3$, and $\mathcal{V}_{ij} = \sum_{k=1}^d \epsilon_{ikj}\beta^k$. Assume that ν and σ are positive functions on Ω uniformly bounded away from zero and that there is $\alpha_0 > 0$ such that a.e. in Ω, $2\left(\frac{\nu}{\sigma}\right)^{\frac{1}{2}} - \mu\|\beta\|_{L^\infty(\Omega)} \ge \alpha_0$. In the above framework, one readily verifies that hypotheses (A1)–(A6) hold with $m_\sigma = 3$ and $m_u = 3$. In the full MHD equations, the off-diagonal term induced by β is compensated by a term originating from the conservation of momentum in the Navier–Stokes equations so that the condition for (A4) to hold is simply that ν and σ be uniformly bounded away from zero.

The graph space is $W = H(\mathrm{curl};\Omega)\times H(\mathrm{curl};\Omega)$ and for all $(H, E), (h, e) \in W$,

$$\begin{aligned} \langle D(H, E), (h, e)\rangle_{W',W} = {} & (\nabla\times E, h)_{[L^2(\Omega)]^3} - (E, \nabla\times h)_{[L^2(\Omega)]^3} \\ & + (H, \nabla\times e)_{[L^2(\Omega)]^3} - (\nabla\times H, e)_{[L^2(\Omega)]^3}. \end{aligned} \tag{3.19}$$

When (H, E) and (h, e) are smooth, the above duality product can be interpreted as the boundary integral $\int_{\partial\Omega}[(n\times E)\cdot h + (n\times e)\cdot H]$.

An acceptable boundary condition for (3.17) consists of setting

$$\begin{aligned} \langle M(H, E), (h, e)\rangle_{W',W} = {} & -(\nabla\times E, h)_{[L^2(\Omega)]^3} + (E, \nabla\times h)_{[L^2(\Omega)]^3} \\ & + (H, \nabla\times e)_{[L^2(\Omega)]^3} - (\nabla\times H, e)_{[L^2(\Omega)]^3}, \end{aligned} \tag{3.20}$$

for all $(H,E),(h,e)\in W$. Assuming $[H^1(\Omega)]^3$ is dense in $H(\mathrm{curl};\Omega)$, this choice yields $V=V^*=H(\mathrm{curl};\Omega)\times H_0(\mathrm{curl};\Omega)$, i.e., the tangential component of the electric field is set to zero; see [7] for the analysis.

4. Discontinuous Galerkin method for $2{\times}2$ systems. In this section we present the discrete setting to design a DG method to approximate the Friedrichs' systems with the $2{\times}2$ block structure presented in §2.2.

4.1. The discrete setting. Let $\{\mathcal{T}_h\}_{h>0}$ be a family of meshes of Ω. The meshes are assumed to be affine to avoid unnecessary technicalities, i.e., Ω is assumed to be a polyhedron. We denote by $\mathcal{F}_h^{\mathrm{i}}$ the set of interfaces, i.e., $F\in\mathcal{F}_h^{\mathrm{i}}$ if F is a $(d{-}1)$-manifold and there are $K_1(F)$ and $K_2(F)\in\mathcal{T}_h$ such that $F=K_1(F)\cap K_2(F)$. For $F\in\mathcal{F}_h^{\mathrm{i}}$, we set $\mathcal{T}(F)=K_1(F)\cup K_2(F)$. We denote by $\mathcal{F}_h^{\partial}$ the set of the faces that separate the mesh from the exterior of Ω, i.e., $F\in\mathcal{F}_h^{\partial}$ if F is a $(d{-}1)$-manifold and there is $K(F)\in\mathcal{T}_h$ such that $F=K(F)\cap\partial\Omega$. For $F\in\mathcal{F}_h^{\partial}$, we set $\mathcal{T}(F)=K(F)$. Finally, we set $\mathcal{F}_h=\mathcal{F}_h^{\mathrm{i}}\cup\mathcal{F}_h^{\partial}$. We assume that the mesh family $\{\mathcal{T}_h\}_{h>0}$ is such that there is a positive constant c, independent of h, such that for all $F\in\mathcal{F}_h$,

$$c_1 h_{\mathcal{T}(F)}\le h_F, \tag{4.1}$$

where $h_{\mathcal{T}(F)}$ denotes the diameter of $\mathcal{T}(F)$ and h_F that of F. No other particular assumption than (4.1) is made on the matching of element faces.

For a non-negative integer p, consider the finite element space of scalar-valued functions

$$P_{h,p}=\{v_h\in L^2(\Omega);\,\forall K\in\mathcal{T}_h,\ v_h|_K\in\mathbb{P}_p\}, \tag{4.2}$$

where $\mathbb{P}_p$ denotes the vector space of polynomials with real coefficients and with total degree less than or equal to p. The mesh family $\{\mathcal{T}_h\}_{h>0}$ is assumed to be regular enough so that there is a constant c, independent of h, such that for all $v_h\in P_{h,p}$,

$$\forall K\in\mathcal{T}_h,\quad \|\nabla v_h\|_{[L^2(K)]^d}\le c\,h_K^{-1}\|v_h\|_{L^2(K)}, \tag{4.3}$$

$$\forall F\in\mathcal{F}_h,\quad \|v_h\|_{L^2(F)}\le c\,h_F^{-\frac12}\|v_h\|_{L^2(\mathcal{T}(F))}. \tag{4.4}$$

Let p_u and p_σ be two non-negative integers such that

$$p_u-1\le p_\sigma\le p_u. \tag{4.5}$$

Define the following vector spaces:

$$U_h=[P_{h,p_u}]^{m_u},\qquad \Sigma_h=[P_{h,p_\sigma}]^{m_\sigma},\qquad W_h=U_h{\times}\Sigma_h, \tag{4.6}$$

and set $U(h)=[H^1(\Omega)]^{m_u}{+}U_h$, $\Sigma(h)=[H^1(\Omega)]^{m_\sigma}{+}\Sigma_h$, and $W(h)=[H^1(\Omega)]^m{+}W_h$. Obviously, inequalities (4.3) and (4.4) can be applied componentwise to all functions in U_h and in Σ_h. Moreover, since every function v in $U(h)$ has a (possibly two-valued) trace almost everywhere on $F\in\mathcal{F}_h^{\mathrm{i}}$, we set

$$v^1(x)=\lim_{\substack{y\to x\\ y\in K_1(F)}}v(y)\qquad v^2(x)=\lim_{\substack{y\to x\\ y\in K_2(F)}}v(y),\quad\text{for a.e. } x\in F, \tag{4.7}$$

$$[\![v]\!]=v^1-v^2,\qquad\qquad \{v\}=\tfrac12(v^1+v^2),\quad\text{a.e. on } F. \tag{4.8}$$

We define τ^1, τ^2, and $[\![\tau]\!]$ similarly for all τ in $\Sigma(h)$. The arbitrariness in the choice of $K_1(F)$ and $K_2(F)$ could be avoided by choosing intrinsic notations that would, however, unnecessarily complicate the presentation; nothing that is said hereafter depends on this choice.

For any measurable subset of Ω or $\mathcal{F}_\Omega = \{x \in \Omega; \exists F \in \mathcal{F}_h,\, x \in F\}$, say E, we denote by $(\cdot,\cdot)_{L,E}$ the scalar product induced by $[L^2(\Omega)]^m$ or $[L^2(\mathcal{F}_\Omega)]^m$ on E, respectively. We define similarly $(\cdot,\cdot)_{L_u,E}$ and $(\cdot,\cdot)_{L_\sigma,E}$.

4.2. Boundary and interface operators. Let $n = (n_1, \dots, n_d)^t$ be the unit outward normal to $\partial\Omega$. Henceforth, we assume that the fields $\{\mathcal{A}^k\}_{1\le k\le d}$ are sufficiently smooth for the matrix $\mathcal{D}_{\partial\Omega} = \sum_{k=1}^d n_k \mathcal{A}^k$ to be meaningful at the boundary. Hence, the following representation holds

$$\langle Dz, y\rangle_{W',W} = \int_{\partial\Omega} y^t \mathcal{D}_{\partial\Omega} z, \tag{4.9}$$

whenever z and y and smooth functions. Owing to (2.13), $\mathcal{D}_{\partial\Omega}$ has a $2{\times}2$ block structure with $\mathcal{D}_{\partial\Omega}^{\sigma u} = \sum_{k=1}^d n_k \mathcal{B}^k$, $\mathcal{D}_{\partial\Omega}^{u\sigma} = [\mathcal{D}_{\partial\Omega}^{\sigma u}]^t$, $\mathcal{D}_{\partial\Omega}^{uu} = \sum_{k=1}^d n_k \mathcal{C}^k$, and

$$\mathcal{D}_{\partial\Omega}^{\sigma\sigma} = 0. \tag{4.10}$$

Likewise, we assume that the boundary operator M has an integral representation, i.e., there exists a matrix-valued field $\mathcal{M} : \partial\Omega \longrightarrow \mathbb{R}^{m,m}$ such that

$$\langle Mz, y\rangle_{W',W} = \int_{\partial\Omega} y^t \mathcal{M} z, \tag{4.11}$$

whenever z and y and smooth functions. We denote by $\mathcal{M}^{\sigma u}$, $\mathcal{M}^{u\sigma}$, and $\mathcal{M}^{uu}$ the top right, bottom left, and bottom right blocks of $\mathcal{M}$, respectively. For the sake of simplicity, we hereafter assume that

$$\mathcal{M}^{\sigma\sigma} = 0. \tag{4.12}$$

This simplifying assumption holds for advection–diffusion–reaction equations with Dirichlet, Neumann, or Robin boundary conditions, the linear elasticity equations, and the simplified MHD equations; see §3. Indeed, assuming $\mathrm{Ker}(\mathcal{D}^{\sigma u}) = \{0\}$, the Dirichlet boundary condition $z^u = 0$ can be enforced by taking

$$\mathcal{M} = \left[\begin{array}{c:c} 0 & -\mathcal{D}^{\sigma u} \\ \hdashline \mathcal{D}^{u\sigma} & \mathcal{M}^{uu} \end{array}\right], \tag{4.13}$$

where $\mathcal{M}^{uu}$ is a suitable positive matrix in $\mathbb{R}^{m_u,m_u}$. Similarly, taking

$$\mathcal{M} = \left[\begin{array}{c:c} 0 & \mathcal{D}^{\sigma u} \\ \hdashline -\mathcal{D}^{u\sigma} & \mathcal{M}^{uu} \end{array}\right], \tag{4.14}$$

where $\mathcal{M}^{uu}$ is a positive matrix in $\mathbb{R}^{m_u,m_u}$ yields the Robin boundary condition $2\mathcal{D}^{u\sigma}z^\sigma + (\mathcal{D}^{uu} - \mathcal{M}^{uu})z^u = 0$; if $\mathcal{M}^{uu} = \mathcal{D}^{uu}$, the boundary condition is of Neumann-type.

For all $K \in \mathcal{T}_h$, we define the matrix-valued field $\mathcal{D}_{\partial K} : \partial K \to \mathbb{R}^{m,m}$ by

$$\mathcal{D}_{\partial K}(x) = \sum_{k=1}^d n_{K,k} \mathcal{A}^k(x) \quad \text{a.e. on } \partial K, \tag{4.15}$$

where $n_K = (n_{K,1}, \dots, n_{K,d})^t$ is the unit outward normal to ∂K. Owing to (2.13), $\mathcal{D}_{\partial K}$ has a 2×2 block structure with $\mathcal{D}^{\sigma u}_{\partial K} = \sum_{k=1}^d n_{K,k}\mathcal{B}^k$, $\mathcal{D}^{u\sigma}_{\partial K} = [\mathcal{D}^{\sigma u}_{\partial K}]^t$, $\mathcal{D}^{uu}_{\partial K} = (\mathcal{D}^{uu}_{\partial K})^t = \sum_{k=1}^d n_{K,k}\mathcal{C}^k$, and

$$\mathcal{D}^{\sigma\sigma}_{\partial K} = 0. \tag{4.16}$$

The definition (4.15) is clearly compatible with that of $\mathcal{D}_{\partial\Omega}$ if $\partial K \subset \partial\Omega$. Moreover, observe that for all z, y in $W(h)$ and for all $K \in \mathcal{T}_h$,

$$(\mathcal{D}_{\partial K} z, y)_{L,\partial K} = (Tz, y)_{L,K} - (z, \tilde{T}y)_{L,K}. \tag{4.17}$$

We now define on $\mathcal{F}_h$ a matrix-valued field $\mathcal{D}$ as follows. On $\mathcal{F}_h^\partial$, $\mathcal{D}$ is single-valued and coincides with $\mathcal{D}_{\partial\Omega}$. On $\mathcal{F}_h^{\mathrm{i}}$, $\mathcal{D}$ is two-valued and for all $F \in \mathcal{F}_h^{\mathrm{i}}$, the two values taken by $\mathcal{D}$ are $\mathcal{D}_{\partial K_1(F)}$ and $\mathcal{D}_{\partial K_2(F)}$. Note that $\{\mathcal{D}\} = 0$ a.e. on $\mathcal{F}_h^{\mathrm{i}}$ since $\sum_{k=1}^d \partial_k \mathcal{A}^k$ is bounded owing to (A2).

To weakly enforce boundary conditions, we introduce for all $F \in \mathcal{F}_h^\partial$ a linear operator

$$M_F = \left[\begin{array}{c:c} M_F^{\sigma\sigma} & M_F^{\sigma u} \\ \hdashline M_F^{u\sigma} & M_F^{uu} \end{array}\right] \in \mathcal{L}([L^2(F)]^m; [L^2(F)]^m). \tag{4.18}$$

Similarly, to penalize interface jumps, we introduce for all $F \in \mathcal{F}_h^{\mathrm{i}}$ a linear operator

$$S_F = \left[\begin{array}{c:c} S_F^{\sigma\sigma} & S_F^{\sigma u} \\ \hdashline S_F^{u\sigma} & S_F^{uu} \end{array}\right] \in \mathcal{L}([L^2(F)]^m; [L^2(F)]^m). \tag{4.19}$$

Star superscripts denote the L^2-adjoint of M_F, S_F, or any block thereof. For instance, $(M_F^{u\sigma})^* \in \mathcal{L}([L^2(F)]^{m_u}; [L^2(F)]^{m_\sigma})$ is defined s.t. $((M_F^{u\sigma})^*(v), \tau)_{L_\sigma,F} = (M_F^{u\sigma}(\tau), v)_{L_u,F}$ for all $v \in [L^2(F)]^{m_u}$ and for all $\tau \in [L^2(F)]^{m_\sigma}$.

The operators M_F and S_F satisfy various design criteria which are collected in §5.1. For the time being, we solely mention the important assumption

$$M_F^{\sigma\sigma} = 0 \qquad \text{and} \qquad S_F^{\sigma\sigma} = 0. \tag{4.20}$$

This assumption is essential to eliminate the σ-component of the discrete solution by solving local problems; see §4.4. Observe that owing to (4.20), the jumps across interfaces of the σ-component of the unknown are no longer controlled; this is an important difference with respect to the DG method analyzed in [6].

4.3. The discrete problem and the notion of fluxes. Drawing inspiration from (2.11), we introduce the bilinear form a_h such that for all z, y in $W(h)$,

$$\begin{aligned} a_h(z, y) = & \sum_{K \in \mathcal{T}_h} (Tz, y)_{L,K} + \sum_{F \in \mathcal{F}_h^\partial} \tfrac{1}{2}(M_F(z) - \mathcal{D}z, y)_{L,F} \\ & - \sum_{F \in \mathcal{F}_h^{\mathrm{i}}} 2(\{\mathcal{D}z\}, \{y\})_{L,F} + \sum_{F \in \mathcal{F}_h^{\mathrm{i}}} (S_F([\![z]\!]), [\![y]\!])_{L,F}. \end{aligned} \tag{4.21}$$

The first and second term in the right-hand-side directly come from (2.9); the third term is zero whenever z is smooth and is meant to ensure that a_h satisfies a coercivity property (see Lemma 5.4); the last term will be used to control the jump of the discrete solution across interfaces.

The discrete counterpart of (2.11) is: For $f = (f^\sigma, f^u) \in L$,

$$\begin{cases} \text{Seek } z_h = (z_h^\sigma, z_h^u) \in W_h \text{ such that} \\ a_h(z_h, y_h) = (f, y_h)_L, \quad \forall y_h = (y_h^\sigma, y_h^u) \in W_h. \end{cases} \tag{4.22}$$

As in [6], the discrete problem (4.22) can be localized by using the notion of flux.

Let K be a mesh element in $\mathcal{T}_h$. For $z \in W(h)$ and $x \in \partial K$, set

$$z^i(x) = \lim_{\substack{s \to x \\ s \in K}} z(s), \qquad z^e(x) = \lim_{\substack{s \to x \\ s \notin K}} z(s), \tag{4.23}$$

$$[\![z]\!]_{\partial K}(x) = z^i(x) - z^e(x), \qquad \{z\}_{\partial K}(x) = \tfrac{1}{2}(z^i(x) + z^e(x)), \tag{4.24}$$

with the convention that $z^e(x) = 0$ if $x \in \partial\Omega$. Then we define the element flux of z on ∂K, say $\phi_{\partial K}(z) \in [L^2(\partial K)]^m$, by its restriction to the faces or interfaces F of ∂K as follows:

$$\phi_{\partial K}(z)|_F = \begin{cases} \frac{1}{2}(M_F(z) + \mathcal{D}z) & \text{if } F \subset \partial K^\partial, \\ S_F([\![z]\!]_{\partial K}|_F) + \mathcal{D}_{\partial K}\{z\}_{\partial K} & \text{if } F \subset \partial K^{\mathrm{i}}, \end{cases} \tag{4.25}$$

where ∂K^{i} denotes that part of ∂K that lies in Ω and ∂K^∂ that part of ∂K that lies on $\partial\Omega$. The discrete problem (4.22) is equivalently reformulated in terms of the following local problems posed for all $K \in \mathcal{T}_h$,

$$\begin{cases} \text{Seek } z_h \in W_h \text{ such that } \forall q = (q^\sigma, q^u) \in [\mathbb{P}_{p_\sigma}(K)]^{m_\sigma} \times [\mathbb{P}_{p_u}(K)]^{m_u}, \\ (Tz_h, q)_{L,K} + (\phi_{\partial K}(z_h) - \mathcal{D}_{\partial K} z_h^i, q)_{L,\partial K} = (f, q)_{L,K}, \end{cases} \tag{4.26}$$

or equivalently using the local integration by parts formula (4.17),

$$\begin{cases} \text{Seek } z_h \in W_h \text{ such that } \forall q = (q^\sigma, q^u) \in [\mathbb{P}_{p_\sigma}(K)]^{m_\sigma} \times [\mathbb{P}_{p_u}(K)]^{m_u}, \\ (z_h, \tilde{T}q)_{L,K} + (\phi_{\partial K}(z_h), q)_{L,\partial K} = (f, q)_{L,K}. \end{cases} \tag{4.27}$$

Remark 4.1. Observe that owing to (4.20), the jumps across interfaces of the σ-component of the unknown are not controlled (this is the key property that allows for the local elimination of the σ-component of the discrete solution z_h, see §4.4). This is an important difference with respect to the DG method analyzed in [6].

4.4. Eliminating the σ-component. We now rewrite (4.26) by making use of the 2×2 block structure, and we show how the unknown z_h^σ can be locally eliminated. To this end, we introduce the σ-component of the element flux

$$\phi_{\partial K}^\sigma(z^u)|_F = \begin{cases} \frac{1}{2}(M_F^{\sigma u}(z^u) + \mathcal{D}^{\sigma u} z^u) & \text{if } F \subset \partial K^\partial, \\ S_F^{\sigma u}([\![z^u]\!]_{\partial K}|_F) + \mathcal{D}_{\partial K}^{\sigma u}\{z^u\}_{\partial K} & \text{if } F \subset \partial K^{\mathrm{i}}, \end{cases} \tag{4.28}$$

where we stress that $\phi_{\partial K}^\sigma$ solely depends on z^u owing to (4.20). Then, (4.26) implies that z_h^σ solves the following local problems: For all $q^\sigma \in [\mathbb{P}_{p_\sigma}(K)]^{m_\sigma}$,

$$(\mathcal{K}^{\sigma\sigma} z_h^\sigma + \mathcal{K}^{\sigma u} z_h^u + B z_h^u, q^\sigma)_{L_\sigma,K} + (\phi_{\partial K}^\sigma(z_h^u) - \mathcal{D}_{\partial K}^{\sigma u}(z_h^u)^i, q^\sigma)_{L_\sigma,\partial K} = (f^\sigma, q^\sigma)_{L_\sigma,K}. \tag{4.29}$$

We now define the mapping $\theta_h^1 : U(h) \longrightarrow \Sigma_h$ such that for all $z^u \in U(h)$ and for all $K \in \mathcal{T}_h$, $\theta_h^1(z^u)|_K$ solves the following problem: For all $q^\sigma \in [\mathbb{P}_{p_\sigma}(K)]^{m_\sigma}$,

$$\begin{aligned} (\mathcal{K}^{\sigma\sigma}\theta_h^1(z^u), q^\sigma)_{L_\sigma,K} = &- (\mathcal{K}^{\sigma u} z^u + B z^u, q^\sigma)_{L_\sigma,K} \\ &- (\phi_{\partial K}^\sigma(z^u) - \mathcal{D}_{\partial K}^{\sigma u}(z^u)^i, q^\sigma)_{L_\sigma,\partial K}. \end{aligned} \tag{4.30}$$

Owing to (A5), this problem is well–posed. Similarly, we define the mapping $\theta_h^2 : L_\sigma \longrightarrow \Sigma_h$ such that for all $f^\sigma \in L_\sigma$ and for all $K \in \mathcal{T}_h$, $\theta_h^2(f^\sigma)|_K$ solves the following local problem: For all $q^\sigma \in [\mathbb{P}_{p_\sigma}(K)]^{m_\sigma}$,

$$(\mathcal{K}^{\sigma\sigma}\theta_h^2(f^\sigma), q^\sigma)_{L_\sigma,K} = (f^\sigma, q^\sigma)_{L_\sigma,K}. \tag{4.31}$$

Finally, define the bilinear form ϕ_h on $U(h) \times U(h)$ by

$$\phi_h(z^u, y^u) = a_h((\theta_h^1(z^u), z^u), (0, y^u)), \tag{4.32}$$

and the linear form ψ_h on $U(h)$ by

$$\psi_h(y^u) = a_h((\theta_h^2(f^\sigma), 0), (0, y^u)). \tag{4.33}$$

This readily leads to the following

PROPOSITION 4.1. *If the pair* (z_h^σ, z_h^u) *solves* (4.22)*, then,*

$$z_h^\sigma = \theta_h^1(z_h^u) + \theta_h^2(f^\sigma), \tag{4.34}$$

and z_h^u *solves the following problem:*

$$\begin{cases} \text{Seek } z_h^u \in U_h \text{ such that} \\ \phi_h(z_h^u, y_h^u) = (f^u, y_h^u)_{L_u} - \psi_h(y_h^u), \quad \forall y_h^u \in U_h. \end{cases} \tag{4.35}$$

Conversely, if z_h^u *solves* (4.35) *and if* z_h^σ *is defined by* (4.34)*, then the pair* (z_h^σ, z_h^u) *solves* (4.22).

Remark 4.2. The bilinear form ϕ_h and the linear form ψ_h are easy to compute in practice since they involve the solution of local problems, namely (4.30) and (4.31), that can be solved elementwise.

5. Convergence analysis. In this section, we present the design criteria for the above DG method and perform the error analysis. The mains results are Theorem 5.8, which estimates the error in the norm (5.9), and Theorem 5.14, which improves the L_u-estimate of the u-component of the error by means of a duality argument. Throughout this section, we assume that:

- For all $k \in \{1, \dots, d\}$, $\mathcal{B}^k \in [\mathcal{C}^{0,1}(\overline{\Omega})]^{m_\sigma, m_u}$.
- The mesh family $\{\mathcal{T}_h\}_{h>0}$ is such that (4.1), (4.3), and (4.4) hold.
- The approximation spaces are defined according to (4.2), (4.5), and (4.6).

5.1. The design criteria for the boundary and interface operators. For all $F \in \mathcal{F}_h^\partial$, for all $v, w \in [L^2(F)]^{m_u}$, and for all $\tau \in [L^2(F)]^{m_\sigma}$, we assume that

$$M_F^{\sigma\sigma} = 0, \tag{DG1}$$

$$M_F^{\sigma u} + (M_F^{u\sigma})^* = 0, \tag{DG2}$$

$$(M_F^{uu}(v), v)_{L_u,F} \geq 0, \tag{DG3}$$

$$|(M_F^{\sigma u}(v) - \mathcal{D}^{\sigma u}v, \tau)_{L_\sigma,F}| \leq c h_F^{\frac{1}{2}} |v|_{M,F} \|\tau\|_{L_\sigma,F}, \tag{DG4}$$

$$|(M_F^{uu}(v) + \mathcal{D}^{uu}v, w)_{L_u,F}| \leq c h_F^{-\frac{1}{2}} \|v\|_{L_u,F} |w|_{M,F}, \tag{DG5}$$

$$|(M_F^{uu}(v) - \mathcal{D}^{uu}v, w)_{L_u,F}| \leq c h_F^{-\frac{1}{2}} |v|_{M,F} \|w\|_{L_u,F}, \tag{DG6}$$

$$\forall y \in [L^2(F)]^m, \quad (\mathcal{M}y - \mathcal{D}y = 0) \implies (M_F(y) - \mathcal{D}y = 0), \tag{DG7}$$

$$\forall y \in [L^2(F)]^m, \quad (\mathcal{M}^t y + \mathcal{D}y = 0) \implies (M_F^*(y) + \mathcal{D}y = 0), \tag{DG8}$$

where c is a constant independent of h and where we have introduced the following semi-norms:

$$\forall v \in U(h), \qquad |v|_M^2 = \sum_{F\in\mathcal{F}_h^\partial} |v|_{M,F}^2 \qquad \text{with} \qquad |v|_{M,F}^2 = (M_F^{uu}(v), v)_{L_u,F}. \quad (5.1)$$

For all $F \in \mathcal{F}_h^{\mathrm{i}}$, for all $v, w \in [L^2(F)]^{m_u}$, and for all $\tau \in [L^2(F)]^{m_\sigma}$, we assume that

$$S_F^{\sigma\sigma} = 0, \qquad (\text{DG}9)$$

$$S_F^{\sigma u} + (S_F^{u\sigma})^* = 0, \qquad (\text{DG}10)$$

$$(S_F^{uu}(v), v)_{L_u,F} \geq 0, \qquad (\text{DG}11)$$

$$|(S_F^{uu}(v), w)_{L_u,F}| \leq c h_F^{-\frac{1}{2}} \|v\|_{L_u,F} |w|_{S,F}, \qquad (\text{DG}12)$$

$$|(S_F^{uu}(v), w)_{L_u,F}| \leq c h_F^{-\frac{1}{2}} |v|_{S,F} \|w\|_{L_u,F}, \qquad (\text{DG}13)$$

$$|(S_F^{\sigma u}(v), \tau)_{L_\sigma,F}| \leq c h_F^{\frac{1}{2}} |v|_{S,F} \|\tau\|_{L_\sigma,F}, \qquad (\text{DG}14)$$

$$|(\mathcal{D}^{\sigma u} v, \tau)_{L_\sigma,F}| \leq c h_F^{\frac{1}{2}} |v|_{S,F} \|\tau\|_{L_\sigma,F}, \qquad (\text{DG}15)$$

$$|(\mathcal{D}^{uu} v, w)_{L_u,F}| \leq c h_F^{-\frac{1}{2}} |v|_{S,F} \|w\|_{L_u,F}, \qquad (\text{DG}16)$$

where c is a constant independent of h and where we have introduced the following semi-norms:

$$\forall v \in U(h), \qquad |v|_S^2 = \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} |v|_{S,F}^2 \qquad \text{with} \qquad |v|_{S,F}^2 = (S_F^{uu}(v), v)_{L_u,F}. \quad (5.2)$$

Theorem 5.8 relies only on assumptions (DG1)–(DG5), (DG7), (DG9)–(DG12), and (DG14)–(DG15) which are collectively referred to as (DG$^\flat$). The additional assumptions (DG6), (DG8), (DG13), and (DG16) are needed to prove Theorem 5.14. Assumptions (DG1)–(DG16) are collectively referred to as (DG$^\sharp$).

Remark 5.1. Assumptions (DG7) and (DG8) are consistency hypotheses which trivially hold if $M_F(z) = \mathcal{M}z$. However, it is not always possible to make this simple choice because it is often necessary to penalize the boundary values of the u-component of the unknown. For instance, when Dirichlet boundary conditions are enforced, i.e., $\mathcal{M}^{\sigma u} = -\mathcal{D}^{\sigma u}$, it may happen that $\mathcal{M}^{uu} = 0$ (see the examples discussed in §3). In this circumstance, (DG4) (see also (5.3) below) cannot be satisfied if we set $M_F^{uu}(v) = \mathcal{M}^{uu} v = 0$. Instead, it is necessary that M_F^{uu} scales like h_F^{-1}. The consistency hypotheses (DG7) and (DG8) then mean that the extra control required by (DG4) is compatible with the way the boundary condition is enforced.

While assumptions (DG$^\sharp$) are just what it takes to prove Theorems 5.8 and 5.14, it is simpler in practice to work with a simplified set of assumptions. These are summarized in the following lemmas. The proofs, which are straightforward, are omitted for brevity. Lemma 5.1 is tailored for the case when Dirichlet boundary conditions are enforced, while Lemma 5.2 is tailored for the case when Neumann or Robin boundary conditions are enforced.

LEMMA 5.1. *Assume that $M_F^{\sigma\sigma} = 0$, $M_F^{\sigma u}(v) = -\mathcal{D}^{\sigma u} v$ for all $v \in [L^2(F)]^{m_u}$, $M_F^{u\sigma} = -(M_F^{\sigma u})^*$ and that M_F^{uu} is self-adjoint and such that for all $v \in [L^2(F)]^{m_u}$,*

$$c_1 (h_F \|\mathcal{D}^{uu} v\|_{L_u,F}^2 + h_F^{-1} \|\mathcal{D}^{\sigma u} v\|_{L_\sigma,F}^2) \leq (M_F^{uu}(v), v)_{L_u,F} \leq c_2 h_F^{-1} \|v\|_{L_u,F}^2, \quad (5.3)$$

where c_1 and c_2 are independent of h. Then, (DG1)–(DG6) *hold.*

LEMMA 5.2. *Assume that* $M_F^{\sigma\sigma} = 0$*,* $M_F^{\sigma u}(v) = \mathcal{D}^{\sigma u}v$ *for all* $v \in [L^2(F)]^{m_u}$*,* $M_F^{u\sigma} = -(M_F^{\sigma u})^*$ *and that* M_F^{uu} *is self-adjoint and such that for all* $v \in [L^2(F)]^{m_u}$*,*

$$c_1\|\mathcal{D}^{uu}v\|_{L_u,F}^2 \leq (M_F^{uu}(v), v)_{L_u,F} \leq c_2\|v\|_{L_u,F}^2, \tag{5.4}$$

where c_1 *and* c_2 *are independent of* h*. Then,* (DG1)–(DG6) *hold.*

LEMMA 5.3. *Assume that* $S_F^{\sigma\sigma} = 0$*,* $S_F^{u\sigma} = 0$*,* $S_F^{\sigma u} = 0$ *and that* S_F^{uu} *is self-adjoint and such that for all* $v \in [L^2(F)]^{m_u}$*,*

$$c_1(h_F\|\mathcal{D}^{uu}v\|_{L_u,F}^2 + h_F^{-1}\|\mathcal{D}^{\sigma u}v\|_{L_\sigma,F}^2) \leq (S_F^{uu}(v), v)_{L_u,F} \leq c_2h_F^{-1}\|v\|_{L_u,F}^2, \tag{5.5}$$

where c_1 *and* c_2 *are independent of* h*. Then,* (DG9)–(DG16) *hold.*

Remark 5.2.

(i) Assumptions (DG1)–(DG4) imply that there is c, independent of h, such that for all $v \in [L^2(F)]^{m_u}$ and for all $\tau \in [L^2(F)]^{m_\sigma}$,

$$|v|_{M,F} \leq ch_F^{-\frac{1}{2}}\|v\|_{L_u,F}, \tag{5.6}$$
$$|(M_F^{\sigma u}(v), \tau)_{L_\sigma,F}| \leq c\|v\|_{L_u,F}\|\tau\|_{L_\sigma,F}, \tag{5.7}$$
$$|(M_F^{u\sigma}(\tau) + \mathcal{D}^{u\sigma}\tau, v)_{L_u,F}| \leq ch_F^{\frac{1}{2}}|v|_{M,F}\|\tau\|_{L_\sigma,F}. \tag{5.8}$$

These properties will be used in the sequel.

(ii) Conditions (5.3) and (5.5) generally imply that S_F^{uu} and M_F^{uu} are of order h_F^{-1}; this differs from the condition derived in [6] where S_F and M_F are of order 1. Roughly speaking, to be able to eliminate the discrete σ-component, it is necessary to have a stronger control of the interface jumps and of the boundary values of the discrete u-component.

(iii) Condition (5.4) can be weakened to $c_1h_F\|\mathcal{D}^{uu}v\|_{L_u,F}^2 \leq (M_F^{uu}(v), v)_{L_u,F} \leq c_2h_F^{-1}\|v\|_{L_u,F}^2$.

5.2. The direct argument. To perform the error analysis we introduce the following two discrete norms on $W(h)$,

$$\|z\|_{h,A}^2 = \|z^\sigma\|_{L_\sigma}^2 + \|z^u\|_{L_u}^2 + |z^u|_J^2 + |z^u|_M^2 + \sum_{K\in\mathcal{T}_h}\|Bz^u\|_{L_\sigma,K}^2, \tag{5.9}$$

$$\|z\|_{h,1}^2 = \|z\|_{h,A}^2 + \sum_{K\in\mathcal{T}_h}[h_K^{-2}\|z^u\|_{L_u,K}^2 + h_K^{-1}\|z^u\|_{L_u,\partial K}^2 + h_K\|z^\sigma\|_{L_\sigma,\partial K}^2], \tag{5.10}$$

where for all $z^u \in U(h)$ we have introduced the jump semi-norm

$$|z^u|_J^2 = \sum_{F\in\mathcal{F}_h^{\mathrm{i}}}|z^u|_{J,F}^2 \qquad \text{with} \qquad |z^u|_{J,F} = |[\![z^u]\!]|_{S,F}. \tag{5.11}$$

The norm $\|\cdot\|_{h,A}$ is used to measure the approximation error, and the norm $\|\cdot\|_{h,1}$ serves to measure the interpolation properties of the discrete space W_h. In this section, it is implicitly assumed that (DG$^\flat$) holds.

LEMMA 5.4 (L-coercivity). *For all* h *and for all* $z = (z^\sigma, z^u)$ *in* $W(h)$*,*

$$a_h(z, z) \geq \mu_0(\|z^\sigma\|_{L_\sigma}^2 + \|z^u\|_{L_u}^2) + |z^u|_J^2 + \tfrac{1}{2}|z^u|_M^2. \tag{5.12}$$

Proof. Let $z = (z^\sigma, z^u)$ in $W(h)$. Using (4.17) and summing over the mesh elements, we infer

$$\sum_{F\in\mathcal{F}_h^\partial} \tfrac{1}{2}(\mathcal{D}z, z)_{L,F} + \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} \int_F \{z^t\mathcal{D}z\} = \tfrac{1}{2}\sum_{K\in\mathcal{T}_h} [(Tz, z)_{L,K} - (z, \tilde{T}z)_{L,K}].$$

Subtracting this equation from (4.21) and using $\{z^t\mathcal{D}z\} = 2\,\{z^t\}\,\{\mathcal{D}z\}$, together with the skew-symmetry assumptions (DG2) and (DG10), yields

$$a_h(z,z) = |z^u|_J^2 + \tfrac{1}{2}|z^u|_M^2 + \tfrac{1}{2}\sum_{K\in\mathcal{T}_h} [(Tz,z)_{L,K} + (z,\tilde{T}z)_{L,K}].$$

Then, the desired result follows from (A4). ∎

Lemma 5.5 (Stability). *There is $c > 0$, independent of h, such that*

$$\inf_{z_h\in W_h\setminus\{0\}} \sup_{y_h\in W_h\setminus\{0\}} \frac{a_h(z_h,y_h)}{\|z_h\|_{h,A}\|y_h\|_{h,A}} \geq c. \tag{5.13}$$

Proof. (1) Let $z_h = (z_h^\sigma, z_h^u)$ be an arbitrary element in W_h. Let $K\in\mathcal{T}_h$. Denote by $\overline{\mathcal{B}_K^k}$ the mean-value of $\mathcal{B}^k$ over K; then,

$$\|\mathcal{B}^k - \overline{\mathcal{B}_K^k}\|_{[L^\infty(K)]^{m_\sigma,m_u}} \leq h_K\|\mathcal{B}^k\|_{[\mathcal{C}^{0,1}(\overline{\Omega})]^{m_\sigma,m_u}}. \tag{5.14}$$

Now, define the field π_h such that $\pi_h|_K = \sum_{k=1}^d \overline{\mathcal{B}_K^k}\partial_k z_h^u$. Set $\varpi_h = (\pi_h, 0)$. It is clear that $\pi_h\in\Sigma_h$ since $p_u - 1 \leq p_\sigma$; hence, $\varpi_h\in W_h$. Using (5.14), together with the inverse inequalities (4.3) and (4.4), leads, for all $F\subset\partial K$, to

$$\begin{cases} \|\pi_h\|_{L_\sigma,F} \leq c\,h_F^{-\frac{1}{2}}\|\pi_h\|_{L_\sigma,\mathcal{T}(F)}, & \text{if } F\in\mathcal{F}_h^\partial,\\ \|\{\pi_h\}\|_{L_\sigma,F} + \|[\![\pi_h]\!]\|_{L_\sigma,F} \leq c\,h_F^{-\frac{1}{2}}\|\pi_h\|_{L_\sigma,\mathcal{T}(F)}, & \text{if } F\in\mathcal{F}_h^{\mathrm{i}}, \end{cases} \tag{5.15}$$

$$\|\pi_h\|_{L_\sigma,K} \leq \|Bz_h^u\|_{L_\sigma,K} + c\,\|z_h^u\|_{L_u,K}. \tag{5.16}$$

From the definition of a_h it follows that

$$\begin{aligned}\sum_{K\in\mathcal{T}_h} \|Bz_h^u\|_{L_\sigma,K}^2 &= a_h(z_h,\varpi_h) + \sum_{K\in\mathcal{T}_h}(Bz_h^u, Bz_h^u - \pi_h)_{L_\sigma,K}\\ &\quad - (\mathcal{K}^{\sigma\sigma}z_h^\sigma + \mathcal{K}^{\sigma u}z_h^u, \pi_h)_{L_\sigma} - \sum_{F\in\mathcal{F}_h^\partial}\tfrac{1}{2}(M_F^{\sigma u}(z_h^u) - \mathcal{D}^{\sigma u}z_h^u, \pi_h)_{L_\sigma,F}\\ &\quad + \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} 2(\{\mathcal{D}^{\sigma u}z_h^u\}, \{\pi_h\})_{L_\sigma,F} - \sum_{F\in\mathcal{F}_h^{\mathrm{i}}}(S_F^{\sigma u}([\![z_h^u]\!]), [\![\pi_h]\!])_{L_\sigma,F}\\ &= a_h(z_h,\varpi_h) + R_1 + R_2 + R_3 + R_4 + R_5,\end{aligned}$$

where R_1, R_2, R_3, R_4, and R_5 denote the second, third, fourth, fifth, and sixth term in the right-hand side of the above equation, respectively. Each of these terms is bounded from above as follows. For the first term we have

$$\begin{aligned}|R_1| &\leq \sum_{K\in\mathcal{T}_h}|(Bz_h^u, Bz_h^u - \pi_h)_{L_\sigma,K}| \leq \sum_{K\in\mathcal{T}_h}\|Bz_h^u\|_{L_\sigma,K}\|Bz_h^u - \pi_h\|_{L_\sigma,K}\\ &\leq c_\gamma\|z_h^u\|_{L_u}^2 + \gamma\sum_{K\in\mathcal{T}_h}\|Bz_h^u\|_{L_\sigma,K}^2,\end{aligned}$$

where $\gamma > 0$ can be chosen as small as needed. Using (5.16) yields

$$|R_2| = |(\mathcal{K}^{\sigma\sigma} z_h^\sigma + \mathcal{K}^{\sigma u} z_h^u, \pi_h)_{L_\sigma}| \le c(\|z_h^\sigma\|_{L_\sigma} + \|z_h^u\|_{L_u})\|\pi_h\|_{L_\sigma}$$
$$\le c_\gamma(\|z_h^\sigma\|_{L_\sigma}^2 + \|z_h^u\|_{L_u}^2) + \gamma \sum_{K\in\mathcal{T}_h} \|Bz_h^u\|_{L_\sigma,K}^2.$$

Using (DG4) together with (5.15) and (5.16) leads to

$$|R_3| \le \tfrac{1}{2} \sum_{F\in\mathcal{F}_h^\partial} |(M_F^{\sigma u}(z_h^u) - \mathcal{D}^{\sigma u} z_h^u, \pi_h)_{L_\sigma,F}| \le c \sum_{F\in\mathcal{F}_h^\partial} h_F^{\frac{1}{2}} |z_h^u|_{M,F} \|\pi_h\|_{L_\sigma,F}$$
$$\le c_\gamma(\|z_h^u\|_{L_u}^2 + |z_h^u|_M^2) + \gamma \sum_{K\in\mathcal{T}_h} \|Bz_h^u\|_{L_\sigma,K}^2.$$

To bound R_4, observe that $2\{\mathcal{D}^{\sigma u} z_h^u\} = \mathcal{D}^{\sigma u}_{\partial K_1(F)} [\![z_h^u]\!]$ and use (DG15) to infer

$$|R_4| \le 2 \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} |(\{\mathcal{D}^{\sigma u} z_h^u\}, \{\pi_h\})_{L_\sigma,F}| \le c \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} |(\mathcal{D}^{\sigma u}_{\partial K_1(F)} [\![z_h^u]\!], \{\pi_h\})_{L_\sigma,F}|$$
$$\le c \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} h_F^{\frac{1}{2}} |z_h^u|_{J,F} \|\{\pi_h\}\|_{L_\sigma,F} \le c_\gamma(\|z_h^u\|_{L_u}^2 + |z_h^u|_J^2) + \gamma \sum_{K\in\mathcal{T}_h} \|Bz_h^u\|_{L_\sigma,K}^2.$$

To bound $|R_5|$, use (DG14) to infer

$$|R_5| \le \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} |(S_F^{\sigma u}([\![z_h^u]\!]), [\![\pi_h]\!])_{L_\sigma,F}| \le c \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} h_F^{\frac{1}{2}} |z_h^u|_{J,F} \|[\![\pi_h]\!]\|_{L_\sigma,F}$$
$$\le c_\gamma(\|z_h^u\|_{L_u}^2 + |z_h^u|_J^2) + \gamma \sum_{K\in\mathcal{T}_h} \|Bz_h^u\|_{L_\sigma,K}^2.$$

Using the above five bounds, $\gamma = \frac{1}{10}$, and Lemma 5.4 leads to

$$\tfrac{1}{2} \sum_{K\in\mathcal{T}_h} \|Bz_h^u\|_{L_\sigma,K}^2 \le a_h(z_h, \varpi_h) + c\, a_h(z_h, z_h). \tag{5.17}$$

(2) Let us now prove that $\|(\pi_h, 0)\|_{h,A} \le c\|z_h\|_{h,A}$. Owing to (5.16), we infer $\|\pi_h\|_{L_\sigma} \le c(\|z_h^u\|_{L_u} + (\sum_{K\in\mathcal{T}_h} \|Bz_h^u\|_{L_\sigma,K}^2)^{\frac{1}{2}})$. Hence,

$$\|\varpi_h\|_{h,A} = \|\pi_h\|_{L_\sigma} \le c\|z_h\|_{h,A}. \tag{5.18}$$

(3) Owing to (5.12) and (5.17), there is $c_1 > 0$ such that

$$\|z_h\|_{h,A}^2 \le c_1 a_h(z_h, z_h) + a_h(z_h, \varpi_h) = a_h(z_h, c_1 z_h + \varpi_h).$$

Then, setting $y_h = c_1 z_h + \varpi_h$ and using (5.18) yields

$$\|z_h\|_{h,A} \|y_h\|_{h,A} \le c\|z_h\|_{h,A}^2 \le c\, a_h(z_h, y_h).$$

The conclusion follows readily. □

Lemma 5.6 (Continuity). *There is c, independent of h, such that*

$$\forall (z, y_h) \in W(h) \times W_h, \quad a_h(z, y_h) \le c\|z\|_{h,1} \|y_h\|_{h,A}. \tag{5.19}$$

Proof. The main idea is to integrate by parts $a_h(z, y_h)$ by using the formal adjoint $\tilde{T}$. Observing that

$$\sum_{K\in\mathcal{T}_h} [(Tz, y_h)_{L,K} - (z, \tilde{T}y_h)_{L,K}] = \sum_{F\in\mathcal{F}_h^\partial} (\mathcal{D}z, y_h)_{L,F} + \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} \int_F 2\left\{y_h^t \mathcal{D}z\right\},$$

and using $2\left\{y_h^t\mathcal{D}z\right\} = 2\left\{y_h^t\right\}\left\{\mathcal{D}z\right\} + \frac{1}{2}[\![y_h^t]\!][\![\mathcal{D}z]\!]$, we infer

$$\begin{aligned} a_h(z, y_h) = \sum_{K\in\mathcal{T}_h} (z, \tilde{T}y_h)_{L,K} &+ \sum_{F\in\mathcal{F}_h^\partial} \tfrac{1}{2}(M_F(z) + \mathcal{D}z, y_h)_{L,F} \\ &+ \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} \tfrac{1}{2}([\![\mathcal{D}z]\!], [\![y_h]\!])_{L,F} + \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} (S_F([\![z]\!]), [\![y_h]\!])_{L,F}. \end{aligned} \tag{5.20}$$

Let R_1 to R_4 be the four terms in the right-hand side.
(1) Using the Cauchy-Schwarz inequality and inverse inequalities, we obtain

$$|R_1| \le c \sum_{K\in\mathcal{T}_h} \|z\|_{L,K}\|y_h\|_{L,K} + \|z^\sigma\|_{L_\sigma,K}\|By_h^u\|_{L_\sigma,K} + h_K^{-1}\|z^u\|_{L_u,K}\|y_h\|_{L,K}.$$

Hence, $|R_1| \le c\,\|z\|_{h,1}\|y_h\|_{h,A}$.
(2) For the second term, we have

$$\begin{aligned} |R_2| \le \tfrac{1}{2} \sum_{F\in\mathcal{F}_h^\partial} |(M_F^{\sigma u}(z^u) + \mathcal{D}^{\sigma u}z^u, y_h^\sigma)_{L_\sigma,F} &+ (M_F^{uu}(z^u) + \mathcal{D}^{uu}z^u, y_h^u)_{L_u,F} \\ &+ (M_F^{u\sigma}(z^\sigma) + \mathcal{D}^{u\sigma}z^\sigma, y_h^u)_{L_u,F}|. \end{aligned}$$

Using (5.7), (DG5), the boundedness of $\mathcal{D}$, (5.8), and the inverse inequality (4.4), each term in the above equality is bounded as follows:

$$\begin{aligned} &|(M_F^{\sigma u}(z^u) + \mathcal{D}^{\sigma u}z^u, y_h^\sigma)_{L_\sigma,F}| \le c\|z^u\|_{L_u,F}\|y_h^\sigma\|_{L_\sigma,F} \le ch_F^{-\frac{1}{2}}\|z^u\|_{L_u,F}\|y_h^\sigma\|_{L_\sigma,\mathcal{T}(F)}, \\ &|(M_F^{uu}(z^u) + \mathcal{D}^{uu}z^u, y_h^u)_{L_u,F}| \le ch_F^{-\frac{1}{2}}\|z^u\|_{L_u,F}|y_h^u|_{M,F}, \\ &|(M_F^{u\sigma}(z^\sigma) + \mathcal{D}^{u\sigma}z^\sigma, y_h^u)_{L_u,F}| \le ch_F^{\frac{1}{2}}\|z^\sigma\|_{L_\sigma,F}|y_h^u|_{M,F}. \end{aligned}$$

As a result, $|R_2| \le c\,\|z\|_{h,1}\|y_h\|_{h,A}$.
(3) For the third term, we have

$$|R_3| \le \tfrac{1}{2} \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} |([\![\mathcal{D}^{\sigma u}z^u]\!], [\![y_h^\sigma]\!])_{L_\sigma,F} + ([\![\mathcal{D}^{uu}z^u]\!], [\![y_h^u]\!])_{L_u,F} + ([\![\mathcal{D}^{u\sigma}z^\sigma]\!], [\![y_h^u]\!])_{L_u,F}|.$$

Using the boundedness of $\mathcal{D}$, the inverse inequality (4.4), and (DG15), each term in the above equality is bounded as follows:

$$\begin{aligned} &|([\![\mathcal{D}^{\sigma u}z^u]\!], [\![y_h^\sigma]\!])_{L_\sigma,F}| \le c\|\{z^u\}\|_{L_u,F}\|[\![y_h^\sigma]\!]\|_{L_\sigma,F} \le ch_F^{-\frac{1}{2}}\|\{z^u\}\|_{L_u,F}\|y_h^\sigma\|_{L_\sigma,\mathcal{T}(F)}, \\ &|([\![\mathcal{D}^{uu}z^u]\!], [\![y_h^u]\!])_{L_u,F}| \le c\|\{z^u\}\|_{L_u,F}\|[\![y_h^u]\!]\|_{L_u,F} \le ch_F^{-\frac{1}{2}}\|\{z^u\}\|_{L_u,F}\|y_h^u\|_{L_u,\mathcal{T}(F)}, \\ &|([\![\mathcal{D}^{u\sigma}z^\sigma]\!], [\![y_h^u]\!])_{L_u,F}| = |(\{z^\sigma\}, \mathcal{D}^{\sigma u}_{\partial K_1(F)}[\![y_h^u]\!])_{L_\sigma,F}| \le ch_F^{\frac{1}{2}}\|\{z^\sigma\}\|_{L_\sigma,F}|y_h^u|_{J,F}. \end{aligned}$$

As a result, $|R_3| \le c\,\|z\|_{h,1}\|y_h\|_{h,A}$.
(4) For the fourth term, we have

$$|R_4| \le \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} |(S_F^{\sigma u}([\![z^u]\!]),[\![y_h^\sigma]\!])_{L_\sigma,F} + (S_F^{uu}([\![z^u]\!]),[\![y_h^u]\!])_{L_u,F} + (S_F^{u\sigma}([\![z^\sigma]\!]),[\![y_h^u]\!])_{L_u,F}|.$$

Using (DG12) and (DG14), together with the inverse inequality (4.4), each term in the above equality is bounded as follows:

$$\begin{aligned}
|(S_F^{\sigma u}([\![z^u]\!]),[\![y_h^\sigma]\!])_{L_\sigma,F}| &\le c h_F^{\frac12}|z^u|_{J,F}\|[\![y_h^\sigma]\!]\|_{L_\sigma,F} \le c\,|z^u|_{J,F}\|y_h^\sigma\|_{L_\sigma,\mathcal{T}(F)},\\
|(S_F^{uu}([\![z^u]\!]),[\![y_h^u]\!])_{L_u,F}| &\le c h_F^{-\frac12}\|[\![z^u]\!]\|_{L_u,F}|y_h^u|_{J,F},\\
|(S_F^{u\sigma}([\![z^\sigma]\!]),[\![y_h^u]\!])_{L_u,F}| &\le c h_F^{\frac12}\|[\![z^\sigma]\!]\|_{L_\sigma,F}|y_h^u|_{J,F}.
\end{aligned}$$

As a result, $|R_4| \le c\,\|z\|_{h,1}\|y_h\|_{h,A}$. The proof is complete. ☐

LEMMA 5.7 (Consistency). *Let z solve* (2.7) *and let z_h solve* (4.22)*. Assume that $z\in[H^1(\Omega)]^m$. Then,*

$$\forall y_h\in W_h,\quad a_h(z-z_h,y_h)=0. \tag{5.21}$$

Proof. Since z solves (2.7) and $z\in[H^1(\Omega)]^m$, the following properties hold: (i) $Tz=f$ in L, (ii) $\mathcal{M}z=\mathcal{D}z$ a.e. on $\partial\Omega$, and (iii) $\{\mathcal{D}z\}=0$ and $[\![z]\!]=0$ a.e. on $\mathcal{F}_h^{\mathrm{i}}$. Owing to (DG7), property (ii) implies that for all $F\in\mathcal{F}_h^{\partial}$, $M_F(z|_F)=\mathcal{D}z|_F$. As a result, for all $y_h\in W_h$, $a_h(z,y_h)=(Tz,y_h)_L=(f,y_h)_L=a_h(z_h,y_h)$. ☐

THEOREM 5.8 (Convergence). *Let z solve* (2.7) *and let z_h solve* (4.22)*. Assume that $z\in[H^1(\Omega)]^m$. Then, there is c, independent of h, such that*

$$\|z-z_h\|_{h,A} \le c \inf_{y_h\in W_h}\|z-y_h\|_{h,1}. \tag{5.22}$$

Proof. Simple application of the Second Strang Lemma. ☐

Owing to the definition of U_h and Σ_h, i.e., (4.6), and the regularity of the mesh family $\{\mathcal{T}_h\}_{h>0}$, the following interpolation property holds: There is c, independent of h, such that for all $z\in[H^{p_\sigma+1}(\Omega)]^{m_\sigma}\times[H^{p_u+1}(\Omega)]^{m_u}$, there is $y_h\in W_h$ satisfying

$$\|z-y_h\|_{h,1} \le c(h^{p_\sigma+1}+h^{p_u})\big(\|z^\sigma\|_{[H^{p_\sigma+1}(\Omega)]^{m_\sigma}} + \|z^u\|_{[H^{p_u+1}(\Omega)]^{m_u}}\big). \tag{5.23}$$

Since $p_u-1\le p_\sigma\le p_u$, the above interpolation error is of order h^{p_u}.

COROLLARY 5.9. *Let z solve* (2.7) *and let z_h solve* (4.22)*. Then, there is c, independent of h, such that if $z\in[H^{p_\sigma+1}(\Omega)]^{m_\sigma}\times[H^{p_u+1}(\Omega)]^{m_u}$,*

$$\|z-z_h\|_{h,A} \le c\,h^{p_u}\big(\|z^\sigma\|_{[H^{p_\sigma+1}(\Omega)]^{m_\sigma}} + \|z^u\|_{[H^{p_u+1}(\Omega)]^{m_u}}\big). \tag{5.24}$$

Remark 5.3. For both the σ- and the u-component of the solution, the error estimate in the L^2-norm is $\mathcal{O}(h^{p_u})$. If $p_\sigma=p_u=p$ this result is suboptimal when compared with that obtained using the DG method analyzed in [6], which yields $\mathcal{O}(h^{p+\frac12})$ error estimates. The reason for this slight optimality loss is that in the present method the interface jumps of the σ-component are not controlled to allow for this component to be locally eliminated and the jumps on the u-component are penalized with an $\mathcal{O}(h^{-1})$ weight. If $p_\sigma=p_u-1$, the result is still suboptimal for the u-component, but is optimal for the σ-component.

Finally, when the exact solution is not smooth enough to be in $[H^1(\Omega)]^m$ but only in the graph space W, we use a density argument to infer the convergence of the DG approximation. For $z \in W + W_h$, define the norm

$$\|z\|_{W^-} = \|z\|_L + \left(\sum_{K \in \mathcal{T}_h} \|Bz^u\|^2_{L_\sigma} \right)^{\frac{1}{2}}. \tag{5.25}$$

Observe that $\|z\|_{W^-} \leq \|z\|_{h,A}$.

COROLLARY 5.10. *Assume there is $\gamma > 0$ such that $[H^\gamma(\Omega)]^{m_\sigma} \times [H^{1+\gamma}(\Omega)]^{m_u} \cap V$ is dense in V. Let z solve* (2.7) *and let z_h solve* (4.22)*. Then,*

$$\lim_{h \to 0} \|z - z_h\|_{W^-} = 0. \tag{5.26}$$

Proof. Let $\epsilon > 0$. There is $z_\epsilon \in [H^\gamma(\Omega)]^{m_\sigma} \times [H^{1+\gamma}(\Omega)]^{m_u} \cap V$ such that $\|z - z_\epsilon\|_W \leq \frac{\epsilon}{2}$. Let $z_{\epsilon h}$ be the unique solution in W_h such that $a_h(z_{\epsilon h}, y_h) = a(z_\epsilon, y_h)$ for all $y_h \in W_h$. From the regularity of z_ϵ together with Theorem 5.8 and Corollary 5.9, it is inferred that $\lim_{h\to 0} \|z_{\epsilon h} - z_\epsilon\|_{h,A} = 0$. Furthermore, using the discrete inf-sup condition (5.13) yields

$$\begin{aligned}\|z_{\epsilon h} - z_h\|_{W^-} &\leq \sup_{y_h \in W_h \setminus \{0\}} \frac{a_h(z_{\epsilon h}, y_h) - a_h(z_h, y_h)}{\|y_h\|_{h,A}} = \sup_{y_h \in W_h \setminus \{0\}} \frac{a(z_\epsilon - z, y_h)}{\|y_h\|_{h,A}} \\ &\leq \|T(z_\epsilon - z)\|_L \sup_{y_h \in W_h \setminus \{0\}} \frac{\|y_h\|_L}{\|y_h\|_{h,A}} \leq \|z - z_\epsilon\|_W \leq \frac{\epsilon}{2},\end{aligned}$$

where we have used the fact that for all $y_h \in W_h$, $a_h(z_h, y_h) = a(z, y_h)$. Finally, using the triangle inequality $\|z - z_h\|_{W^-} \leq \|z - z_\epsilon\|_{W^-} + \|z_\epsilon - z_{\epsilon h}\|_{W^-} + \|z_{\epsilon h} - z_h\|_{W^-}$, it is deduced that $\limsup_{h \to 0} \|z - z_h\|_{W^-} \leq \epsilon$. ☐

5.3. The duality argument. We now improve the error estimate on the L^2-norm of the u-component of the solution by using a duality argument. In this section, it is implicitly assumed that (DG$^\sharp$) holds.

Let z solve (2.7) and let z_h solve (4.22). Let $\psi \in V^*$ solve

$$\tilde{T}\psi = (0, z^u - z^u_h). \tag{5.27}$$

Assume that the above problem yields (elliptic) regularity, i.e., there is c, independent of h, such that

$$\|\psi^u\|_{[H^2(\Omega)]^{m_u}} + \|\psi^\sigma\|_{[H^1(\Omega)]^{m_\sigma}} \leq c\|z^u - z^u_h\|_{L_u}. \tag{5.28}$$

LEMMA 5.11. *Under the above hypotheses, the following holds:*

$$a_h(y, \psi) = (y^u, z^u - z^u_h)_{L_u}, \quad \forall y \in W(h). \tag{5.29}$$

Proof. Let $y \in W(h)$. By integrating by parts (i.e., using (5.20)) and using the fact that ψ is continuous across interfaces, we obtain

$$a_h(y, \psi) = \sum_{K \in \mathcal{T}_h} (y, \tilde{T}\psi)_{L,K} + \sum_{F \in \mathcal{F}^\partial_h} \tfrac{1}{2}(M_F(y) + \mathcal{D}y, \psi)_{L,F}.$$

Since $\psi \in V^*{:=}\mathrm{Ker}(M^*+D)$ (see (2.6)) and $\psi \in [H^1(\Omega)]^m$, it is inferred that $\mathcal{M}^t\psi + \mathcal{D}\psi = 0$ a.e. on $\partial\Omega$. Hence, owing to (DG8), $(M_F(y)+\mathcal{D}y,\psi)_{L,F} = 0$ for all $F \in \mathcal{F}_h^\partial$. The conclusion is straightforward since ψ solves (5.27). ∎

To avoid lengthy technicalities, we introduce the following norms:

$$\|y^\sigma\|_{h,\widetilde{1}} = \Big(\sum_{K\in\mathcal{T}_h}[h_K^2\|y^\sigma\|^2_{[H^1(K)]^{m_\sigma}} + h_K\|y^\sigma\|^2_{L_\sigma,\partial K}]\Big)^{\frac12}, \tag{5.30}$$

$$\|y\|_{h,A^+} = \|y\|_{h,A} + \|y^\sigma\|_{h,\widetilde{1}}, \tag{5.31}$$

$$\|y\|_{h,1^+} = \|y\|_{h,1} + \|y^\sigma\|_{h,\widetilde{1}}. \tag{5.32}$$

The DG method converges optimally in the $\|\cdot\|_{h,A^+}$-norm as stated in the following

Corollary 5.12. *Let $z \in [H^1(\Omega)]^m$ solve (2.7) and let z_h solve (4.22). Then, there is c, independent of h, such that*

$$\|z-z_h\|_{h,A^+} \le c \inf_{y_h\in W_h}\|z-y_h\|_{h,1^+}. \tag{5.33}$$

Proof. Let y_h be an arbitrary element in W_h. Using inverse inequalities yields

$$\begin{aligned}
\|z^\sigma - z_h^\sigma\|_{h,\widetilde{1}} &\le \|z^\sigma - y_h^\sigma\|_{h,\widetilde{1}} + \|y_h^\sigma - z_h^\sigma\|_{h,\widetilde{1}} \le \|z^\sigma - y_h^\sigma\|_{h,\widetilde{1}} + c\|y_h^\sigma - z_h^\sigma\|_{L_\sigma}\\
&\le \|z^\sigma - y_h^\sigma\|_{h,\widetilde{1}} + c(\|y_h^\sigma - z^\sigma\|_{L_\sigma} + \|z^\sigma - z_h^\sigma\|_{L_\sigma})\\
&\le \|z^\sigma - y_h^\sigma\|_{h,\widetilde{1}} + c(\|z-y_h\|_{h,A} + \|z-z_h\|_{h,A})\\
&\le c(\|z-y_h\|_{h,A^+} + \|z-z_h\|_{h,A}).
\end{aligned}$$

Hence, using the above inequality along with (5.22) leads to

$$\|z-z_h\|_{h,A^+} \le c(\|z-y_h\|_{h,A^+} + \|z-y_h\|_{h,1}) \le c\|z-y_h\|_{h,1^+}.$$

That concludes the proof since y_h is arbitrary in W_h. ∎

Lemma 5.13 (Continuity). *There is c, independent of h, such that for all (r,y) in $W(h){\times}W(h)$,*

$$a_h(r,y) \le c\,\|r\|_{h,A^+}\|y\|_{h,1}. \tag{5.34}$$

Proof. Let us use (4.21) and bound all the terms in the right-hand side.
(1) For the first term, say R_1, we proceed as follows:

$$\begin{aligned}
|(Tr,y)_{L,K}| &\le |(Kr,y)_{L,K}| + |(Br^u,y^\sigma)_{L_\sigma,K}| + |((\tilde{B}+C)r^\sigma,y^u)_{L_u,K}|\\
&\le c\|r\|_{L,K}\|y\|_{L,K} + \|Br^u\|_{L_\sigma,K}\|y^\sigma\|_{L_\sigma,K} + c\|r^\sigma\|_{[H^1(K)]^{m_\sigma}}\|y^u\|_{L_u,K}\\
&\le c(\|r\|^2_{L,K} + \|Br^u\|^2_{L_\sigma,K} + h_K^2\|r^\sigma\|^2_{[H^1(K)]^{m_\sigma}})^{\frac12}(\|y\|^2_{L,K} + h_K^{-2}\|y^u\|^2_{L_u,K})^{\frac12}.
\end{aligned}$$

Hence, $|R_1| \le c\,\|r\|_{h,A^+}\|y\|_{h,1}$.
(2) To bound the second term, say R_2, use (DG4), (DG6), (5.7), and the boundedness of $\mathcal{D}$ to infer

$$\begin{aligned}
&|(M_F^{\sigma u}(r^u) - \mathcal{D}^{\sigma u}r^u, y^\sigma)_{L_\sigma,F}| \le c|r^u|_{M,F}\,h_F^{\frac12}\|y^\sigma\|_{L_\sigma,F},\\
&|(M_F^{uu}(r^u) - \mathcal{D}^{uu}r^u, y^u)_{L_u,F}| \le c|r^u|_{M,F}\,h_F^{-\frac12}\|y^u\|_{L_u,F},\\
&|(M_F^{u\sigma}(r^\sigma) - \mathcal{D}^{u\sigma}r^\sigma, y^u)_{L_u,F}| \le c\|r^\sigma\|_{L_\sigma,F}\|y^u\|_{L_u,F} \le ch_F^{\frac12}\|r^\sigma\|_{L_\sigma,F}\,h_F^{-\frac12}\|y^u\|_{L_u,F}.
\end{aligned}$$

As a result, $|R_2| \leq c\, \|r\|_{h,A^+} \|y\|_{h,1}$.
(3) To bound the third term, say R_3, use (DG15), (DG16), and the boundedness of $\mathcal{D}$ to infer

$$
\begin{aligned}
|(\{\mathcal{D}^{\sigma u} r^u\}, \{y^\sigma\})_{L_\sigma,F}| &= |2(\mathcal{D}^{\sigma u}_{\partial K_1(F)} [\![r^u]\!], \{y^\sigma\})_{L_\sigma,F}| \leq c|r^u|_{J,F}\, h_F^{\frac{1}{2}} \|\{y^\sigma\}\|_{L_\sigma,F},\\
|(\{\mathcal{D}^{uu} r^u\}, \{y^u\})_{L_u,F}| &= |2(\mathcal{D}^{uu}_{\partial K_1(F)} [\![r^u]\!], \{y^u\})_{L_u,F}| \leq c|r^u|_{J,F}\, h_F^{-\frac{1}{2}} \|\{y^u\}\|_{L_u,F},\\
|(\{\mathcal{D}^{u\sigma} r^\sigma\}, \{y^u\})_{L_u,F}| &\leq c\|[\![r^\sigma]\!]\|_{L_\sigma,F} \|\{y^u\}\|_{L_u,F} \leq c h_F^{\frac{1}{2}} \|[\![r^\sigma]\!]\|_{L_\sigma,F}\, h_F^{-\frac{1}{2}} \|\{y^u\}\|_{L_u,F}.
\end{aligned}
$$

These bounds yield $|R_3| \leq c\, \|r\|_{h,A^+} \|y\|_{h,1}$.
(4) To bound the fourth term, say R_4, use (DG10), (DG13), and (DG14) to infer

$$
\begin{aligned}
|(S_F^{\sigma u}([\![r^u]\!]), [\![y^\sigma]\!])_{L_\sigma,F}| &\leq c|r^u|_{J,F}\, h_F^{\frac{1}{2}} \|[\![y^\sigma]\!]\|_{L_\sigma,F},\\
|(S_F^{uu}([\![r^u]\!]), [\![y^u]\!])_{L_u,F}| &\leq c|r^u|_{J,F}\, h_F^{-\frac{1}{2}} \|[\![y^u]\!]\|_{L_u,F},\\
|(S_F^{u\sigma}([\![r^\sigma]\!]), [\![y^u]\!])_{L_u,F}| &\leq c h_F^{\frac{1}{2}} \|[\![r^\sigma]\!]\|_{L_\sigma,F}\, |y^u|_{J,F}.
\end{aligned}
$$

Hence, $|R_4| \leq c\, \|r\|_{h,A^+} \|y\|_{h,1}$. The proof is complete. □

THEOREM 5.14 (Convergence). *Let $z \in [H^1(\Omega)]^m$ solve* (2.7) *and let z_h solve* (4.22)*. Then, there is c, independent of h, such that*

$$
\|z^u - z_h^u\|_{L_u} \leq ch \inf_{y_h \in W_h} \|z - y_h\|_{h,1^+}. \tag{5.35}
$$

Proof. Using $z - z_h$ as test function in (5.29) we infer $a_h(z - z_h, \psi) = \|z^u - z_h^u\|^2_{L_u}$. Then, using the consistency property stated in Lemma 5.7, this yields for all $\psi_h \in W_h$, $a_h(z - z_h, \psi - \psi_h) = \|z^u - z_h^u\|^2_{L_u}$. Lemma 5.13 in turn implies

$$
\|z^u - z_h^u\|^2_{L_u} \leq c\|z - z_h\|_{h,A^+} \|\psi - \psi_h\|_{h,1}, \quad \forall \psi_h \in W_h.
$$

Then, using the elliptic regularity (5.28) leads to

$$
\begin{aligned}
\|z^u - z_h^u\|^2_{L_u} &\leq c\|z - z_h\|_{h,A^+} \inf_{\psi_h \in W_h} \|\psi - \psi_h\|_{h,1}\\
&\leq ch\|z - z_h\|_{h,A^+} (\|\psi^u\|_{[H^2(\Omega)]^{m_u}} + \|\psi^\sigma\|_{[H^1(\Omega)]^{m_\sigma}})\\
&\leq ch\|z - z_h\|_{h,A^+} \|z^u - z_h^u\|_{L_u}.
\end{aligned}
$$

The conclusion follows readily using Corollary 5.12. □

Remark 5.4. Stability and convergence in the $\|\cdot\|_{h,A^+}$-norm could have been proved directly by adding the quantity $(\sum_{K\in\mathcal{T}_h} h_K^2 \|\tilde{B} y^\sigma + C y^u\|^2_{L_u,K})^{\frac{1}{2}}$ in the definition of the $\|\cdot\|_{h,A}$-norm, but this significantly lengthens the proof of Lemma 5.5.

6. Applications.

In this section we apply the DG method designed in §4 and analyzed in §5 to the Friedrichs systems presented in §3.

6.1. Advection–diffusion–reaction.

We describe various DG methods that can be used to approximate the advection–diffusion–reaction equation introduced in §3.1 and in which the σ-component of the unknown can be eliminated locally. Comparisons with the unified approached developed by Arnold et al. [1] are presented to illustrate the fact that the present DG method generalizes some of the DG methods that have been previously developed in the literature for the Poisson equation.

6.1.1. A first example (The LDG method). Consider first Dirichlet boundary conditions. Owing to (3.5) and (3.6), the integral representations (4.9) and (4.11) hold with the $\mathbb{R}^{d+1,d+1}$-valued boundary fields

$$\mathcal{D}_{\partial\Omega} = \left[\begin{array}{c|c} 0 & n \\ \hline n^t & \beta{\cdot}n \end{array}\right] \qquad \text{and} \qquad \mathcal{M} = \left[\begin{array}{c|c} 0 & -n \\ \hline n^t & 0 \end{array}\right], \tag{6.1}$$

where n is the unit outward normal to $\partial\Omega$. Let $\varsigma > 0$ and $\eta > 0$ (these design parameters can vary from face to face). For all $F \in \mathcal{F}_h$, set

$$\mathcal{M}_F = \left[\begin{array}{c|c} 0 & -n \\ \hline n^t & \varsigma h_F^{-1} \end{array}\right] \qquad \text{and} \qquad \mathcal{S}_F = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \eta h_F^{-1} \end{array}\right] \tag{6.2}$$

and define for all $y \in [L^2(F)]^{d+1}$, $M_F(y) = \mathcal{M}_F y$ and $S_F(y) = \mathcal{S}_F y$.

LEMMA 6.1. *Let M_F and S_F be defined as above. Then, properties* (DG$^\sharp$) *hold.*

Proof. The consistency properties (DG7) and (DG8) are readily verified. The remaining properties are a direct consequence of Lemmas 5.1 and 5.3. ∎

Remark 6.1. Let $\delta \in \mathbb{R}^d$. A slightly more general choice for the interface operator consists of setting for all $F \in \mathcal{F}_h^{\mathrm{i}}$, $\mathcal{S}_F^{\sigma u} = (\delta{\cdot}n_F)n_F$ where n_F is any of the two unit normal vectors to F. This choice leads to the so-called LDG method of Cockburn and Shu [4] as considered in the unified approach of [1] for the Poisson equation.

When Neumann and Robin boundary conditions are enforced, the integral representation (4.11) holds for the $\mathbb{R}^{d+1,d+1}$-valued boundary field

$$\mathcal{M} = \left[\begin{array}{c|c} 0 & n \\ \hline -n^t & 2\varrho + \beta{\cdot}n \end{array}\right]. \tag{6.3}$$

For all $F \in \mathcal{F}_h^{\partial}$, choose $\mathcal{M}_F = \mathcal{M}$ and for all $y \in [L^2(F)]^{d+1}$, define $M_F(y) = \mathcal{M}_F y$. Then, it is easily verified that (5.4) holds for Neumann boundary conditions and also for Robin boundary conditions provided $\varrho \geq (\beta{\cdot}n)^-$, the negative part of $\beta{\cdot}n$ (this is not restrictive since the usual Robin condition at an inflow boundary uses $\varrho = -\beta{\cdot}n \geq 0$). Hence, Lemma 5.2 implies that assumptions (DG1)–(DG6) hold. Moreover, the consistency assumptions (DG7) and (DG8) trivially hold.

Remark 6.2. Observe that the scalings of the block $\mathcal{M}_F^{uu}$ are radically different whether Dirichlet boundary conditions or Robin/Neumann are enforced.

6.1.2. Comparison with other methods. In this section we restrict the setting to the equation $u - \Delta u = f$ and to the homogeneous Dirichlet boundary conditions so as to make comparisons with the unified approach developed in [1] where it is shown that most of the DG methods amount to solving the following problem:

$$\begin{cases} \text{Seek } z_h = (\sigma_h, u_h) \in W_h \text{ such that } \forall y_h \in [\mathbb{P}_{p_\sigma}(K)]^d \times \mathbb{P}_{p_u}(K), \\ (Tz_h, y_h)_{L,K} + (\widehat{\phi}_{\partial K}(z_h) - \mathcal{D}_{\partial K} z_h^i, y_h)_{L,\partial K} = (f, y_h)_{L,K}, \end{cases} \tag{6.4}$$

where the so-called numerical fluxes $\widehat{\phi}_{\partial K}(z_h)$ depend on the method under consideration. In view of (4.25) and (4.26), the link between the present formalism and that of [1] is based on following identification:

$$\widehat{\phi}_{\partial K}(z_h)|_F = \phi_{\partial K}(z_h)|_F := \begin{cases} \frac{1}{2}(M_F(z) + \mathcal{D}z) & \text{if } F \subset \partial K^{\partial}, \\ S_F([\![z]\!]_{\partial K}|_F) + \mathcal{D}_{\partial K}\,\{z\}_{\partial K} & \text{if } F \subset \partial K^{\mathrm{i}}. \end{cases} \tag{6.5}$$

For the purpose of comparison, we restrict ourselves to boundary and interface operators such that for all $F \in \mathcal{F}_h$, for all $v \in L^2(F)$, and for all $\tau \in [L^2(F)]^d$,

$$M_F^{\sigma u}(v) = -nv, \qquad M_F^{u\sigma}(\tau) = \tau{\cdot}n, \tag{6.6}$$
$$S_F^{\sigma u}(v) = 0, \qquad S_F^{u\sigma}(\tau) = 0. \tag{6.7}$$

Therefore, the methods that can be constructed from this set of assumptions only differ in the design of M_F^{uu} and S_F^{uu}.

Following the notation in [1], we set $\widehat{\phi}_{\partial K}(z_h) = (\widehat{u}_{\partial K}, \widehat{\sigma}_{\partial K})$ (note that $\widehat{u}$ is $\mathbb{R}^d$-valued whereas $\widehat{\sigma}$ is $\mathbb{R}$-valued). Then, the identification (6.5) is possible if the DG method under consideration is such that

$$(\widehat{u}_{\partial K}, \widehat{\sigma}_{\partial K})|_F = \begin{cases} (0, \sigma_h{\cdot}n_K + \frac{1}{2}M_F^{uu}(u_h|_F)) & \text{if } F \subset \partial K^{\partial}, \\ (n_K\left\{u_h\right\}_{\partial K}, \left\{\sigma_h\right\}_{\partial K}|_F{\cdot}n_K + S_F^{uu}([\![u_h]\!]_{\partial K}|_F)) & \text{if } F \subset \partial K^{\mathrm{i}}, \end{cases} \tag{6.8}$$

The DG methods that belong to this class are those from [2, 3, 5].

Comparison with the method of Brezzi et al. Let $F \in \mathcal{F}_h$. Define the mapping $r_F : [L^2(F)]^d \longrightarrow \Sigma_h$ so that for all $z^\sigma \in [L^2(F)]^d$, $r_F(z^\sigma)$ solves

$$(r_F(z^\sigma), y_h^\sigma)_{L_\sigma} = (z^\sigma, \{y_h^\sigma\})_{L_\sigma,F}, \quad \forall y_h^\sigma \in \Sigma_h. \tag{6.9}$$

Note that the support of $r_F(z^\sigma)$ is contained in $\mathcal{T}(F)$. Then, the method described by Brezzi *et al.* [3] is such that for all $v \in L^2(F)$,

$$M_F^{uu}(v) = \zeta r_F(vn_F){\cdot}n_F, \qquad S_F^{uu}(v) = \kappa\left\{r_F(vn_F)\right\}{\cdot}n_F, \tag{6.10}$$

where n_F is any of the two unit normal vectors to F and where ζ and κ are positive constants; see also [1]. The operator r_F is endowed with the following property.

LEMMA 6.2. *There are c_1 and c_2, independent of h, such that for all $F \in \mathcal{F}_h$ and for all $\tau_h \in [\mathbb{P}_{p_\sigma}(F)]^d$, $c_1 h_F^{-\frac{1}{2}}\|\tau_h\|_{L_\sigma,F} \le \|r_F(\tau_h)\|_{L_\sigma} \le c_2 h_F^{-\frac{1}{2}}\|\tau_h\|_{L_\sigma,F}$.*

Then, it is easily deduced from Lemma 6.2 and the definition of r_F that there are c_1 and c_2, independent of h, such that for all $F \in \mathcal{F}_h$ and for all $v_h \in \mathbb{P}_{p_u}(F)$,

$$c_1 h_F^{-1}\|v_h\|^2_{L_u,F} \le (\left\{r_F(v_h n_F)\right\}{\cdot}n_F, v_h)_{L_u,F} \le c_2 h_F^{-1}\|v_h\|^2_{L_u,F}. \tag{6.11}$$

These inequalities are just what is takes to prove that if the boundary and interface operators are defined using (6.6), (6.7), and (6.10), properties (DG$^\sharp$) hold. Therefore, the conclusions of Theorem 5.8 and of Theorem 5.14 hold.

Comparison with the method of Douglas and Dupont. The method introduced by Douglas and Dupont [5] is the so-called Interior Penalty (IP) method. Using the same definition for the mapping r_F as in (6.9), the IP method consists of setting for all $v \in L^2(F)$,

$$M_F^{uu}(v) = \tfrac{\zeta}{h_F}v - r_F(vn_F){\cdot}n_F, \qquad S_F^{uu}(v) = \tfrac{\kappa}{h_F}v - \left\{r_F(vn_F)\right\}{\cdot}n_F, \tag{6.12}$$

where ζ and κ are positive constants; see also [1]. Then, by using the same arguments as above, we infer that the IP method satisfies all the required properties, i.e., (DG$^\sharp$), provided the constants ζ and κ are large enough.

Comparison with the method of Bassi and Rebay. The method proposed by Bassi and Rebay [2] corresponds to the choice $M_F^{uu} = 0$ and $S_F^{uu} = 0$. Our analysis needs to be revised to account for this situation. Obviously, the L^2-coercivity still holds in the form $a_h(y, y) \geq c\,\|y\|_L^2$ for all $y \in W(h)$. Moreover, one easily derives the following continuity estimate: For all $(y, y_h) \in W(h) \times W_h$,

$$|a_h(y, y_h)| \leq c \left(\sum_{K \in \mathcal{T}_h} [\|Ty\|_{L,K}^2 + h_K^{-1} \|y\|_{L,\partial K}^2] \right)^{\frac{1}{2}} \|y_h\|_L. \tag{6.13}$$

Then, provided $p_\sigma = p_u := p$, the second Strang Lemma implies $\|z - z_h\|_L \leq ch^p \|z\|_{[H^{p+1}(\Omega)]^m}$. Although this estimate is not optimal, it shows that the method of Bassi and Rebay is (possibly non-optimally) convergent.

6.2. Linear elasticity. Consider the linear elasticity equations introduced in §3.2 and let us describe a DG method where the $(\overline{\sigma}, p)$-component of the unknown can be eliminated locally. Owing to (3.14) and (3.15), the integral representations (4.9) and (4.11) hold with the $\mathbb{R}^{m,m}$-valued boundary fields (recall that $m = d^2 + 1 + d$)

$$\mathcal{D}_{\partial\Omega} = \left[\begin{array}{c:c} 0 & \mathcal{H} \\ \hdashline \mathcal{H}^t & 0 \end{array}\right] \qquad \text{and} \qquad \mathcal{M} = \left[\begin{array}{c:c} 0 & -\mathcal{H} \\ \hdashline \mathcal{H}^t & 0 \end{array}\right], \tag{6.14}$$

where $\mathcal{H} = \sum_{k=1}^d n_k (\mathcal{E}^k, 0)^t \in \mathbb{R}^{d^2+1,d}$. Observe that for all $\xi \in \mathbb{R}^d$, $\mathcal{H}\xi = (-\frac{1}{2}(n \otimes \xi + \xi \otimes n), 0)$. Let $\varsigma > 0$ and $\eta > 0$ (these design parameters can vary from face to face). For all $F \in \mathcal{F}_h$, set

$$\mathcal{M}_F = \left[\begin{array}{c:c} 0 & -\mathcal{H} \\ \hdashline \mathcal{H}^t & \varsigma h_F^{-1} \mathcal{I}_d \end{array}\right] \qquad \text{and} \qquad \mathcal{S}_F = \left[\begin{array}{c:c} 0 & 0 \\ \hdashline 0 & \eta h_F^{-1} \mathcal{I}_d \end{array}\right], \tag{6.15}$$

and define for all $y \in [L^2(F)]^m$, $M_F(y) = \mathcal{M}_F y$ and $S_F(y) = \mathcal{S}_F y$. Then, using Lemmas 5.1 and 5.3, one readily verifies that properties (DG$^\sharp$) hold.

6.3. Simplified MHD. Consider the simplified MHD equations introduced in §3.3 and let us describe a DG method where the H-component of the unknown can be eliminated locally (the derivation of a DG method where the E-component of the unknown can be eliminated locally is similar). To apply the setting of §5, set $\sigma \equiv H$ and $u \equiv E$. Owing to (3.19) and (3.20), the integral representations (4.9) and (4.11) hold with the $\mathbb{R}^{6,6}$-valued boundary fields

$$\mathcal{D}_{\partial\Omega} = \left[\begin{array}{c:c} 0 & \mathcal{N} \\ \hdashline \mathcal{N}^t & 0 \end{array}\right] \qquad \text{and} \qquad \mathcal{M} = \left[\begin{array}{c:c} 0 & -\mathcal{N} \\ \hdashline \mathcal{N}^t & 0 \end{array}\right], \tag{6.16}$$

where $\mathcal{N} = \sum_{k=1}^3 n_k \mathcal{R}^k$, and the $\mathbb{R}^{3,3}$-valued fields $\mathcal{R}^1$, $\mathcal{R}^2$, and $\mathcal{R}^3$ are defined in §3.3. Observe that for all $\xi \in \mathbb{R}^3$, $\mathcal{N}\xi = n \times \xi$. Let $\varsigma > 0$ and $\eta > 0$ (these design parameters can vary from face to face). For all $F \in \mathcal{F}_h$, set

$$\mathcal{M}_F = \left[\begin{array}{c:c} 0 & -\mathcal{N} \\ \hdashline \mathcal{N}^t & \varsigma h_F^{-1} \mathcal{N}^t \mathcal{N} \end{array}\right] \qquad \text{and} \qquad \mathcal{S}_F = \left[\begin{array}{c:c} 0 & 0 \\ \hdashline 0 & \eta h_F^{-1} \mathcal{N}_F^t \mathcal{N}_F \end{array}\right], \tag{6.17}$$

where $\mathcal{N}_F$ is defined as $\mathcal{N}$ by using n_F instead of n. For all $y \in [L^2(F)]^6$, let $M_F(y) = \mathcal{M}_F y$ and $S_F(y) = \mathcal{S}_F y$. Then, using Lemmas 5.1 and 5.3, one readily verifies that properties (DG$^\sharp$) hold.

Remark 6.3. As opposed to advection–diffusion–reaction equations, the upper bound in (5.3) and (5.5) is not sharp for the simplified MHD equations since the operators M_F and S_F do not need to control the whole L^2-norm of the electric field but only that of its tangential component.

7. Conclusions. It happens sometimes that (A4) does not hold; instead, the following weaker inequality holds:

$$\exists \mu_0 > 0, \quad \forall z \in W, \quad (Tz, z)_L + (z, \tilde{T}z)_L \geq 2\mu_0 \|\pi z^\sigma\|^2_{L_\sigma}, \tag{7.1}$$

where $\pi \in \mathcal{L}(L_\sigma; L_\sigma)$. In other words, coercivity no longer holds for the u-component of the unknown and holds only for some part of the σ-component, namely πz^σ. The equation $-\Delta u = f$ corresponds to this situation with π equal to the identity. The Stokes equations and the linear elasticity equations in the incompressible limit fall also in this framework with a nontrivial operator π. It will be shown in a forthcoming third part, that provided additional mild assumptions are made on the differential operators and the DG setting, all that has been said herein in the fully L-coercive case remains valid in the situation with partial coercivity.

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