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Abstract

This paper deals with the control of nonlinear systems in the presence of state and control constraints for discrete time dynamics in finite dimensional spaces. The viability kernel is known to play a basic role for the analysis of such problems and the design of viable control feedbacks. Unfortunately, this kernel may display very non regular geometry and its computation is not an easy task in general. In the present paper, we show how monotonic properties of both dynamics and constraints allow for relevant analytical upper and lower approximations of the viability kernel through weakly and strongly invariant sets. An example on fish harvesting management illustrates some of the assertions.

Key words: control, state constraints, viability, invariance, monotonicity.

1 Introduction

Let us consider a nonlinear control system described in discrete time by the difference equation

$$\begin{cases} x_{t+1} = f(x_t, u_t), \quad t \geq 0, \\ x_0 \quad \text{given}, \end{cases} \tag{1}$$

where the *state variable* x_t belongs to the finite dimensional state space $\mathbb{X} = \mathbb{R}^{n_{\mathbb{X}}}$, the *control variable* u_t is an element of the *control set* $\mathbb{U} = \mathbb{R}^{n_{\mathbb{U}}}$ while the dynamics f maps $\mathbb{X} \times \mathbb{U}$ into $\mathbb{X}$.

A controller or a decision maker describes “desirable configurations of the system” through a set $\mathcal{K} \subset \mathbb{X} \times \mathbb{U}$ termed the *desirable set*

$$(x_t, u_t) \in \mathcal{K}, \quad t \geq 0, \tag{2}$$

where $\mathcal{K}$ includes both system states and controls constraints. Typical instances of such a desirable set are given by inequalities requirements:

$$\mathcal{K} = \{(x, u) \in \mathbb{X} \times \mathbb{U} \mid \forall i = 1, \ldots, p, \quad g_i(x, u) \geq 0\}.$$

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The *state constraints set* associated with $\mathcal{K}$ is obtained by projecting the desirable set $\mathcal{K}$ onto the state space $\mathbb{X}$:

$$\mathbb{V}^0 \stackrel{\text{def}}{=} \text{Proj}_{\mathbb{X}}(\mathcal{K}) = \{x \in \mathbb{X} \mid \exists u \in \mathbb{U}\,, \quad (x,u) \in \mathcal{K}\}. \tag{3}$$

Such problems of dynamic control under constraints refers to viability [1] or invariance [8] framework. Basically, such an approach focuses on inter-temporal feasible paths. It has been applied for instance to models related to the sustainable management of resource and bio-economic modeling as in [10, 3, 5, 4, 9, 11, 14]. From the mathematical viewpoint, most of viability and weak invariance results are addressed in the time continuous case. However, some mathematical works deal with the discrete-time case. This includes the study of numerical schemes for the approximation of the viability problems of the continuous dynamics as in [1, 12]. Important contributions for discrete time case are also captured by the study of the positivity for linear systems as in [6] or by the hybrid control as in [2, 15].

Viability is defined as the ability to choose, at each time step t, a control $u_t \in \mathbb{U}$ such that the system configuration remains desirable. More precisely, the system is viable if the following feasible set is not empty:

$$\mathbb{V}(f,\mathcal{K}) \stackrel{\text{def}}{=} \left\{ x_0 \in \mathbb{X} \;\middle|\; \begin{array}{c} \exists\, (u_0, u_1, \dots) \text{ and } (x_0, x_1, \dots) \\ \text{satisfying (1) and (2)} \end{array} \right\}. \tag{4}$$

The set $\mathbb{V}(f,\mathcal{K})$ is called the *viability kernel* associated with the dynamics f and the desirable set $\mathcal{K}$. By definition, we have $\mathbb{V}(f,\mathcal{K}) \subset \mathbb{V}^0$ but, in general, the inclusion is strict. For a decision maker or control designer, knowing the viability kernel has practical interest since it describes the states from which controls can be found that maintain the system in a desirable configuration forever. However, computing this kernel is not an easy task in general.

The present paper aims at giving explicit upper and lower approximations of this kernel using weak (viable) or strong invariant domains in the specific context of monotonic properties of both constraints and dynamics. To achieve this let us recall what is meant by weakly or strong invariant domains.

A subset $\mathbb{V}$ of the state space $\mathbb{X}$ is said to be *strongly invariant* for the dynamics f in the desirable set $\mathcal{K}$ if

$$\forall x \in \mathbb{V}\,, \quad \forall u \in \mathbb{U}\,, \quad (x,u) \in \mathcal{K} \Longrightarrow f(x,u) \in \mathbb{V}. \tag{5}$$

That is, if one starts from $\mathbb{V}$, any control may transfer the state in $\mathbb{V}$ into a desirable configuration. This is generally a too demanding requirement.

Similarly, a subset $\widetilde{\mathbb{V}}$ is said to be *weakly invariant* for the dynamics f in the desirable set $\mathcal{K}$, or a *viability domain* of f in $\mathcal{K}$, if

$$\forall x \in \widetilde{\mathbb{V}}\,, \quad \exists u \in \mathbb{U}\,, \quad (x,u) \in \mathcal{K} \text{ and } f(x,u) \in \widetilde{\mathbb{V}}. \tag{6}$$

That is, if one starts from $\widetilde{\mathbb{V}}$, a suitable control may transfer the state in $\widetilde{\mathbb{V}}$ and the system into a desirable configuration. In particular, it is worth pointing out that any desirable equilibrium is a viability domain of f in $\mathcal{K}$.

A *desirable equilibrium* is an equilibrium of the system that belongs to $\mathcal{K}$, that is a pair $(\bar{x}, \bar{u}) \in \mathcal{K}$ such that $\bar{x} = f(\bar{x}, \bar{u})$. Moreover, according to viability theory [1], the viability kernel $\mathbb{V}(f,\mathcal{K})$ turns out to be the union of all viability domains, that is the largest set such that

$$\mathbb{V}(f,\mathcal{K}) = \bigcup \left\{ \widetilde{\mathbb{V}},\ \widetilde{\mathbb{V}} \subset \mathbb{V}^0,\ \widetilde{\mathbb{V}} \text{ weakly invariant for } f \text{ in } \mathcal{K} \right\}. \tag{7}$$

For the sake of completeness, we recall briefly the proof in the Appendix (see Proposition 9). A major interest of such a property lies in the fact that any weakly invariant set for the dynamics f in the desirable set $\mathcal{K}$ provides a *lower approximation* of the viability kernel. The characterization of weakly invariant domains also gives hints for an algorithmic method of computation. Consider the following decreasing sequence of subsets of $\mathbb{X}$ by

$$\mathbb{V}_0 \stackrel{\mathrm{def}}{=} \mathbb{V}^0 \quad \text{and} \quad \mathbb{V}_{k+1} \stackrel{\mathrm{def}}{=} \{x \in \mathbb{V}_k \mid \exists u \in \mathbb{U}, \quad f(x,u) \in \mathbb{V}_k \text{ and } (x,u) \in \mathcal{K}\}. \tag{8}$$

Such a dynamic programming algorithm provides approximation from above of the viability kernel as follows:

$$\mathbb{V}(f,\mathcal{K}) \subset \bigcap_{k \in \mathbb{N}} \mathbb{V}_k = \lim_{k \to \infty} \downarrow \mathbb{V}_k. \tag{9}$$

In [1], conditions for the equality to hold true are exposed. Are required the compacity for the constraints and upper semicontinuity with closed images for the set-valued map associated with the controlled dynamics.

It may be proved by induction that the above upper approximation $\mathbb{V}_k$ coincides with the so called *viability kernel until time k associated with f in $\mathcal{K}$*:

$$\mathbb{V}_k = \left\{ x_0 \in \mathbb{X} \;\middle|\; \begin{array}{l} \exists\ (u_0, u_1, \dots, u_k) \text{ and } (x_0, x_1, \dots, x_k) \\ \text{satisfying (1) and (2)} \end{array} \right\}. \tag{10}$$

We have

$$\forall k > 0 \quad \mathbb{V}(f,\mathcal{K}) \subset \mathbb{V}_{k+1} \subset \mathbb{V}_k \subset \mathbb{V}^0. \tag{11}$$

Once the viability kernel, or any approximation, or a viability viability domain is known, we have to consider the management or control issue, that is the problem of selecting suitable controls at each time step. For any viability domain $\widetilde{\mathbb{V}}$ and any state $x \in \widetilde{\mathbb{V}}$, the following subset $\mathbb{U}_{\widetilde{\mathbb{V}}}(x)$ of the decision set $\mathbb{U}$ is not empty:

$$\mathbb{U}_{\widetilde{\mathbb{V}}}(x) = \{u \in \mathbb{U} \mid (x,u) \in \mathcal{K} \text{ and } f(x,u) \in \widetilde{\mathbb{V}}\}. \tag{12}$$

Therefore $\mathbb{U}_{\mathbb{V}(f,\mathcal{K})}(x)$ stands for largest set of *viable controls associated with x*. Given a state x of the system, the decision consists in the choice of a viable *feedback* control, namely any selection $\Psi : \mathbb{X} \to \mathbb{U}$ which associates with each state $x \in \mathbb{V}(f,\mathcal{K})$ a control $u = \Psi(x)$ satisfying $\Psi(x) \in \mathbb{U}_{\mathbb{V}(f,\mathcal{K})}(x)$.

The paper is organized as follows. Section 2 is devoted to the definitions of monotonicity for both the dynamics and constraints. Then, Section 3 exhibits lower and upper approximations of the viability kernel in this monotonicity context. An example is exposed in Section 4 to illustrate some of the main findings.

2 Monotonicity properties

In this section we define what is meant by monotonicity of the desirable set $\mathcal{K}$ together with the dynamics f both with respect to state x and control u.

2.1 Set monotonicity

Let us assume that the state space $\mathbb{X}$ and the control space $\mathbb{U}$ are ordered sets. In practice, $\mathbb{X} \subset \mathbb{R}^{n_{\mathbb{X}}}$ and $\mathbb{U} \subset \mathbb{R}^{n_{\mathbb{U}}}$ are supplied with the componentwise order: $x' \geq x$ if and only if each component of x' is greater than or equal to the corresponding component of x:

$$x' \geq x \iff x'_i \geq x_i,\ i = 1, \ldots, n.$$

We also define the maximum $x \bigvee x'$ of (x, x') as follows:

$$x \bigvee x' \stackrel{\text{def}}{=} (x_1 \bigvee x'_1, \ldots, x_n \bigvee x'_n) = (\max(x_1, x'_1), \ldots, \max(x_n, x'_n))$$

We now define the monotonicity of constraint sets.

Definition 1 **[Set monotonicity]** *We say that a set $S \subset \mathbb{X}$ is* increasing *if it satisfies the following property:*

$$\forall x \in S \quad x' \geq x \Rightarrow x' \in S.$$

We say that $K \subset \mathbb{X} \times \mathbb{U}$ is increasing *if it satisfies the following property:*

$$\forall (x, u) \in K \quad x' \geq x \Rightarrow (x', u) \in K.$$

A geometric characterization of set monotonicity is given equivalently by $S + \mathbb{R}_+^{n_{\mathbb{X}}} \subset S$ in the first case, and by $K + \mathbb{R}_+^{n_{\mathbb{X}}} \times \{0_{\mathbb{R}^{n_{\mathbb{U}}}}\} \subset K$ in the second case (where state and control do not play the same role in this definition).

2.2 Dynamics monotonicity

Similarly, we define monotonic characterization for the dynamics as follows:

Definition 2 **[Mapping monotonicity]** *We say that $f : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$ is* increasing with respect to the state *if it satisfies*

$$\forall (x, u) \in \mathbb{X} \times \mathbb{U} \quad x' \geq x \Rightarrow f(x', u) \geq f(x, u),$$

and is decreasing with respect to the control *if*

$$\forall (x, u) \in \mathbb{X} \times \mathbb{U} \quad u' \geq u \Rightarrow f(x, u') \leq f(x, u).$$

2.3 Saturated dynamics

Since the dynamics f has several components, $\bigvee_{u \in \mathbb{U}, (x,u) \in \mathcal{K}} f(x, u)$ is generally not achieved by a common $\bar{u}$. This is why we introduce the notion of function "saturated at x".

Definition 3 *The* maximal dynamics $\check{f}$ *is defined by*

$$\forall x \in \mathbb{V}^0, \quad \check{f}(x) \stackrel{\text{def}}{=} \bigvee_{u \in \mathbb{U}, (x,u) \in \mathcal{K}} f(x, u)\,. \tag{13}$$

We say that the maximal dynamics $\check{f}$ is saturated at $x \in \mathbb{V}^0$ *if there exists $u \in \mathbb{U}$ such that $(x, u) \in \mathcal{K}$ and $\check{f}(x) = f(x, u)$.*

When $\mathbb{U} \subset \mathbb{R}$ and the dynamics f is decreasing with respect to the control, this is the case under reasonable topological assumptions on $\mathbb{U}$ and $\mathcal{K}$.

3 Viability results under monotonicity properties

This section exhibits lower and upper approximations of the viability kernel in this monotonicity context. We show that some monotonicity properties of the dynamics f and the desirable set $\mathcal{K}$ are transmitted to the associated viability kernel. This allows to compute or approximate the viability kernel through weakly invariant domains under suitable assumptions.

A first obvious monotonicity property is that any viability domain associated with f in $\mathcal{K}$ is a viability domain for any $\mathcal{K}'$ such that $\mathcal{K} \subset \mathcal{K}'$.

Proposition 1 *If the desirable set $\mathcal{K}$ is increasing and the dynamics f is increasing with respect to the state, then the associated viability kernel $\mathbb{V}(f, \mathcal{K})$ is also increasing, as well as all the sets $\mathbb{V}_k$, $k \in \mathbb{N}$ given by* (8)*.*

Proof. Let us prove that $\mathbb{V}(f, \mathcal{K})$ is increasing. Consider $x \in \mathbb{V}(f, \mathcal{K})$ and $x' \geq x$. By definition (4), there exists two sequences $(u_t)_{t\in\mathbb{N}}$ in $\mathbb{U}$ and $(x_t)_{t\in\mathbb{N}}$ in $\mathbb{X}$ such that

$$x_0 = x, \quad x_{t+1} = f(x_t, u_t) \quad \text{and} \quad (x_t, u_t) \in \mathcal{K}.$$

Since f is increasing with respect to the state, we can show by induction that the trajectory $(x'_t)_{t\in\mathbb{N}}$ defined by

$$x'_0 = x' \quad \text{and} \quad x'_{t+1} = f(x'_t, u_t)$$

satisfies $x'_t \geq x_t$, $\forall t \in \mathbb{N}$. Since $\mathcal{K}$ is increasing, we obtain that $(x'_t, u_t) \in \mathcal{K}$, $\forall t \in \mathbb{N}$. Thus $x' \in \mathbb{V}(f, \mathcal{K})$ and, finally, $\mathbb{V}(f, \mathcal{K})$ is increasing.

For the rest, the proof uses the definition (8) of the sets $(\mathbb{V}_k)_{k\in\mathbb{N}}$. The set $\mathbb{V}_0 = \mathbb{V}^0$ is increasing since $\mathcal{K}$ is supposed to be increasing. Now, assume that $\mathbb{V}_k$ is increasing and consider $x \in \mathbb{V}_{k+1}$ and $x' \geq x$. By the definition of $\mathbb{V}_{k+1}$, there exists $u \in \mathbb{U}$ such that $(x, u) \in \mathcal{K}$ and $f(x, u) \in \mathbb{V}_k$. Since $\mathcal{K}$ is increasing, $(x', u) \in \mathcal{K}$. Since f is increasing with respect to the state and $\mathbb{V}_k$ is increasing then $f(x', u) \in \mathcal{K}$, meaning that $x' \in \mathbb{V}_{k+1}$. Thus, $\mathbb{V}_{k+1}$ is increasing and, by induction, we conclude that $\mathbb{V}_k$ is increasing for all $k \in \mathbb{N}$.

□

3.1 A first lower approximation of the viability kernel

As an application of Proposition 1, if there exists a desirable equilibrium $(\bar{x}, \bar{u})$, then $\mathbb{V} = \{x \in \mathbb{X} \mid x \geq \bar{x}\}$ is a viability domain.

Proposition 2 *If the desirable set $\mathcal{K}$ is increasing and the dynamics f is increasing with respect to the state, and if there exists a desirable equilibrium $(\bar{x}, \bar{u})$, then*

1. *the domain* $\left\{x \in \mathbb{X} \mid x \geq \bar{x}\right\}$ *is a viability domain for f in $\mathcal{K}$;*

2. *Consequently* $\left\{x \in \mathbb{X} \mid x \geq \bar{x}\right\} \subset \mathbb{V}(f, \mathcal{K})$.

3.2 A first upper approximation of the viability kernel

We combine strongly invariant domains and desirable equilibria to derive an upper approximation of the viability kernel.

Proposition 3 *Assume that the desirable set $\mathcal{K}$ is increasing and that the dynamics f is increasing with respect to the state. Assume also that the maximal dynamics $\check{f}$ is saturated at all $x \in \mathbb{V}^0$. Then*

1. *the domain $\{x \in \mathbb{X} \mid \check{f}(x) \leq x\}$ is strongly invariant;*
2. *the domain $\{x \in \mathbb{X} \mid x \leq \bar{x}\}$ is strongly invariant, whenever $\bar{x}$ is a fixed point of $\check{f}$.*

Proof. We first prove that the maximal dynamics $\check{f}$ is increasing with respect to the state:

$$\begin{aligned} x \leq x' \quad &\Rightarrow \quad f(x,u) \leq f(x',u)\,, \quad \forall u \in \mathbb{U} \text{ since } f \text{ is increasing with } x \\ &\Rightarrow \quad \bigvee_{u\in\mathbb{U},(x,u)\in\mathcal{K}} f(x,u) \leq \bigvee_{u\in\mathbb{U},(x,u)\in\mathcal{K}} f(x',u) \\ &\Rightarrow \quad \bigvee_{u\in\mathbb{U},(x,u)\in\mathcal{K}} f(x,u) \leq \bigvee_{u\in\mathbb{U},(x',u)\in\mathcal{K}} f(x',u) \\ &\qquad \text{since } \mathcal{K} \text{ is increasing and thus } (x,u) \in \mathcal{K} \Rightarrow (x',u) \in \mathcal{K} \\ &\Rightarrow \quad \check{f}(x) \leq \check{f}(x')\,. \end{aligned}$$

Thus, we deduce that

1. $\check{f}(x) \leq x \Rightarrow \check{f}(\check{f}(x)) \leq \check{f}(x) \leq x$;
2. whenever $\check{f}(\bar{x}) = \bar{x}$, $x \leq \bar{x} \Rightarrow \check{f}(x) \leq \check{f}(\bar{x}) = \bar{x}$, that is,

$$x \leq \bar{x} \Rightarrow \forall u \in \mathbb{U}\,, \quad f(x,u) \leq \check{f}(x) \leq \check{f}(\bar{x}) = \bar{x}\,.$$

□

Proposition 4 *Assume that the desirable set $\mathcal{K}$ is increasing and that the dynamics f is increasing with respect to the state. Assume also that the maximal dynamics $\check{f}$ is continuous, and that $\mathbb{V}^0$ is bounded from below. Define $\mathbb{M}$ as the set of those elements which are larger than at least one fixed point of $\check{f}$ in the closure $\overline{\mathbb{V}^0}$ of the state constraints set:*

$$\mathbb{M} \overset{\text{def}}{=} \{x \in \mathbb{X} \mid \exists x' \in \overline{\mathbb{V}^0}\,, \quad \check{f}(x') = x'\,, \quad x \geq x'\}\,. \tag{14}$$

Then

$$\mathbb{V}(f,\mathcal{K}) \subset \mathbb{V}^0 \backslash \{x \in \mathbb{V}^0 \mid \check{f}(x) \leq x \text{ and } x \notin \mathbb{M}\}\,. \tag{15}$$

Proof. We know that the maximal dynamics $\check{f}$ is increasing with respect to the state.

Now assume that there exists $x_0 \in \mathbb{V}(f,\mathcal{K})$ such that $\check{f}(x_0) \leq x_0$. We shall prove that necessarily $x_0 \in \mathbb{M}$. Let $(u_0, u_1, \ldots)$ and $(x_0, x_1, \ldots)$ be such that $x_{t+1} = f(x_t, u_t)$ and $(x_t, u_t) \in \mathcal{K}$. Define $\check{x}(0) = x_0$ and $\check{x}_{t+1} = \check{f}(\check{x}_t)$, for $t = 1, 2 \ldots$. We easily see that $x_t \leq \check{x}_t$, and thus $\check{x}_t \in \mathbb{V}^0$ since $x_t \in \mathbb{V}^0$ and $\mathbb{V}^0$ is an increasing set by Proposition 1. The decreasing sequence $(\check{x}_t)_{t\in\mathbb{N}}$ being in $\overline{\mathbb{V}^0}$, it is bounded below and therefore converges to $\underline{x} \in \overline{\mathbb{V}^0}$. By continuity of $\check{f}$, we have $\check{f}(\underline{x}) = \underline{x}$. Thus, by $x_0 \geq \underline{x}$, we deduce that $x_0 \in \mathbb{M}$.

□

3.3 A second upper approximation of the viability kernel

We adapt the dynamic programming algorithm (8) to the case of saturated dynamics. We stress that under nice monotonicity properties, we obtain an algorithm converging exactly to the viability kernel $\mathbb{V}(f,\mathcal{K})$.

Proposition 5 *Assume that the desirable set $\mathcal{K}$ is increasing and that the dynamics f is increasing with respect to the state. Assume also that the maximal dynamics $\check{f}$ is saturated at all $x \in \mathbb{V}^0$. Then*

1. *the decreasing sequence* (8) *is now given by*

$$\mathbb{V}_0 = \mathbb{V}^0 \quad \textit{and} \quad \mathbb{V}_{k+1} = \mathbb{V}_k \bigcap \check{f}^{-1}(\mathbb{V}_k)\,, \quad k \in \mathbb{N}\,, \tag{16}$$

2. *any $\mathbb{V}_k$ is an upper approximation of the viability kernel for $k \in \mathbb{N}$: $\mathbb{V}(f,\mathcal{K}) \subset \mathbb{V}_k$*

3. *the decreasing sequence $(\mathbb{V}_k)_{k\in\mathbb{N}}$ converges to $\mathbb{V}(f,\mathcal{K})$:*

$$\mathbb{V}(f,\mathcal{K}) = \bigcap_{k\in\mathbb{N}} \mathbb{V}_k = \lim_{k\to\infty} \downarrow \mathbb{V}_k .$$

Proof. Let $x \in \mathbb{V}_k$ given by (8). By the assumptions on $\mathcal{K}$ and f, $\mathbb{V}_k$ is an increasing set by Proposition 1. Thus, we have

$$\begin{aligned}
\exists u \in \mathbb{U}\,,\ f(x,u) \in \mathbb{V}_k \text{ and } (x,u) \in \mathcal{K} \quad &\iff \quad \exists \bar{u} \in \mathbb{U}\,,\ \check{f}(x) = f(x,\bar{u}) \in \mathbb{V}_k \text{ and } (x,\bar{u}) \in \mathcal{K} \\
&\qquad\ \text{since } \mathbb{V}_k \text{ is increasing} \\
&\iff \quad x \in \check{f}^{-1}(\mathbb{V}_k) \text{ and } \exists \bar{u} \in \mathbb{U}\,,\ (x,\bar{u}) \in \mathcal{K} \\
&\iff \quad x \in \check{f}^{-1}(\mathbb{V}_k) \text{ and } x \in \mathbb{V}^0 \\
&\iff \quad x \in \check{f}^{-1}(\mathbb{V}_k) \text{ since } x \in \mathbb{V}_k \subset \mathbb{V}^0 .
\end{aligned}$$

Thus, (8) is equivalent to (16).

Now, we prove that $\mathbb{V}(f,\mathcal{K}) = \lim_{k\to\infty} \downarrow \mathbb{V}_k$. Thanks to Proposition 8, we already know that $\mathbb{V}(f,\mathcal{K}) \subset \lim_{k\to\infty} \downarrow \mathbb{V}_k$. We obtain the reverse inclusion $\mathbb{V}(f,\mathcal{K}) \supset \lim_{k\to\infty} \downarrow \mathbb{V}_k$ by showing that $\lim_{k\to\infty} \downarrow \mathbb{V}_k$ is a viability domain associated with f in $\mathcal{K}$ (see Proposition 9). Consider a fixed $x \in \lim_{k\to\infty} \downarrow \mathbb{V}_k$. Since the maximal dynamics $\check{f}$ is saturated at x, there exists $u \in \mathbb{U}$ such that $(x,u) \in \mathcal{K}$ and $\check{f}(x) = f(x,u)$. We claim that $f(x,u) \in \mathbb{V}_k$ for all $k \in \mathbb{N}$. Indeed, or all $k \in \mathbb{N}$, there exists $u_k \in \mathcal{K}$ such that $(x,u_k) \in \mathcal{K}$ and $f(x,u_k) \in \mathbb{V}_k$. On the one hand, by definition of the maximal dynamics $\check{f}$, we have $f(x,u_k) \leq \check{f}(x) = f(x,u)$. On the other hand, recall that $\mathbb{V}_k$ is an increasing set. Thus, $f(x,u) \in \mathbb{V}_k$. Since $\lim_{k\to\infty} \downarrow \mathbb{V}_k = \bigcap_{k\in\mathbb{N}} \mathbb{V}_k$, we obtain that $f(x,u) \in \lim_{k\to\infty} \downarrow \mathbb{V}_k$. Hence $\lim_{k\to\infty} \downarrow \mathbb{V}_k$ is a weakly invariant domain. □

3.4 A second lower approximation of the viability kernel

If we do not start the dynamic programming algorithm (8) from $\mathbb{V}^0$ but from a weakly invariant domain, we obtain a lower approximation of the viability kernel as follows.

Proposition 6 *If $\widetilde{\mathbb{V}}$ is a weakly invariant domain of f in $\mathcal{K}$, then*

$$\mathbb{V} = \{x \in \mathbb{X} \mid \exists u \in \mathbb{U}, \quad (x,u) \in \mathcal{K} \quad \text{and} \quad f(x,u) \in \widetilde{\mathbb{V}}\} \tag{17}$$

is a weakly invariant domain which contains $\widetilde{\mathbb{V}}$.

Proof. By the definition of a weakly invariant domain, we have $\widetilde{\mathbb{V}} \subset \mathbb{V}$. For $x \in \mathbb{V}$, by definition, there exists $u \in \mathbb{U}$ such that $f(x,u) \in \widetilde{\mathbb{V}} \subset \mathbb{V}$; that is, $\mathbb{V}$ is a weakly invariant domain of f in $\mathcal{K}$. □

The following Proposition 7 is a consequence of Proposition 6 and of Proposition 9.

Proposition 7 *Assume that the desirable set $\mathcal{K}$ is increasing and that the dynamics f is increasing with respect to the state. Assume also that the maximal dynamics $\check{f}$ is saturated at all $x \in \mathbb{V}^0$. Then, for all weakly invariant domain $\widetilde{\mathbb{V}}$ of f in $\mathcal{K}$,*

1. *the induction $\widetilde{\mathbb{V}}_0 = \widetilde{\mathbb{V}}$ and $\widetilde{\mathbb{V}}_{k+1} = \check{f}^{-1}(\widetilde{\mathbb{V}}_k)$ is increasing;*
2. *and its limit is included in the viability kernel:*

$$\bigcup_{k \in \mathbb{N}} \widetilde{\mathbb{V}}_k \subset \mathbb{V}(f, \mathcal{K}).$$

3.5 A third lower approximation of the viability kernel

A lower approximation of the viability kernel may be obtained by a lower approximation of the dynamics as follows.

Proposition 8 *Assume that the desirable set $\mathcal{K}$ is increasing and that the dynamics f is bounded below by an increasing $f^\flat : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$:*

$$f^\flat \leq f \quad \text{and} \quad f^\flat \text{ is increasing with respect to the state.}$$

Then, $\mathbb{V}(f^\flat, \mathcal{K})$ is a weakly invariant domain associated with f in $\mathcal{K}$, and thus

$$\mathbb{V}(f^\flat, \mathcal{K}) \subset \mathbb{V}(f, \mathcal{K}). \tag{18}$$

Proof. The assumptions that $f^\flat$ and $\mathcal{K}$ are increasing ensure that $\mathbb{V}(f^\flat, \mathcal{K})$ is increasing, according to Proposition 1. Consider $x \in \mathbb{V}(f^\flat, \mathcal{K})$. By the definition of $\mathbb{V}(f^\flat, \mathcal{K})$, there exists $u \in \mathbb{U}$ such that $(x,u) \in \mathcal{K}$ and $f^\flat(x,u) \in \mathbb{V}(f^\flat, \mathcal{K})$. By assumption $f(x,u) \geq f^\flat(x,u)$. Since $\mathbb{V}(f^\flat, \mathcal{K})$ is increasing, we deduce that $f(x,u) \in \mathbb{V}(f^\flat, \mathcal{K})$: $\mathbb{V}(f^\flat, \mathcal{K})$ is a weakly invariant domain associated with f in $\mathcal{K}$. □

Consequently, if f is not well known, it can be replaced by an increasing sub-approximation which gives a weakly invariant domain associated with f in $\mathcal{K}$. This may provide a deterministic and precautionary way of taking some uncertainty into account.

4 An example

In this section we apply various results to a model inspired by the management of an age structured abundance population model with a possibly non linear stock-recruitment relationship. This model is derived from fish stock management [13].

We introduce a desirable set $\mathcal{K}$ based on yield. The algorithm of Proposition 5 to obtain the viability kernel being not practicable, we instead make use of the method of Proposition 7 to compute useful weakly invariant sets associated with the dynamics.

The state variable is $x = (x^a)a = 1, \dots, A \in \mathbb{R}_+^A$, the *abundances* at age, and the control variable is $u \in \mathbb{R}_+$, the *exploitation pattern multiplier* $u_t \geq 0$, where t is the time index and a is the age class index ($a \in \{1, \dots, A\}$), so that $\mathbb{X} = \mathbb{R}_+^A$ and $\mathbb{U} = \mathbb{R}_+$.

The following dynamical relations relate the hereabove variables and define a *dynamics* f

$$\begin{cases} x_{t+1}^1 & = & \varphi(SSB(x_t)) \\ x_{t+1}^{a+1} & = & e^{-(M+u_t F^a)} x_t^a, \qquad a \in \{1, \dots, A-1\} \end{cases} \tag{19}$$

where SSB is the *spawning stock biomass*, defined by

$$SSB(x) = \sum_{a=1}^{A} p^a w^a x^a \tag{20}$$

and φ describes a stock-recruitment relationship. We assume that φ is increasing with respect to its argument. Typically, φ can be a Beverton-Holt like relationship [7], defined by $\varphi(x) = \frac{x}{\alpha+\beta x}$. For each time step, the exploitation is described by catch-at-age C^a and yield Y, respectively defined for a given vector of abundance x and a given multiplier u by

$$C^a(x^a, u) = \frac{uF^a}{uF^a + M}\left(1 - e^{-(M+uF^a)}\right) x^a \quad \text{and} \quad Y(x,u) = \sum_{a=1}^{A} w^a \, C^a(x^a, u). \tag{21}$$

State constraints set. The desirable set $\mathcal{K}$ we consider is simply defined by a minimum threshold $y_{\min}$ on the yield:

$$\mathcal{K} = \{(x,u) \in \mathbb{X} \times \mathbb{U} \mid Y(x,u) \geq y_{\min}\}.$$

We first compute $\mathbb{V}^0 = \mathrm{Proj}_{\mathbb{X}}(\mathcal{K})$ and, since $u \mapsto C^a(x^a, u)$ is increasing, we find that

$$\mathbb{V}^0 = \{x \in \mathbb{R}_+^A \mid \lim_{u \to +\infty} Y(x,u) > y_{\min}\} = \{x \in \mathbb{R}_+^A \mid \sum_{a \in A_e} w^a x^a > y_{\min}\},$$

Notice that $\mathbb{V}^0$ is not weakly invariant.

Monotonicity properties. The desirable set is increasing, since $Y(x,u)$ is increasing with respect to x. The dynamics f is increasing with respect to the state, with the assumption that φ is an increasing function. The dynamics f is decreasing with respect to the control and we have

$$\check{f}(x) = \bigvee_{u \in \mathbb{U}, (x,u) \in \mathcal{K}} f(x,u) = f(x, u_x),$$

where the control u_x is defined by the implicit equation $\sum_{a=1}^{A} C^a(x^a, u_x) w^a = y_{\min}$.

Computation of desirable equilibria. Let u be fixed. Introducing $D^a(u) = \mathrm{e}^{-((a-1)M+u(F_1+\ldots+F_{a-1}))}$ $(a = 2, \ldots, A)$, the *proportion of equilibrium recruits which survive up to age a* $(D_1(u) = 1)$, the *equilibrium spawners per recruits* $\mathrm{spr}(u) \stackrel{\mathrm{def}}{=} \sum_{a=1}^{A} p^a w^a D^a(u)$, and assuming that the increasing function $r \mapsto \varphi(r \sum_{a=1}^{A} p^a w^a D^a(u))$ has a nonnegative fixed point $R_\varphi(u)$, then

$$x^*(u) = (R_\varphi(u), D^1(u)R_\varphi(u), \ldots, D^A(u)R_\varphi(u))$$

is such that $f(x^*(u), u) = x^*(u)$. There remains to find conditions under which such $(x^*(u), u)$ is not only an equilibrium but a desirable equilibrium, that is $Y(x^*(u), u) \geq y_{\min}$.

For the sake of simplicity, we assume that the function $Y^* : \ u \mapsto Y(x(u), u)$ (yield at equilibrium) is continuous and goes to zero at infinity. Then, the function Y^* admits a maximum value, commonly called the *maximum sustainable yield*, since it is the maximum equilibrium yield. We denote it by y_{msy}. Moreover, it is attained for a fishing effort multiplier u_{msy}:

$$y_{\mathrm{msy}} \stackrel{\mathrm{def}}{=} \max_{u \geq 0} Y^*(u) = Y^*(u_{\mathrm{msy}}).$$

By definition, y_{msy} is the maximum value for $y_{\min}$ such that there exists a desirable equilibrium.

Approximation of the viability kernel. Let us consider $y_{\min} \in]0, y_{msy}[$. By the intermediate value theorem, there exist $u^1_{\min} < u^2_{\min}$ such that

$$x(u^1_{\min}) < x(u^2_{\min}) \quad \text{and} \quad Y^*(u^1_{\min}) = Y^*(u^2_{\min}) = y_{\min}.$$

We define

$$\widetilde{\mathbb{V}}^2 = \{x \in \mathbb{R}^A_+ \mid x \geq x(u^2_{\min})\} \subset \widetilde{\mathbb{V}}^1 = \{x \in \mathbb{R}^A_+ \mid x \geq x(u^1_{\min})\}.$$

According to Proposition 1, both sets $\widetilde{\mathbb{V}}^1$ and $\widetilde{\mathbb{V}}^2$ are weakly invariant sets associated with f in $\mathcal{K}$, defined with the threshold $y_{\min}$. We obtain the lower approximation of the viability kernel

$$\widetilde{\mathbb{V}}^1 \subset \mathbb{V}(f, \mathcal{K})$$

The resulting patterns are shown in Figure 1 in the two-class case, that is with $A = 2$. Using Proposition 6, the weakly invariant domain obtained one step backward from $\widetilde{\mathbb{V}}^2$ is sketched in Figure 2.

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A Appendix

Proposition 9 *The viability kernel $\mathbb{V}(f, \mathcal{K})$ is the union of all viability domains, that is the largest set such that* $\widetilde{\mathbb{V}} \subset \{x \in \mathbb{X} \mid \exists u \in \mathbb{U}, \quad (x, u) \in \mathcal{K} \quad \textit{and} \quad f(x, u) \in \widetilde{\mathbb{V}}\}$.

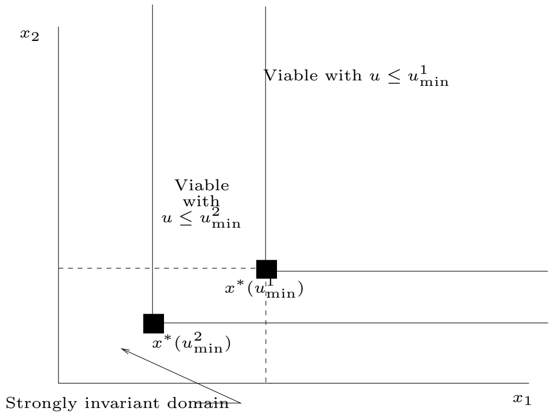

Figure 1: Example of two weakly invariant domains defined with two desirable equilibria.

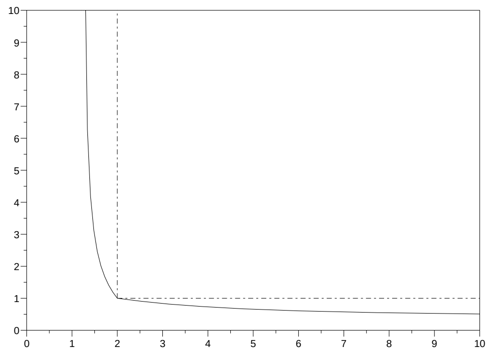

Figure 2: Enlargement of a viable orthant in the plan (x^1, x^2).

Proof. First, we prove that any viability domain $\widetilde{\mathbb{V}}$ associated with f in $\mathcal{K}$ is a subset of $\mathbb{V}(f, \mathcal{K})$. For $x \in \widetilde{\mathbb{V}}$, let us put $x_0 = x$ and $u_0 \in \mathbb{U}$ be such that $x_1 = f(x_0, u_0)$ and $(x_0, u_0) \in \mathcal{K}$. Starting from x_1, we proceed in the same way to obtain u_1. Going on, we find a sequence $(u_0, u_1, \dots)$ such that $x_{t+1} = f(x_t, u_t)$ and $(x_t, u_t) \in \mathcal{K}$ for all $t \in \mathbb{N}$. Thus, $x \in \mathbb{V}(f, \mathcal{K})$.

Second, we prove that $\mathbb{V}(f, \mathcal{K})$ is a viability domain. By definition, for all $x \in \mathbb{V}(f, \mathcal{K})$, there exists decisions $(u_0, u_1, \dots, u_t, \dots)$ and states starting from x at time 0 satisfying for all times $t \in \mathbb{N}$, $(x_t, u_t) \in \mathcal{K}$ and $x_{t+1} = f(x_t, u_t)$. Set $y = f(x, u_0)$. With the states $y_t = x_{t+1}$ and the decisions $v_t = u_{t+1}$, we obtain that $y \in \mathbb{V}(f, \mathcal{K})$. Then, there exists u $(= u_0) \in \mathbb{U}$ such that $(x, u) \in \mathcal{K}$ and $f(x, u) \in \mathbb{V}(f, \mathcal{K})$. Thus, $\mathbb{V}(f, \mathcal{K})$ is a viability domain.

We conclude that $\mathbb{V}(f, \mathcal{K})$ is the largest viability domain associated with f in $\mathcal{K}$ and that all the viability domains are included within $\mathbb{V}(f, \mathcal{K})$. $\square$

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