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Convergence of a non-local eikonal equation to anisotropic mean curvature motion. Application to dislocations dynamics.

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Abstract

In this paper we prove the convergence at a large scale of a non-local first order equation to an anisotropic mean curvature motion. This is an eikonal-type equation with a velocity depending in a non-local way on the solution itself, that arises in the theory of dislocations dynamics. We show that if a mean curvature motion is approximated by this type of equations then it is always of variational type, whereas the converse is true only in dimension two.

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1 Introduction

1.1 Physical motivation

In this paper, we study the asymptotic behaviour of an equation modelling dislocations dynamics. More precisely, we show that, in a large scale, dislocations dynamics is given by a mean curvature motion (we refer to Subsection 1.3 for the exact setting of the result). Dislocations are line defects in crystals whose typical length in metallic alloys is of the order of $10^{-6}m$ and thickness of the order of $10^{-9}m$. The concept of dislocations in crystals was put forward in the XXth century, as the main microscopic explanation of the macroscopic plastic behaviour of metallic crystals (see the physical monograph Hirth, Lothe [31]). Since the beginning of the 90's, the research field of dislocations is enjoying a new boom, in particular thanks to the power of computers which allows simulations with a large number of dislocations.

Recently Rodney, Le Bouar, Finel introduced in [37] a new model called *the phase field model of dislocation*. In this model, the dislocation line in the crystal moves in its slip plane with a normal velocity which is proportional to the Peach-Koeller force acting on this line. In the case where there are no exterior stress, this force is simply the self-force created by the elastic field generated by the dislocation line itself. In [5], [4], Alvarez, Hoch, Le Bouar and Monneau proposed to rewrite this model as a non-local Hamilton-Jacobi equation. Using viscosity solutions (we refer to the monographs of Barles [7] and Bardi and Capuzzo-Dolcetta [6] and to the paper of Crandall, Ishii and Lions [21] for a good introduction to this theory), Alvarez *et al.* [5], [4] proved a short time existence and uniqueness result. Then, Alvarez, Cardaliaguet and Monneau [1] and Barles and Ley [10] proved a long time result under certain assumptions. We also refer to Forcadel [27] for a uniqueness and existence result for dislocations dynamics with a mean curvature term. This equation was also numerically studied by Alvarez, Carlini, Monneau, Rouy [2], [3].

Mathematically, a dislocation line is represented by the boundary of a bounded domain $\Omega \subset \mathbb{R}^2$ which moves with normal speed given by

$$V_n = \overline{c_0} \star \rho$$

where the kernel $\overline{c_0} = \overline{c_0}(x)$ depends only on the space variables, $\star$ denotes the convolution in space and ρ is the characteristic function of the set Ω, *i.e.*

$$\rho(x) = \begin{cases} 1 & \text{if} \quad x \in \Omega \\ 0 & \text{if} \quad x \notin \Omega. \end{cases}$$

In this paper, we consider a simplified model of the one proposed by Alvarez *et al.* [5], [4]. Here, we assume that the negative part of the kernel $\overline{c_0}$ is concentrated on one point, *i.e.*, $\overline{c_0} = c_0 - \left(\int_{\mathbb{R}^2} c_0\right)\delta_0$ where c_0 is now a positive kernel. Because of the formal half contribution of the Dirac mass to $\overline{c_0} \star \rho$ on the dislocation line $\partial\Omega$, we can rewrite (formally on the dislocation line)

$$V_n = c_0 \star \rho - \frac{1}{2}\int_{\mathbb{R}^2} c_0.$$

For this model, we will be able to prove, in the framework of a Slepčev level set formulation (see [39]), a long time existence and uniqueness result for the solution of this equation (see Section 2).

Physically, the kernel c_0 is assumed to behave like $\frac{1}{|x|^3}$ at infinity. For this reason, we can rescale the characteristic function ρ, defining

$$\rho^\varepsilon(x,t) = \rho\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2|\ln\varepsilon|}\right).$$

This is almost the parabolic scaling. Here the presence of the logarithm is a well-known factor in physics (see for instance Brown [18]). We then show that in a large scale (*i.e.* $\varepsilon \to 0$), the normal speed of the dislocation line associated to ρ^ε is given by anisotropic mean curvature of the line. More precisely, we show that the solution of the non-local Hamilton-Jacobi equation modelling dislocations dynamics converges, at a large scale, to the solution of a mean curvature motion. We also study the link between the energy of dislocations and the energy associated to the mean curvature motion and we prove a formal convergence of the energies. We show that the mean curvature motion we can approach with this type of non-local eikonal equations is always of variational type. Finally, we show that in the two dimensional case, essentially all mean curvature motion of variational type can be approximated, which is not true in higher dimensions.

This result is very natural for dislocation dynamics. Indeed, in many references in physics, the authors describes dislocations dynamics by line tension terms deriving from an energy associated to the dislocation line. See for instance Brown [18] and Barnet Gavazza [14] for physical references and Garroni, Müller [29], [28] for a variational approach. As far as we know, our result is the first rigorous proof for the convergence of dislocation dynamics to mean curvature motion.

Similar results have already been proved for general kernels in relation with the Merriman, Bence, Osher algorithm for computing mean curvature motion [36]. We refer to Barles, Georgelin [9] Evans [25], Ishii [33] and Ishii, Pires, Souganidis [34] for such kind of results. We also refer to Souganidis [40] for example where the kernels are fractional laplacian. Nevertheless, our kernel does not satisfy the assumptions of these papers. We refer to Subsection 4.1 for a comparison with other related works. Moreover, we show in Section 7 that the limit mean curvature motion obtained by convolution is of variational type.

1.2 Mathematical setting of the problem

Given a function g defined on the unit sphere $\mathbf{S}^{n-1}$of $\mathbb{R}^n$ by

$$g \in C^0(\mathbf{S}^{n-1}), \quad g(-\theta) = g(\theta) \geq 0, \quad \forall \theta \in \mathbf{S}^{n-1} \tag{1.1}$$

we consider kernels $c_0 \in L^\infty(\mathbb{R}^n)$ satisfying

$$\begin{cases} c_0(x) = \dfrac{1}{|x|^{n+1}} \; g\left(\dfrac{x}{|x|}\right) & \text{if} \quad |x| \geq 1, \\ \\ c_0(-x) = c_0(x) \geq 0, \quad \forall x \in \mathbb{R}^n. \end{cases} \tag{1.2}$$

We want to look what happen for large dislocation, *i.e.*, in a large scale. Up to a change of variable, this is equivalent to concentrate the kernel. Since c_0 behaves like $\frac{1}{|x|^{n+1}}$ at infinity (see (1.2)), the "natural scaling" is then the following one for $0 < \varepsilon < 1$

$$c_0^\varepsilon(x) = \frac{1}{\varepsilon^{n+1}|\ln \varepsilon|} \; c_0\left(\frac{x}{\varepsilon}\right). \tag{1.3}$$

The presence of the logarithm comes out naturally in the proofs (see Subsection 4.1) but is also expected from a physical point of view.

We will use the level set formulation in the sense that the dislocation line (here in any dimension $n \geq 1$) is represented by any level set of a continuous function u^ε, solving the following equation (in the sense of Definition 2.1)

$$\begin{cases} u_t^\varepsilon(x,t) = \left((c_0^\varepsilon \star 1_{\{u^\varepsilon(\cdot,t) > u^\varepsilon(x,t)\}})(x) - \dfrac{1}{2}\displaystyle\int_{\mathbb{R}^n} c_0^\varepsilon\right) |Du^\varepsilon(x,t)| & \text{in} \quad \mathbb{R}^n \times (0,T), \\ \\ u^\varepsilon(\cdot,0) = u_0(\cdot) & \text{in} \quad \mathbb{R}^n \end{cases} \tag{1.4}$$

where Du^ε indicates the gradient of u^ε with respect to the space variables, the convolution is done in space only and $1_{\{u^\varepsilon(\cdot,t) > u^\varepsilon(x,t)\}}$ is the characteristic function of the set $\{u^\varepsilon(\cdot,t) > u^\varepsilon(x,t)\}$. Here, we consider the simultaneous evolutions of all the level sets of the function u^ε. This approach has been introduced by Slepčev [39] (see also Da Lio, Kim, Slepčev [23]).

We will prove that the unique viscosity solution of (1.4) converges to the unique solution of a mean curvature-type equation.

1.3 Main results

We denote by $C^{1,1/2}_{x,t}(\mathbb{R}^n \times [0,T])$ the set of continuous functions satisfying a Lipschitz condition in x and a Hölder condition in t of exponent $1/2$ and by $\mathrm{Lip}(\mathbb{R}^n)$ the set of Lipschitz continuous functions.

Theorem 1.1 (Existence, uniqueness and regularity for the ε-problem)
Let $n \geq 1$. Assume that the initial data $u_0 \in \mathrm{Lip}(\mathbb{R}^n)$ and that $c_0 \in W^{1,1}(\mathbb{R}^n)$. Then for all $\varepsilon \in (0,1)$, there exists a unique viscosity solution u^ε of (1.4) in the sense of Definition 2.1. Moreover, u^ε is $C^{1,1/2}_{x,t}(\mathbb{R}^n \times [0,T])$ uniformly in ε for $\varepsilon \in (0, \frac{1}{2})$. Namely, we have the following estimates for $\varepsilon \in (0, \frac{1}{2})$:

$$|Du^\varepsilon(\cdot,t)|_{L^\infty(\mathbb{R}^n)} \leq |Du_0|_{L^\infty(\mathbb{R}^n)}, \quad \forall t \geq 0$$

and
$$|u^\varepsilon(x,t+h)-u^\varepsilon(x,t)|\le C|Du_0|_{L^\infty(\mathbb{R}^n)}\sqrt{h},\quad \forall x\in\mathbb{R}^n,\ \forall t\ge 0,\ h\in[0,1],$$
where the constant C depends only on n and $\sup\limits_{\mathbb{R}^n} c_0$.

We are interested in the limit problem satisfied by the limit u^0 of u^ε as ε goes to zero. To this purpose, we consider the following problem

$$\text{(1.5)}\qquad \begin{cases} u^0_t(x,t)+F(D^2u^0,Du^0)=0 & \text{in } \mathbb{R}^n\times(0,T)\\ \\ u^0(\cdot,0)=u_0(\cdot) & \text{in } \mathbb{R}^n\end{cases}$$

with

$$\text{(1.6)}\qquad F(M,p)=-\text{trace }\left(M\cdot A\left(\frac{p}{|p|}\right)\right)$$

with

$$\text{(1.7)}\qquad A\left(\frac{p}{|p|}\right)=\int_{\theta\in\mathbf{S}^{n-2}=\mathbf{S}^{n-1}\cap\left\{\langle x,\frac{p}{|p|}\rangle=0\right\}}\left(\frac{1}{2}g(\theta)\theta\otimes\theta\right)\,d\theta$$

Hereafter $M\cdot A$ and $\langle\cdot,\cdot\rangle$ denote respectively the product between the two matrices and the usual scalar product.

Remark 1.2 *In particular F is geometric (see Barles, Soner, Souganidis [12]) because $M\mapsto F(M,p)$ is linear and*

$$F(M,p)=F\left(\left(Id-\frac{p}{|p|}\otimes\frac{p}{|p|}\right)\cdot M,\frac{p}{|p|}\right)$$

Remark 1.3 *In the particular case where $g\equiv 1$, we get $A=\dfrac{|\mathbf{S}^{n-2}|}{2(n-1)}Id_{\{x,\ <x,p>=0\}}$ where $|\mathbf{S}^{n-2}|$ is the Lebesgue measure of $\mathbf{S}^{n-2}$, and then*

$$F(M,p)=\frac{-|\mathbf{S}^{n-2}|}{2(n-1)}\text{ trace}\left(\left(Id-\frac{p}{|p|}\otimes\frac{p}{|p|}\right)\cdot M\right)$$

We recover the classical mean curvature motion up to the factor $|\mathbf{S}^{n-2}|/2(n-1)$.

We prove the following result

Theorem 1.4 (Convergence of dislocations dynamics to mean curvature motion) *Let $n\ge 1$. Given $u_0\in\mathrm{Lip}(\mathbb{R}^n)$ and $c_0\in W^{1,1}(\mathbb{R}^n)$, we consider the solution u^ε of problem (1.4) with the kernel c_0^ε defined in (1.1)-(1.2)-(1.3). Then the solution u^ε converges locally uniformly on compact sets of $\mathbb{R}^n\times[0,+\infty)$ to the unique viscosity solution u^0 of (1.5)-(1.6)-(1.7).*

Remark 1.5 *This result also suggests a natural scheme to compute numerically mean curvature motion. This is the subject of a paper in preparation [22].*

From expression (1.6)-(1.7) it is not clear if the anisotropic mean curvature motion (1.5) is of a variational type or not. Theorem 1.7 below will show that this mean curvature motion is indeed of variational type. Before to state Theorem 1.7, we need the following definition:

Definition 1.6 *Let $g \in C^0(\mathbb{R}^n\backslash\{0\})$ satisfy $g(\lambda p) = \dfrac{g(p)}{|\lambda|^{n+1}}$, $\forall \lambda \in \mathbb{R}\backslash\{0\}, p \in \mathbb{R}^n\backslash\{0\}$. We then associate to g a temperate distribution L_g defined by*

$$\langle L_g, \varphi\rangle = \int_{\mathbb{R}^n} dx\, \frac{g\left(\frac{x}{|x|}\right)}{|x|^{n+1}} \left(\varphi(x) - \varphi(0) - x\cdot D\varphi(0)\, 1_{B_1(0)}(x)\right)$$

for $\varphi \in \mathcal{S}(\mathbb{R}^n)$, where $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of test functions, and $B_1(0)$ denotes the unit ball centered in zero.

We define the Fourier transform of $\varphi \in \mathcal{S}(\mathbb{R}^n)$ as

$$\mathcal{F}(\varphi)(\xi) = \int_{\mathbb{R}^n} dx\, \varphi(x)e^{-i\xi\cdot x}.$$

We have the following theorem:

Theorem 1.7 (Variational origin of the anisotropic mean curvature motion)
Let $n \geq 2$. Let $g \in C^0(\mathbb{R}^n\backslash\{0\})$ satisfy $g(\lambda p) = \dfrac{g(p)}{|\lambda|^{n+1}}$, $\forall\lambda \in \mathbb{R}\backslash\{0\}$, $\forall p \in \mathbb{R}^n\backslash\{0\}$. We have

$$(1.8)\qquad \int_{S^{n-1}\cap\left\{\langle x, \frac{p}{|p|}\rangle = 0\right\}} \frac{1}{2} g(\theta)\theta\otimes\theta d\theta = D^2G\left(\frac{p}{|p|}\right) \quad \text{with} \quad G := -\frac{1}{2\pi}\mathcal{F}(L_g)$$

where $\mathcal{F}(L_g)$ is the Fourier transform of L_g. Moreover $G(\lambda p) = |\lambda| G(p)$, $\forall \lambda \in \mathbb{R}\backslash\{0\}$, $\forall p \in \mathbb{R}^n$ and, with A defined in (1.7), if $u^0 \in C^2(\mathbb{R}^n)$ with $|Du^0| \neq 0$, then the following holds:

$$(1.9)\qquad \frac{1}{|Du^0|} \mathit{trace}\, \left(A\left(\frac{Du^0}{|Du^0|}\right)\cdot D^2u^0\right) = \mathit{div}\left(\nabla G\left(\frac{Du^0}{|Du^0|}\right)\right),$$

which means that the mean curvature motion derives from the following energy: $\int G(Du^0)$.

Moreover, if $g \geq 0$, then G is convex.

The converse is true in the two dimensional case, namely, if $G \in C^0(\mathbb{R}^2)\cap C^2(\mathbb{R}^2\backslash\{0\})$ is convex and satisfies $G(\lambda p) = |\lambda|G(p)$ $\forall \lambda \in \mathbb{R}\backslash\{0\}$, $p \in \mathbb{R}^2$, then there exists a non-negative function g such that $L_g := -2\pi\mathcal{F}(G)$.

A different non-local equation for a mean field model describing a spin flip dynamics has been studied in De Masi, Orlandi, Presutti, Triolo [24], Katsoulakis, Souganidis [35] and Barles, Souganidis [13]. In [15], Bellettini, Buttà and Presutti have proved that the limit dynamics is related to the Hessian of an energy.

Proposition 1.8 (Counter-example)
The converse of Theorem 1.7 is false in dimension $n \geq 3$, i.e., there exists g which changes its sign such that $A(p) = D^2G(p) \geq 0$.

Remark 1.9 *If g is a positive measure, we can formally approximate crystalline curvature by our non-local eikonal equation.*

Remark 1.10 *Physically, only $\mathcal{F}(L_g)$ is known. We see that formula (1.8) allows easily to compute g in dimension $n = 2$ and then to check that $g \geq 0$ or not. See Hirth and Lothe [31] Chapter 13-8 for an example where g is not non-negative, and Head [30] for examples in cubic elasticity.*

In the simplest case of applications for dislocations dynamics, the crystal is described by isotropic elasticity (see [31]). When the Burgers vector is along the x_1 direction, we have

$$G(p) = \frac{p_2^2 + \frac{1}{1-\nu}\, p_1^2}{|p|} \quad \textit{with} \quad \nu \in (-1, \frac{1}{2})$$

where ν is the Poisson ratio of the material, and

$$g(\theta) = \frac{(2\gamma - 1)(\theta_1)^2 + (2-\gamma)(\theta_2)^2}{|\theta|^5} \geq 0 \quad \textit{with} \quad \gamma = \frac{1}{1-\nu} \in (\frac{1}{2}, 2).$$

It is well-known that we can approach mean curvature motion with Merriman Bence Osher [36] construction with a general kernel K_0 satisfying $K_0(-x) = K_0(x)$ and for every $p \in \mathbf{S}^{n-1}$

$$\int_{\mathbb{R}^n \cap \{p^\perp\}} K_0(x)|x|^2 < \infty \tag{1.10}$$

where $\{p^\perp\} = \left\{\langle x, \frac{p}{|p|}\rangle = 0\right\}$ and with the "parabolic scaling" $K_0^\varepsilon = \dfrac{1}{\varepsilon^{n+1}} K_0\left(\dfrac{x}{\varepsilon}\right)$. We refer, for instance to Barles Georgelin [9], Evans [25], Ishii [33] and Ishii, Pires, Souganidis [34] (we also refer to Subsection 4.1 for a formal proof).

More precisely, the limit motion is (1.5)-(1.6), with (1.7) replaced by

$$A\left(\frac{p}{|p|}\right) = \int_{\theta \in \mathbb{R}^{n-1} = \mathbb{R}^n \cap \left\{\langle x, \frac{p}{|p|}\rangle = 0\right\}} \left(\frac{1}{2} K_0(x) \cdot x \otimes x\right)\, dx \tag{1.11}$$

Up to our knowledge, it was not known in this general setting if the limit mean curvature motion associated to (1.11) is of variational type (*cf* (1.9)). It turns out that this is a simple consequence of our Theorem 1.7:

Theorem 1.11 (Variational property of the limit motion)
Every mean curvature motion of the form of (1.5)-(1.6) with A defined in (1.11) is of variational type.

The problem we consider is formally associated to the following energy:

$$\mathcal{E}^\varepsilon(u^\varepsilon) = \int_\lambda \overline{\mathcal{E}^\varepsilon}(\lambda) d\lambda \tag{1.12}$$

where

$$\overline{\mathcal{E}^\varepsilon}(\lambda) = \int_{\mathbb{R}^n} -\frac{1}{2} \left(\overline{c_0}^\varepsilon \star \rho_\lambda^\varepsilon\right) \rho_\lambda^\varepsilon$$

with

$$\rho_\lambda^\varepsilon = 1_{\{u^\varepsilon > \lambda\}}, \quad \overline{c_0}^\varepsilon = c_0^\varepsilon - \left(\int_{\mathbb{R}^n} c_0^\varepsilon\right) \delta_0.$$

We will show formally in Section 8 that this energy is non increasing in time and that there is a convex function G such that $\mathcal{E}^\varepsilon(u^\varepsilon) \to \int G(Du^0)$ which is the energy associated to a mean curvature motion of the limit solution u^0.

1.4 Organisation of the paper

Let us now explain how this paper is organised: Section 2 is devoted to the study of the ε-problem. In Section 3, we give some results on the limit problem. Then, we give, in Section 4, a result on the convergence of the velocity for a test function. The regularity result of Theorem 1.1 is proved in Section 5 (see Corollary 5.3) as well as estimates at initial time. The convergence result Theorem 1.4 is proved in Section 6. The variational property of the limit motion Theorem 1.7 and Theorem 1.11 and the counter-example Proposition 1.8 are proved in Section 7. In Section 8, we study very formally the link between energy and mean curvature motion. Finally, in an appendix, we give some technical *lemmata* on Fourier transform.

2 Existence and uniqueness for the ε-problem

In the sequel we will denote by $B_{\text{loc}}USC(\mathbb{R}^n \times [0,T])$ and $B_{\text{loc}}LSC(\mathbb{R}^n \times [0,T])$ respectively the set of locally bounded upper semicontinuous and lower semicontinuous functions in $\mathbb{R}^n \times [0,T]$.

Definition 2.1 (Viscosity sub/super/solution for the non-local eikonal equation) *A function $u^\varepsilon \in B_{\text{loc}}USC(\mathbb{R}^n \times [0,T])$ is a viscosity subsolution of (1.4) if it satisfies:*

(i) $u^\varepsilon(x,0) \le u_0(x)$ in $\mathbb{R}^n$,

(ii) for every $(x_0,t_0) \in \mathbb{R}^n \times (0,T)$ and for every test function $\Phi \in C^\infty(\mathbb{R}^n \times [0,T))$ such that $u^\varepsilon - \Phi$ has a maximum at (x_0,t_0), the following holds:

$$\Phi^\varepsilon_t(x_0,t_0) \le \left((c_0^\varepsilon \star 1_{\{u^\varepsilon(\cdot,t_0) \ge u^\varepsilon(x_0,t_0)\}})(x_0) - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon \right) |D\Phi^\varepsilon(x_0,t_0)|. \tag{2.13}$$

A function $u^\varepsilon \in B_{\text{loc}}LSC(\mathbb{R}^n \times [0,T])$ is a viscosity supersolution of (1.4) if it satisfies:

(i) $u^\varepsilon(x,0) \ge u_0(x)$ in $\mathbb{R}^n$,

(ii) for every $(x_0,t_0) \in \mathbb{R}^n \times (0,T)$ and for every test function $\Phi \in C^\infty(\mathbb{R}^n \times [0,T))$ such that $u^\varepsilon - \Phi$ has a minimum at (x_0,t_0), the following holds:

$$\Phi^\varepsilon_t(x_0,t_0) \ge \left((c_0^\varepsilon \star 1_{\{u^\varepsilon(\cdot,t_0) > u^\varepsilon(x_0,t_0)\}})(x_0) - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon \right) |D\Phi^\varepsilon(x_0,t_0)|. \tag{2.14}$$

A continuous function u^ε is a viscosity solution of (1.4) if, and only if, it is a sub and a supersolution of (1.4).

This definition comes from the definition of viscosity solution for nonlocal equation given by Slepčev [39] (see also Da Lio, Kim, Slepčev [23]) and it permits to extend to non-local equations all properties enjoyed by viscosity solutions of local equations.

Note the difference in the choice of the set in the indicatrice function in the definition of a subsolution and a supersolution. This is crucial to extend all the properties of viscosity solutions to nonlocal, geometric parabolic equations (see Slepčev [39]), in particular for the stability of the solution, *i.e.*, the lim sup of subsolution is a subsolution (and so the existence by Perron's method).

Next we prove a comparison result between locally bounded semicontinuous viscosity sub and supersolutions to the equation (1.4).

Theorem 2.2 (Comparison principle for the ε-problem)
Assume $c_0 \in W^{1,1}(\mathbb{R}^n)$. Let $u \in B_{\rm loc}USC(\mathbb{R}^n \times [0,T])$, $v \in B_{\rm loc}LSC(\mathbb{R}^n \times [0,T])$ be respectively viscosity sub and supersolution of (1.4). If $u(x,0) \leq v(x,0)$ for all $x \in \mathbb{R}^n$ then $u(x,t) \leq v(x,t)$ for all $(x,t) \in \mathbb{R}^n \times [0,T]$.

To prove this result, we need the analogous of the Ishii's *Lemma* for non-local equations. We first recall the definition of the limit sub and super-differentials:

$$\bar{\mathcal{P}}^+u(x,t) = \left\{ \begin{array}{l} (p,a) \in \mathbb{R}^n \times \mathbb{R},\ \exists\ (x_n,t_n,p_n,a_n) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \\ \text{ such that } (p_n,a_n) \in \mathcal{P}^+u(x_n,t_n) \\ \text{ and } (x_n,t_n,u(x_n,t_n),p_n,a_n) \to (x,t,u(x,t),p,a) \end{array} \right\}$$

where $\mathcal{P}^+$ is the classical super-differentials. The set $\bar{\mathcal{P}}^-u(x,t)$ is defined in a similar way. It is well known that we have an equivalent definition for viscosity solution by using sub and super-differentials (cf Crandall, Ishii, Lions [21]). We claim that the definition remains equivalent if we replace the classical sub and super-differentials by the limit ones. Indeed, let $u \in B_{\rm loc}USC(\mathbb{R}^n \times [0,T])$ be a viscosity subsolution of (1.4). We will show that

$$(p,a) \in \bar{\mathcal{P}}^+u(x,t) \Rightarrow a \leq \left(c_0^\varepsilon \star 1_{\{u(\cdot,t)\geq u(x,t)\}}(x) - \frac{1}{2}\int c_0^\varepsilon \right)|p|. \tag{2.15}$$

Let $(x_n,t_n,p_n,a_n) \in \mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}$ such that $(p_n,a_n) \in \mathcal{P}^+u(x_n,t_n)$ and $(x_n,t_n,u(x_n,t_n),p_n,a_n) \to (x,t,u(x,t),p,a)$. We then have, by definition,

$$\begin{aligned} a_n &\leq \left(c_0^\varepsilon \star 1_{\{u(\cdot,t_n)\geq u(x_n,t_n)\}}(x_n) - \frac{1}{2}\int c_0^\varepsilon \right)|p_n| \\ &\leq \left(c_0^\varepsilon \star 1_{\{u(\cdot,t_n)\geq u(x_n,t_n)\}\cup\{u(\cdot,t)\geq u(x,t)\}}(x_n) - \frac{1}{2}\int c_0^\varepsilon \right)|p_n|. \end{aligned}$$

We just have to show that

$$c_0^\varepsilon \star 1_{\{u(\cdot,t_n)\geq u(x_n,t_n)\}\cup\{u(\cdot,t)\geq u(x,t)\}}(x_n) \to c_0^\varepsilon \star 1_{\{u(\cdot,t)\geq u(x,t)\}}(x).$$

To do this, we use the following decomposition:

$$\begin{aligned} &c_0^\varepsilon \star 1_{\{u(\cdot,t_n)\geq u(x_n,t_n)\}\cup\{u(\cdot,t)\geq u(x,t)\}}(x_n) - c_0^\varepsilon \star 1_{\{u(\cdot,t)\geq u(x,t)\}}(x) \\ =&c_0^\varepsilon \star 1_{\{u(\cdot,t_n)\geq u(x_n,t_n)\}\cup\{u(\cdot,t)\geq u(x,t)\}}(x_n) - c_0^\varepsilon \star 1_{\{u(\cdot,t_n)\geq u(x_n,t_n)\}\cup\{u(\cdot,t)\geq u(x,t)\}}(x) \\ &+ c_0^\varepsilon \star 1_{\{u(\cdot,t_n)\geq u(x_n,t_n)\}\cup\{u(\cdot,t)\geq u(x,t)\}\setminus\{u(\cdot,t)\geq u(x,t)\}}(x). \end{aligned}$$

The first part clearly goes to zero as n goes to infinity. For the second part, we need the following *Lemma*:

***Lemma* 2.3** *Let f_n be a sequence of measurable functions on $\mathbb{R}^n$ and*

$$f \geq \limsup{}^* f_n(x) := \sup\left\{ \limsup_{n\to 0} f_n(y) : y \to x \right\}.$$

Let a_n be a sequence converging to zero. Then

$$\mathcal{L}\left(\{f_n \geq a_n\}\backslash\{f \geq 0\}\right) \to 0 \quad \text{as n} \to \infty.$$

where, for any measurable set A, $\mathcal{L}(A)$ denotes the Lebesgue measure of A.

For the proof of this *lemma*, we refer to Slepčev [39].

Applying this *Lemma* with $f_n = u(\cdot, t_n) - u(x,t)$, $a_n = u(x_n, t_n) - u(x,t)$ and $f = u(\cdot, t) - u(x,t)$ yields the result. The proof for supersolution is analogous.

Using (2.15), we can rewrite the Ishii's *Lemma* (see Crandall, Ishii, Lions [21] *Lemma* 8.3) for non-local equations:

***Lemma* 2.4 (Ishii's *lemma* for non-local equations)**
Let U and V be open sets of $\mathbb{R}^n$, and for $T > 0$, $u \in B_{\mathrm{loc}}USC(U \times (0,T))$ and $v \in B_{\mathrm{loc}}LSC(V \times (0,T))$ be respectively subsolution and supersolution of (1.4). Let $\phi : U \times V \times (0,T) \to (0, \infty)$ of class C^∞. Assume that $(x,y,t) \mapsto u(x,t) - v(y,t) - \phi(x,y,t)$ reaches a local maximum in $(\bar{x}, \bar{y}, \bar{t}) \in U \times V \times (0,T)$. We set $\tau = \partial_t \phi(\bar{x}, \bar{y}, \bar{t})$, $p_1 = D_x\phi(\bar{x}, \bar{y}, \bar{t})$, and $p_2 = -D_y\phi(\bar{x}, \bar{y}, \bar{t})$ Then, there exists τ_1, $\tau_2 \in \mathbb{R}$ such that:

$$\begin{gathered}\tau = \tau_1 - \tau_2,\\ (p_1, \tau_1) \in \bar{\mathcal{P}}^+ u(\bar{x}, \bar{t}),\ (p_2, \tau_2) \in \bar{\mathcal{P}}^- v(\bar{y}, \bar{t}),\end{gathered}$$

and then

$$\tau_1 \leq \left(c_0^\varepsilon \star 1_{\{u(\cdot, \bar{t}) \geq u(\bar{x}, \bar{t})\}}(\bar{x}) - \frac{1}{2} \int c_0^\varepsilon \right) |p|$$

and

$$\tau_2 \geq \left(c_0^\varepsilon \star 1_{\{v(\cdot, \bar{t}) > v(\bar{y}, \bar{t})\}}(\bar{y}) - \frac{1}{2} \int c_0^\varepsilon \right) |q|.$$

Proof of Theorem 2.2
The proof of this Theorem is inspired by Barles, Cardaliaguet, Ley and Monneau [8].
Let $u \in B_{\mathrm{loc}}USC(\mathbb{R}^n \times [0,T])$, $v \in B_{\mathrm{loc}}LSC(\mathbb{R}^n \times [0,T])$ be respectively viscosity sub and supersolution of (1.4). Since the equation is geometric we may assume without loss of generality that u and v are bounded (see Slepčev [39], property (P1)). Suppose by contradiction that $M = \sup_{\mathbb{R}^n \times [0,T]}(u(x,t) - v(x,t)) > 0$. Then for $\eta \in (0,1)$ small enough we have $M_\eta = \sup_{t \in [0,T]} \limsup_{|x-y| \to 0}(u(x,t) - v(y,t) - \eta t) > 0$ as well.

For all $\gamma > 0$ and $\alpha > 0$ with $\alpha << \gamma$, we introduce the auxiliary function $\Phi_{\gamma,\alpha} \colon \mathbb{R}^n \times \mathbb{R}^n \times [0,T] \to \mathbb{R}$ defined by

$$\Phi_{\gamma,\alpha}(x,y,t) = u(x,t) - v(y,t) - \eta t - \frac{|x-y|^2}{\gamma^2} - \alpha(|x|^2 + |y|^2). \tag{2.16}$$

We observe that $\limsup_{|x|,|y| \to +\infty} \Phi_{\gamma,\alpha}(x,y,t) = -\infty$, thus $\Phi_{\gamma,\alpha}(x,y,t)$ reaches its maximum at a point $(x_{\gamma,\alpha}, y_{\gamma,\alpha}, t_{\gamma,\alpha}) \in \mathbb{R}^n \times \mathbb{R}^n \times [0,T]$. Standard arguments show that

$$\alpha(|x_{\gamma,\alpha}|^2 + |y_{\gamma,\alpha}|^2),\ \frac{|x_{\gamma,\alpha} - y_{\gamma,\alpha}|^2}{\gamma^2} \leq C_0, \tag{2.17}$$

with $C_0 > 0$ depending on $||u||_\infty, ||v||_\infty$. In particular we get that

$$\lim_{\gamma \to 0} \limsup_{\alpha \to 0} |x_{\gamma,\alpha} - y_{\gamma,\alpha}| = 0\,.$$

Then, the following estimate holds

$$\limsup_{\gamma\to 0}\limsup_{\alpha\to 0}\Phi_{\gamma,\alpha}(x_{\gamma,\alpha},y_{\gamma,\alpha},t_{\gamma,\alpha}) \le \limsup_{\gamma\to 0}\limsup_{\alpha\to 0}(u(x_{\gamma,\alpha},t_{\gamma,\alpha})-v(y_{\gamma,\alpha},t_{\gamma,\alpha})-\eta t_{\gamma,\alpha})$$
$$\le M_\eta\,. \tag{2.18}$$

We also have

$$\liminf_{\gamma\to 0}\liminf_{\alpha\to 0}\Phi_{\gamma,\alpha}(x_{\gamma,\alpha},y_{\gamma,\alpha},t_{\gamma,\alpha}) \ge M_\eta\,. \tag{2.19}$$

Indeed, by definition, we have for all $(x,y,t)\in\mathbb{R}^n\times\mathbb{R}^n\times[0,T]$

$$\begin{aligned} u(x,t)-v(y,t)-\eta t-\frac{|x-y|^2}{\gamma^2}-\alpha(|x|^2+|y|^2) \le&\Phi_{\gamma,\alpha}(x_{\gamma,\alpha},y_{\gamma,\alpha},t_{\gamma,\alpha})\\ \le& u(x_{\gamma,\alpha},t_{\gamma,\alpha})-v(y_{\gamma,\alpha},t_{\gamma,\alpha})-\eta t_{\gamma,\alpha}. \end{aligned}$$

We first take $\liminf\limits_{\alpha\to 0}$. We get

$$\begin{aligned} u(x,t)-v(y,t)-\eta t-\frac{|x-y|^2}{\gamma^2} \le&\liminf_{\alpha\to 0}\Phi_{\gamma,\alpha}(x_{\gamma,\alpha},y_{\gamma,\alpha},t_{\gamma,\alpha})\\ \le&\liminf_{\alpha\to 0}(u(x_{\gamma,\alpha},t_{\gamma,\alpha})-v(y_{\gamma,\alpha},t_{\gamma,\alpha})-\eta t_{\gamma,\alpha}). \end{aligned} \tag{2.20}$$

We then take $\limsup\limits_{|x-y|\to 0}$ and get

$$\sup_{t\in[0,T]}\limsup_{|x-y|\to 0}(u(x,t)-v(y,t)-\eta t)\le\liminf_{\alpha\to 0}\Phi_{\gamma,\alpha}(x_{\gamma,\alpha},y_{\gamma,\alpha},t_{\gamma,\alpha}),$$

and finally take $\liminf\limits_{\gamma\to 0}$ and get (2.19).

By combining (2.19) and (2.18) we get

$$\begin{aligned} M_\eta \le&\liminf_{\gamma\to 0}\liminf_{\alpha\to 0}\Phi_{\gamma,\alpha}(x_{\gamma,\alpha},y_{\gamma,\alpha},t_{\gamma,\alpha})\\ \le&\limsup_{\gamma\to 0}\limsup_{\alpha\to 0}\Phi_{\gamma,\alpha}(x_{\gamma,\alpha},y_{\gamma,\alpha},t_{\gamma,\alpha})\\ \le& M_\eta\,. \end{aligned}$$

Therefore

$$\lim_{\gamma\to 0}\liminf_{\alpha\to 0}\Phi_{\gamma,\alpha}(x_{\gamma,\alpha},y_{\gamma,\alpha},t_{\gamma,\alpha})=\lim_{\gamma\to 0}\limsup_{\alpha\to 0}\Phi_{\gamma,\alpha}(x_{\gamma,\alpha},y_{\gamma,\alpha},t_{\gamma,\alpha})=M_\eta.$$

In a analogous way, we can deduce that (using (2.18) and (2.20))

$$\begin{aligned} M_\eta =&\lim_{\gamma\to 0}\liminf_{\alpha\to 0}(u(x_{\gamma,\alpha},t_{\gamma,\alpha})-v(y_{\gamma,\alpha},t_{\gamma,\alpha})-\eta t_{\gamma,\alpha})\\ =&\lim_{\gamma\to 0}\limsup_{\alpha\to 0}(u(x_{\gamma,\alpha},t_{\gamma,\alpha})-v(y_{\gamma,\alpha},t_{\gamma,\alpha})-\eta t_{\gamma,\alpha})\,. \end{aligned}$$

We then get

$$\lim_{\gamma\to 0}\limsup_{\alpha\to 0}\left(\frac{|x_{\gamma,\alpha}-y_{\gamma,\alpha}|^2}{\gamma^2}+\alpha(|x_{\gamma,\alpha}|^2+|y_{\gamma,\alpha}|^2)\right)=0\,. \tag{2.21}$$

Let us fix $\gamma_0 > 0$ such that for all $\gamma \leq \gamma_0$, and for all α small enough we have

$$M_{\gamma,\alpha} = \Phi_{\gamma,\alpha}(x_{\gamma,\alpha}, y_{\gamma,\alpha}, t_{\gamma,\alpha}) > \frac{M_\eta}{2}$$

and

$$\limsup_{\alpha\to 0}\Bigg(\|Dc_0^\varepsilon\|_1\left(2\frac{|x_{\gamma,\alpha}-y_{\gamma,\alpha}|^2}{\gamma^2} + \alpha|y_{\gamma,\alpha}|^2|x_{\gamma,\alpha}-y_{\gamma,\alpha}| + \alpha|x_{\gamma,\alpha}-y_{\gamma,\alpha}|\right) + \frac{3}{2}\|c_0\|_1\alpha\left(2+|x_{\gamma,\alpha}|^2+|y_{\gamma,\alpha}|^2\right)\Bigg) \leq \frac{\eta}{3}. \tag{2.22}$$

We claim that there is $\gamma \leq \gamma_0$ such that for all α small enough $t_{\gamma,\alpha} > 0$. Indeed if, for all $\gamma \leq \gamma_0$, there is $\alpha \in (0,\gamma)$ such that $t_{\gamma,\alpha} = 0$, then the following estimate holds

$$\begin{aligned}\frac{M_\eta}{2} < M_{\gamma,\alpha} &\leq u(x_{\gamma,\alpha},0) - v(y_{\gamma,\alpha},0)\\ &\leq u_0(x_{\gamma,\alpha}) - u_0(y_{\gamma,\alpha})\\ &\leq \|Du_0\||x_{\gamma,\alpha}-y_{\gamma,\alpha}|\\ &\leq C\|Du_0\|\gamma,\end{aligned}$$

where we have use (2.17). Thus we get a contradiction if γ is small enough and we prove the claim. Hence, by *Lemma* 2.4 (if $t_{\gamma,\alpha} = T$, we use the fact that u (resp. v) is subsolution (resp. supersolution) in $(0,T]$, see *Lemma* 2.8 of Barles [7]), there are $(a,p) \in \bar{D}^+u(x_{\gamma,\alpha}, t_{\gamma,\alpha})$ and $(b,q) \in \bar{D}^-v(y_{\gamma,\alpha}, t_{\gamma,\alpha})$ such that

$$a - b = \eta\,;$$

$$p = 2\frac{(x_{\gamma,\alpha}-y_{\gamma,\alpha})}{\gamma^2} + 2\alpha x_{\gamma,\alpha}\,;$$

$$q = 2\frac{(x_{\gamma,\alpha}-y_{\gamma,\alpha})}{\gamma^2} - 2\alpha y_{\gamma,\alpha}\,;$$

$$a - \left((c_0^\varepsilon \star 1_{\{u(\cdot,t_{\gamma,\alpha})\geq u(x_{\gamma,\alpha},t_{\gamma,\alpha})\}})(x_{\gamma,\alpha}) - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon\right)|p| \leq 0\,; \tag{2.23}$$

$$b - \left((c_0^\varepsilon \star 1_{\{v(\cdot,t_{\gamma,\alpha})> v(y_{\gamma,\alpha},t_{\gamma,\alpha})\}})(y_{\gamma,\alpha}) - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon\right)|q| \geq 0\,. \tag{2.24}$$

By subtracting (2.24) to (2.23) we get

$$\begin{aligned}\eta &+ \left((c_0^\varepsilon \star 1_{\{v(\cdot,t_{\gamma,\alpha})> v(y_{\gamma,\alpha},t_{\gamma,\alpha})\}})(y_{\gamma,\alpha}) - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon\right)|q|\\ &- \left((c_0^\varepsilon \star 1_{\{u(\cdot,t_{\gamma,\alpha})\geq u(x_{\gamma,\alpha},t_{\gamma,\alpha})\}})(x_{\gamma,\alpha}) - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon\right)|p| \leq 0\,.\end{aligned} \tag{2.25}$$

From the fact that $\Phi_{\gamma,\alpha}(x_{\gamma,\alpha}, y_{\gamma,\alpha}, t_{\gamma,\alpha}) \geq \Phi_{\gamma,\alpha}(x, x, t_{\gamma,\alpha})$ it follows that

$$\begin{aligned}v(x,t_{\gamma,\alpha}) - v(y_{\gamma,\alpha},t_{\gamma,\alpha}) &\geq u(x,t_{\gamma,\alpha}) - u(x_{\gamma,\alpha},t_{\gamma,\alpha}) - 2\alpha|x|^2\\ &+ \frac{|x_{\gamma,\alpha}-y_{\gamma,\alpha}|^2}{\gamma^2} + \alpha(|x_{\gamma,\alpha}|^2 + |y_{\gamma,\alpha}|^2)\,.\end{aligned}$$

In particular from the above inequality we deduce that

$$\{u(\cdot,t_{\gamma,\alpha}) \geq u(x_{\gamma,\alpha},t_{\gamma,\alpha})\} \cap \{v(\cdot,t_{\gamma,\alpha}) \leq v(y_{\gamma,\alpha},t_{\gamma,\alpha})\} \subset \{|x|^2 \geq R^2_{\alpha,\gamma}\}\,,$$

where $R^2_{\alpha,\gamma} = \dfrac{1}{2\alpha}\left(\dfrac{|x_{\gamma,\alpha}-y_{\gamma,\alpha}|^2}{\gamma^2} + \alpha(|x_{\gamma,\alpha}|^2 + |y_{\gamma,\alpha}|^2\right)$.

Thus

$$\{u(\cdot,t_{\gamma,\alpha}) \geq u(x_{\gamma,\alpha},t_{\gamma,\alpha})\} \subset \{v(\cdot,t_{\gamma,\alpha}) > v(y_{\gamma,\alpha},t_{\gamma,\alpha})\} \cup \{|x|^2 \geq R^2_{\alpha,\gamma}\}\,. \tag{2.26}$$

Given $\gamma \leq \gamma_0$ the following two cases may occur.

Case 1. For all α small and for some $\tilde{C}_\gamma > 0$ we have

$$\frac{|x_{\gamma,\alpha}-y_{\gamma,\alpha}|^2}{\gamma^2} \geq \tilde{C}^2_\gamma\,.$$

In this case we have

$$\{|x - x_{\alpha,\gamma}| \geq R_{\alpha,\gamma}\} \subset \{|x| \geq \tilde{R}_{\alpha,\gamma}\}, \tag{2.27}$$

where $\tilde{R}_{\alpha,\gamma} = -|x_{\alpha,\gamma}| + R_{\alpha,\gamma}$ satisfies the following *lemma* which proof is postponed

***Lemma* 2.5** *We have the following estimate on $\tilde{R}_{\alpha,\gamma}$*

$$\tilde{R}_{\alpha,\gamma} = R_{\alpha,\gamma} - |x_{\alpha,\gamma}| \geq \frac{\tilde{C}^2_\gamma}{8\sqrt{C_0}\sqrt{\alpha}}.$$

Now let us choose $\delta > 0$ such that $\delta C_\gamma \leq \dfrac{\eta}{3}$, $C_\gamma > 0$ being an upper bound of $|p|, |q|$ depending on γ and independent of α small enough. Since $c_0^\varepsilon \in W^{1,1}(\mathbb{R}^n)$, we have for α small

$$\int_{B^c(0,\tilde{R}_{\alpha,\gamma})} c_0^\varepsilon(x)\,dx \leq \delta\,.$$

and

$$|(c_0^\varepsilon \star 1_{\{v(\cdot,t_{\gamma,\alpha})>v(y_{\gamma,\alpha},t_{\gamma,\alpha})\}})(x_{\gamma,\alpha}) - (c_0^\varepsilon \star 1_{\{v(\cdot,t_{\gamma,\alpha})>v(y_{\gamma,\alpha},t_{\gamma,\alpha})\}})(y_{\gamma,\alpha})| \leq ||Dc_0^\varepsilon||_1|x_{\gamma,\alpha} - y_{\gamma,\alpha}|\,.$$

By using the inclusions (2.26) and (2.27) from (2.25) we get

$$\begin{aligned}
0 &\geq \eta + |q|c_0^\varepsilon \star 1_{\{v(\cdot,t_{\gamma,\alpha})>v(y_{\gamma,\alpha},t_{\gamma,\alpha})\}}(y_{\gamma,\alpha}) - |p|c_0^\varepsilon \star 1_{\{u(\cdot,t_{\gamma,\alpha})\geq u(x_{\gamma,\alpha},t_{\gamma,\alpha})\}}(x_{\gamma,\alpha})\\
&- \frac{1}{2}\int c_0^\varepsilon(x)dx(|q| - |p|)\\
&\geq \eta + |q|c_0^\varepsilon \star 1_{\{v(\cdot,t_{\gamma,\alpha})>v(y_{\gamma,\alpha},t_{\gamma,\alpha})\}}(y_{\gamma,\alpha}) - |p|c_0^\varepsilon \star 1_{\{v(\cdot,t_{\gamma,\alpha})>v(y_{\gamma,\alpha},t_{\gamma,\alpha})\}}(x_{\gamma,\alpha})\\
&- |p|c_0^\varepsilon \star 1_{B^c(0,R_{\alpha,\gamma})}(x_{\gamma,\alpha}) - \frac{1}{2}||c_0^\varepsilon||_1(|p-q|)\\
&\geq \eta - ||Dc_0^\varepsilon||_1|x_{\gamma,\alpha} - y_{\gamma,\alpha}|(2\frac{|x_{\gamma,\alpha}-y_{\gamma,\alpha}|}{\gamma^2} + \alpha + \alpha|y_{\gamma,\alpha}|^2)\\
&- \frac{3}{2}||c_0||_1\{2\alpha + \alpha(|x_{\gamma,\alpha}|^2 + |y_{\gamma,\alpha}|^2)\} - |p|\int_{B^c(0,\tilde{R}_{\alpha,\gamma})} c_0^\varepsilon(x)\,dx\\
&\geq \eta - ||Dc_0^\varepsilon||_1|x_{\gamma,\alpha} - y_{\gamma,\alpha}|(2\frac{|x_{\gamma,\alpha}-y_{\gamma,\alpha}|}{\gamma^2} + \alpha + \alpha|y_{\gamma,\alpha}|^2)\\
&- \frac{3}{2}||c_0||_1\{2\alpha + \alpha(|x_{\gamma,\alpha}|^2 + |y_{\gamma,\alpha}|^2)\} - \delta C_\gamma\,.
\end{aligned} \tag{2.28}$$

By taking in (2.28) the $\limsup_{\alpha\to 0}$ and using (2.22) we get a contradiction and we can conclude.

Case 2. There is a subsequence $\alpha_n > 0$ which we still denote by α such that

$$\frac{|x_{\gamma,\alpha} - y_{\gamma,\alpha}|^2}{\gamma^2} \to 0, \quad \text{as } \alpha \to 0\,.$$

In this case we have $\lim_{\alpha\to 0} |p| = 0$ and $\lim_{\alpha\to 0} |q| = 0$. On the other hand, from (2.25) we have the following estimate

$$0 \geq \eta - \frac{1}{2}\|c_0^\varepsilon\|_{L^1}(|p| + |q|). \tag{2.29}$$

By letting in (2.29) $\alpha \to 0$, we get a contradiction and we can conclude.

Proof of *Lemma* 2.5
By assumptions, we have

$$\frac{|x_{\gamma,\alpha} - y_{\gamma,\alpha}|^2}{\gamma^2} \geq \tilde{C}_\gamma^2.$$

We then deduce

$$\begin{aligned}
R_{\gamma,\alpha}^2 - |x_{\gamma,\alpha}|^2 &\geq \frac{\tilde{C}_\gamma^2}{2\alpha} - \frac{1}{2}(|x_{\gamma,\alpha}|^2 - |y_{\gamma,\alpha}|^2) \\
&\geq \frac{\tilde{C}_\gamma^2}{2\alpha} - \frac{1}{2}\left(|x_{\gamma,\alpha} - y_{\gamma,\alpha}|(|x_{\gamma,\alpha}| + |y_{\gamma,\alpha}|)\right) \\
&\geq \frac{\tilde{C}_\gamma^2}{2\alpha} - \frac{\gamma C_0}{\sqrt{\alpha}} \\
&\geq \frac{\tilde{C}_\gamma^2}{4\alpha} \quad \text{if } \alpha \text{ is small enough}
\end{aligned}$$

where we have used (2.17) for the third line. Moreover, using (2.17), we deduce

$$R_{\gamma,\alpha} \leq \sqrt{\frac{C_0}{\alpha}}$$

so

$$\tilde{R}_{\gamma,\alpha} = R_{\gamma,\alpha} - |x_{\gamma,\alpha}| = \frac{R_{\gamma,\alpha}^2 - |x_{\gamma,\alpha}|^2}{R_{\gamma,\alpha} + |x_{\gamma,\alpha}|} \geq \frac{\tilde{C}_\gamma^2}{4\alpha}\,\frac{1}{2\sqrt{\frac{C_0}{\alpha}}} \geq \frac{\tilde{C}_\gamma^2}{8\sqrt{C_0}\sqrt{\alpha}}.$$

This ends the proof of the *lemma*.

Theorem 2.6 (Existence and uniqueness for the ε-problem)
Let $u_0 \in \mathrm{Lip}(\mathbb{R}^n)$ such that

$$|Du_0| < B_0 \quad \text{in } \mathbb{R}^n \tag{2.30}$$

then there is a unique solution of (1.4).

Proof of Theorem 2.6
The uniqueness comes from the comparison principle and the existence is a straightforward consequence of Perron's method (see Da Lio, Kim, Slepčev [23] Theorem 1.2). Indeed, it suffices to remark that $u^{\pm}(x,t) = u_0(x) \pm \|c_0^\varepsilon\|_1 B_0 t$ are respectively super and subsolution of (1.4).

Proposition 2.7 (Lipschitz estimates in space)
The unique solution of (1.4) is Lipschitz continuous:

$$|Du^\varepsilon(\cdot,t)|_{L^\infty(\mathbb{R}^n)} \leq |Du^\varepsilon(\cdot,0)|_{L^\infty(\mathbb{R}^n)} \tag{2.31}$$

Proof of Proposition 2.7
The estimate (2.31) follows from the fact that the equation is invariant by space translation. Indeed, if we set $v(x,t) = u^\varepsilon(x+h,t) + |Du_0|_{L^\infty(\mathbb{R}^n)}|h|$, then it is easy to check that v is still a supersolution to the problem (1.4). Moreover, $v(x,0) \geq u(x,0)$, so, by comparison principle, $v(x,t) \geq u(x,t)$ for all $t \in [0,\infty)$ *i.e.* $u(x,t) - u(x+h,t) \leq |Du_0|_{L^\infty(\mathbb{R}^n)}|h|$. Using similarly a subsolution, we deduce the result.

3 The limit problem

Definition 3.1 (Viscosity sub/super/solution for mean curvature type motions)
A function $u^0 \in B_{\mathrm{loc}}USC(\mathbb{R}^n \times [0,T])$ is a viscosity subsolution of (1.5)-(1.6)-(1.7) if it satisfies:

(i) $u^0(x,0) \leq u_0(x)$ *in* $\mathbb{R}^n$,

(ii) for every $(x_0,t_0) \in \mathbb{R}^n \times (0,\infty)$ *and for every test function* $\Phi \in C^\infty(\mathbb{R}^n \times [0,\infty))$ *such that* $u^0 - \Phi$ *has a maximum at* (x_0,t_0)*, the following holds:*

$$\frac{\partial \Phi}{\partial t}(x_0,t_0) + F_*\left(D\Phi, D^2\Phi\right) \leq 0. \tag{3.32}$$

A function $u^0 \in B_{\mathrm{loc}}LSC(\mathbb{R}^n \times [0,T])$ is a viscosity supersolution of (1.5)-(1.6)-(1.7) if it satisfies:

(i) $u^0(x,0) \geq u_0(x)$ *in* $\mathbb{R}^n$,

(ii) for every $(x_0,t_0) \in \mathbb{R}^n \times (0,\infty)$ *and for every test function* $\Phi \in C^\infty(\mathbb{R}^n \times [0,\infty))$ *such that* $u^0 - \Phi$ *has a minimum at* (x_0,t_0)*, the following holds:*

$$\frac{\partial \Phi}{\partial t}(x_0,t_0) + F^*\left(D\Phi, D^2\Phi\right) \geq 0. \tag{3.33}$$

A continuous function u^0 is a viscosity solution of (1.5)-(1.6)-(1.7) if, and only if, it is a sub and a supersolution of (1.5)-(1.6)-(1.7).

This definition comes from the general definition of viscosity solution for discontinuous Hamiltonians first given by Ishii [32] (see also Crandall, Ishii, Lions [21]). We need an equivalent definition which eliminates, at least partially, the difficulty related to the fact that $D\Phi$ may be equal to zero.

Theorem 3.2 (Equivalent definition for mean curvature type motions)
We can replace in Definition 3.1 Condition (3.32) by

$$\frac{\partial \Phi}{\partial t}(x_0,t_0) + F\left(D\Phi, D^2\Phi\right) \leq 0 \text{ if } D\Phi(x_0,t_0) \neq 0$$

or

$$\frac{\partial \Phi}{\partial t}(x_0,t_0) \leq 0 \text{ if } D\Phi(x_0,t_0)=0 \text{ and } D^2\Phi(x_0,t_0)=0$$

and Condition (3.33) by

$$\frac{\partial \Phi}{\partial t}(x_0,t_0) + F\left(D\Phi, D^2\Phi\right) \geq 0. \text{ if } D\Phi(x_0,t_0) \neq 0$$

or

$$\frac{\partial \Phi}{\partial t}(x_0,t_0) \leq 0 \text{ if } D\Phi(x_0,t_0)=0 \text{ and } D^2\Phi(x_0,t_0)=0$$

and the definition remains equivalent.

The equivalence between these two definitions was first proved by Barles, Georgelin [9] for the isotropic mean curvature motion and their proof adapts here without any difficulty.

It's well known that this problem admits a unique viscosity solution. See for instance Bellettini, Novaga [16] [17], Chen, Giga, Goto [20] and Evans, Spruck [26]. Moreover, we have the following comparison principle:

Theorem 3.3 (Comparison principle for the limit problem)
If $u \in B_{\mathrm{loc}}USC(\mathbb{R}^n \times [0,T])$ is a subsolution of (1.5) and $v \in B_{\mathrm{loc}}LSC(\mathbb{R}^n \times [0,T])$ is a supersolution of (1.5) satisfying $u(x,0) \leq v(x,0)$ $\forall x \in \mathbb{R}^n$, then $u(x,t) \leq v(x,t)$ $\forall (x,t) \in \mathbb{R}^n \times (0,T)$.

In this Theorem, we do not need any assumption on the behaviour of the solution at infinity, since the equation is geometric.

4 Convergence of the velocity for a test function

4.1 Link with other works

In this subsection, we show in an heuristic way the links and the differences between our result and previous strongly related works as Barles, Georgelin [9], Chambolle, Novaga [19], Evans [25], Ishii [33] and Ishii, Pires, Souganidis [34]. In particular, we explain the term $1/|\ln \varepsilon|$ in our scaling. We make the computation formally for a general kernel K_0 with the parabolic scaling, *i.e.*

$$K_0^\varepsilon(x) = \frac{1}{\varepsilon^{n+1}} K_0\left(\frac{x}{\varepsilon}\right).$$

We assume that K_0 is symmetric, *i.e.*, $K_0(-x) = K_0(x)$ and admits a moment of order two for every section, *i.e.*, for every $p \in \mathbf{S}^{n-1}$

$$\int_{\mathbb{R}^n \cap \{p^\perp\}} K_0(x)|x|^2 < \infty \tag{4.34}$$

We want to show formally that for every regular function φ, the velocity

$$c^\varepsilon = K_0^\varepsilon \star 1_{\{\varphi \geq 0\}}(0) - \frac{1}{2}\int K_0^\varepsilon$$

converges to anisotropic mean curvature. To simplify the computation, we finally assume that the zero level set of φ is the graph of a function h, *i.e.*, more precisely that $\varphi(x', x_n) = h(x') - x_n$ where $x = (x', x_n)$, $x' \in \mathbb{R}^{n-1}$ and $D_{x'}h(0) = 0$. We have

$$\begin{aligned}
c^\varepsilon =& \int_{\{x_n \le h(x')\}} K_0^\varepsilon - \int_{\{x_n \le 0\}} K_0^\varepsilon \\
=& \frac{1}{\varepsilon} \int_{\{0 \le x_n \le \frac{h(\varepsilon x')}{\varepsilon}\}} K_0(x) dx \\
\simeq& \int_{x' \in \mathbb{R}^{n-1}} \left(\frac{1}{\varepsilon} \int_0^{\frac{\varepsilon}{2} D^2 h(0)(x',x')} K_0(x', x_n)\, dx_n \right) dx' \\
\simeq& \int_{x' \in \mathbb{R}^{n-1}} \frac{1}{2} K_0(x', 0) D^2 h(0)(x', x')\, dx' \\
=& \mathrm{trace}\left(A(p)(Id - p \otimes p)(D^2 \varphi) \right) \qquad \text{with } |D\varphi(0)| = 1,
\end{aligned}$$

where $p = \dfrac{D\varphi}{|D\varphi|}$ and $A(p) = \int_{x \in \{p^\perp\} \simeq \mathbb{R}^{n-1}} \frac{1}{2} K_0(x)\ x \otimes x\ dx$. So, formally, if (4.34) holds, then the velocity c^ε converges to anisotropic mean curvature. Barles, Georgelin [9] and Evans [25] used this result to prove the convergence of the Merriman, Bence, Osher scheme [36]. For the proof, they used the kernel

$$K_0(x) = \frac{1}{(4\pi)^{n/2}} e^{-\frac{x^2}{4}}$$

which satisfied the assumptions. This result was then generalised by Ishii [33] and Ishii, Pires, Souganidis [34] to more general kernels assuming also the symmetry of the kernel and (4.34). A by-product of our work shows that for general kernels, the limit mean curvature motion is of variational type (see Theorem 1.11).

The main difference in our case is that c_0 behaves like $\dfrac{1}{|x|^{n+1}}$ and so (4.34) does not hold. This explain the term $\dfrac{1}{|\ln \varepsilon|}$ in our scaling. Indeed to make a renormalization of the integral $\displaystyle\int_{x' \in \mathbb{R}^{n-1}} \frac{1}{2} K_0(x', 0) D^2 h(0)(x', x')\, dx'$ finite, we have to multiply by a term going to zero faster. We denote by $J(\varepsilon)$ this term (*i.e.*, we use the scaling $c_0^\varepsilon(x) = \frac{J(\varepsilon)}{\varepsilon^{n+1}} c_0\left(\frac{x}{\varepsilon}\right)$). Using the same computation as above, we obtain:

$$\begin{aligned}
c^\varepsilon =& J(\varepsilon) \frac{1}{\varepsilon} \int_{\{0 \le x_n \le \frac{h(\varepsilon x')}{\varepsilon}\} \cap \{|x'| \le \delta/\varepsilon\}} c_0(x) dx + J(\varepsilon) \mathcal{I}_1 \\
\simeq& J(\varepsilon) \int_{\{|x'| \le \delta/\varepsilon\}} \frac{1}{2} c_0(x', 0) D^2 h(0)(x', x')\, dx' + J(\varepsilon) \mathcal{I}_1
\end{aligned}$$

where

$$\mathcal{I}_1 = \frac{1}{\varepsilon} \int_{\{0 \le x_n \le \frac{h(\varepsilon x')}{\varepsilon}\} \cap \{|x'| \ge \delta/\varepsilon\}} c_0(x) dx \le \frac{1}{\varepsilon} \int_{\left(B_{\delta/\varepsilon}(0)\right)^c} c_0(x) dx$$

Using the particular form of c_0 for $|x| \ge 1$, we deduce that $\mathcal{I}_1 \le \frac{1}{\varepsilon} \int_{\delta/\varepsilon}^\infty dr \frac{1}{r^2} \int_{\theta \in \mathbf{S}^{n-1}} d\theta\ g(\theta)$ and so $\mathcal{I}_1$ is finite. This implies that the last term $J(\varepsilon)\mathcal{I}_1$ goes to zero as $\varepsilon \to 0$. We then decompose the first integral in two terms:

$$\begin{aligned}
&J(\varepsilon)\int_{\{|x'|\le\delta/\varepsilon\}}\frac{1}{2}c_0(x',0)D^2h(0)(x',x')\,dx'\\
=&J(\varepsilon)\int_{|x'|\le 1}\frac{1}{2}c_0(x',0)D^2h(0)(x',x')\,dx'+J(\varepsilon)\int_{|x'|\in(1,\delta/\varepsilon)}\frac{1}{2}c_0(x',0)D^2h(0)(x',x')\,dx'.
\end{aligned}$$

Since c_0 is bounded, we remark that the first term goes to zero as ε goes to zero. Then, the only interesting term is the second one. Using again the particular form of c_0 for $|x|\ge 1$, we deduce that

$$\begin{aligned}
&J(\varepsilon)\int_{|x'|\in(1,\delta/\varepsilon)}\frac{1}{2}c_0(x',0)D^2h(0)(x',x')\,dx'\\
=&J(\varepsilon)\int_{\theta\in\mathbf{S}^{n-2}}d\theta\frac{1}{2}D^2h(0)(\theta,\theta)g(\theta)\int_1^{\delta/\varepsilon}\frac{1}{r}dr\\
=&J(\varepsilon)(\ln\frac{\delta}{\varepsilon})\int_{\theta\in\mathbf{S}^{n-2}}\frac{1}{2}g(\theta)D^2h(0)(\theta,\theta)d\theta\\
=&J(\varepsilon)(\ln\frac{\delta}{\varepsilon})\ \mathrm{trace}\left(A(p)D^2\varphi\right).
\end{aligned}$$

So the correct scaling is to take $J(\varepsilon)=|\ln\varepsilon|$ and we finally obtain

$$c^\varepsilon\to\mathrm{trace}\left(A(p)D^2\varphi\right)\quad\text{when } |D\varphi(0)|=1.$$

4.2 Proof of convergence

In this section, we prove rigorously the convergence result for test functions.

Let us define (for $M=D^2\varphi$, $p=D\varphi$)

$$G(M,p)=\frac{-1}{|p|}F(M,p).$$

For a $n\times n$ matrix M we set the norm

$$|M|=\sup_{\xi\in B_1(0)}|M\cdot\xi|. \tag{4.35}$$

We define the modulus of continuity of the function g by

$$\omega_g(r)=\sup_{|\theta'-\theta|\le r,\quad \theta,\theta'\in\mathbf{S}^{n-1}}|g(\theta')-g(\theta)|.$$

Then we have the following fundamental estimate for test function independent on time:

Proposition 4.1 (Error estimate on the velocity for a test function)
Let us assume that $\varphi\in C^2(\mathbb{R}^n)$ and that $D\varphi(x_0)\neq 0$. For $c_0^\varepsilon(\cdot)=\dfrac{1}{\varepsilon^{n+1}|\ln\varepsilon|}\ c_0\left(\dfrac{\cdot}{\varepsilon}\right)$, let us define

$$c^\varepsilon=(c_0^\varepsilon\star 1_{\{\varphi(\cdot)>\varphi(x_0)\}})(x_0)-\frac{1}{2}\int_{\mathbb{R}^n}c_0^\varepsilon.$$

Let us call $b = |D\varphi(x_0)|$, *and for any* $a \geq |D^2\varphi|_{L^\infty(B_1(x_0))}$, *let us introduce the relative modulus of continuity of* $D^2\varphi$ *at* x_0, *defined for* $0 < r < 1$ *by*

$$\omega(r) = \begin{cases} \displaystyle\sup_{x \in B_r(x_0)} \frac{|D^2\varphi(x) - D^2\varphi(x_0)|}{a} & \text{if } \; a \neq 0 \\ \\ 0 & \text{if } \; a = 0. \end{cases}$$

We fix $\delta_1 \leq 1$ *such that*

$$\omega(\delta_1) \leq 1.$$

We define $\delta_0 = \min(1, \dfrac{b}{3a}, \delta_1)$. *There exists a constant* $C = C(n, \sup_{\mathbb{R}^n} c_0) > 0$ *such that for* $0 < \varepsilon < \delta$ *with* $0 < \delta \leq \delta_0/2$, *we have*

$$|c^\varepsilon - G(D^2\varphi(x_0), D\varphi(x_0))| \leq C \cdot e\,(\varepsilon, \delta, \delta_0)$$

with

$$e\,(\varepsilon, \delta, \delta_0) = \frac{1}{|\ln \varepsilon|}\left(1 + \frac{1}{\delta_0}|\ln \delta|\right) + \frac{1}{\delta_0}\left(\omega_g\left(\frac{\delta}{\delta_0}\right) + \omega(2\delta) + \frac{\delta}{\delta_0}\right).$$

Before to prove proposition 4.1, let us give a corollary.

Corollary 4.2 (Convergence of the velocity for a test function)
Let us assume that $\varphi \in C^2(\mathbb{R}^n \times (0, +\infty))$ *and that* $D\varphi(x_0, t_0) \neq 0$. *If* $(x_\varepsilon, t_\varepsilon) \longrightarrow (x_0, t_0)$, *then*

$$c^\varepsilon := \left((c_0^\varepsilon \star 1_{\{\varphi(\cdot, t_\varepsilon) > \varphi(x_\varepsilon, t_\varepsilon)\}})(x_\varepsilon, t_\varepsilon) - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon\right) \quad \longrightarrow \quad G(D^2\varphi(x_0, t_0), D\varphi(x_0, t_0))$$

Proof of Corollary 4.2
This is a straightforward consequence of the fact that we can choose the relative modulus of continuity ω uniformly in a neighbourhood of (x_0, t_0) and then estimate $c^\varepsilon - G(D^2\varphi(x_\varepsilon, t_\varepsilon), D\varphi(x_\varepsilon, t_\varepsilon))$ using Proposition 4.1. We conclude choosing a suitable sequence $\delta = \delta(\varepsilon)$.

Proof of Proposition 4.1
Up to change the coordinates, we can assume that $x_0 = 0$, $\varphi(x_0) = 0$, $D\varphi(x_0) = be_n$ with $b > 0$. We denote $x' = (x_1, ..., x_{n-1})$ a point of $\mathbb{R}^{n-1}$ and $x = (x', x_n) \in \mathbb{R}^n$. Then using the implicit function Theorem, we can assume that there exists a neighbourhood

$$Q_\delta = B_\delta^{n-1} \times (-\delta, \delta) \subset \mathbb{R}^n$$

of the origin such that the level set $\{\varphi = 0\}$ can be written

$$\{\varphi = 0\} \cap Q_\delta = \{(x', x_n) \in Q_\delta, \quad x_n = h(x')\}$$

for a suitable function $h \in C^2(B_\delta^{n-1}; (-\delta, \delta))$.

Then we have the following result which will be proved later:

Lemma 4.3 *Let δ_0 as defined in Proposition 4.1. For $0<\delta\le\delta_0/2$, we have*

$$\forall x'\in B^{n-1}_\delta,\quad (x',h(x'))\in Q_\delta \quad \textit{and} \quad \left|\frac{h(x')-\frac{1}{2}D^2h(0)\cdot(x',x')}{|x'|^2}\right| \quad \le \quad \frac{a}{b}\left(\omega(2\delta)+8\frac{\delta}{\delta_0}\right).$$

Moreover

$$\frac{\partial^2 h}{\partial x_i\partial x_j}(0)=-\frac{1}{|D\varphi(0)|}\ \frac{\partial^2\varphi}{\partial x_i\partial x_j}(0),\qquad i,j=1,...,n-1$$

and

$$|h(x')|\le 6\frac{a}{b}|x'|^2\quad \text{for } x'\in B^{n-1}_\delta.$$

We have

$$\begin{aligned}
c^\varepsilon\ &=(c_0^\varepsilon\star 1_{\{\varphi(\cdot)>0\}})(0)-\frac{1}{2}\int_{\mathbb{R}^n}c_0^\varepsilon\\
&=(c_0^\varepsilon\star 1_{\{\varphi(\cdot)>0\}})(0)-(c_0^\varepsilon\star 1_{\{x_n>0\}})(0)\\
&=-(c_0^\varepsilon\star 1_{\{\varphi(\cdot)\le 0\}\cap\{x_n>0\}})(0)+(c_0^\varepsilon\star 1_{\{\varphi(\cdot)>0\}\cap\{x_n<0\}})(0)\\
&=-\left\{(I)_\varepsilon+(II)_\varepsilon\right\}
\end{aligned}$$

where

$$(I)_\varepsilon=(c_0^\varepsilon\star 1_{Q_\delta\cap\{\varphi(\cdot)\le 0\}\cap\{x_n>0\}})(0)-(c_0^\varepsilon\star 1_{Q_\delta\cap\{\varphi(\cdot)>0\}\cap\{x_n<0\}})(0)$$

and

$$(II)_\varepsilon=(c_0^\varepsilon\star 1_{(\mathbb{R}^n\setminus Q_\delta)\cap\{\varphi(\cdot)\le 0\}\cap\{x_n>0\}})(0)-(c_0^\varepsilon\star 1_{(\mathbb{R}^n\setminus Q_\delta)\cap\{\varphi(\cdot)>0\}\cap\{x_n<0\}})(0).$$

We have for $\delta>\varepsilon$

$$\begin{aligned}
|(II)_\varepsilon|\ &\le\int_{\mathbb{R}^n\setminus Q_\delta}c_0^\varepsilon\\
&=\frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}^n\setminus Q_{\frac{\delta}{\varepsilon}}}c_0\\
&\le\frac{1}{|\ln\varepsilon|}\left(\int_{\mathbb{R}^n\setminus Q_1}c_0\right).
\end{aligned}$$

Let us now compute the term $(I)_\varepsilon$. We have for $\delta\le\delta_0/2$

$$(I)_\varepsilon=\int_{B^{n-1}_\delta}dx'\int_0^{h(x')}dx_n\,\frac{1}{\varepsilon^{n+1}|\ln\varepsilon|}\ \ c_0\left(\frac{x}{\varepsilon}\right).$$

Let us define (with $x=(x',x_n)=|x|\theta$)

$$(I)'_\varepsilon=\int_{B^{n-1}_\delta\setminus B^{n-1}_\varepsilon}dx'\int_0^{h(x')}dx_n\,\frac{1}{|\ln\varepsilon|}\ \ \frac{g(\theta)}{(|x'|^2+|x_n|^2)^{\frac{n+1}{2}}}.$$

Then

$$|(I)_\varepsilon - (I)'_\varepsilon| \le \int_{B_\varepsilon^{n-1}} dx' \int_0^{|h(x')|} dx_n \, \frac{1}{\varepsilon^{n+1}|\ln\varepsilon|} \;\; c_0\left(\frac{x}{\varepsilon}\right)$$

$$\le \int_{B_\varepsilon^{n-1}} dx' \int_0^{6\frac{a}{b}|x'|^2} dx_n \, \frac{1}{\varepsilon^{n+1}|\ln\varepsilon|} \;\; \left(\sup_{\mathbb{R}^n} c_0\right)$$

$$\le \frac{6a}{b|\ln\varepsilon|} \;\; \left(\frac{\sup_{\mathbb{R}^n} c_0}{n+1}\right)$$

where we have used the fact that $|h(x')| \le 6\frac{a}{b}|x'|^2$ for $|x'| \le \delta \le \delta_0/2$. We now compute $(I)'_\varepsilon$

$$(I)'_\varepsilon = \int_{B_\delta^{n-1}\setminus B_\varepsilon^{n-1}} dx' \int_0^{\frac{h(x')}{|x'|^2}} d\zeta \, \frac{|x'|^2}{|\ln\varepsilon|} \;\; \frac{g\left(\frac{(x',|x'|^2\zeta)}{\sqrt{|x'|^2+(|x'|^2\zeta)^2}}\right)}{\left(|x'|^2+(|x'|^2\zeta)^2\right)^{\frac{n+1}{2}}}$$

$$= \int_{B_\delta^{n-1}\setminus B_\varepsilon^{n-1}} \frac{1}{|\ln\varepsilon|}\frac{dx'}{|x'|^{n-1}} \left(\int_0^{\frac{h(x')}{|x'|^2}} d\zeta \;\; \frac{g\left(\frac{\left(\frac{x'}{|x'|},|x'|\zeta\right)}{\sqrt{1+|x'|^2\zeta^2}}\right)}{(1+|x'|^2\zeta^2)^{\frac{n+1}{2}}}\right).$$

Let us define

$$(I)''_\varepsilon = \int_{B_\delta^{n-1}\setminus B_\varepsilon^{n-1}} \frac{1}{|\ln\varepsilon|}\frac{dx'}{|x'|^{n-1}} \left(\int_0^{\frac{1}{2}D^2h(0)\cdot\left(\frac{x'}{|x'|},\frac{x'}{|x'|}\right)} d\zeta \;\; g\left(\frac{x'}{|x'|}\right)\right)$$

$$= \frac{\ln(\delta/\varepsilon)}{|\ln\varepsilon|}\left(\int_{\theta\in\mathbf{S}^{n-2}\subset\{x_n=0\}} d\theta \;\; \frac{1}{2}\,g(\theta)\cdot D^2h(0)\cdot(\theta,\theta)\right).$$

We define

$$(I)''_0 = \int_{\theta\in\mathbf{S}^{n-2}\subset\{x_n=0\}} d\theta \;\; \left(\frac{1}{2}\,g(\theta)\cdot D^2h(0)\cdot(\theta,\theta)\right)$$

i.e. we have from Lemma 4.3

$$-(I)''_0 = \int_{\theta\in\mathbf{S}^{n-2}\subset\{x_n=0\}} d\theta \;\; \left(\frac{1}{2}\,g(\theta)\cdot\frac{1}{|D\varphi(0)|}D^2\varphi(0)\cdot(\theta,\theta)\right)$$

$$= \frac{1}{|D\varphi(0)|}\text{trace}\;\left(D^2\varphi(0)\cdot A\left(\frac{D\varphi(0)}{|D\varphi(0)|}\right)\right)$$

$$= G(D^2\varphi(0), D\varphi(0))$$

where A is defined in (1.7).
Then we have

$$|(I)''_\varepsilon - (I)''_0| \le \frac{|\ln\delta|}{|\ln\varepsilon|}\left(\int_{\theta\in\mathbf{S}^{n-2}} d\theta\right) \;\; \left(\sup_{\mathbf{S}^{n-1}} g\right)\frac{1}{2}|D^2h(0)|$$

(4.36)

$$\le \frac{|\ln\delta|}{|\ln\varepsilon|}(n-1)|B_1^{n-1}| \;\; \left(\sup_{\mathbf{S}^{n-1}} g\right)\frac{a}{2b}.$$

We now want to estimate the difference between $(I)'_\varepsilon$ and $(I)''_\varepsilon$. To this end, we first set $v = (\frac{x'}{|x'|}, |x'|\zeta)$, $\theta = \left(\frac{x'}{|x'|}, 0\right)$. Then using only the fact that $|\theta| = 1$ and the identity $\langle v-\theta, \theta\rangle = 0$ for the scalar product, we get $0 \le |v| - 1 \le |v-\theta|$, and $\left|\frac{v}{|v|} - \theta\right| \le 2|v-\theta|$. We then estimate

$$\begin{aligned}
\left|\frac{g\left(\frac{v}{|v|}\right)}{|v|^{n+1}} - g(\theta)\right| &\le \left|g\left(\frac{v}{|v|}\right) - g(\theta)\right| + g(\theta)\left(|v|^{n+1} - 1\right) \\
&\le \omega_g\left(\left|\frac{v}{|v|} - \theta\right|\right) + \left(\sup_{\mathbf{S}^{n-1}} g\right)(n+1)|v|^n\left(|v| - 1\right) \\
&\le \omega_g(2|v-\theta|) + \left(\sup_{\mathbf{S}^{n-1}} g\right)(n+1)\left(1 + |v-\theta|\right)^n |v-\theta| \\
&\le \omega_g(2|v-\theta|) + \left(\sup_{\mathbf{S}^{n-1}} g\right)(n+1)2^n|v-\theta|
\end{aligned}$$

where for the last line, we have moreover used the fact that $|v-\theta| \le 1$ when $|x'| \le \delta$, $|\zeta| \le \frac{1}{2}\left|D^2h(0)\cdot(\theta,\theta)\right| \le \frac{a}{2b}$, and $\delta \le \delta_0/2$.
Using $|v-\theta| \le \frac{\delta a}{2b}$, we bound the last term by the quantity

$$e_1 = \omega_g\left(\frac{\delta a}{b}\right) + \left(\sup_{\mathbf{S}^{n-1}} g\right)(n+1)2^{n-1}\frac{\delta a}{b}.$$

Using Lemma 4.3 with

$$e_2 = \frac{a}{b}\left(\omega(2\delta) + 8\frac{\delta}{\delta_0}\right)$$

we then estimate

$$\begin{aligned}
|(I)''_\varepsilon - (I)'_\varepsilon| &\le \int_{B^{n-1}_\delta \setminus B^{n-1}_\varepsilon} \frac{1}{|\ln\varepsilon|}\frac{dx'}{|x'|^{n-1}}\left\{e_2\cdot\left(\sup_{\mathbf{S}^{n-1}} g\right) + \frac{a}{2b}\cdot e_1\right\} \\
&\le \frac{\ln(\delta/\varepsilon)}{|\ln\varepsilon|}\left(\int_{\theta\in\mathbf{S}^{n-2}} d\theta\right)\left\{e_2\cdot\left(\sup_{\mathbf{S}^{n-1}} g\right) + \frac{a}{2b}\cdot e_1\right\} \\
&\le (n-1)|B^{n-1}_1|\left\{e_2\cdot\left(\sup_{\mathbf{S}^{n-1}} g\right) + \frac{a}{2b}\cdot e_1\right\}.
\end{aligned}$$

Finally we get (using $\frac{3a}{b} \leq \frac{1}{\delta_0}$, $\delta \leq \frac{\delta_0}{2} \leq \frac{1}{2}$),

$$\begin{aligned} |c^\varepsilon + (I)''_0| &\leq |(II)_\varepsilon| + |(I)_\varepsilon - (I)'_\varepsilon| + |(I)'_\varepsilon - (I)''_\varepsilon| + |(I)''_\varepsilon - (I)''_0| \\ &\leq \frac{C}{|\ln \varepsilon|}\left(1 + \frac{a}{b}|\ln \delta|\right) + C\left(\frac{a}{b}\omega_g\left(\frac{\delta a}{b}\right) + \frac{a}{b}\omega(2\delta) + \frac{a}{b}\frac{\delta}{\delta_0}\right) \\ &\leq \frac{C}{|\ln \varepsilon|}\left(1 + \frac{1}{\delta_0}|\ln \delta|\right) + C\frac{1}{\delta_0}\left(\omega_g\left(\frac{\delta}{\delta_0}\right) + \omega(2\delta) + \frac{\delta}{\delta_0}\right) \end{aligned}$$

where the constant C only depends on the dimension n and c_0. More precisely we have $C = C\left(n,\ \int_{\mathbb{R}^n \setminus B_1} c_0,\ \sup_{\mathbb{R}^n} c_0,\ \sup_{\mathbf{S}^{n-1}} g\right) = C(n,\ \sup_{\mathbb{R}^n} c_0)$. This ends the proof of Proposition 4.1.

Proof of *Lemma* 4.3

Using the notations $\varphi_i = \frac{\partial \varphi}{\partial x_i}$, and $\varphi_{ij} = \frac{\partial^2 \varphi}{\partial x_i \partial x_j}$, and taking the derivatives of the relation $\varphi(x', h(x')) = 0$, we get

$$\begin{cases} h_i = -\dfrac{\varphi_i}{\varphi_n}, \quad i = 1, ..., n-1 \\ \\ h_{ij} = -\dfrac{1}{\varphi_n}\left(\varphi_{ij} + \varphi_{in}h_j + \varphi_{jn}h_i + \varphi_{nn}h_i h_j\right), \quad i,j = 1, ..., n-1. \end{cases}$$

Now, by definition of a, we have $|D^2\varphi(x)| \leq a$ for $x \in B_1$. Therefore for $0 < \delta \leq 1$, we get

$$|D\varphi(x) - D\varphi(0)| \leq a\delta \quad \text{for} \quad x \in B_\delta.$$

Let us define $\delta''_0 \in (0, +\infty]$ such that $a\delta''_0 = \frac{1}{2}|D\varphi(0)|$ and $\delta'_0 = \min(1, \delta''_0)$. Then for $b = |D\varphi(0)| = \varphi_n(0)$ and $0 < \delta \leq \delta'_0$ we get

$$a\delta_0 \leq a\delta'_0 \leq \frac{b}{2} \leq \varphi_n(x) \leq |D\varphi(x)| \quad \text{for} \quad x \in B_\delta.$$

Using the elementary estimate

(4.37)
$$\forall x \in B_\delta, \quad \left|\frac{f(x)}{g(x)} - \frac{f(0)}{g(0)}\right| \quad \leq \quad \frac{1}{g(0)(\inf_{B_\delta} g)}\left(|f(x) - f(0)|\, g(0) + |f(0)|\, |g(x) - g(0)|\right)$$

and using the fact that $\varphi_i(0) = 0$ for $i = 1, ..., n-1$, we get

$$Dh(0) = 0 \quad \text{and} \quad |Dh(x')| \leq \frac{\delta}{\delta_0} \quad \text{for} \quad (x', h(x')) \in B_\delta.$$

Still using (4.37), we get for $(x', h(x')) \in B_\delta$ and $0 < \delta \leq \delta_0$

$$\begin{aligned} |D^2h(x') - D^2h(0)| &\leq \frac{2}{b^2}\left((a\ \omega(\delta)) \cdot b + a \cdot (a\delta)\right) + \frac{2}{b}\left(2a\frac{\delta}{\delta_0} + a\left(\frac{\delta}{\delta_0}\right)^2\right) \\ &\leq \frac{2a}{b}\left(\omega(\delta) + 4\frac{\delta}{\delta_0}\right) \end{aligned}$$

where we have used the fact that $\frac{a}{b} \le \frac{1}{2\delta_0}$.
Using the Taylor formula with $h(0) = 0 = Dh(0)$, we get

$$\left| h(x') - \frac{1}{2} D^2 h(0) \cdot (x', x') \right| \le \int_0^1 dt \int_0^t ds \, \left| D^2 h(sx') - D^2 h(0) \right| \cdot |x'|^2$$

and then for $(x', h(x')) \in B_\delta$

$$\left| \frac{h(x') - \frac{1}{2} D^2 h(0) \cdot (x', x')}{|x'|^2} \right| \le \frac{a}{b} \left(\omega(\delta) + 4 \frac{\delta}{\delta_0} \right) =: J(\delta).$$

Let us now assume that $0 < 2\delta \le \delta_0$. Then $Q_\delta = B_\delta^{n-1} \times (-\delta, \delta) \subset B_{2\delta}$, and for $x' \in B_\delta^{n-1}$ we have (using $|D^2 h(0)| \le a/b$)

$$|h(x')| \le \delta^2 \left(\frac{1}{2} \frac{a}{b} + J(2\delta) \right) < \delta$$

while $\omega(2\delta) \le 1$ and $\frac{6a\delta}{b} \le 1$. Therefore for $0 < 2\delta \le \delta_0$, we get that $(x', h(x')) \in Q_\delta \subset B_{2\delta}$ if $x' \in B_\delta^{n-1}$, and then

$$\left| \frac{h(x') - \frac{1}{2} D^2 h(0) \cdot (x', x')}{|x'|^2} \right| \le \frac{a}{b} \left(\omega(2\delta) + 8 \frac{\delta}{\delta_0} \right).$$

We then deduce

$$|h(x')| \le |x'|^2 \frac{a}{b} \left(\omega(2\delta) + 8 \frac{\delta}{\delta_0} + \frac{1}{2} \right) \le 6 \frac{a}{b} |x'|^2,$$

which ends the proof of Lemma 4.3.

Corollary 4.4 (Error estimate for a particular test function)
For $B, \eta > 0$, we consider the function

$$\varphi(x) = B \sqrt{\eta^2 + |x|^2}.$$

Then, there exists a constant $C' = C'(n, \sup_{\mathbb{R}^n} c_0) > 0$ such that for

$$c^\varepsilon(x) = (c_0^\varepsilon \star 1_{\{\varphi(\cdot) > \varphi(x)\}})(x) - \frac{1}{2} \int_{\mathbb{R}^n} c_0^\varepsilon$$

we have pointwise, for $|x_0|, \eta \ge 6\sqrt{2}\,\varepsilon$:

$$|c^\varepsilon(x_0) |D\varphi(x_0)| + F(D^2\varphi(x_0), D\varphi(x_0))| \le C' \cdot B \left(\frac{1}{\eta} + \frac{1}{|\ln \varepsilon|} \right).$$

Proof of Corollary 4.4
Let us first remark that we do not change the result if we divide φ by B (because F is geometric), so we can assume that $B = 1$.

For all x, we have

$$D\varphi(x) = \frac{x}{\sqrt{\eta^2 + |x|^2}}, \qquad D^2\varphi(x) = \frac{1}{\sqrt{\eta^2 + |x|^2}} \left(Id - p(x) \otimes p(x) \right)$$

where $p(x) = D\varphi(x),\ |p(x)| \leq 1,\ \forall x.$

We have, for all $x,\ x_0$:

$$\begin{aligned} D^2\varphi(x) - D^2\varphi(x_0) &= \left(\frac{1}{\sqrt{\eta^2+|x|^2}} - \frac{1}{\sqrt{\eta^2+|x_0|^2}}\right)(Id - p(x)\otimes p(x)) \\ &\quad - \frac{1}{\sqrt{\eta^2+|x_0|^2}}\,(p(x)\otimes(p(x)-p(x_0)) + (p(x)-p(x_0))\otimes p(x_0)). \end{aligned}$$

Moreover, the following holds

$$\begin{aligned} \left|\frac{1}{\sqrt{\eta^2+|x|^2}} - \frac{1}{\sqrt{\eta^2+|x_0|^2}}\right| &\leq \frac{\left|\sqrt{\eta^2+|x|^2} - \sqrt{\eta^2+|x_0|^2}\right|}{\eta^2} \\ &\leq \frac{\left||x|^2 - |x_0|^2\right|}{\eta^2\left(\sqrt{\eta^2+|x|^2}+\sqrt{\eta^2+|x_0|^2}\right)} \\ &\leq \frac{\left||x|-|x_0|\right|(|x|+|x_0|)}{\eta^2(|x|+|x_0|)} \\ &\leq \frac{|x-x_0|}{\eta^2} \end{aligned}$$

and, using the bound $|D^2\varphi| \leq \frac{1}{\eta}$, we get

$$|p(x)-p(x_0)| = |D\varphi(x) - D\varphi(x_0)| \leq \frac{|x-x_0|}{\eta}.$$

We set $a = \frac{1}{\eta} \geq |D^2\varphi|$. We then get, with the notation of Proposition 4.1:

$$\frac{|D^2\varphi(x) - D^2\varphi(x_0)|}{a} \leq \frac{3|x-x_0|}{\eta}, \qquad \omega(r) \leq \frac{3r}{\eta}.$$

Then we can apply Proposition 4.1 with $a = \dfrac{1}{\eta}$, $b = |D\varphi(x_0)| > 0$, $\delta_1 = \frac{\eta}{3}$,
$2\delta = \delta_0 = \min\left(\dfrac{b}{3a},\ \delta_1\right) = \dfrac{b}{3a}$ (because $b \leq 1$).
We deduce that there exists a constant $C' = C'(n, \sup_{\mathbb{R}^n} c_0) > 0$ such that for $\delta > \varepsilon > 0$:

$$|c^\varepsilon(x_0)|D\varphi(x_0) + F(D^2\varphi(x_0), D\varphi(x_0))| \leq C'\left(\frac{1}{\eta} + \frac{1}{|\ln\varepsilon|}\right).$$

Moreover, the condition $\delta > \varepsilon$ is equivalent to $b > \dfrac{6\varepsilon}{\eta}$. We then deduce conditions on $|x_0|$ and η:

1. If $|x_0| \leq \eta$, then $b \geq \dfrac{|x_0|}{\sqrt{2}\eta}$ and it suffices to take $|x_0| > 6\sqrt{2}\varepsilon$.

2. If $|x_0| \geq \eta$, then $b \geq \dfrac{1}{\sqrt{2}}$ and it suffices to take $\eta > 6\sqrt{2}\varepsilon$.

5 *A priori* estimate at initial time

Proposition 5.1 (Modulus of continuity in time)
There is a constant $C'' = C''(n, \sup_{\mathbb{R}^n} c_0) > 0$ *such that for every* $x_0 \in \mathbb{R}^n$ *and* $t > 0$ *we have, for* $\eta > 6\sqrt{2}\varepsilon$*, and* $\varepsilon \in (0, 1/2)$

$$|u^\varepsilon(x_0, t) - u_0(x_0)| \le |Du_0|_{L^\infty(\mathbb{R}^n)} \cdot \left\{ \eta + t \cdot C'' \left(\frac{1}{\eta} + \frac{1}{|\ln \varepsilon|} \right) \right\}.$$

Remark 5.2 *Since* $|Du^\varepsilon(\cdot, t)|_{L^\infty(\mathbb{R}^n)} \le |Du_0|_{L^\infty(\mathbb{R}^n)}$ *(see Proposition 2.7), we also have, for* $\varepsilon \in (0, 1/2)$ *and* $\forall \eta > 6\sqrt{2}\varepsilon$

$$|u^\varepsilon(x_0, t+s) - u^\varepsilon(x_0, s)| \le |Du_0|_{L^\infty(\mathbb{R}^n)} \cdot \left\{ \eta + t \cdot C'' \left(\frac{1}{\eta} + \frac{1}{|\ln \varepsilon|} \right) \right\}.$$

Proof of Proposition 5.1
We consider the following function

$$\varphi(x, t) = B_0 \sqrt{\eta^2 + |x|^2} + u_0(x_0) - B_0 |x_0| + L \cdot t$$

with $B_0 = |Du_0|_{L^\infty(\mathbb{R}^n)}$ and L that will be precised later. To prove the result, it suffices to show that for $L = B_0 C'' \left(\frac{1}{\eta} + \frac{1}{|\ln \varepsilon|} \right)$ and C'' large enough, then φ is a supersolution of (1.4). Indeed, by comparison principle (Theorem 2.2), we will then have

$$u^\varepsilon(x_0, t) \le \varphi(x_0, t) \le B_0 \left(\eta + t \cdot C'' \left(\frac{1}{\eta} + \frac{1}{|\ln \varepsilon|} \right) \right) + u_0(x_0).$$

Let $(x, t) \in \mathbb{R}^n \times (0, \infty)$. To prove that φ is a supersolution of (1.4) at (x, t), since φ is $C^\infty(\mathbb{R}^n \times (0, \infty))$, it suffices to show that φ satisfies the equation pointwise, *i.e.*

$$\varphi_t(x, t) \ge c^\varepsilon |D\varphi(x, t)|.$$

The proof is now decomposed into two cases:

1. $|x| \le 6\sqrt{2}\,\varepsilon$. In this case, we have

$$c^\varepsilon |D\varphi(x, t)| \le \frac{\|c_0\|_{L^1}}{\varepsilon |\ln \varepsilon|} \frac{B_0 |x|}{\eta} \le \frac{6\sqrt{2} \|c_0\|_{L^1} B_0}{|\ln \varepsilon| \eta}.$$

   So it suffices to take $L \ge \dfrac{6\sqrt{2}\|c_0\|_{L^1}}{\left|\ln \frac{1}{2}\right|} \dfrac{B_0}{\eta}$.

2. $|x| \ge 6\sqrt{2}\,\varepsilon$. In this case we will show that φ is a supersolution of

$$\varphi_t + F(D^2\varphi, D\varphi) \ge L - L_0 \tag{5.38}$$

   for $L_0 = \dfrac{B_0}{\eta} \sup_{q \in \mathbf{S}^{n-1}} \operatorname{trace} \left(A \left(\dfrac{q}{|q|} \right) \right)$ and then we will use Corollary 4.4.

We set $M = D^2\varphi$. We can choose a basis such that

$$A\left(\frac{p}{|p|}\right) = \left(\begin{array}{cc} A_{n-1}\left(\frac{p}{|p|}\right) & 0 \\ 0 & 0 \end{array}\right),$$

where the last vector of the basis is $\frac{p}{|p|}$, with $p = D\varphi$. We set

$$M = B_0 \left(\begin{array}{cc} M_{n-1} & M_n \\ {}^tM_n & M_{nn} \end{array}\right),$$

where $M_{n-1} = \dfrac{1}{\sqrt{\eta^2+|x|^2}} Id$, M_n is a vector and $M_{nn} = \dfrac{\eta^2}{(\eta^2+|x|^2)^{\frac{3}{2}}}$. We then deduce that

$$\text{trace}\left(M.A\left(\frac{p}{|p|}\right)\right) = \frac{B_0}{\sqrt{\eta^2+|x|^2}}\text{trace}\left(A_{n-1}\right) \leq \frac{B_0}{\eta}\ \text{trace}\left(\mathrm{A}\left(\frac{\mathrm{p}}{|\mathrm{p}|}\right)\right).$$

We then deduce that

$$\begin{aligned}\varphi_t(x,t) + F(D^2\varphi, D\varphi) =& L - \text{trace}\left(MA\left(\frac{p}{|p|}\right)\right) \\ \geq& L - \frac{B_0}{\eta}\sup_{\mathbf{S}^{n-1}}\text{trace}\left(A\left(\frac{p}{|p|}\right)\right) \\ =& L - L_0.\end{aligned}$$

We now prove that φ is a supersolution of (1.4), *i.e.*

$$\varphi_t(x,t) \geq c^\varepsilon |D\varphi(x,t)|,$$

where $c^\varepsilon = (c_0^\varepsilon \star 1_{\{\varphi(\cdot,t)>\varphi(x,t)\}})(x,t) - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon$. We have pointwise

$$\begin{aligned}\varphi_t \geq& -F(D^2\varphi, D\varphi) + L - L_0 \\ \geq& c^\varepsilon|D\varphi| + L - L_0 - F(D^2\varphi, D\varphi) - c^\varepsilon|D\varphi| \\ \geq& c^\varepsilon|D\varphi| + L - L_0 - C'\cdot B_0\left(\frac{1}{\eta} + \frac{1}{|\ln\varepsilon|}\right),\end{aligned}$$

where we have used Corollary 4.4. It is sufficient to take

$$L \geq B_0 C''\left(\frac{1}{\eta} + \frac{1}{|\ln\varepsilon|}\right)$$

with

$$C'' = \sup_{q\in\mathbf{S}^{n-1}}\ \text{trace}\left(A\left(\frac{q}{|q|}\right)\right) + C' + \frac{6\sqrt{2}|c_0|_{L^1}}{\ln\frac{1}{2}}. \tag{5.39}$$

Moreover, trace(A) is bounded by $|g|_{L^\infty}$ which is controlled by $|c_0|_{L^\infty}$ (since $c_0(x) = g(x)$ if $|x| = 1$). So, by Corollary 4.4, $C'' = C''(n, \sup_{\mathbb{R}^n} c_0)$.

Using similarly a subsolution, we deduce the result. This ends the proof of the proposition.

Corollary 5.3 *The solution u^ε of (1.4) is Holder continuous of exponent $1/2$ with respect to t, uniformly in ε for $\varepsilon \leq \frac{1}{2}$.*

Proof of Corollary 5.3
We can optimise the estimate of Remark 5.2 to obtain:

$$|u^\varepsilon(x_0, t+s) - u^\varepsilon(x_0, s)| \leq |Du_0|_{L^\infty(\mathbb{R}^n)} \cdot \left(2\sqrt{C''}\sqrt{t} + \frac{C''}{|\ln \varepsilon|}t\right) \qquad \text{if } \sqrt{t} > \frac{6\sqrt{2}\varepsilon}{\sqrt{C''}}.$$

Moreover, for all ε, we have:

$$|u^\varepsilon(x_0, t+s) - u^\varepsilon(x_0, s)| \leq \frac{t}{\varepsilon|\ln \varepsilon|}|c_0|_{L^1}|Du_0|_{L^\infty(\mathbb{R}^n)}.$$

But, for $\sqrt{t} \leq \frac{6\sqrt{2}\varepsilon}{\sqrt{C''}}$ and $\varepsilon \leq \frac{1}{2}$, the following holds (using (5.39))

$$\frac{t}{\varepsilon|\ln \varepsilon|}|c_0|_{L^1}|Du_0|_{L^\infty(\mathbb{R}^n)} \leq |Du_0|_{L^\infty(\mathbb{R}^n)}\sqrt{t}\frac{6\sqrt{2}}{\sqrt{C''}}\frac{|c_0|_{L^1}}{|\ln \frac{1}{2}|} \leq |Du_0|_{L^\infty(\mathbb{R}^n)}\sqrt{t}\sqrt{C''},$$

so, $\forall t,\ s$, we have

$$|u^\varepsilon(x_0, t+s) - u^\varepsilon(x_0, s)| \leq |Du_0|_{L^\infty(\mathbb{R}^n)} \left(2\sqrt{C''}\sqrt{t} + \frac{C''}{|\ln \varepsilon|}t\right).$$

This ends the proof of the corollary.

6 Proof of the convergence Theorem

Proof of Theorem 1.4
We use the half-relaxed limits introduced by Barles, Perthame [11], and defined by:

$$\overline{u}(x,t) = \limsup_{\varepsilon \to 0,\, y \to x,\, s \to t} u^\varepsilon(y,s)$$

and

$$\underline{u}(x,t) = \liminf_{\varepsilon \to 0,\, y \to x,\, s \to t} u^\varepsilon(y,s).$$

We will show that $\overline{u}$ (resp. $\underline{u}$) is a viscosity subsolution (resp. supersolution) of (1.5)-(1.6)-(1.7).

We argue by contradiction. Assume that there exists $\phi \in C^2$ such that $\overline{u} - \phi$ reaches a global strict maximum at (x_0, t_0) and such that

$$\phi_t(x_0, t_0) + F_*(D^2\phi, D\phi) = \theta > 0. \tag{6.40}$$

Two cases may occur:

1. $|D\phi(x_0,t_0)| \neq 0$.

   We then deduce that there exists $(x_\varepsilon,t_\varepsilon) \to (x_0,t_0)$ such that $u^\varepsilon - \phi$ reaches a maximum at $(x_\varepsilon,t_\varepsilon)$. Using the fact that u^ε has linear growth, we can assume (by adding a term like $|x-x_0|^4 + |t-t_0|^2$ to ϕ if necessary) that this maximum is global. Since u^ε is a solution of (1.4), the following holds:

$$\phi_t(x_\varepsilon,t_\varepsilon) \leq \left((c_0^\varepsilon \star 1_{\{u^\varepsilon(\cdot,t_\varepsilon) \geq u^\varepsilon(x_\varepsilon,t_\varepsilon)\}})(x_\varepsilon) - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon \right) |D\phi(x_\varepsilon,t_\varepsilon)|.$$

   Moreover, $\forall x \neq x_\varepsilon$, we have $u^\varepsilon(x,t_\varepsilon) - \phi(x,t_\varepsilon) < u^\varepsilon(x_\varepsilon,t_\varepsilon) - \phi(x_\varepsilon,t_\varepsilon)$. So $\{u^\varepsilon(\cdot,t_\varepsilon) \geq u^\varepsilon(x_\varepsilon,t_\varepsilon)\} \subset \{\phi(\cdot,t_\varepsilon) > \phi(x_\varepsilon,t_\varepsilon)\} \cup \{x_\varepsilon\}$. We then deduce:

$$\phi_t(x_\varepsilon,t_\varepsilon) \leq \left((c_0^\varepsilon \star 1_{\{\phi(\cdot,t_\varepsilon) > \phi(x_\varepsilon,t_\varepsilon)\}})(x_\varepsilon) - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon \right) |D\phi(x_\varepsilon,t_\varepsilon)|.$$

   We can use Corollary 4.2 and pass to the limit in ε. The following holds:

$$\phi_t(x_0,t_0) \leq G(D^2\phi(x_0,t_0), D\phi(x_0,t_0))|D\phi(x_0,t_0)| = -F(D^2\phi(x_0,t_0), D\phi(x_0,t_0)),$$

   what contradicts (6.40) (since $F(M,p) = F_*(M,p)$ for $p \neq 0$).

2. $|D\phi(x_0,t_0)| = 0$ and $|D^2\phi(x_0,t_0)| = 0$. As in the first case, there exist $(x_\varepsilon,t_\varepsilon) \to (x_0,t_0)$ such that $u^\varepsilon - \phi$ reaches a global maximum at $(x_\varepsilon,t_\varepsilon)$ (up to add a term like $|x-x_0|^4 + |t-t_0|^2$ to ϕ if necessary). We set

$$c^\varepsilon[\phi](x_\varepsilon,t_\varepsilon) = \left((c_0^\varepsilon \star 1_{\{\phi(\cdot,t_\varepsilon) > \phi(x_\varepsilon,t_\varepsilon)\}})(x_\varepsilon) - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon \right).$$

   By assumptions, there exists $\eta > 0$, $r > 0$ such that

$$|D^2\phi(x,t)| \leq \eta \quad \text{if } (x,t) \in Q_{2r}(x_0,t_0)$$

   where $Q_r(x_0,t_0) = B_r(x_0) \times (t_0 - r, t_0 + r)$.

Subcase A: $|D\phi(x_\varepsilon,t_\varepsilon)| > 12\varepsilon\eta r$.

We set

$$\mathcal{I}(x,t) = c^\varepsilon[\phi](x,t)|D\phi| + F_*(D^2\phi, D\phi)$$

and

$$\phi^r(x,t) = \frac{1}{r^2}\phi(x_0 + rx, t_0 + rt).$$

Straightforward computations give with $\bar{x}_\varepsilon = \frac{x_\varepsilon}{r}$, $\bar{t}_\varepsilon = \frac{t_\varepsilon}{r}$

$$\begin{aligned}
\mathcal{I}(x_\varepsilon,t_\varepsilon) &= F_*(D^2\phi^r, D\phi^r) + \frac{|\ln \frac{\varepsilon}{r}|}{|\ln \varepsilon|}|D\phi^r|c^{\varepsilon/r}[\phi^r](\bar{x}_\varepsilon,\bar{t}_\varepsilon) \\
&= F_*(D^2\phi^r, D\phi^r) + \left(1 - \frac{|\ln r|}{|\ln \varepsilon|}\right)|D\phi^r|c^{\varepsilon/r}[\phi^r](\bar{x}_\varepsilon,\bar{t}_\varepsilon) \\
&= \left(1 - \frac{|\ln r|}{|\ln \varepsilon|}\right)\mathcal{I}_1 + \mathcal{I}_2
\end{aligned}$$

where

$$\mathcal{I}_1 = F_*(D^2\phi^r, D\phi^r) + |D\phi^r| c^{\varepsilon/r}[\phi^r](\bar{x}_\varepsilon, \bar{t}_\varepsilon) \quad \text{and} \quad \mathcal{I}_2 = \frac{|\ln r|}{|\ln \varepsilon|} F_*(D^2\phi^r, D\phi^r).$$

We can then apply Proposition 4.1 to $\mathcal{I}_1$ with

$$a = 2\eta \geq |D^2\phi^r|, \quad b = |D\phi^r(\bar{x}_\varepsilon, \bar{t}_\varepsilon)| \to 0, \quad 2\delta = \delta_0 = \frac{b}{6\eta}$$

and get (with an abuse of notation for a generic constant C)

$$\begin{aligned}
|\mathcal{I}_1| \leq & Cb\left\{\frac{1}{\delta_0} + \frac{1}{\delta_0}\frac{|\ln \delta|}{|\ln \varepsilon|} + \frac{1}{|\ln \varepsilon|}\right\} \\
\leq & C\left\{\eta + \eta + \frac{b}{|\ln \varepsilon|}\right\} \\
\leq & C\eta
\end{aligned}$$

for ε small enough to get b small enough. We then deduce that for ε small enough we have

$$|\mathcal{I}(x_\varepsilon, t_\varepsilon)| \leq C\eta$$

and so

$$\begin{aligned}
\phi_t(x_\varepsilon, t_\varepsilon) + F_*(D^2\phi, D\phi) = & \phi_t(x_\varepsilon, t_\varepsilon) - c^\varepsilon[\phi](x_\varepsilon, t_\varepsilon) + F_*(D^2\phi, D\phi) + c^\varepsilon[\phi](x_\varepsilon, t_\varepsilon) \\
\leq & |\mathcal{I}(x_\varepsilon, t_\varepsilon)| \\
\leq & C\eta.
\end{aligned}$$

Subcase B: $|D\phi(x_\varepsilon, t_\varepsilon)| \leq 12\varepsilon\eta r$.
Then we have

$$c^\varepsilon[\phi](x_\varepsilon, t_\varepsilon)|D\phi| \leq \frac{|c_0|_{L^1}}{\varepsilon|\ln \varepsilon|}|D\phi| \leq \frac{12\eta r}{|\ln \varepsilon|}|c_0|_{L^1}$$

and using $F_*(D^2\phi, D\phi) = 0$ in (x_0, t_0), we also deduce that for ε small enough we have

$$\phi_t(x_\varepsilon, t_\varepsilon) + F_*(D^2\phi, D\phi) \leq C\eta.$$

Sending $\varepsilon \to 0$, we get

$$\phi(x_0, t_0) + F_*(D^2\phi, D\phi) \leq C\eta$$

and so

$$\theta \leq C\eta$$

which is a contradiction for η small enough.

Finally, we have shown that $\overline{u}$ is a subsolution. The proof to show that $\underline{u}$ is a supersolution is exactly the same.

Moreover, by corollary 5.3, we have:

$$|u^\varepsilon(\cdot, t) - u_0(\cdot)| \leq C|t|^{\frac{1}{2}}, \quad \text{for } 0 \leq t \leq 1$$

where C is a constant which depends only on n, $\sup_{\mathbb{R}^n} c_0$ and $|Du_0|_{L^\infty}$. So $\overline{u}(\cdot,0) = \underline{u}(\cdot,0) = u_0(\cdot)$. Since $\overline{u}$ is a subsolution and $\underline{u}$ is a supersolution, we deduce by the comparison principle (Theorem 3.3) that

$$\overline{u}(x,t) \leq \underline{u}(x,t) \quad \forall (x,t)$$

and so $\overline{u} = \underline{u} = u^0$, *i.e.* u^ε converges locally uniformly on compact sets of $\mathbb{R}^n \times [0,\infty)$ to u^0 which is the unique solution of (1.5)-(1.6)-(1.7). This ends the proof of the Theorem.

7 Proof of Theorem 1.7

We now prove Theorem 1.7. We need the following proposition:

Proposition 7.1 (The matrix A is an hessian)
Let $n \geq 2$. Let $g \in C^0(\mathbb{R}^n\backslash\{0\})$ such that $g(\lambda p) = \dfrac{g(p)}{|\lambda|^{n+1}}$. We set

$$A\left(\frac{p}{|p|}\right) = \int_{\theta \in \mathbf{S}^{n-2} = \mathbf{S}^{n-1} \cap \left\{\langle x, \frac{p}{|p|}\rangle = 0\right\}} \left(\frac{1}{2} g(\theta)\theta \otimes \theta\right) \, d\theta$$

with $A(\lambda p) = \dfrac{1}{|\lambda|}A(p)$ for $\lambda \neq 0$. Then, the function $G := -\dfrac{1}{2\pi}\mathcal{F}(L_g)$ (where L_g and the Fourier transform are given in definition 1.6) is such that $G(\lambda p) = |\lambda| G(p)$ and satisfies

$$A(p) = D^2 G(p).$$

For the proof of this proposition, we will need the following *lemma*

***Lemma* 7.2 (The Curl of the matrix A)**
Under the assumptions of Proposition 7.1, the Curl of A, defined by $Curl(A) = (\partial_k A_{ij} - \partial_i A_{jk})_{i,j,k}$ is zero, and there exists a distribution Φ such that $A(p) = D^2\Phi(p)$. Moreover, $\Phi \in C^0(\mathbb{R}^n) \cap C^2(\mathbb{R}^n\backslash\{0\})$, and Φ is unique if we assume $\Phi(-p) = \Phi(p)$ and $\Phi(0) = 0$. We then have $\Phi(\lambda p) = |\lambda|\Phi(p)$, $\forall \lambda \in \mathbb{R}\backslash\{0\}$, $\forall p \in \mathbb{R}^n$.

Proof of *Lemma* 7.2
In this proof, we denote by $e \cdot f$ the scalar product between e and f.
First, we compute $\partial_k A_{ij}(p)$ for $p \neq 0$ and where $g \in C^1(\mathbb{R}^n\backslash\{0\})$ and ∂_k indicates the derivation in the direction e_k. Two cases may occur:

1. e_k is parallell to p ($e_k \parallel p$). Then,

$$\partial_k A_{ij}(p) = -\frac{p \cdot e_k}{|p|^2} A_{ij}(p).$$

2. e_k is perpendicular to p ($e_k \perp p$). In this case (see Figure 1), we have to consider variations at the first order of the integral defining $A(p)$ for $\theta \in \{p^\perp\} \cap \mathbf{S}^{n-1}$ to $\theta \in \{(p+\varepsilon e_k)^\perp\} \cap \mathbf{S}^{n-1}$ for ε arbitrarily small. Let us consider a unit vector $\theta \in \{p^\perp\} \cap \mathbf{S}^{n-1}$ that we write

$$\theta = (\cos\alpha)e' + (\sin\alpha)e_k$$

with $\sin\alpha = \theta\cdot e_k$ and $e' \perp p,\ e' \perp e_k$. At the first order, this vector becomes (by infinitesimal rotation)

$$(\cos\alpha)e' + (\sin\alpha)(e_k + \varepsilon e_k^\perp) \in \{(p+\varepsilon e_k)^\perp\} \cap \mathbf{S}^{n-1}.$$

Then the following holds

$$\partial_k A_{ij}(p) = \frac{1}{|p|^2}\int_{\mathbf{S}^{n-1}\cap\{p^\perp\}} d\theta\ \frac{1}{2} e_k^\perp \cdot \nabla\bar{g}(\theta)(\theta\cdot e_k)(e_i,e_j),$$

where

$$e_k^\perp = \frac{-p}{|p|},\quad \bar{g}(\theta) = g(\theta)\theta\otimes\theta.$$

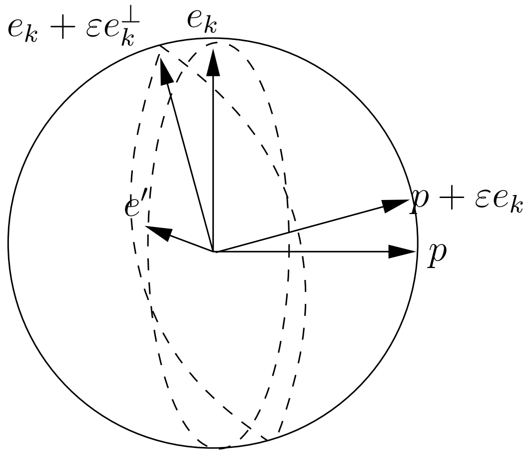

Figure 1: Computation at the first order of $\partial_k A_{ij}(p)$, case $|p| = 1$.

Moreover,

$$\begin{aligned}
&e_k^\perp \cdot \nabla\bar{g}(\theta)(\theta\cdot e_k)(e_i,e_j))\\
=&(e_k^\perp\cdot\nabla g(\theta))(\theta\cdot e_k)(\theta\cdot e_i)(\theta\cdot e_j) + g(\theta)(\theta\cdot e_k)\left(e_k^\perp\cdot e_i(\theta\cdot e_j) + (\theta\cdot e_i)e_k^\perp\cdot e_j\right)\\
=&e_k^\perp\cdot\nabla g(\theta)(\theta\cdot e_k)(\theta\cdot e_i)(\theta\cdot e_j) + (e_k^\perp\cdot e_i)\bar{g}(\theta)(e_k,e_j) + (e_k^\perp\cdot e_j)\bar{g}(\theta)(e_k,e_i).
\end{aligned}$$

We are now able to compute the Curl of A. To do this we separate in several cases:

1. $e_k,\ e_i,\ e_j \parallel p$. Then, $A_{ij}(p) = A_{kj}(p) = 0$ and so $\partial_k A_{ij} - \partial_i A_{kj} = 0$.
2. $e_k,\ e_i \parallel p,\ e_j \perp p$. In the same way, $\partial_k A_{ij} - \partial_i A_{kj} = 0$
3. $e_k,\ e_j \parallel p,\ e_i \perp p$. Then $\partial_k A_{ij} = 0$ and $\partial_i A_{kj} = 0$ (since $\theta\cdot e_j = \theta\cdot e_k = 0$).
4. $e_k \parallel p,\ e_i,\ e_j \perp p$. Then $\partial_k A_{ij} = -\frac{1}{|p|}A_{ij}$ (if $e_k = \frac{p}{|p|}$)and

$$\begin{aligned}
\partial_i A_{kj} =&\frac{1}{|p|^2}\int_{\mathbf{S}^{n-1}\cap\{p^\perp\}} d\theta\ \frac{1}{2}e_i^\perp\cdot e_k\bar{g}(\theta)(e_i,e_j)\\
=&-\frac{1}{|p|^2}\int_{\mathbf{S}^{n-1}\cap\{p^\perp\}} d\theta\ \frac{1}{2}\bar{g}(\theta)(e_i,e_j)\\
=&-\frac{1}{|p|}A_{ij}(p).
\end{aligned}$$

We have used the fact that $e_i^\perp \cdot e_k = \frac{-p}{|p|} \cdot \frac{p}{|p|} = -1$. So $\partial_k A_{ij} - \partial_i A_{kj} = 0$.

5. $e_k,\ e_i,\ e_j \perp p$. Then

$$\partial_k A_{ij}(p) = \frac{1}{|p|^2} \int_{S^{n-1}\cap\{p^\perp\}} d\theta\ \frac{1}{2}(e_k^\perp \cdot \nabla g(\theta))(\theta \cdot e_k)(\theta \cdot e_i)(\theta \cdot e_j) = \partial_i A_{kj}(p).$$

We have used the fact that $e_k^\perp = e_i^\perp$ and $e_k^\perp \cdot e_i = e_k^\perp \cdot e_j = e_i^\perp \cdot e_k = e_i^\perp \cdot e_j = 0$.

6. $e_k,\ e_i \perp p,\ e_j \parallel p$. Then

$$\partial_k A_{ij}(p) = \frac{1}{|p|^2} \int_{S^{n-1}\cap\{p^\perp\}} d\theta\ (e_k^\perp \cdot e_j)\bar{g}(\theta)(e_k, e_i) = \partial_i A_{kj}(p).$$

7. $e_k,\ e_j \perp p,\ e_i \parallel p$. It is the same case as 4.

8. $e_k \perp p,\ e_i,\ e_j \parallel p$. It is the same case as 3.

We then deduce that $\mathrm{Curl}(A) = 0$ on $\mathbb{R}^n \backslash \{0\}$. We now remark that

$$\begin{aligned}
\langle -(\mathrm{CurlA})_{\mathrm{i,j,k}}, \varphi \rangle &= \int_{\mathbb{R}^n} A_{ij}\partial_k \varphi - A_{kj}\partial_i \varphi \\
&= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n\backslash B_\varepsilon} A_{ij}\partial_k \varphi - A_{kj}\partial_i \varphi \\
&= \lim_{\varepsilon\to 0} \left(\int_{\mathbb{R}^n\backslash B_\varepsilon} -(\partial_k A_{ij} - \partial_i A_{kj})\varphi + \int_{\partial B_\varepsilon} (A_{ij} n_k - A_{kj} n_i)\varphi \right) \\
&= \lim_{\varepsilon\to 0} \varepsilon^{n-2} \int_{\partial B_1} (A_{ij}(\theta)\theta_k - A_{kj}(\theta)\theta_i)\varphi(\varepsilon\theta) d\theta \\
&= \begin{cases} \varphi(0)\int_{\mathbf{S}^1} (A_{ij}(\theta)\theta_k - A_{kj}(\theta)\theta_i) d\theta & \text{if} \quad n = 2 \\ 0 & \text{if} \quad n \neq 2 \end{cases}
\end{aligned}$$

In particular, we have used the fact that for $n = 1$, $A \equiv 0$. Now, using the symmetry of g, we deduce that $A(-\theta) = A(\theta)$ and then by antisymmetry the last integral on $\mathbf{S}^1$ vanishes. Therefore

$$\mathrm{Curl}(A) = 0 \quad \text{on } \mathbb{R}^n.$$

By a passage to the limit, this is still true if $g \in C^0$ (and not only $g \in C^1$).

To deduce that there exists Φ such that $A = D^2\Phi$, we use the following *Lemma*:

***Lemma* 7.3 (Vectors fields with zero Curl are gradients)**
Let $f = (f_1, ..., f_n)(x) \in \mathcal{D}'(\mathbb{R}^n)$ be such that $Curl(f) = (\partial_k f_i - \partial_i f_k)_{i,k} = 0$, then there is $h \in \mathcal{D}'(\mathbb{R}^n)$ such that $f_i = \partial_i h$.

For the proof of this *Lemma*, we refer to Schwartz [38] Chapter II, Paragraph 6, Theorem VI p59.

We denote by $f_j = (f_{j1}, ..., f_{jn}) = (A_{j1}, ..., A_{jn})$. Using the fact that $Curl(A) = 0$, we deduce that for all $j \in \{1, .., n\}$, $Curl(f_j) = 0$. Then, by *Lemma* 7.3 there are h_j such that $f_j = \nabla h_j$. Using the fact that A is symmetric, we deduce that $\partial_j h_i - \partial_i h_j = 0$. Applying again *Lemma* 7.3, we deduce that there is Φ such that $h = \nabla\Phi$ and so $A = D^2\Phi$. Let us remark that Φ is unique up to a polynomial of degree 1. Let $\Phi^s(p) = \frac{1}{2}(\Phi(p) + \Phi(-p))$. Then $A = D^2\Phi^s$ and then Φ^s is unique up to a constant. Moreover, $D^2\Phi(p)$ behaves like $\frac{1}{|p|}$ for small p and then $D^2\Phi \in L^{n-\varepsilon}$ for every $\varepsilon > 0$. Therefore $\Phi \in W^{2,n-\varepsilon}_{Loc}$ and by Sobolev injections $\Phi \in C^0(\mathbb{R}^n)$. We deduce that there is a unique Φ such that

$$\Phi(-p) = \Phi(p) \quad \text{and} \quad \Phi(0) = 0. \tag{7.41}$$

Finally, we remark that

$$D^2\left(\frac{\Phi(\lambda p)}{|\lambda|}\right) = |\lambda|(D^2\Phi)(\lambda p) = |\lambda| A(\lambda p) = A(p) = D^2\Phi(p)$$

Therefore $\Phi(\lambda p) = |\lambda|\Phi(p)$ if Φ satisfies (7.41).

Proof of Proposition 7.1

We show that $\Phi = -\dfrac{1}{2\pi}\mathcal{F}(L_g)$ (where Φ is defined in *Lemma* 7.2). Let $\varphi \in \mathcal{S}$. The following holds:

$$\begin{aligned}
\langle -D^2_{\xi\xi}\mathcal{F}(L_g)(\xi), \varphi\rangle(\zeta,\zeta) =& \langle \mathcal{F}(-ix \otimes ixL_g(x)), \varphi\rangle(\zeta,\zeta) \\
=& \langle L_g, (x \otimes x)\mathcal{F}(\varphi)\rangle(\zeta,\zeta) \\
=& \langle L_g, (x\cdot\zeta)^2\mathcal{F}(\varphi)(x)\rangle \\
=& \int_{\mathbb{R}^n} dx\, \frac{g\left(\frac{x}{|x|}\right)}{|x|^{n+1}}(x\cdot\zeta)^2\mathcal{F}(\varphi)(x) \\
=& \langle \mathcal{F}\left(\frac{g\left(\frac{x}{|x|}\right)}{|x|^{n+1}}(x\cdot\zeta)^2\right), \varphi\rangle.
\end{aligned}$$

We then have the following *lemma*

***Lemma* 7.4** *Let $n \geq 2$. Let $g \in C^0(\mathbb{R}^n\backslash\{0\})$ such that $g(\lambda p) = \dfrac{g(p)}{|\lambda|^{n+1}}$. Then, the following holds*

$$\mathcal{F}\left(\frac{g\left(\frac{x}{|x|}\right)}{|x|^{n+1}}(x\cdot\zeta)^2\right)(\xi) = 2\pi A(\xi).$$

We just give here a formal proof. The complete proof is given in Appendix.

By definition of Fourier transform, we have formally for $\xi \neq 0$, with $\theta = \dfrac{x}{|x|}$, $r = |x|$

$$\begin{aligned}
\mathcal{F}\left(\frac{g\left(\frac{x}{|x|}\right)}{|x|^{n+1}}(x\cdot\zeta)^2\right) &= \int_{\mathbb{R}^n} \frac{g\left(\frac{x}{|x|}\right)(x\cdot\zeta)^2 e^{-i\xi\cdot x}}{|x|^{n+1}}dx \\
&= \int_{\mathbb{R}^n} \frac{g(\theta)(\theta\cdot\zeta)^2 e^{-i\xi\cdot x}}{|x|^{n-1}}dx \\
&= \int_{\mathbf{S}^{n-1}\times(0,\infty)} g(\theta)(\theta\cdot\zeta)^2 e^{-i\xi\cdot\theta r} d\theta dr \\
&= \int_{\mathbf{S}^{n-1}} d\theta\; g(\theta)(\theta\cdot\zeta)^2 \int_0^\infty dr \; \left(\frac{e^{i\xi\cdot\theta r}+e^{-i\xi\cdot\theta r}}{2}\right) \\
&= \int_{\mathbf{S}^{n-1}} d\theta g(\theta)(\theta\cdot\zeta)^2 \int_{-\infty}^{\infty} dr\frac{e^{i\xi\cdot\theta r}}{2} \\
&= \frac{2\pi}{|\xi|}\int_{\mathbf{S}^{n-1}\cap\{\xi^\perp\}} d\theta\; \frac{1}{2}g(\theta)(\theta\cdot\zeta)^2 \\
&= 2\pi A(\xi)(\zeta,\zeta),
\end{aligned}$$

where we have used the fact that $\mathcal{F}(1) = 2\pi\delta_0$ in 1D, that formally gives

$$\int_{-\infty}^{+\infty} dr\; e^{i\xi\cdot\theta r} = 2\pi\delta_0(\xi\cdot\theta) = \frac{2\pi}{|\xi|}\delta_0\left(\frac{\xi}{|\xi|}\cdot\theta\right).$$

This achieves the formal proof of Lemma 7.4.

We then get

$$-D^2\mathcal{F}(L_g)(\xi) = 2\pi A(\xi) = 2\pi D^2\Phi.$$

Moreover $\mathcal{F}(L_g)(-\xi) = \mathcal{F}(L_g)(\xi)$ and $\mathcal{F}(L_g)(0) = 0$. Therefore, by *Lemma* 7.2 we deduce that

$$\Phi = -\frac{1}{2\pi}\mathcal{F}(L_g)$$

and $\Phi(\lambda p) = |\lambda|\Phi(p)$. This achieves the proof of the proposition.

We now prove Theorem 1.7:

Proof of Theorem 1.7

Let us first compute $\mathrm{div}\nabla G\left(\frac{Du}{|Du|}\right)$. We set $p = Du$. The following holds:

$$\begin{aligned}
\mathrm{div}\nabla G\left(\frac{Du}{|Du|}\right) &= \sum_i \frac{\partial}{\partial x_i}\left(\frac{\partial G}{\partial x_i}\left(\frac{p}{|p|}\right)\right) \\
&= \sum_{i,j} \frac{\partial^2 G}{\partial x_i \partial x_j}\left(\frac{p}{|p|}\right)\frac{\partial}{\partial x_i}\left(\frac{D_j u}{|Du|}\right) \\
&= \frac{1}{|p|}\sum_{i,j}\frac{\partial^2 G}{\partial x_i\partial x_j}\left(\frac{p}{|p|}\right)\left(D^2_{ij}u - \frac{D^2_{\cdot i}u\cdot p\otimes p_j}{|p|^2}\right) \\
&= \frac{1}{|p|}\mathrm{trace}\left(D^2 G\left(\frac{p}{|p|}\right)\left(I - \frac{p\otimes p}{|p|^2}\right)D^2 u\right).
\end{aligned}$$

Moreover, for $\lambda > 0$, we have $G(\lambda p) = \lambda G(p)$. Then by derivation we get

$$p \cdot \nabla G(\lambda p) = G(p).$$

Taking the gradient, we get

$$\nabla G(p) = \nabla G(\lambda p) + p \cdot D^2 G(\lambda p)\lambda$$

which implies for $\lambda = 1$

$$p \cdot D^2 G(P) = 0.$$

This implies that $D^2 G\left(\frac{p}{|p|}\right)\left(I - \frac{p \otimes p}{p^2}\right) = D^2 G\left(\frac{p}{|p|}\right)$. We then deduce:

$$\mathrm{div}\nabla G\left(\frac{Du}{|Du|}\right) = \frac{1}{|p|}\mathrm{trace}\left(A\left(\frac{p}{|p|}\right) \cdot D^2 u\right).$$

This show the first part of the Theorem.

In the two dimensional case, we simply remark that we have

$$g(\theta)\theta \otimes \theta = D^2 G(\theta^{\perp})$$

which implies the result. This ends the proof of Theorem 1.7

Proof of Theorem 1.11

We can then rewrite $A(p)$ as

$$\begin{aligned} A(p) &= \int_{x \in \{p^{\perp}\}} \frac{1}{2} K_0(x)\; x \otimes x \; dx \\ &= \int_{\theta \in \{p^{\perp}\} \cap \mathbf{S}^{n-1}} d\theta \; \frac{1}{2}\left(\int_{(0,+\infty)} dr \; r^n K_0(r\theta)\right) \theta \otimes \theta \\ &= \int_{\theta \in \{p^{\perp}\} \cap \mathbf{S}^{n-1}} d\theta \; \frac{1}{2} g(\theta)\; \theta \otimes \theta \end{aligned}$$

with $g(\theta) = \int_{(0,+\infty)} dr \; r^n K_0(r\theta)$. So, by applying Theorem 1.7, we see that the mean curvature motion defined by (1.5)-(1.6) using the matrix $A(p)$, is of variational type.

Proof of Proposition 1.8

The idea to build a function g which changes its sign, such that

$$\int_{\mathbf{S}^{n-1} \cap \{p^{\perp}\}} d\theta \; \frac{1}{2} g(\theta)\; \theta \otimes \theta \geq 0$$

for all $p \in \mathbb{R}^n \backslash \{0\}$ is simple. First, we consider the set

$$\mathbf{S} = \cup_{i=1}^{n} (\mathbf{S}^{n-1} \cap \{x_i = 0\})$$

and we remark that any hyperplane Π which contains the origine intersect $\mathbf{S}$ with an angle $\alpha \geq \alpha_0$ with $\alpha_0 > 0$ independant of Π. We then define g on $\mathbf{S}^{n-1}$ as a mollification of $\delta_{\mathbf{S}} - \eta$ for η small enough where $\delta_{\mathbf{S}}$ is a Dirac mass on $\mathbf{S}^{n-1}$ with support the set $\mathbf{S}$.

We now make the rigorous construction. We denote by $(e_i)_{i=1,\dots,n}$ a orthonormal basis of $\mathbb{R}^n$. We use the following *Lemma*

***Lemma* 7.5** *For $\varepsilon \in (0,1]$, there exist $g_i^\varepsilon \in C^\infty(\mathbf{S}^{n-1})$, for $i = 1, ..., n$, such that for all $\Psi^\varepsilon \in C^\infty(\mathbf{S}^{n-1})$, $\forall p_\varepsilon \in \mathbf{S}^{n-1}$, if $p_\varepsilon \longrightarrow p_0$, $\|\Psi^\varepsilon - \Psi^0\|_{L^\infty(\mathbf{S}^{n-1})} \to 0$*

$$\int_{\mathbf{S}^{n-1}\cap\{p_\varepsilon^\perp\}\simeq\mathbf{S}^{n-2}} d\theta\ g_i^\varepsilon(\theta)\Psi^\varepsilon(\theta) \longrightarrow \frac{1}{\sin\widehat{(p_0,e_i)}} \int_{\mathbf{S}^{n-1}\cap\{p_0^\perp\}\cap\{e_i^\perp\}\simeq\mathbf{S}^{n-3}} d\theta\ \Psi^0(\theta) \quad \text{as } \varepsilon \to 0$$

provided p_0 is not parallel to e_i and where $\widehat{(p_0,e_i)} \in [0, \frac{\pi}{2}]$ denotes the angle between p_0 and e_i. Moreover,

$$g_i^\varepsilon(\theta) = 0 \quad \text{if } |\langle\theta, e_i\rangle| \geq \varepsilon. \tag{7.42}$$

The proof is postponed.

We set

$$g^\varepsilon = \sum_{i=1}^n g_i^\varepsilon - \eta$$

with η a small parameter to be precised. We remark that by (7.42) for ε small enough, g^ε is not nonnegative. We want to show that there exists ε_0 such that for all $0 < \varepsilon < \varepsilon_0$, for all $p,\ \xi \in \mathbf{S}^{n-1}$

$$\int_{\mathbf{S}^{n-1}\cap\{p^\perp\}\simeq\mathbf{S}^{n-2}} d\theta\ g^\varepsilon(\theta)\ \langle\theta,\xi\rangle^2 \geq 0. \tag{7.43}$$

We will prove (7.43) by contradiction, using the following *Lemma*:

***Lemma* 7.6** *There exists $C_0 > 0$ such that $\forall p \in \mathbf{S}^{n-1}$, $\forall \xi \in \mathbf{S}^{n-1} \cap \{p^\perp\}$, $\exists\ i_0 \in \{1, ..., n\}$ such that*

$$\int_{\mathbf{S}^{n-1}\cap\{p^\perp\}\cap\{e_{i_0}^\perp\}\simeq\mathbf{S}^{n-3}} d\theta\ \langle\theta,\xi\rangle^2 \geq C_0$$

and

$$\widehat{(p, e_{i_0})} \geq C_0$$

where $\widehat{(p, e_{i_0})} \in [0, \frac{\pi}{2}]$ denotes the angle between p and e_{i_0}.

The proof is postponed.

We now prove (7.43) by contradiction assuming that there exists a subsequence $\varepsilon_k \to 0$ such that there exists $p_k \in \mathbf{S}^{n-1}$, $\xi_k \in \mathbf{S}^{n-1} \cap \{p_k^\perp\}$ such that

$$\int_{\mathbf{S}^{n-1}\cap\{p_k^\perp\}\simeq\mathbf{S}^{n-2}} d\theta\ g^{\varepsilon_k}(\theta)\ \langle\theta,\xi_k\rangle^2 \leq 0.$$

Up to extract a subsequence, we can assume that $p_k \to p_\infty$ and $\xi_k \to \xi_\infty$ with $p_\infty,\ \xi_\infty \in \mathbf{S}^{n-1}$. We then have with the index i_0 given by *Lemma* 7.6 for $p = p_\infty,\ \xi = \xi_\infty$

$$\begin{aligned} 0 \geq \int_{\mathbf{S}^{n-1}\cap\{p_k^\perp\}} d\theta\ g^{\varepsilon_k}(\theta)\ \langle\theta,\xi_k\rangle^2 &\geq \int_{\mathbf{S}^{n-1}\cap\{p_k^\perp\}} d\theta\ g_{i_0}^{\varepsilon_k}(\theta)\ \langle\theta,\xi_k\rangle^2 - \eta \int_{\mathbf{S}^{n-1}\cap\{p_k^\perp\}} d\theta \\ &\geq \int_{\mathbf{S}^{n-1}\cap\{p_k^\perp\}} d\theta\ g_{i_0}^{\varepsilon_k}(\theta)\ \langle\theta,\xi_k\rangle^2 - \eta|\mathbf{S}^{n-2}|. \end{aligned}$$

By passing to the limit, using *Lemma* 7.5, we then obtain

$$0 \geq \frac{1}{\sin(\widehat{p_\infty, e_{i_0}})} \int_{\mathbf{S}^{n-1}\cap\{p_\infty^\perp\}\cap\{e_{i_0}^\perp\}} d\theta \, \langle \theta, \xi_\infty \rangle^2 - \eta|\mathbf{S}^{n-2}| \geq \frac{C_0}{\sin C_0} - \eta|\mathbf{S}^{n-2}|$$

where C_0 is given in *Lemma* 7.6. This is a contradiction for η small enough.

We now prove *Lemma* 7.6

Proof of *Lemma* 7.6
We perform the proof by contradiction. If the result is false, then $\exists\ C_k \to 0$, $\exists\ p_k \in \mathbf{S}^{n-1}$, $\exists\ \xi_k \in \mathbf{S}^{n-1} \cap \{p_k^\perp\}$ such that for all $i \in \{1, ..., n\}$

$$0 \leq (\widehat{p_k, e_i}) \leq C_k \tag{7.44}$$

or

$$\int_{\mathbf{S}^{n-1}\cap\{p_k^\perp\}\cap\{e_i^\perp\}} d\theta \, (\xi_k \cdot \theta)^2 \leq C_k \quad \text{if } (\widehat{p_k, e_i}) \neq 0. \tag{7.45}$$

We distinguish two cases:

Case 1. There exist two indices i such that (7.44) holds. Up to reorganise the indices, we can assume that (7.44) holds for $i = 1, 2$. We deduce by extracting a subsequence and passing to the limit that there exists $p_\infty = \lim p_k$ such that $(\widehat{p_\infty, e_i}) = 0$ for $i = 1, 2$, which is a contradiction.

Case 2. There exists two indices i such that (7.45) holds. Up to reorganise the indices, we can assume that (7.45) holds for $i = 1, 2$. In this case, by passing to the limit, up to extract a subsequence, we obtain

$$\int_{\mathbf{S}^{n-1}\cap\{p_\infty^\perp\}\cap\{e_i^\perp\}} d\theta \, (\xi_\infty \cdot \theta)^2 = 0 \quad \forall i = 1, 2.$$

We then deduce that $\xi_\infty \in \mathbf{S}^{n-1} \cap \{p_\infty^\perp\}$ is parallel to e_i, for $i = 1, 2$ which is a contradiction.

Finally, in dimension $n \geq 3$, we are either in case 1 or case 2, so we obtained a contradiction.
Proof of *Lemma* 7.5
We set $\tilde{g}_i^\varepsilon(x) = \frac{1}{\varepsilon}\rho\left(\frac{x \cdot e_i}{\varepsilon}\right)$ where $\rho \in C_c^\infty(\mathbb{R}, \mathbb{R})$ and satisfies:

$$\rho \geq 0, \quad \mathrm{supp}(\rho) \subset [-1, 1], \quad \int_{\mathbb{R}} \rho(x) dx = 1.$$

We then set

$$g_i^\varepsilon(\theta) = \int_0^\infty r^{n-1} \tilde{g}_i^\varepsilon(r\theta) f(r) \, dr$$

with $f \in C_c^\infty((0,\infty), \mathbb{R})$ satisfying $\int_0^\infty f(r) r^{n-2} \, dr = 1$. For all $\Psi^\varepsilon \in C^\infty(\mathbf{S}^{n-1})$ and $p_\varepsilon \in \mathbf{S}^{n-1}$, let us define

$$\mathcal{I}^\varepsilon = \int_{\mathbf{S}^{n-1}\cap\{p_\varepsilon^\perp\}\simeq\mathbf{S}^{n-2}} d\theta \, g_i^\varepsilon(\theta) \Psi^\varepsilon(\theta).$$

To simplify the notations, let us set $\Psi = \Psi^\varepsilon$ and $p = p_\varepsilon$. We then have, if p is not parallel to e_i

$$\begin{aligned}\mathcal{I}^\varepsilon &= \int_{\mathbf{S}^{n-1}\cap\{p^\perp\}\simeq\mathbf{S}^{n-2}} d\theta\ \Psi(\theta) \int_0^\infty dr\ r^{n-1}\tilde{g}_i^\varepsilon(r\theta)f(r) \\ &= \int_{\mathbb{R}^n\cap\{p^\perp\}} dx\ \tilde{g}_i^\varepsilon(x)\tilde{\Psi}(x)\end{aligned}$$

where

$$\tilde{\Psi}(x) = f(|x|)\Psi\left(\frac{x}{|x|}\right)|x|.$$

Using the definition of $\tilde{g}_i^\varepsilon$, we then have, by denoting $\alpha_i = \widehat{(p,e_i)}$ the angle between p and e_i and using the change of coordinates $x = (y', y_n)$ with $y' \in \{p^\perp\}$ and $y_n \in \mathbb{R}$, that

$$\begin{aligned}\mathcal{I}^\varepsilon &= \int_{\mathbb{R}^n\cap\{p^\perp\}} dx\ \frac{1}{\varepsilon}\rho\left(\frac{x\cdot e_i}{\varepsilon}\right)\tilde{\Psi}(x) \\ &= \int_{\mathbb{R}^n\cap\{p^\perp\}} dy'\ \frac{1}{\sin\alpha_i}\left(\frac{\sin\alpha_i}{\varepsilon}\right)\rho\left(\frac{y'.e_i'}{\left(\frac{\varepsilon}{\sin\alpha_i}\right)\sin\alpha_i}\right)\tilde{\Psi}(y',0) \\ &= \frac{1}{\sin\alpha_i}\int_{\mathbb{R}^n\cap\{p^\perp\}} dy'\ \frac{1}{\varepsilon'}\rho\left(\frac{y'\cdot\frac{e_i'}{|e_i'|}}{\varepsilon'}\right)\tilde{\Psi}(y',0)\end{aligned}$$

where $\varepsilon' = \dfrac{\varepsilon}{\sin\hat{\theta}_i}$ and e_i' is the orthogonal projection of e_i onto the hyperplane $\{p^\perp\}$. In particular, e_i' satisfies $|e_i'| = \sin\alpha_i$. Passing to the limit in ε, with $p_\varepsilon \to p_0$, $\Psi^\varepsilon \to \Psi^0$, $\alpha_i = \alpha_i^\varepsilon = \widehat{(p_\varepsilon, e_i)} \to \alpha_i^0 = \widehat{(p_0, e_i)}$ and $\tilde{\Psi}^\varepsilon = f(|x|)\Psi^\varepsilon\left(\frac{x}{|x|}\right)|x| \to \tilde{\Psi}^0 = f(|x|)\Psi^0\left(\frac{x}{|x|}\right)|x|$, yields

$$\begin{aligned}\mathcal{I}^\varepsilon &\longrightarrow \frac{1}{\sin\alpha_i^0}\int_{\mathbb{R}^n\cap\{p^\perp\}\cap\{e_i'^\perp\}} dy'\ \tilde{\Psi}^0(y',0) \\ &= \frac{1}{\sin\alpha_i^0}\int_{\mathbb{R}^n\cap\{p^\perp\}\cap\{e_i^\perp\}} dy'\ \tilde{\Psi}^0(y',0) \\ &= \frac{1}{\sin\alpha_i^0}\int_{\mathbf{S}^{n-3}\simeq\mathbf{S}^{n-1}\cap\{p^\perp\}\cap\{e_i^\perp\}} d\theta\ \left(\int_0^\infty dr\ r^{n-3}f(r)r\right)\Psi^0(\theta) \\ &= \frac{1}{\sin\alpha_i^0}\int_{\mathbf{S}^{n-3}\simeq\mathbf{S}^{n-1}\cap\{p^\perp\}\cap\{e_i^\perp\}} d\theta\ \Psi^0(\theta).\end{aligned}$$

This ends the proof of the *Lemma*.

8 Heuristical convergence and properties of the energies

8.1 Monotonicity of the energy

We begin this section by showing that the energy associated to (1.4) is nonincreasing in time. We recall that (1.4) is formally associated to the following energy:

$$\mathcal{E}^\varepsilon(u^\varepsilon) = \int_\lambda \overline{\mathcal{E}^\varepsilon}(\lambda) d\lambda \tag{8.46}$$

where

$$\overline{\mathcal{E}^\varepsilon}(\lambda) = \int_{\mathbb{R}^n} -\frac{1}{2} \left(\overline{c_0}^\varepsilon \star \rho_\lambda^\varepsilon\right) \rho_\lambda^\varepsilon$$

with

$$\rho_\lambda^\varepsilon = 1_{\{u^\varepsilon > \lambda\}}, \quad \overline{c_0}^\varepsilon = c_0^\varepsilon - \left(\int_{\mathbb{R}^n} c_0^\varepsilon\right) \delta_0.$$

Formally, we have:

$$\frac{d\overline{\mathcal{E}^\varepsilon}(\lambda)}{dt} = \int_{\mathbb{R}^n} - \left(\overline{c_0}^\varepsilon \star \rho_\lambda^\varepsilon\right) (\rho_\lambda^\varepsilon)_t$$

which is defined only on the support of $|D\rho_\lambda^\varepsilon|$ (since (1.4) formally implies $(\rho_\lambda^\varepsilon)_t = (\overline{c_0}^\varepsilon \star \rho_\lambda^\varepsilon)|D\rho_\lambda^\varepsilon|$). Moreover, $\overline{c_0}^\varepsilon \star \rho_\lambda^\varepsilon = c_0^\varepsilon \star \rho_\lambda^\varepsilon - \left(\int c_0^\varepsilon\right) \delta_0 \star \rho_\lambda^\varepsilon$. If we set η_n a regularisation of the Dirac mass, we then have $\eta_n \star \rho_\lambda^\varepsilon = \frac{1}{2}$ on the support of $|D\rho_\lambda^\varepsilon|$ (see Figure 2).

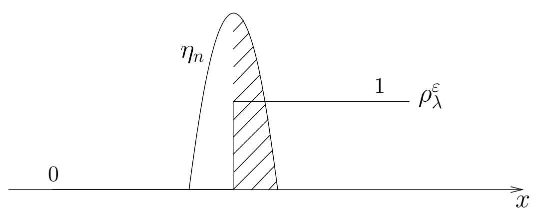

Figure 2: The convolution of ρ_λ^ε with the Dirac.

So, we can assume that $\overline{c_0}^\varepsilon \star \rho_\lambda^\varepsilon = c_0^\varepsilon \star \rho_\lambda^\varepsilon - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon$ on the support of $|D\rho_\lambda^\varepsilon|$. We then deduce that

$$\frac{d\overline{\mathcal{E}^\varepsilon}(\lambda)}{dt} = \int_{\mathbb{R}^n} - \left(c_0^\varepsilon \star \rho_\lambda^\varepsilon - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon\right)^2 |D\rho_\lambda^\varepsilon|.$$

This implies:

$$\frac{d\mathcal{E}^\varepsilon(u^\varepsilon)}{dt} = \int d\lambda \int_{\mathbb{R}^n} - \left(c_0^\varepsilon \star \rho_\lambda^\varepsilon - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon\right)^2 |D\rho_\lambda^\varepsilon| \le 0.$$

So the energy is nonincreasing in time.

8.2 Formal convergence of the energy

We set $\mathcal{E}(u^0) = \int G(Du^0)$, the energy associated to the mean curvature motion. We have formally

$$\frac{d}{dt}\mathcal{E}(u^0) = \int \nabla G\left(\frac{Du^0}{|Du^0|}\right)\cdot Du^0_t = \int -\left(\mathrm{div}\nabla G\left(\frac{Du^0}{|Du^0|}\right)\right)^2 |Du^0|.$$

Moreover, still formally we have

$$\begin{array}{rcl}
\frac{d}{dt}\mathcal{E}^\varepsilon(u^\varepsilon) & = & \int d\lambda \int -\left(c_0^\varepsilon \star \rho_\lambda^\varepsilon - \frac{1}{2}\int c_0^\varepsilon\right)^2 |D\rho_\lambda^\varepsilon| \\
& \to & \int d\lambda \int -\left(\mathrm{trace}\left(A\left(\frac{Du^0}{|Du^0|}\right)D^2u^0\right)\right)^2 |D\rho_\lambda^0| \\
& = & \int d\lambda \int -\left(\mathrm{div}\left(\nabla G\left(\frac{Du^0}{|Du^0|}\right)\right)\right)^2 |D\rho_\lambda^0| \\
& = & \int -\left(\mathrm{div}\left(\nabla G\left(\frac{Du^0}{|Du^0|}\right)\right)\right)^2 |Du^0|.
\end{array}$$

So, formally,

$$\frac{d}{dt}\mathcal{E}^\varepsilon(u^\varepsilon) \to \frac{d}{dt}\mathcal{E}(u^0).$$

The work of Garroni, Müller [28], suggests that we should have $\int_{x,\lambda}\frac{1}{2}(c_0^\varepsilon \star \rho_\lambda^\varepsilon)\rho_\lambda^\varepsilon \to \int d\lambda \int_{\Gamma_\lambda} G\left(\frac{Du^0}{|Du^0|}\right)$, where Γ_λ is the λ level set of u^0. We deduce that (using formally the coarea formula for BV functions)

$$\begin{array}{rcl}
\int_{x,\lambda}\frac{1}{2}(c_0^\varepsilon \star \rho_\lambda^\varepsilon)\rho_\lambda^\varepsilon & \to & \int d\lambda \int_x G\left(\frac{Du^0}{|Du^0|}\right)|D\rho_\lambda^0| \\
& = & \int_x G\left(\frac{Du^0}{|Du^0|}\right)|Du^0| \\
& = & \int G(Du^0)
\end{array}$$

and so, formally

$$\mathcal{E}^\varepsilon(u^\varepsilon) \to \mathcal{E}(u^0).$$

9 Appendix: some *lemmata* on Fourier transform

***Lemma* 9.1** *The distribution L_g associated to g (see definition 1.6) satisfies the following properties:*

$$L_g(\lambda\cdot) = \frac{1}{\lambda^{n+1}}L_g \quad \forall \lambda > 0, \tag{9.47}$$

$$\mathcal{F}(L_g)(\lambda\cdot) = \lambda\mathcal{F}(L_g) \quad \forall\lambda > 0, \tag{9.48}$$

where $\mathcal{F}(L_g)$ is the Fourier transform of L_g defined by

$$\forall\varphi \in \mathcal{S},\ \langle\mathcal{F}(L_g),\varphi\rangle = \langle L_g, \mathcal{F}(\varphi)\rangle.$$

Proof of *Lemma* 9.1
Equation (9.47) results from the definition of L_g which by construction is of homogeneity of degree $-(n+1)$. This can be rigorously shown using the general definition for a distribution $u \in \mathcal{D}'(\mathbb{R}^n)$

$$\forall \varphi \in C_c^\infty(\mathbb{R}^n), \quad \langle u(\lambda \cdot, \varphi\rangle := \frac{1}{\lambda^n}\langle u, \varphi\left(\frac{\cdot}{\lambda}\right)\rangle. \tag{9.49}$$

We now prove (9.48). A straightforward computation for $\varphi \in \mathcal{S}$ gives

$$\mathcal{F}(\varphi)(\lambda \cdot) = \mathcal{F}\left(\frac{1}{\lambda^n}\varphi\left(\frac{\cdot}{\lambda}\right)\right)(\cdot). \tag{9.50}$$

Using the definition (9.49), one can show that (9.50) is still true for element of $\mathcal{S}'$. Hence, we have

$$\begin{aligned}
\mathcal{F}(L_g)(\lambda \cdot) =& \mathcal{F}\left(\frac{1}{\lambda^n} L_g\left(\frac{\cdot}{\lambda}\right)\right)(\cdot) \\
=& \mathcal{F}\left(\frac{\lambda^{n+1}}{\lambda^n} L_g(\cdot)\right)(\cdot) \\
=& \lambda \mathcal{F}(L_g(\cdot))(\cdot)
\end{aligned}$$

where we have use (9.47). This ends the proof of the *Lemma*.

Proof of *Lemma* 7.4
Let $R_0 > r_0 > 0$ and $\varphi \in C^\infty(\mathbb{R}^n)$ with Supp $\varphi \subset B_{R_0}(0)\backslash B_{r_0}(0)$. Let $\Psi_\lambda(y) = \Psi(\lambda y)$ for $y \in \mathbb{R}$ with $\Psi \in C_c^\infty(\mathbb{R})$ such that

$$\text{Supp } \Psi \subset [-1,1], \quad \Psi \equiv 1 \text{ on } [-\frac{1}{2}, \frac{1}{2}], \quad 0 \leq \Psi \leq 1, \quad \Psi(-y) = \Psi(y).$$

Let us consider $f \in C_c^\infty([0,+\infty))$ with Supp $f \subset [r_0, R_0]$ and such that

$$\int_0^\infty f(\bar{r})\bar{r}^n d\bar{r} = 1.$$

Let us assume first that $g \in C^\infty(\mathbf{S}^{n-1})$. Let us compute for $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$\begin{aligned}
\mathcal{I} &= \left\langle \mathcal{F}\left(\frac{g\left(\frac{x}{|x|}\right)}{|x|^{n+1}}(x\cdot\zeta)^2\right), \varphi \right\rangle \\
&= \left\langle \frac{g\left(\frac{x}{|x|}\right)}{|x|^{n+1}}(x\cdot\zeta)^2, \mathcal{F}(\varphi) \right\rangle \\
&= \int_{\mathbb{R}^n} dx\, \frac{g\left(\frac{x}{|x|}\right)}{|x|^{n+1}}(x\cdot\zeta)^2 \left(\int_{\mathbb{R}^n} d\xi\, e^{-i\xi\cdot x}\varphi(\xi)\right)\left(\int_0^\infty f(\bar{r})\bar{r}^n d\bar{r}\right)
\end{aligned}$$

Since $\left|\Psi_\lambda\left(\frac{|x|}{\bar r}\right)\right| \leq 1$ and $\Psi_\lambda\left(\frac{|x|}{\bar r}\right) \to 1$ as $\lambda \to 0$, we deduce by Dominated Convergence Theorem that

$$\begin{aligned}
\mathcal{I} &= \lim_{\lambda\to 0}\int_{\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}_+} dx\ d\xi\ d\bar r \quad \frac{g\left(\frac{x}{|x|}\right)}{|x|^{n+1}}(x\cdot\zeta)^2\Psi_\lambda\left(\frac{|x|}{\bar r}\right)e^{-i\xi\cdot x}f(\bar r)\bar r^n\varphi(\xi)\\
&= \lim_{\lambda\to 0}\int_{\mathbf{S}^{n-1}\times\mathbb{R}^n\times\mathbb{R}_+\times\mathbb{R}_+} d\theta\ d\xi\ d\bar r\ dr \quad g(\theta)(\theta\cdot\zeta)^2\Psi_\lambda\left(\frac{r}{\bar r}\right)e^{-i\xi\cdot\theta r}f(\bar r)\bar r^n\varphi(\xi)
\end{aligned}$$

where $\theta = \dfrac{x}{|x|}$, $r = |x|$. We set $r = \bar r s$, $\bar x = \theta\bar r$, $\bar s = |\xi| s$ and we get

$$\begin{aligned}
\mathcal{I} &= \lim_{\lambda\to 0}\int_{\mathbf{S}^{n-1}\times\mathbb{R}^n\times\mathbb{R}_+\times\mathbb{R}_+} d\theta\ d\xi\ d\bar r\ \bar r ds \quad g(\theta)(\theta\cdot\zeta)^2\Psi_\lambda(s)\, e^{-i\xi\cdot\theta\bar r s}f(\bar r)\bar r^n\varphi(\xi)\\
&= \lim_{\lambda\to 0}\int_{\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}_+} d\bar x\ d\xi\ ds \quad f(|\bar x|)g\left(\frac{\bar x}{|\bar x|}\right)(\bar x\cdot\zeta)^2\Psi_\lambda(s)e^{-i\xi\cdot\bar x s}\varphi(\xi)\\
&= \lim_{\lambda\to 0}\int_{\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}_+} d\bar x\ d\xi\ d\bar s \quad f(|\bar x|)\ g\left(\frac{\bar x}{|\bar x|}\right)(\bar x\cdot\zeta)^2\Psi_{\frac{\lambda}{|\xi|}}(\bar s)\ e^{-i\frac{\xi}{|\xi|}\cdot\bar x\bar s}\ \frac{\varphi(\xi)}{|\xi|}\\
&= \lim_{\lambda\to 0}\int_{\mathbb{R}^n} d\xi\ \frac{\varphi(\xi)}{|\xi|}\int_{\mathbb{R}^n} d\bar x\ \Phi(\bar x)\int_{R_+} d\bar s\ \Psi_{\frac{\lambda}{|\xi|}}(\bar s)\ e^{-i\frac{\xi}{|\xi|}\cdot\bar x\bar s}
\end{aligned}$$

where $\Phi(\bar x) = f(|\bar x|)g\left(\dfrac{\bar x}{|\bar x|}\right)(\bar x\cdot\zeta)^2 \in C_c^\infty(\mathbb{R})$ and $\mathrm{Supp}\ \Phi \subset B_{R_0}(0)\backslash B_{r_0}(0)$. Using the fact that $\Phi(-\bar x) = \Phi(\bar x)$, we deduce

$$\begin{aligned}
\mathcal{I} &= \lim_{\lambda\to 0}\int_{\mathbb{R}^n} d\xi\ \frac{\varphi(\xi)}{|\xi|}\int_{\mathbb{R}^n} d\bar x\ \Phi(\bar x)\int_{R_+} d\bar s\ \Psi_{\frac{\lambda}{|\xi|}}(\bar s)\ \frac{e^{-i\frac{\xi}{|\xi|}\cdot\bar x\bar s} + e^{i\frac{\xi}{|\xi|}\cdot\bar x\bar s}}{2}\\
&= \lim_{\lambda\to 0}\int_{\mathbb{R}^n} d\xi\ \frac{\varphi(\xi)}{2|\xi|}\int_{\mathbb{R}^n} d\bar x\ \Phi(\bar x)\int_{R} d\bar s\ \Psi_{\frac{\lambda}{|\xi|}}(\bar s)\ e^{-i\frac{\xi}{|\xi|}\cdot\bar x\bar s}
\end{aligned}$$

We set $\bar x = \bar x' + \bar y e_\xi$ with $\bar x' \in e_\xi^\perp$, $\bar y \in \mathbb{R}$ and $e_\xi = \dfrac{\xi}{|\xi|}$ and get

$$\begin{aligned}
\mathcal{I} &= \lim_{\lambda\to 0}\int_{\mathbb{R}^n} d\xi\ \frac{\varphi(\xi)}{2|\xi|}\int_{\mathbb{R}^{n-1}} d\bar x'\int_{\mathbb{R}} d\bar y\ \Phi(\bar x', \bar y)\int_{\mathbb{R}} d\bar s\ \Psi_{\frac{\lambda}{|\xi|}}(\bar s)\ e^{-i\bar y\bar s}\\
\mathcal{I} &= \lim_{\lambda\to 0}\int_{\mathbb{R}^n} d\xi\ \frac{\varphi(\xi)}{2|\xi|}\int_{\mathbb{R}^{n-1}} d\bar x'\ J_{\frac{\lambda}{|\xi|}}(\bar x')
\end{aligned}$$

where

$$\begin{aligned}
J_{\frac{\lambda}{|\xi|}}(\bar x') &= \int_R d\bar y\ \Phi(\bar x', \bar y)\mathcal{F}\left(\Psi_{\frac{\lambda}{|\xi|}}\right)(\bar y)\\
&= \left\langle \mathcal{F}\left(\Psi_{\frac{\lambda}{|\xi|}}\right), \Phi(\bar x', \cdot)\right\rangle
\end{aligned}$$

We claim the following whose proof is postponed

***Lemma* 9.2** *We have*

$$\mathcal{F}(\Psi_\mu) \to 2\pi\delta_0$$

in $\mathcal{S}'(\mathbb{R})$ as $\mu \to 0$.

Using this result and the fact that $\left|J_{\frac{\lambda}{|\xi|}}(\bar{x}')\right| \leq |\mathcal{F}(\Phi(x',\cdot))|_{L^1(\mathbb{R})}$, we deduce that

$$\begin{aligned}
\mathcal{I} =& 2\pi \int_R d\xi \, \frac{\varphi(\xi)}{2|\xi|} \int_{\mathbb{R}^{n-1}} d\bar{x}' \; \Phi(\bar{x}',0) \\
=& 2\pi \int_R d\xi \, \frac{\varphi(\xi)}{2|\xi|} \int_{\mathbb{R}^{n-1}} d\bar{x}' \; f(|\bar{x}|) g\left(\frac{\bar{x}}{|\bar{x}|}\right)(\bar{x}\cdot\zeta)^2 \\
=& 2\pi \int_R d\xi \, \frac{\varphi(\xi)}{2|\xi|} \int_{\mathbf{S}^{n-1}\cap\{\xi^\perp\}} d\theta \; g(\theta)(\theta\cdot\zeta)^2 \int_{R_+} d\bar{r} \; f(\bar{r})\bar{r}^n \\
=& 2\pi \int_R d\xi \, \frac{\varphi(\xi)}{|\xi|} A\left(\frac{\xi}{|\xi|}\right)(\zeta,\zeta) \\
& 2\pi \int_R d\xi \; \varphi(\xi) A(\xi)(\zeta,\zeta) \\
=& 2\pi \langle A(\xi)(\zeta,\zeta), \varphi\rangle
\end{aligned}$$

We then have shown that

$$\left\langle \mathcal{F}\left(\frac{g\left(\frac{x}{|x|}\right)}{|x|^{n+1}}(x\cdot\zeta)^2\right), \varphi \right\rangle = 2\pi\langle A(\xi)(\zeta,\zeta),\varphi\rangle.$$

By a passage to the limit, this is still true if $g \in C^0$ (and not only C^∞) and for all $\varphi \in C_c^\infty(\mathbb{R}^n\backslash\{0\})$. We then deduce that

$$\mathcal{F}\left(\frac{g\left(\frac{x}{|x|}\right)}{|x|^{n+1}}(x\cdot\zeta)^2\right) - 2\pi A(\xi)(\zeta,\zeta) = T$$

with Supp $T \subset \{0\}$, and then the distribution T is a finite sum of derivatives of Dirac mass: $T = \sum a_\alpha \delta_0^{(\alpha)}$. Using the fact that $\delta_0^{(\alpha)}(\lambda\xi) = \frac{1}{\lambda^{n+|\alpha|}}\delta_0^{(\alpha)}(\xi)$ with $|\alpha| = \alpha_1 + ... + \alpha_n$, and the homogeneity of degree -1 of $D^2\mathcal{F}(L_g)$, we deduce that for $n \geq 2$, $T = 0$ and

$$\mathcal{F}\left(\frac{g\left(\frac{x}{|x|}\right)}{|x|^{n+1}}(x\cdot\zeta)^2\right) = 2\pi A(\xi)(\zeta,\zeta).$$

This ends the proof of the *lemma*.

Proof of *Lemma* 9.2
Let $\varphi_1 \in \mathcal{S}(\mathbb{R})$. The following holds

$$\begin{aligned}
\langle \mathcal{F}(\Psi_\mu) - 2\pi\delta_0, \varphi_1\rangle =& \langle \mathcal{F}(\Psi_\mu) - \mathcal{F}(1), \varphi_1\rangle \\
=& \langle \Psi_\mu - 1, \mathcal{F}(\varphi_1)\rangle.
\end{aligned}$$

So, it just remains to show that $\Psi_\mu \to 1$ in $\mathcal{S}'(\mathbb{R})$ as $\mu \to 0$. Let $\varphi \in \mathcal{S}(\mathbb{R})$. The following holds

$$\begin{aligned}
\langle \Psi_\mu - 1, \varphi \rangle &= \int_{\mathbb{R}} dx\ \left(\Psi_\mu(x) - 1\right)\varphi(x) \\
&= \int_{\mathbb{R}} dx\ (\Psi(\mu x) - 1)\varphi(x) \\
&= \int_{|x|\geq \frac{1}{2\mu}} dx\ (\Psi(\mu x) - 1)\varphi(x) \\
&\leq \int_{|x|\geq \frac{1}{2\mu}} dx\ |\varphi(x)| \\
&\leq C\mathcal{N}_2(\varphi) \int_{|x|\geq \frac{1}{2\lambda}} dx\ \frac{1}{1+x^2} \\
&\to 0 \quad \text{as } \mu \to 0
\end{aligned}$$

where we have used the definition of $\mathcal{N}_p(\varphi) = \sup_{|\alpha|,|\beta|\leq p} \left| |x|^\alpha \frac{d^\beta \varphi(x)}{dx^\beta} \right|$ and the fact that

$$(1+x^2)|\varphi(x)| \leq C\mathcal{N}_2(\varphi).$$

This ends the proof of the *lemma*.

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