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Well-posedness and numerical analysis of a one-dimensional non-local transport equation modelling dislocations dynamics

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Abstract

We consider a situation where dislocations are parallel lines moving in a single plane. For this simple geometry, dislocations dynamics is modeled by a one-dimensional non-local transport equation. We prove a result of existence and uniqueness for all time of the continuous viscosity solution for this equation. A finite difference scheme is proposed to approximate the continuous viscosity solution. We also prove an error estimate result between the continuous solution and the discrete solution and we provide some simulations.

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1 Introduction

1.1 Physical motivation

In this work, we are interested in the dislocations dynamics in a crystal material (see [18] for a physical description of dislocations). A perfect crystal, for small deformations, is well described by the equations of linear elasticity. The real crystals contain in particular some line defects called dislocations. The dislocations dynamics is one of the main explanation of the plastic deformation of metals. When we apply an exterior stress, these dislocations lines can move in a slip plane of the crystal. We consider here a simple geometry where the dislocations are parallel lines moving in a same plane (xy). This plane is embedded in a three-dimensional elastic crystal. The particular geometry of this problem leads to study a one-dimensional model given by the following non-local transport equation modelling dislocations dynamics:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = c[u](x,t)\,\dfrac{\partial u}{\partial x}(x,t) \quad \text{in } \mathbb{R}\times(0,+\infty) \\ u(x,0) = u^0(x) \quad \text{in } \mathbb{R} \end{cases} \tag{1}$$

where the solution u is a scalar function, $\dfrac{\partial u}{\partial t}$ and $\dfrac{\partial u}{\partial x}$ are respectively its time and space derivatives. Here the dislocations move with a non-local velocity $c[u]$ known as the resolved Peach-Koehler force, see [20]. It is given by

$$\begin{cases} c[u](x,t) = c^{\text{ext}}(x) + c^{\text{int}}[u](x,t) \\ c^{\text{int}}[u](x,t) = \displaystyle\int_{\mathbb{R}} c^0(x-x')\,(E(u(x',t)) - Px')\;dx' \end{cases} \tag{2}$$

where the function E is the floor function defined by $E(v) = k$ if $k \le v < k+1$, $k \in \mathbb{Z}$. The scalar function u has no physical meaning but it is chosen such that the jumps of $E(u)$ correspond to the positions of dislocations (see Figure 1). The velocity $c[u]$ is the sum of two terms. We first assume the existence in the material of obstacles to the motion of dislocations. The term c^{ext} represents the exterior stress created by these obstacles (such as precipitates in the material, other fixed dislocations, other defects, ...). We consider obstacles that are independent on time and periodic in space. Namely we assume that the velocity satisfies

$$c^{\text{ext}} \in W^{1,\infty}(\mathbb{R}) \quad \text{such that} \quad c^{\text{ext}}(x+1) = c^{\text{ext}}(x) \quad \text{in } \mathbb{R}. \tag{3}$$

The second term $c^{\text{int}}[u]$ is a non-local term, given by a convolution with respect to the space variable, and represents the elastic interior stress created by all the dislocations in the material. This term $c^{\text{int}}[u]$ is obtained by the resolution of the equations of linear elasticity. For instance, in

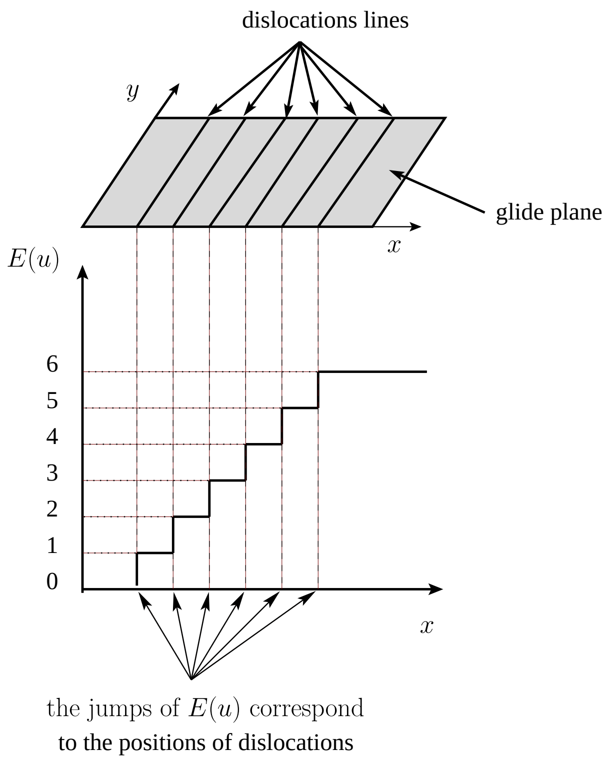

Figure 1: Representation of dislocations with the function $E(u)$

the model of Peierls-Nabarro (see [5]), we have in the case of edge dislocation (see [18])

$$c^0(x) = \frac{-\mu b^2}{2\pi(1-\nu)} \frac{x^2 - \zeta^2}{(x^2+\zeta^2)^2} \quad \text{on } \mathbb{R}\,, \tag{4}$$

where $\nu = \dfrac{\lambda}{2(\lambda+\mu)}$ is the Poisson ratio and λ and $\mu > 0$ are the Lamé coefficients for isotropic elasticity. The Burgers vector $\vec{b}$ is equal to $b\vec{e}_x$, with $b > 0$ and $\vec{e}_x$ the unit vector in the direction of x of Figure 1. There is a physical parameter $\zeta \neq 0$ (depending on the material) which represents the size of the core of the dislocation.

1.2 Main results

In the sequel of this paper, we use some adapted norms introduced in the following definition.

Definition 1.1 (Adapted norms)
Let two functions $f \in L^1_{loc}(\mathbb{R})$ and $g \in L^\infty_{loc}(\mathbb{R})$. We define the quantities

$$|f|_{L^1_{unif}(\mathbb{R})} = \sup_{x\in\mathbb{R}} \int_{I(x)} |f(y)|\, dy \quad \text{and} \quad |g|_{L^\infty_{int}(\mathbb{R})} = \int_{\mathbb{R}} |g|_{L^\infty(I(x))}\, dx$$

where $I(x) = (x - \frac{1}{2}, x + \frac{1}{2})$. We denote respectively $L^1_{unif}(\mathbb{R})$ and $L^\infty_{int}(\mathbb{R})$ spaces that consist of functions for which these quantities are finite.

Remark 1.2 *These spaces are motivated by the following fact. For $c^0 \in L^\infty_{int}(\mathbb{R})$ and $f \in L^1_{unif}(\mathbb{R})$, we will show later that the convolution product $c^0 \star f$ is well defined. This will be applied to define $c^{\text{int}}[u]$ with $f(x) = E(u(x,t)) - Px$.*

We denote $\text{Lip}(\mathbb{R})$ the space of Lipschitz continuous functions on $\mathbb{R}$.

1.2.1 Existence and uniqueness of a continuous solution

We consider the following assumptions for the kernel c^0:

$$\left\{\begin{array}{l} c^0 \in W^{1,1}(\mathbb{R}) \cap L^\infty_{\text{int}}(\mathbb{R}) \\ c^0(x) = c^0(-x) \text{ and } \displaystyle\int_{\mathbb{R}} c^0(x)\, dx = 0\,. \end{array}\right. \tag{5}$$

One can check easily that the kernel given in (4) satisfies (5). We consider the initial condition $u^0 \in \mathrm{Lip}(\mathbb{R})$ such that for $x \in \mathbb{R}$

$$u^0(x+1) = u^0(x) + P \quad \text{and} \quad 0 < b_0 \leq u^0_x \leq B_0 < +\infty \quad a.e. \tag{6}$$

with b_0 and B_0 some constants and $P \in \mathbb{N}\backslash\{0\}$. This condition means in particular that dislocations are periodically distributed. As mentioned above, in order to study the solutions of (1), we use the theory of continuous viscosity solutions (see [7, 10]). Our first main result is:

Theorem 1.3 (Long time existence and uniqueness of the solution)
Under Assumptions (3), (5), (6), there exists a unique continuous viscosity solution $u \in W^{1,\infty}_{loc}(\mathbb{R} \times [0, +\infty))$ of (1), (2) satisfying $u(x+1,t) = u(x,t) + P$.

In [5], a short time existence and uniqueness result is given for a 2D problem for a single dislocation line. Because in the present work our problem is one-dimensional, we are able to get a refined result for the dynamics of several dislocations in interaction, namely the existence and uniqueness of a solution for all time.
Let us mention that under the more restrictive assumptions that the velocity $c[u]$ is nonnegative it is proved in [1, 9] the existence and uniqueness of a solution for all time.
In the special case where the kernel c^0 is assumed nonnegative some existence and uniqueness results for all time in any dimension, are available in a "Slepčev formulation" (see [8, 14]). The previous theorem will be proved in two steps. First, we will prove the result for short time (see for instance [17]) using a fixed point theorem. Secondly, we will repeat this short time result on a sequence of time intervals of lengths T_n decreasing to zero, such that $\displaystyle\sum_{n\in\mathbb{N}} T_n = +\infty$.

Remark 1.4 *Let us mention three remaining open problems.*

- *When the initial data u^0 is not monotone, the existence and uniqueness of the solution for all time is an open question.*
- *We do not know the behavior of the solution as $t \to +\infty$.*
- *If we replace $\dfrac{\partial u}{\partial x}$ in Equation (1) by its absolute value, then we have a non-local Hamilton-Jacobi equation. Physically, the absolute value would allow to consider the possible annihilation of two dislocations associated to opposed jumps of $E(u)$. The existence and uniqueness of a solution for all time is an open question in the general case. Nevertheless, in the whole paper we will only consider the case of solutions u monotone in space which allows to forget the absolute value.*

1.2.2 Convergence of a numerical scheme

We build a finite difference scheme of order one in space and time

- by assuming that it is upwind,
- by approximating the non-local term $c^0 \star E\,(u(\cdot,t))$ by a discrete convolution,
- using an explicit Euler scheme in time.

Given a mesh size Δx, Δt and a lattice $I_d = \{(i\Delta x, n\Delta t);\ i \in \mathbb{Z},\ n \in \mathbb{N}\}$, (x_i, t_n) denotes the node $(i\Delta x, n\Delta t)$ and $v^n = (v_i^n)_i$ the values of the numerical approximation of the continuous solution $u(x_i, t_n)$. We then consider the following numerical scheme:

$$v_i^0 = u^0(x_i), \quad v_i^{n+1} = v_i^n + \Delta t\, c_i(v^n) \times \begin{cases} D_x^+ v_i^n & \text{if } c_i(v^n) \geq 0 \\ D_x^- v_i^n & \text{if } c_i(v^n) < 0 \end{cases} \tag{7}$$

with

$$D_x^+ v_i^n = \frac{v_{i+1}^n - v_i^n}{\Delta x}, \quad D_x^- v_i^n = \frac{v_i^n - v_{i-1}^n}{\Delta x} \tag{8}$$

and $c_i(v^n)$ is defined below.
We choose $\Delta x = \dfrac{1}{K}$, $K \in \mathbb{N} \setminus \{0\}$ because of the 1-periodicity in space. We denote $c_i^{\text{ext}} = c^{\text{ext}}(x_i)$ which satisfies $c_{i+K}^{\text{ext}} = c_i^{\text{ext}}$. The discrete velocity is

$$c_i(v^n) = c_i^{\text{ext}} + c_i^{\text{int, n}} \ \text{ with } \ c_i^{\text{int, n}} = \sum_{l\in\mathbb{Z}} c_l^0\, E(v_{i-l}^n)\,\Delta x \tag{9}$$

where

$$c_i^0 = \frac{1}{\Delta x}\int_{I_i} c^0(x)\,dx \quad \text{and} \quad I_i = \left[x_i - \frac{\Delta x}{2}, x_i + \frac{\Delta x}{2}\right]. \tag{10}$$

We are interested in solutions v^n satisfying $v_{i+K}^n = v_i^n + P$ for all $i \in \mathbb{Z}$. Then we can check that the discrete velocity satisfies $c_{i+K}(v^n) = c_i(v^n)$.
Note that the global scheme $v^{n+1} = S(v^n)$ given by (7) is not monotone in general because the velocity $c_i(v^n)$ depends non-monotonically on the solution v^n itself (here some c_i^0 are nonpositive because we assumed that $\int_{\mathbb{R}} c^0(x)\,dx = 0$).
We assume that the mesh satisfies the following CFL (Courant, Friedrichs, Lévy) condition

$$\Delta t < \frac{\Delta x}{4\left(\left|c^{\text{ext}}\right|_{L^\infty(\mathbb{R})} + P\left|c^0\right|_{L^1(\mathbb{R})}\right)}. \tag{11}$$

Our second main result is

Theorem 1.5 (Error estimate)
Let u be the continuous viscosity solution of Problem (1), (2) under Assumptions (3), (5), (6). Let v be the discrete solution of the associated finite difference scheme (7)-(10). Assume that the time step Δt satisfies

$$\Delta t = \alpha\Delta x \quad \textit{with} \quad 0 < \alpha < \frac{1}{4\left(\left|c^{\text{ext}}\right|_{L^\infty(\mathbb{R})} + P\left|c^0\right|_{L^1(\mathbb{R})}\right)}. \tag{12}$$

Then, there exists two constants T_1, $C > 0$, depending on α, $|c^{\text{ext}}|_{W^{1,\infty}(\mathbb{R})}$, P, $\left|c^0\right|_{W^{1,1}(\mathbb{R})}$, $\left|c^0\right|_{L^\infty_{int}(\mathbb{R})}$, B^0 and b^0 in (6), such that:

$$\sup_{i\in\mathbb{Z}} |u(i\Delta x, n\Delta t) - v_i^n| \leq C\,|\Delta x|^{1/2} \quad \textit{for all } n \leq \frac{T_1}{\Delta t} \quad \textit{if} \quad \Delta x \leq \frac{T_1}{C}.$$

The proof of this theorem is based on the ideas of Crandall and Lions [13] adapted to the case of non-local equations (see [2]).
Extensive simulations of dislocations dynamics will be provided in [16].

1.3 Brief review of the literature

Let us recall that, in the 1980's, the notion of viscosity solution was first introduced by Crandall and Lions in [11] for first order Hamilton-Jacobi equations. For an introduction to this notion, see in particular the books of Barles [7], and of Bardi and Capuzzo-Dolcetta [6].
Recently, Alvarez, Hoch, Le Bouar and Monneau [4, 5] used this theory for the resolution of a non-local Hamilton-Jacobi equation modelling dislocation dynamics. They proved results of short time existence and uniqueness of a discontinuous viscosity solution. Their results are mainly valid for dislocations with the shape of graphs and loops and they used the level set approach, which was introduced by Osher and Sethian [19]. As already mentioned, in the situation where the non-local velocity is nonnegative, Barles and Ley [9] proved that the existence and uniqueness is valid for any time interval for a level set formulation. Still in the case of nonnegative velocity an approach for discontinuous viscosity solution was developped by Alvarez, Cardaliaguet and Monneau [1]. Let us mention, for dislocations dynamics with mean curvature terms, Forcadel in [15] proved a short time existence and uniqueness result.
A numerical analysis was done by Crandall and Lions [13], for approximations of solutions of Hamilton-Jacobi equations. Convergence of a first order scheme for an abstract non-local eikonal equation was proved by Alvarez, Carlini, Monneau and Rouy [2]. They also applied this convergence result for the numerical analysis of a non-local Hamilton-Jacobi equation in [3] describing the dynamics of a single dislocation in 2D.

1.3.1 Organization of the paper

In Section 2, we give some properties of the solution of an auxiliary local equation, *i.e.* an eikonal equation where the velocity is assumed to be a given function independent on the solution. In Section 3, we give some properties of the non-local velocity. The existence and uniqueness result of a continuous solution, *i.e.* Theorem 1.3, is then proved in Section 4. We give preliminary results for the discrete local problem in Section 5 and for the discrete non-local velocity in Section 6. Theorem 1.5 about the error estimate is proved in Section 7. Finally in Section 8 we give some simulations.

2 Preliminary results for the eikonal equation with prescribed velocity

In this section, we start by recalling the notion of viscosity solution of an eikonal equation. We then give some properties of the solution of a such equation.
Let $T > 0$. Consider the following problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) & = c(x,t)\left|\dfrac{\partial u}{\partial x}(x,t)\right| \quad \text{in} \quad \mathbb{R}\times(0,T)\,, \\ u(x,0) & = u_0(x) \quad\quad\quad\quad\quad \text{on} \quad \mathbb{R}\,. \end{cases} \tag{13}$$

We make the following assumptions:

a) the velocity $c : \mathbb{R}\times(0,T) \longrightarrow \mathbb{R}$ is bounded, Lipschitz continuous in space and in time,

b) the initial data $u_0 \in \mathrm{Lip}(\mathbb{R})$.

We recall the notions of viscosity subsolutions, supersolutions and solutions for (13) (see [10]). We denote

$$\mathrm{USC}(\mathbb{R}\times[0,T)) = \{u : \mathbb{R}\times[0,T) \longrightarrow \mathbb{R}\,,\ \text{locally bounded, upper semicontinuous}\}$$

$$\text{and } \mathrm{LSC}(\mathbb{R}\times[0,T)) = \{u : \mathbb{R}\times[0,T) \longrightarrow \mathbb{R}\,,\ \text{locally bounded, lower semicontinuous}\}.$$

We then define

Definition 2.1 (Viscosity subsolution, supersolution and solution)

1) *A function $u \in USC(\mathbb{R}\times[0,T))$ is a viscosity subsolution of (13) if the following properties hold:*

i) $u(x,0) \le u_0(x)$ *in* $\mathbb{R}$,

ii) *for every $(x_0,t_0) \in \mathbb{R}\times(0,T)$ and for every test function $\varphi \in C^1(\mathbb{R}\times(0,T))$ such that $u-\varphi$ has a local maximum at (x_0,t_0), we have*

$$\frac{\partial \varphi}{\partial t}(x_0,t_0) \le c(x_0,t_0)\left|\frac{\partial \varphi}{\partial x}(x_0,t_0)\right| .$$

2) *A function $u \in LSC(\mathbb{R}\times[0,T))$ is a viscosity supersolution of (13) if the following properties hold:*

i) $u(x,0) \ge u_0(x)$ *in* $\mathbb{R}$,

ii) *for every $(x_0,t_0) \in \mathbb{R}\times(0,T)$ and for every test function $\phi \in C^1(\mathbb{R}\times(0,T))$ such that $u-\phi$ has a local minimum at (x_0,t_0), we have*

$$\frac{\partial \phi}{\partial t}(x_0,t_0) \ge c(x_0,t_0)\left|\frac{\partial \phi}{\partial x}(x_0,t_0)\right| .$$

3) *A function $u \in C(\mathbb{R}\times[0,T))$ is a continuous viscosity solution of (13) if it is both a viscosity subsolution and a viscosity supersolution of (13).*

We have the following *a priori* estimates for the solution of the eikonal equation. These estimates are may be quite classical, and part of them is already proved in [5], but we give a proof for sake of completeness.

Proposition 2.2 (*a priori* estimates for the solution of the eikonal equation)
Assume that $c \in W^{1,\infty}(\mathbb{R}\times[0,T])$ and $u_0 \in Lip(\mathbb{R})$ such that $|(u_0)_x| \le B_0$ a.e. and $(u_0)_x \ge b_0$ a.e. for some $B_0 > b_0 > 0$. Then, there exists a unique continuous viscosity solution u on $\mathbb{R}\times[0,T)$ of problem (13). Moreover, $u \in Lip(\mathbb{R}\times[0,T))$. With $L_c := L_c(t) = |c_x(\cdot,t)|_{L^\infty(\mathbb{R})}$, $B(t) = B_0\ e^{L_c t}$ and $b(t) = b_0\ e^{-L_c t}$, we have the following estimates

i) *for every $0 \le t < T$,*

$$|u_x(x,t)| \le B(t) \quad a.e.$$

and

$$u_x(x,t) \ge b(t) \quad a.e.$$

ii) *Moreover*

$$|u_t(x,t)| \le |c|_{L^\infty(\mathbb{R}\times(0,T))}\ B(t) \quad a.e.$$

Proof of Proposition 2.2
We refer to [7, Theorem 2.8, page 38] for the proof of existence and uniqueness of a solution u. We introduce the double variables $(x,y) \in \mathbb{R}^2$ and set the half-plane $\Omega = \{x \ge y\}$. Consider the following problem

$$\left\{\begin{array}{lll} w_t(x,y,t) & = c(x,t)\,|w_x(x,y,t)| - c(y,t)\,|w_y(x,y,t)| & \text{in} \quad \Omega\times(0,T), \\ w(x,y,0) & = u(x,0) - u(y,0) & \text{in} \quad \Omega, \\ w(x,x,t) & = 0 & \text{on} \quad \partial\Omega\times(0,T). \end{array}\right. \tag{14}$$

Then, $w(x,y,t) = u(x,t) - u(y,t)$ is a continuous viscosity solution of Problem (14) (we refer to [12, Lemma 2, page 357] for a proof).
Let $\Phi(x,y,t) = B(t)(x-y)$. Then, we have

Claim 1: Φ is a (viscosity) supersolution of Problem (14).

As a matter of fact, since Φ is smooth, Φ is a classical supersolution of Problem (14). Indeed, on the one hand, we have

$$w(x,y,0) = u(x,0) - u(y,0) \leq B_0(x-y) = \Phi(x,y,0)$$

and

$$w(x,x,t) = 0 = \Phi(x,x,t)\,.$$

On the other hand, we have

$$\begin{aligned}\Phi_t - c(x,t)\,|\Phi_x| + c(y,t)\,|\Phi_y| &= L_c B_0\, e^{L_c t}(x-y) - c(x,t)B_0\, e^{L_c t} + c(y,t)B_0\, e^{L_c t}\\ &= B_0\, e^{L_c t}\left(-c(x,t) + c(y,t) + L_c(x-y)\right)\,.\end{aligned}$$

Moreover,

$$|c(x,t) - c(y,t)| \leq L_c\,|x-y| \quad \text{and} \quad x \geq y$$

implies

$$c(x,t) - c(y,t) \leq\; L_c(x-y).$$

We then obtain

$$\Phi_t - c(x,t)\,|\Phi_x| + c(y,t)\,|\Phi_y| \geq 0\,.$$

This proves claim 1.
Let $\varphi(x,y,t) = b_0\, e^{-L_c t}(x-y)$. Then we have

Claim 2: φ is a (viscosity) subsolution of the Problem (14).

The proof is similar to the proof of claim 1 and we skip it.
By the comparison principle (see [7, Theorem 2.10, page 47]):

a)

$$w(x,y,t) \leq \Phi(x,y,t)$$

i.e.

$$u(x,t) - u(y,t) \leq B(t)(x-y) \quad \text{for } (x,y,t) \in \Omega \times (0,T) \tag{15}$$

b) and

$$\varphi(x,y,t) \leq w(x,y,t)$$

i.e.

$$b(t)(x-y) \leq u(x,t) - u(y,t) \quad \text{for } (x,y,t) \in \Omega \times (0,T) \tag{16}$$

We deduce that

$$0 \leq b(t)(x-y) \leq u(x,t) - u(y,t) \leq B(t)(x-y) \leq B(t)\,|x-y| \quad \text{for all } (x,y,t) \in \Omega \times (0,T)\,. \tag{17}$$

Passing to the limit in (17), by Rademacher's Theorem [7], we get

$$0 \leq b(t) \leq u_x(x,t) \leq B(t) \quad \text{for a.e. } (x,t) \in \Omega \times [0,T)\,.$$

We now prove the Lipschitz in time estimate. Let $(x_0,t_0) \in \mathbb{R} \times (0,T)$ and $\varphi \in C^1(\mathbb{R} \times (0,T))$ such that $u - \varphi$ has a local maximum at (x_0,t_0). We show that $\varphi_t \leq |c|_{L^\infty(\mathbb{R}\times(0,T))}\, B(t_0)$. From $\dfrac{\varphi(x_0,t_0) - \varphi(x,t_0)}{|x-x_0|} \leq \dfrac{u(x_0,t_0) - u(x,t_0)}{|x-x_0|}$ and (17) we obtain

$$\varphi_x(x_0,t_0) \leq B(t_0)$$

and then

$$\varphi_t(x_0,t_0) \le c(x_0,t_0)\,|\varphi_x(x_0,t_0)| \le |c|_{L^\infty(\mathbb{R}\times(0,T))}\,|\varphi_x(x_0,t_0)| \le |c|_{L^\infty(\mathbb{R}\times(0,T))}\,B(t_0).$$

Let $\Phi \in C^1(\mathbb{R}\times(0,T))$ such that $u-\varphi$ has a local maximum at $(x_0,t_0)\in\mathbb{R}\times(0,T)$. Similarly, we check easily that $\Phi_t \ge -|c|_{L^\infty(\mathbb{R}\times(0,T))}\,B(t_0)$. Therefore, we have

$$\varphi_t \le |c|_{L^\infty(\mathbb{R}\times(0,T))}\,B(t_0) \quad \text{and} \quad \Phi_t \ge -|c|_{L^\infty(\mathbb{R}\times(0,T))}\,B(t_0)\,.$$

We conclude that

$$|u_t| \le |c|_{L^\infty(\mathbb{R}\times(0,T))}\,B(t_0) \quad \text{in the viscosity sense.}$$

□

We now give a stability result.

Proposition 2.3 (Stability of the solution by perturbation of the velocity) *Let v^i, $i=1,2$, be a viscosity solution of the problem*

$$\begin{cases} v^i_t(x,t) &= c^i(x,t)\left|v^i_x(x,t)\right| \quad \text{in} \quad \mathbb{R}\times[0,T)\,,\\ v^i(x,0) &= u_0(x) \qquad\qquad\quad\; \text{on} \quad \mathbb{R}\,, \end{cases} \tag{18}$$

where $c^i \in W^{1,\infty}(\mathbb{R}\times[0,T])$ and $u_0 \in Lip(\mathbb{R})$. Then,

$$\left|v^1-v^2\right|_{L^\infty(\mathbb{R}\times[0,T])} \le \int_0^T ds\,\left|c^2(\cdot,s)-c^1(\cdot,s)\right|_{L^\infty(\mathbb{R})} \max\left(\left|v^1_x(\cdot,s)\right|_{L^\infty(\mathbb{R})}, \left|v^2_x(\cdot,s)\right|_{L^\infty(\mathbb{R})}\right). \tag{19}$$

This result has been proved in [5]. For sake of completeness we give it here.

Proof of Proposition 2.3

We denote $\bar v^2(x,t) = v^2(x,t) - \int_0^t \left|c^2(\cdot,s)-c^1(\cdot,s)\right|_{L^\infty(\mathbb{R})} \left|v^2_x(\cdot,s)\right|_{L^\infty(\mathbb{R})} ds$. We want to prove that $\bar v^2$ is a viscosity subsolution of the equation satisfied by v^1. We denote $I(v) = v_t - c^1(x,t)\,|v_x|$. Formally, $I(v^1)=0$ and $I(v^2) = v^2_t - c^1(x,t)\left|v^2_x\right| = \left(c^2(x,t)-c^1(x,t)\right)\left|v^2_x\right|$. We show that $I(v^2) \le \left|c^2(\cdot,t)-c^1(\cdot,t)\right|_{L^\infty(\mathbb{R})} \left|v^2_x(\cdot,t)\right|_{L^\infty(\mathbb{R})}$ in the viscosity sense. Indeed, let $\varphi \in C^1(\mathbb{R}\times(0,T))$ such that $v^2-\varphi$ has a local maximum at $(x_0,t_0)\in\mathbb{R}\times(0,T)$. Then,

$$I(\varphi) = \varphi_t - c^1(x,t)\,|\varphi_x| \le \left(c^2(x,t)-c^1(x,t)\right)|\varphi_x| \le \left|c^2(\cdot,t)-c^1(\cdot,t)\right|_{L^\infty(\mathbb{R})} |\varphi_x|\,.$$

Similarly, setting $\Phi \in C^1(\mathbb{R}\times(0,T))$ such that $v^2-\Phi$ has a local minimum at $(x_0,t_0)\in\mathbb{R}\times(0,T)$, we have

$$I(\Phi) \ge -\left|c^2(\cdot,t)-c^1(\cdot,t)\right|_{L^\infty(\mathbb{R})} |\Phi_x|\,.$$

Moreover, at $t=0$, we have $\bar v^2 = v^2 = u_0 = v^1$. Hence, we deduce that $\bar v^2$ is a subsolution of the equation satisfied by v^1. Then, by the comparison principle [7], for all $t\in[0,T]$, we have $\bar v^2 \le v^1$ *i.e.*

$$v^2(x,t) - v^1(x,t) \le \int_0^t \left|c^2(\cdot,s)-c^1(\cdot,s)\right|_{L^\infty(\mathbb{R})} \left|v^2_x(\cdot,s)\right|_{L^\infty(\mathbb{R})} ds\,.$$

Similarly we prove the inequality $\bar v^1 \ge v^2$ which leads to

$$v^2(x,t) - v^1(x,t) \ge -\int_0^t \left|c^2(\cdot,s)-c^1(\cdot,s)\right|_{L^\infty(\mathbb{R})} \left|v^1_x(\cdot,s)\right|_{L^\infty(\mathbb{R})} ds\,.$$

We conclude that

$$\left|v^2-v^1\right|_{L^\infty(\mathbb{R}\times[0,T])} \le \int_0^T \left|c^2(\cdot,s)-c^1(\cdot,s)\right|_{L^\infty(\mathbb{R})} \max\left(\left|v^1_x(\cdot,s)\right|_{L^\infty(\mathbb{R})}, \left|v^2_x(\cdot,s)\right|_{L^\infty(\mathbb{R})}\right) ds\,.$$

□

3 Properties of the non-local velocity

The goal of this section is to prove the following estimate, which will be used in Section 4.

Proposition 3.1 (Estimate on the difference of integer parts in the continuous case) *Let $\rho^1 \in C(\mathbb{R})$ such that*

i)
$$\rho^1(x+1) = \rho^1(x) + P \quad \textit{where } P \in \mathbb{N} \setminus \{0\}, \tag{20}$$

ii) *there exists constants $B \geq b > 0$ such that $b \leq \rho^1_x \leq B$ in the distribution sense.*

Let $\rho^2 \in L^\infty_{loc}(\mathbb{R})$ satisfying (20). Then,

$$\left|E\left(\rho^2\right) - E\left(\rho^1\right)\right|_{L^1_{unif}(\mathbb{R})} \leq \frac{2}{b}\left(P + \left|\rho^2 - \rho^1\right|_{L^\infty(\mathbb{R})}\right)\left|\rho^2 - \rho^1\right|_{L^\infty(\mathbb{R})}. \tag{21}$$

Remark 3.2 *Note that if $\left|\rho^2 - \rho^1\right|_{L^\infty(\mathbb{R})} \leq 1 \leq P$ then the previous estimate (21) becomes*

$$\left|E\left(\rho^2\right) - E\left(\rho^1\right)\right|_{L^1_{unif}(\mathbb{R})} \leq \frac{4P}{b}\left|\rho^2 - \rho^1\right|_{L^\infty(\mathbb{R})}.$$

We will use this estimate later.

This result is the generalization of Lemma 4.2 in [3] to the case of several dislocations where the characteristic function $\rho^i > 0$ is replaced with the floor part $E(\rho^i)$. To do the proof of Proposition 3.1 we need to introduce the following notations.
We denote $\Lambda = \left|\rho^2 - \rho^1\right|_{L^\infty(\mathbb{R})}$ and we assume that $\Lambda \in (0, +\infty)$ (other cases are trivial). For $k \in \mathbb{Z}$, we denote, for $i = 1, 2$,

$$E^i_k = \{x \in \mathbb{R}, \rho^i(x) < k+1\}.$$

First, we remark that since $\rho^1_x \geq b > 0$ and ρ^1 is continuous, there exists a unique $a_k \in \mathbb{R}$ such that $\rho^1(a_k) = k+1$ and we have $E^1_k = (-\infty, a_k)$. We will use the following lemma for the proof of Proposition 3.1.

Lemma 3.3 (Estimate of the distance between the sets E^1_k and E^2_k) *With the notations introduced above and the assumptions of Proposition 3.1, we have*

$$E^1_k - \frac{\Lambda}{b} \subset E^2_k \subset E^1_k + \frac{\Lambda}{b}.$$

Proof of Lemma 3.3
The main idea in this proof is to use the minoration of the gradient of the function ρ^1, *i.e.* $\rho^1_x \geq b > 0$.
Let us first check that $E^1_k - \frac{\Lambda}{b} \subset E^2_k$. Let $x \in E^1_k - \frac{\Lambda}{b}$. Then, $x < a_k - \frac{\Lambda}{b}$ *i.e.* $\Lambda < b(a_k - x)$.
Since $\rho^1_x \geq b > 0$ and $a_k - x > 0$, we have

$$\rho^1(a_k) - \rho^1(x) \geq b(a_k - x)$$

which implies (by definition of Λ)

$$k+1 = \rho^1(a_k) > \rho^1(x) + \Lambda \geq \rho^2(x)$$

and therefore

$$k+1 > \rho^2(x).$$

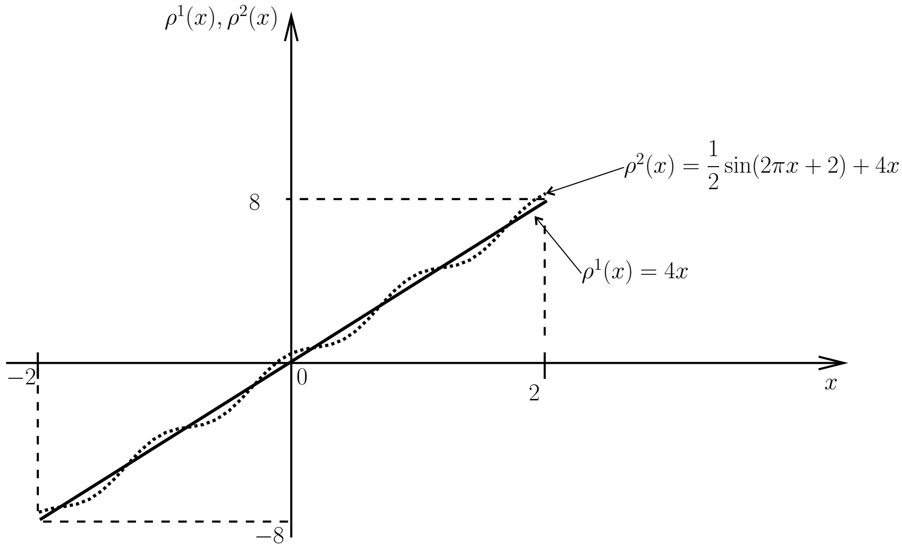

Figure 2: example of functions ρ^1 and ρ^2 satisfying (20)

Thus, $x \in E_k^2$ for every $x \in E_k^1 - \dfrac{\Lambda}{b}$ and therefore $E_k^1 - \dfrac{\Lambda}{b} \subset E_k^2$ (see Figure 2). The second inclusion can be proved similarly. $\square$

Proof of Proposition 3.1

The main idea in this proof is to bound the function $\left|E\left(\rho^2(x)\right) - E\left(\rho^1(x)\right)\right|$ by the characteristic functions of the sets $E_k^2 \vartriangle E_k^1$. We then bound its L^1_{unif}-norm.

From the definition of E_k^i, for $i = 1, 2$, we remark that $E_{k-1}^i \subset E_k^i$ then $E\left(\rho^i(x)\right) = k$ if $x \in E_k^i \setminus E_{k-1}^i$, for $i = 1, 2$. We can write

$$E\left(\rho^i(x)\right) = \sum_{k\in\mathbb{N}} 1_{\left(E_k^i\right)^c}(x) - \sum_{k\in\mathbb{Z}\setminus\mathbb{N}} 1_{E_k^i}(x).$$

Then,

$$E\left(\rho^2(x)\right) - E\left(\rho^1(x)\right) = \sum_{k\in\mathbb{N}} \left(1_{\left(E_k^2\right)^c}(x) - 1_{\left(E_k^1\right)^c}(x)\right) - \sum_{k\in\mathbb{Z}\setminus\mathbb{N}} \left(1_{E_k^2}(x) - 1_{E_k^1}(x)\right).$$

Therefore,

$$\left|E\left(\rho^2(x)\right) - E\left(\rho^1(x)\right)\right| \leq \sum_{k\in\mathbb{N}} 1_{\left(E_k^2\right)^c \vartriangle \left(E_k^1\right)^c}(x) + \sum_{k\in\mathbb{Z}\setminus\mathbb{N}} 1_{E_k^2 \vartriangle E_k^1}(x) \quad \text{a.e.}$$

$$\left|E\left(\rho^2(x)\right) - E\left(\rho^1(x)\right)\right| \leq \sum_{k\in\mathbb{Z}} 1_{E_k^2 \vartriangle E_k^1}(x) \quad \text{a.e.} \tag{22}$$

where $E_k^2 \vartriangle E_k^1 = \left(E_k^2 \setminus E_k^1\right) \cup \left(E_k^1 \setminus E_k^2\right) = \left(E_k^2\right)^c \vartriangle \left(E_k^1\right)^c$. By Lemma 3.3, $\left|E_k^2 \vartriangle E_k^1\right| \leq \dfrac{2\Lambda}{b}$. Then, we estimate for every $x \in \mathbb{R}$ with $I(x) = (x - \frac{1}{2}, x + \frac{1}{2})$.

$$\begin{aligned} \left|E\left(\rho^2\right) - E\left(\rho^1\right)\right|_{L^1(I(x))} &\leq \sum_{k\in\mathbb{Z}} \left|I(x) \cap \left(E_k^2 \vartriangle E_k^1\right)\right| \\ &\leq N' \, \frac{2\Lambda}{b} \end{aligned} \tag{23}$$

where

$$N' = Card\left\{k \in \mathbb{Z},\ \left|I(x) \cap \left(E_k^2 \vartriangle E_k^1\right)\right| \neq 0\right\}.$$

Let us assume that there exists $k \in \mathbb{Z}$ such that $\left|I(x) \cap \left(E_k^2 \vartriangle E_k^1\right)\right| \neq 0$. Then there exists $x^1 \in I(x)$ such either $x^1 \in \left(E_k^1 \setminus E_k^2\right)$ or $x^1 \in \left(E_k^2 \setminus E_k^1\right)$. In the second case ($x^1 \in \left(E_k^2 \setminus E_k^1\right)$), one can check easily on Figure 2 that the number of k is less than $P + \left|\rho^2 - \rho^1\right|_{L^\infty(\mathbb{R})}$. Therefore

$$N' \leq P + \left|\rho^2 - \rho^1\right|_{L^\infty(\mathbb{R})}. \tag{24}$$

Taking the supremum on $x \in \mathbb{R}$, we get

$$\left|E\left(\rho^2\right) - E\left(\rho^1\right)\right|_{L^1_{\mathrm{unif}}(\mathbb{R})} \leq \frac{2}{b}\left(P + \left|\rho^2 - \rho^1\right|_{L^\infty(\mathbb{R})}\right)\left|\rho^2 - \rho^1\right|_{L^\infty(\mathbb{R})}.$$

□

We recall the following result (we refer to [5] for a proof).

Lemma 3.4 (Norm of the product of convolution)
For every $f \in L^1_{unif}(\mathbb{R})$ and $g \in L^\infty_{int}(\mathbb{R})$, the convolution product $f \star g$ is bounded and satisfies

$$|f \star g|_{L^\infty(\mathbb{R})} \leq |f|_{L^1_{unif}(\mathbb{R})}\, |g|_{L^\infty_{int}(\mathbb{R})}. \tag{25}$$

We now present some properties of the non-local velocity.

Lemma 3.5 (Properties of the non-local velocity)
Recall that $c^{\mathrm{int}}[u](x,t) = c^0 \star (E(u(\cdot,t)) - P\,\cdot)(x)$ is a convolution on $\mathbb{R}$. We assume that c^0 is a kernel in $W^{1,1}(\mathbb{R}) \cap L^\infty_{int}(\mathbb{R})$ satisfying $\displaystyle\int_{\mathbb{R}} c^0(x)\,dx = 0$. Then we have the following properties:

1. *the convolution c^{int} is well defined if $u_x \geq 0$ a.e. and if $u(x+1,t) = u(x,t) + P$ with $P \in \mathbb{N} \setminus \{0\}$;*
2. *Moreover, the function c^{int} is 1-periodic in space, i.e. $c^{\mathrm{int}}[u](x+1,t) = c^{\mathrm{int}}[u](x,t)$. We also have $c^{\mathrm{int}} \in L^\infty\left((0,T), W^{1,\infty}(\mathbb{R})\right)$, i.e. more precisely*

$$\left|c^{\mathrm{int}}[u](\cdot,t)\right|_{L^\infty(\mathbb{R})} \leq P\left|c^0\right|_{L^1(\mathbb{R})} \quad \text{and} \quad \left|c^{\mathrm{int}}_x[u](\cdot,t)\right|_{L^\infty(\mathbb{R})} \leq P\left|(c^0)_x\right|_{L^1(\mathbb{R})};$$

3. *if there exists $A > 0$ such that $|u(x,t) - u(x,s)| \leq A\,|t-s|$ for a.e. $t,s \in (0,T)$ and $u_x \geq b$ a.e. then c^{int} is Lipschitz continuous in time with Lipschitz constant $\dfrac{4AP}{b}\left|c^0\right|_{L^\infty_{int}(\mathbb{R})}$ i.e.*

$$\left|c^{\mathrm{int}}(x,t) - c^{\mathrm{int}}(x,s)\right| \leq \frac{4AP}{b}\left|c^0\right|_{L^\infty_{int}(\mathbb{R})}|t-s|.$$

Proof of Lemma 3.5

1. From $u(x+1,t) = u(x,t)+P$ we deduce that $E\left(u(x+1,t)\right) = E\left(u(x,t)\right)+P$ and $E\left(u(x+1,t)\right) - P(x+1) = E\left(u(x,t)\right) - Px$. Since $\displaystyle\int_{\mathbb{R}} c^0(x)\,dx = 0$, $\left(c^0 \star P\right)(x) = 0$ and then $c^0 \star \left(E\left(u(\cdot,t)\right) - P\,\cdot\right)(x+1) = c^0 \star \left(E\left(u(\cdot,t)\right) - P\,\cdot\right)(x)$. Point 1 is therefore proved.
2. Since $u(x+1,t) = u(x,t) + P$ and $u_x \geq 0$ for a.e. $(x,t) \in \mathbb{R} \times [0,+\infty)$, we have

$$u(0,t) \leq u(x,t) \leq u(1,t) = u(0,t) + P \quad \text{for all } (x,t) \in [0,1[\times[0,+\infty).$$

Passing to the floor part, for $x \in [0,1[$ we obtain

$$E\left(u(0,t)\right) \leq E\left(u(x,t)\right) \leq E\left(u(0,t)\right) + P\,.$$

Then

$$-P \leq 0 \leq E\left(u(x,t)\right) - E\left(u(0,t)\right) \leq P$$

and then $\left|E\left(u(x,t)\right) - E\left(u(0,t)\right)\right| \leq P$ for every $x \in [0,1[$ and every $t \geq 0$. Remark first that $c^0 \star E(u(\cdot,t))(x) = c^0 \star \left(\, E\left(u(\cdot,t)\,\right) - E\left(u(0,t)\,\right)\,\right)(x)$ because $\displaystyle\int_{\mathbb{R}} c^0(x)\,dx = 0$. Then

$$\begin{aligned}
&c^0 \star \left(E(u(\cdot,t)) - P\,\cdot\right)(x) = \int_{\mathbb{R}} dy\; c^0(x-y)\left(E\left(u(y,t)\right) - Py - E\left(u(0,t)\right)\right)\\
&= \sum_{k\in\mathbb{Z}} \int_k^{k+1} dy\; c^0(x-y)\left(E\left(u(y,t)\right) - Py - E\left(u(0,t)\right)\right)\\
&= \sum_{k\in\mathbb{Z}} \int_0^{1} dy\; c^0(x-y-k)\left(E\left(u(y+k,t)\right) - P(y+k) - E\left(u(0,t)\right)\right)\\
&= \sum_{k\in\mathbb{Z}} \int_0^{1} dy\; c^0(x-y-k)\left(E\left(u(y,t)\right) - Py - E\left(u(0,t)\right)\right)\,.
\end{aligned}$$

Since $E\left(u(y,t)\right) - E\left(u(0,t)\right) \leq P$ for $y \in [0,1[$, we deduce

$$\begin{aligned}
c^0 \star \left(E(u(\cdot,t)) - P\,\cdot\right)(x) \quad &\leq \sum_{k\in\mathbb{Z}} \int_0^1 dy\; P(1-y)c^0(x-y-k)\\
&\leq P\int_{\mathbb{R}} dy\; c^0(x-y) = P\int_{\mathbb{R}} dy'c^0(y')\\
&\leq P\left|c^0\right|_{L^1(\mathbb{R})}\,.
\end{aligned}$$

where for the last inequality we have used that $c^0(-x) = c^0(x)$ for all $x \in \mathbb{R}$ and $\displaystyle\int_{\mathbb{R}} c^0(x)\,dx = 0$. We now show that c_x^{int} is bounded on $\mathbb{R}\times(0,T)$. Indeed, $c_x^{\mathrm{int}} = (c^0)_x \star E(u)$. Similarly, we get

$$\left|c_x^{\mathrm{int}}\right|_{L^\infty(\mathbb{R})} \leq P\left|(c^0)_x\right|_{L^1(\mathbb{R})}\,.$$

3. We now prove the Lipschitz continuity in time of c^{int}. Let $x \in \mathbb{R}$, $0 < t, s < T$. Then we have

$$\begin{aligned}
\left|c^{\mathrm{int}}[u](x,t) - c^{\mathrm{int}}[u](x,s)\right| \quad &= \left|c^0 \star \left(E\left(u(\cdot,t)\right) - E\left(u(\cdot,s)\right)\right)(x)\right|\\
&\leq \left|c^0\right|_{L^\infty_{\mathrm{int}}(\mathbb{R})}\left|E\left(u(\cdot,t)\right) - E\left(u(\cdot,s)\right)\right|_{L^1_{\mathrm{unif}}(\mathbb{R})}\\
&\leq \frac{4P}{b}\left|c^0\right|_{L^\infty_{\mathrm{int}}(\mathbb{R})}\left|u(\cdot,t) - u(\cdot,s)\right|_{L^\infty(\mathbb{R})}\\
&\leq \frac{4AP}{b}\left|c^0\right|_{L^\infty_{\mathrm{int}}(\mathbb{R})}\left|t-s\right|\,,
\end{aligned}$$

where we have used successively Lemma 3.4, Proposition 3.1 (see Remark 3.2) and the Lipschitz continuity of u we assumed to hold.

□

4 Proof of Theorem 1.3

We prove Theorem 1.3 in two main steps. In a first step (see subsection 4.1), we prove existence and uniqueness for short time, using a fixed point theorem. In a second step (see subsection 4.2), we extend the result for all time, by repeating the argument on successive time intervals. We need to recall Lemma 2.8 of Barles [7].

Lemma 4.1 *Let H be a continuous Hamiltonian. If $u \in C(\bar{\Omega}\times[0,T])$ is a subsolution (respectively supersolution) of the problem*

$$\frac{\partial u}{\partial t} + H(x,t,Du) = 0 \quad in \quad \Omega\times(0,T) \tag{26}$$

then u is a subsolution (respectively supersolution) of the problem

$$\frac{\partial u}{\partial t} + H(x,t,\frac{\partial u}{\partial x}) = 0 \quad in \quad \Omega\times(0,T]. \tag{27}$$

This lemma will be applied for $H(x,t,\frac{\partial u}{\partial x}) = c[u](x,t)\left|\frac{\partial u}{\partial x}\right|$ where u is a solution on $(0,T)$, $u \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}\times[0,T])$ and $c[u]\in C(\mathbb{R}\times[0,T])$.

4.1 Short time existence and uniqueness of the solution

For c^{ext} satisfying (3) and c^0 satisfying (5), we denote

$$K = \left|c^{\mathrm{ext}}\right|_{L^\infty(\mathbb{R})} + P\left|c^0\right|_{L^1(\mathbb{R})}. \tag{28}$$

Consider four constants satisfying $0<b_1<b_0<B_0<B_1$ and for $T>0$, we set

$$X_T = \left\{ u\in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}\times[0,T)) \;\middle|\; \begin{array}{ll} u(x+1,t)=u(x,t)+P & \text{for } (x,t)\in\mathbb{R}\times[0,T), \\ 0<b_1\le u_x\le B_1 & \text{a.e. on } \mathbb{R}\times[0,T), \\ |u_t|\le KB_1 & \text{a.e. on } \mathbb{R}\times[0,T) \end{array} \right\}.$$

Clearly, $X_T - Px$ is a closed set of the Banach space $W^{1,\infty}(\mathbb{R}\times[0,T))$. We want to establish that there exists a unique solution $u\in X_T$ of the following problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \left(c^{\mathrm{ext}}(x) + c^0\star\left(E\left(u(\cdot,t)\right)-P\,\cdot\right)(x)\right)\dfrac{\partial u}{\partial x}(x,t) & \text{in } \mathbb{R}\times(0,T) \\ u(x,0)=u^0(x) \quad \text{on } \mathbb{R}, \end{cases} \tag{29}$$

where u^0 satisfies Assumptions (6). For any $u\in X_T$ such that $u(x,0)=u^0(x)$, we consider the continuous viscosity solution v of the following problem

$$\begin{cases} \dfrac{\partial v}{\partial t}(x,t) &= \left(c^{\mathrm{ext}}(x) + c^0\star\left(E\left(u(\cdot,t)\right)-P\,\cdot\right)(x)\right)\dfrac{\partial v}{\partial x}(x,t) \quad \text{on} \quad \mathbb{R}\times(0,T), \\ v(x,0) &= u^0(x) \quad \text{on} \quad \mathbb{R}. \end{cases} \tag{30}$$

The main idea, in this section, is to show that the map

$$\begin{array}{rl} \varphi: & X_T \longrightarrow X_T \\ & u \longmapsto \varphi(u)=v \quad \text{viscosity solution of (30)} \end{array}$$

is well defined and has a unique fixed point.
We will first show that φ is well defined for T small enough, and then show that φ is a contraction. Let us define

$$L = \left|c^{\mathrm{ext}}_x\right|_{L^\infty(\mathbb{R})} + P\left|c^0_x\right|_{L^1(\mathbb{R})} \quad \text{and} \quad T^* = \frac{1}{L}\min\left(\ln\left(\frac{B_1}{B_0}\right), \ln\left(\frac{b_0}{b_1}\right)\right). \tag{31}$$

1) $\varphi(X_T) \subset X_T$ **for** $0 < T \leq T^*$**.** We first remark that the solution v of (30) is given by Proposition 2.2. Indeed this proposition applies because our initial condition satisfies its assumptions and the velocity $c(x,t) = c^{\text{ext}}(x) + c^0 \star E(u(\cdot,t))(x)$ is in $W^{1,\infty}(\mathbb{R} \times [0,T])$ by Lemma 3.5 and the definition of X_T.

We will now check that $v \in X_T$ for T small enough. By Lemma 3.5, assertion 2), we know that $\left|c^{\text{int}}\right|_{L^\infty(\mathbb{R})} \leq P \left|c^0\right|_{L^1(\mathbb{R})}$ and $\left|c_x^{\text{int}}\right|_{L^\infty(\mathbb{R})} \leq P \left|(c^0)_x\right|_{L^1(\mathbb{R})}$. Therefore

$$|c| \leq K = \left|c^{\text{ext}}\right|_{L^\infty(\mathbb{R})} + P \left|c^0\right|_{L^1(\mathbb{R})}$$

and

$$|c_x| \leq L = \left|c_x^{\text{ext}}\right|_{L^\infty(\mathbb{R})} + P \left|c_x^0\right|_{L^1(\mathbb{R})} .$$

By the *a priori* estimates for the eikonal equation (Proposition 2.2), we see that the function v satisfies for a.e. $(x,t) \in \mathbb{R} \times [0,T)$

$$\begin{cases} |v_x(x,t)| \leq B_0 e^{Lt} = B(t), \\ v_x(x,t) \geq b_0 e^{-Lt} = b(t), \\ |v_t(x,t)| \leq |c|_{L^\infty(\mathbb{R}\times[0,T))} B(t) \end{cases}$$

and we have $B(T^*) \leq B_1$ and $b(T^*) \geq b_1$ with the definition of T^* in (31).

By Lemma 3.5, assertion 2), we know that $c(x+1,t) = c(x,t)$. Let $w(x,t) = v(x+1,t) - P$. Then $w(x,0) = u^0(x+1) - P = u^0(x) = v(x,0)$. Then by the space periodicity of the velocity c and the fact that the eikonal equation "does not see the constants", we deduce that w is still a viscosity solution of (30). By the uniqueness of the solution we get that $w(x,t) = v(x,t)$, and therefore $v(x+1,t) = v(x,t) + P$. We deduce that $v \in X_T$ if $T \leq T^*$.

2) φ **has a unique fixed point.** Let us define T_0 by

$$T_0 = \min\left(\frac{1}{\left|c_x^{\text{ext}}\right|_{L^\infty(\mathbb{R})} + P\left|(c^0)_x\right|_{L^1(\mathbb{R})}}, \ \frac{1}{8P\left|c^0\right|_{L^\infty_{\text{int}}(\mathbb{R})}}\right) \min\left(\ln\frac{b_0}{b_1}, \ \frac{b_1}{B_1}\right). \qquad (32)$$

Indeed, the following proposition shows that φ is a contraction.

Proposition 4.2 (Contraction)
Let $v^i = \varphi(u^i)$ for $i = 1,2$. If $u^i \in X_T$ for $i = 1,2$, and if $\left|u^2 - u^1\right|_{L^\infty(\mathbb{R}\times[0,T))} \leq P$, then

$$\left|v^2 - v^1\right|_{L^\infty(\mathbb{R}\times[0,T))} \leq \frac{1}{2}\left|u^2 - u^1\right|_{L^\infty(\mathbb{R}\times[0,T))} \quad \textit{for all} \quad T \in [0, T_0].$$

A corollary of this contraction property is

Proposition 4.3 (Short time existence and uniqueness of the solution)
We assume that c^{ext} and c^0 satisfy (3) and (5) and that u^0 satisfies (6). There then exists a unique continuous viscosity solution $u \in X_{T_0}$ of (29).

To finish this subsection, we will first prove Proposition 4.3 and then prove Proposition 4.2.

Proof of Proposition 4.3
Note that we can write

$$[0, T_0] = \bigcup_{k=0,\dots,N-1} \left[\frac{kT_0}{N}, \frac{(k+1)T_0}{N}\right]$$

where N will be large enough and fixed later. Let us denote $\tau_k = \left[\frac{kT_0}{N}, \frac{(k+1)T_0}{N}\right]$ for $k \in \{0, \dots, N-1\}$.

Step 1: Let $u^1, u^2 \in X_{T_0}$ such that $u^1(x,0) = u^2(x,0) = u^0(x)$. For all $t \in \tau_0$, and for all $x \in \mathbb{R}$, we compute

$$\begin{aligned} \left|u^2(x,t) - u^1(x,t)\right| \leq & \left|u^2(x,t) - u^2(x,0)\right| + \left|u^1(x,0) - u^1(x,t)\right| \\ \leq & \; 2KB_1\,|t| \\ \leq & \; 2KB_1 \frac{T_0}{N} \\ \leq & \; 1 \leq P \end{aligned} \tag{33}$$

if we choose $N \geq 2KB_1T_0$. Then Proposition 4.2 holds, *i.e.* φ is a contraction on $X_{\frac{T_0}{N}}$. Since $X_{T_0} - Px$ is a closed subset of a Banach space then by the Banach-Picard fixed point theorem, there exists a unique solution $u \in X_{\frac{T_0}{N}}$ such that $u = \varphi(u)$, *i.e.* u is a solution of (29) on $\tau_0 = [0, \dfrac{T_0}{N}]$.

Step 2: First we remark that the solution $u \in X_{\frac{T_0}{N}}$ belongs to $W^{1,\infty}_{\text{loc}}\left(\mathbb{R} \times [0, \frac{T_0}{N}]\right)$ by the *a priori* bounds on u_x and u_t defining $X_{\frac{T_0}{N}}$.

Second, we then apply again Step 1 with the new initial condition $u^0(\cdot) := u\left(\cdot, \frac{T_0}{N}\right)$ and get a solution $v \in X_{T_0}$. We then define

$$u(\cdot,t) = v\left(\cdot, t - \frac{T_0}{N}\right) \quad \text{for} \quad t \in \tau_1.$$

Third by construction, u is a viscosity solution on $\left(0, \dfrac{T_0}{N}\right) \cup \left(\dfrac{T_0}{N}, \dfrac{2T_0}{N}\right)$ and by Lemma 4.1, we see that it also satisfies the viscosity inequalities at time $\dfrac{T_0}{N}$, and therefore u is a viscosity solution of (29) on $\left(0, \dfrac{2T_0}{N}\right)$.

Step 3: We repeat the previous argument on the time intervals τ_k, $k = 2, \ldots, N$, and get the existence of a viscosity solution u of (29) on the time interval $(0, T_0)$.

Step 4: Uniqueness. Let us assume that we have two solutions u^1 and u^2 of (29) on $(0, T_0)$, with $u^1 \neq u^2$ and let us define $T_0^* < T_0$ such that $u^1 = u^2$ on $[0, T_0^*]$ and

$$\forall\, \delta > 0, \exists\, t_\delta \in [T_0^*, T_0^* + \delta] \cap [T_0^*, T_0] \quad \text{such that} \quad u^2(\cdot, t_\delta) \neq u^1(\cdot, t_\delta).$$

Applying again Step 1 with initial condition $u^0(\cdot) := u^1(\cdot, T_0^*) = u^2(\cdot, T_0^*)$ we get by the contraction property that $(u^i = \varphi(u^i),\ i = 1,2)$

$$\left|u^2 - u^1\right|_{L^\infty\left(\mathbb{R}\times[0,T_0^*+\delta)\right)} \leq \frac{1}{2}\left|u^2 - u^1\right|_{L^\infty\left(\mathbb{R}\times[0,T_0^*+\delta)\right)}$$

for $\delta \leq \dfrac{T_0}{N}$ and $T_0^* + \delta \leq T_0$ (using (33)). Contradiction.

□

Proof of Proposition 4.2

Let $v^i = \varphi(u^i)$ for $i = 1,2$. By the stability result (Proposition 2.3), we have

$$\left|v^1 - v^2\right|_{L^\infty(\mathbb{R}\times[0,T])} \leq \int_0^T \left|c^2(\cdot,s) - c^1(\cdot,s)\right|_{L^\infty(\mathbb{R})}\ \max\left(\left|v^1_x(\cdot,s)\right|_{L^\infty(\mathbb{R})}, \left|v^2_x(\cdot,s)\right|_{L^\infty(\mathbb{R})}\right) ds$$

where $c^i(x,t) = c^{\text{ext}}(x) + \left(c^0 \star E(u^i(\cdot,t))\right)(x)$. By Lemma 3.4, we have

$$\left|c^2 - c^1\right|_{L^\infty(\mathbb{R}\times[0,T))} \le \left|c^0\right|_{L^\infty_{\text{int}}(\mathbb{R})} \left|E(u^2) - E(u^1)\right|_{L^\infty([0,T),\, L^1_{\text{unif}}(\mathbb{R}))}.$$

By Proposition 3.1 and Remark 3.2, we know that

$$\left|E(u^2) - E(u^1)\right|_{L^\infty([0,T),\, L^1_{\text{unif}}(\mathbb{R}))} \le \frac{4P}{b} \left|u^2 - u^1\right|_{L^\infty(\mathbb{R}\times[0,T))}$$

then combining Proposition 4.2 and Lemma 3.4, we obtain:

$$\left|v^1 - v^2\right|_{L^\infty(\mathbb{R}\times[0,T])} \le 4\,P\,T\,\frac{B}{b}\,\left|c^0\right|_{L^\infty_{\text{int}}(\mathbb{R})} \left|u^2 - u^1\right|_{L^\infty(\mathbb{R}\times[0,T))}.$$

We set $T^{**} = \dfrac{1}{8\,P\,|c^0|_{L^\infty_{\text{int}}(\mathbb{R})}}\,\dfrac{b}{B}$. For $T_0 = \inf(T^*, T^{**})$, the following holds for $T \le T_0$

$$\left|v^2 - v^1\right|_{L^\infty(\mathbb{R}\times[0,T))} \le \frac{1}{2}\left|u^2 - u^1\right|_{L^\infty(\mathbb{R}\times[0,T))}.$$

□

4.2 Long time existence of the viscosity solution: proof of Theorem 1.3

Proof of Theorem 1.3

We will repeat this short time result on a sequence of time intervals of lengths T_n decreasing to zero, such that $\sum_{n\in\mathbb{N}} T_n = +\infty$. We will do the proof in 3 steps.

Step 1: We rephrase the result of Proposition 4.3. We proved in the previous subsection that given an initial data u^0 such that

$$0 < b_0 \le u^0_x \le B_0$$

and b_1, B_1 (which will be specified later) such that

$$0 < b_1 < b_0 < B_0 < B_1 \quad \text{and} \quad \frac{b_0}{b_1} = \frac{B_1}{B_0},$$

there exists a unique viscosity solution u of Problem (29) up to time T_0 satisfying

$$0 < b_1 \le u_x \le B_1 \quad \text{on} \quad \mathbb{R}\times[0,T_0),$$

where T_0 is defined by

$$T_0 = A\ \min\left\{\ln\mu_0, \frac{1}{\mu_0^2}\bar{b}_0\right\} \tag{34}$$

where

$$\begin{cases} A = \min\left\{\dfrac{1}{|c^{\text{ext}}_x|_{L^\infty(\mathbb{R})} + P\,|c^0_x|_{L^1(\mathbb{R})}}, \dfrac{1}{8\,P\,|c^0|_{L^\infty_{\text{int}}(\mathbb{R})}}\right\}, \\ \mu_0 = \dfrac{b_0}{b_1} = \dfrac{B_1}{B_0} > 1 \quad \text{and} \\ \bar{b}_0 = \dfrac{b_0}{B_0} < 1 \end{cases}$$

For given b_0 and B_0, in order to equalize the two terms in the infimum of (34), we choose μ_0 such that $\ln\mu_0 = \dfrac{1}{\mu_0^2}\bar{b}_0$, in other words, μ_0 is fixed by the relation

$$\bar{b}_0 = \mu_0^2 \ln\mu_0$$

and it determinates b_1 and B_1 as a function of b_0, B_0. Therefore we have

$$T_0 = A \ \ln \mu_0, \tag{35}$$

Step 2: Definition of the recurrence. We apply successively this reasoning on time intervals of length T_n which will be specified below. So, for $n \geq 1$, for b_{n+1}, B_{n+1} (which will be specified later) there exists a unique solution of the problem (29) up to time $T_0 + T_1 + \cdots + T_n$, where

$$\mu_n = \frac{b_n}{b_{n+1}} = \frac{B_{n+1}}{B_n} > 1 \quad \text{for} \quad 0 < b_{n+1} < b_n < B_n < B_{n+1}, \tag{36}$$

$$\bar{b}_n = \frac{b_n}{B_n} < 1 \quad \text{and} \quad \mu_n \quad \text{is fixed by} \quad \bar{b}_n = \mu_n^2 \ln \mu_n. \tag{37}$$

$$T_n = A \ \ln \mu_n \tag{38}$$

Step 3: Divergence of the series $\sum_{n\in\mathbb{N}} T_n$.

Proposition 4.4 *With previous notations and the choice of the sequence $(\mu_n)_n$, the series $\sum_{n\in\mathbb{N}} T_n$ diverges.*

This ends the proof of Theorem 1.3.

$\square$

In the rest of this subsection, we will prove Proposition 4.4. Before proving this proposition we need preliminary results. First, we remark by (36) that

$$\bar{b}_{n+1} = \frac{b_{n+1}}{B_{n+1}} = \frac{1}{\mu_n^2} \bar{b}_n$$

and then by (37) and (38) we get

$$\ln \mu_n = \mu_{n+1}^2 \ln \mu_{n+1} \quad \text{and} \quad T_n = A \ln \mu_n. \tag{39}$$

The recurrence relation defining the sequence $(\mu_n)_n$ can be inverted as $\mu_{n+1} = G(\mu_n)$ with $\mu_n > 1$. Introducing

$$\varepsilon_n = \mu_n - 1 > 0\,,$$

we can rewrite $\mu_{n+1} = G(\mu_n)$ as

$$\varepsilon_{n+1} = F(\varepsilon_n) \tag{40}$$

where by the implicit function theorem $F \in C^2\left(\left(-\frac{1}{2e}, +\infty\right); \mathbb{R}\right)$ and satisfies

$$F(0) = 0,\ F'(0) = 1,\ F''(0) = -4, \text{ and } F' > 0. \tag{41}$$

We have the following lemma

Lemma 4.5 (Subsolution for the sequence)
Let $F \in C^2([0, +\infty); \mathbb{R})$ satisfying (41). For $a \geq 0$, let $\sigma(a) = \int_0^1 dt\, (F''(ta) - F''(0))\,(1-t)$ and $a_0 = \sup\left\{a \geq 0 : \inf_{[0,a]} \sigma \geq -2\right\}$. Then $a_0 > 0$. Let $\rho_a(t)$ be the solution of

$$\begin{cases} \rho_a'(t) = -4\rho_a^2(t), \\ \rho_a(0) = a\,. \end{cases} \tag{42}$$

If $a \in [0, a_0]$ then for all $t \geq 0$, we have

$$\rho_a(t+1) \leq F(\rho_a(t))\ . \tag{43}$$

Corollary 4.6 (A lower bound on the sequence $(\varepsilon_n)_n$)
Under the assumptions of Lemma 4.5, we consider a sequence $(\varepsilon_n)_n > 0$ satisfying $\varepsilon_{n+1} = F(\varepsilon_n)$. If for an integer k, we have $0 < \varepsilon_k \leq a_0$, then for all $n \geq 0$ we have

$$\varepsilon_{k+n} \geq \rho(n)$$

where $\rho = \rho_{\varepsilon_k}$.

Proof of corollary 4.6
Since ρ is decreasing in t ($\rho'(t) < 0$), $\rho(n) \leq \rho(0) = \varepsilon_k$. Applying (43) and using the fact that F is increasing, we get

$$\rho(n) \leq F(\rho(n-1)) \leq \cdots \leq F^n(\rho(0)) = F^n(\varepsilon_k) = \varepsilon_{k+n}.$$

□

Proof of Lemma 4.5
We set $\phi(t) = F(ta)$. Using the Taylor formula with integral remainder, we have

$$\begin{aligned}\phi(1) &= \phi(0) + \phi'(0) + \int_0^1 dt\ \phi''(t)(1-t) \\ &= \phi(0) + \phi'(0) + \frac{\phi''(0)}{2} + \int_0^1 dt\ (\phi''(t) - \phi''(0))(1-t)\end{aligned}$$

with $\phi''(t) = F''(ta)a^2$. Then setting $\sigma(a) = \displaystyle\int_0^1 (F''(ta) - F''(0))(1-t)\,dt$, we get $\sigma \in C^0([0,+\infty),\mathbb{R})$ and $F(a) = a - 2a^2 + a^2\sigma(a)$. Thus,

$$\begin{aligned}F(\rho(t)) - \rho(t+1) &= \rho(t) - 2\rho^2(t) + \rho^2(t)\sigma(\rho(t)) - \rho(t+1) \\ &\geq \rho(t) - \rho(t+1) - 4\rho^2(t)\end{aligned}$$

because $\rho(t) \in [0, a_0]$ from the assumption of the lemma (and the fact that ρ is decreasing in t), which guaranties $\sigma(\rho(t)) \geq -2$. We now estimate

$$\begin{aligned}\rho(t) - \rho(t+1) &= \int_t^{t+1} -\rho'(s)\,ds \\ &= \int_t^{t+1} 4\rho^2(s)\,ds \quad \geq 4\rho^2(t)\,.\end{aligned}$$

We deduce that $F(\rho(t)) - \rho(t+1) \geq 0$. □

Proof of Proposition 4.4
Let us first remark that $\displaystyle\sum_{n\geq 0} T_n = \sum_{n\geq 0} \ln \mu_n = \ln \prod_{n\geq 0}(1+\varepsilon_n) \geq \ln\left(1 + \sum_{n\geq 0}\varepsilon_n\right)$. We will now show that $\displaystyle\sum_{n\geq 0}\varepsilon_n$ diverges. If it is not the case, then $\varepsilon_k \longrightarrow 0$ when $k \to \infty$ and so for k large enough we have $\varepsilon_k \leq a_0$. Therefore by Corollary 4.6, we know that $\varepsilon_{k+n} \geq \rho(n-1)$ for all $n \in \mathbb{N}$ and $\rho(t) = \dfrac{1}{\frac{1}{a} + 4t}$. We deduce that

$$\sum_{n\geq 0}\varepsilon_n \geq \sum_{n\geq 1}\rho(n) \geq \int_1^{+\infty}\rho(t)\,dt = \int_1^{+\infty}\frac{1}{\frac{1}{a}+4t}\,dt = +\infty\,.$$

Then the series $\displaystyle\sum_{n\geq 0}\varepsilon_n$ diverges and $\displaystyle\sum_{n\geq 0} T_n$ also diverges. □

5 Preliminary results for the discrete local problem

As explained in Subsection 1.2, we construct a numerical scheme for the non-local equation by discretising explicitely the time variable by an Euler scheme and the space variable by an upwind scheme. We first study the case of a local equation whose gradient satisfies $\dfrac{\partial u}{\partial x} \geq 0$. This leads to study the following local transport equation:

$$\left\{\begin{array}{llll} \dfrac{\partial u}{\partial t}(x,t) & = c(x,t)\, \dfrac{\partial u}{\partial x}(x,t) & \text{in} & \mathbb{R}\times(0,T)\,, \\ u(x,0) & = u_0(x) & \text{on} & \mathbb{R}\,. \end{array}\right. \tag{44}$$

Given a mesh size Δx, Δt and a lattice $I_d = \left\{(i\Delta x, n\Delta t);\ i\in\mathbb{Z},\ n\leq \dfrac{T}{\Delta t}\right\}$, (x_i, t_n) denotes the node $(i\Delta x, n\Delta t)$ and $v^n = (v_i^n)_i$ the values of the numerical approximation of the continuous solution $u(x_i, t_n)$. We consider an explicite Euler scheme in time, *i.e.*

$$v_i^{n+1} = v_i^n + \Delta t\, H_d\left(v^n, i\right) \tag{45}$$

where the discrete Hamiltonian is chosen so that the scheme is upwind; precisely we choose

$$H_d\left(v^n, i\right) = \left\{\begin{array}{ll} c_i^n D_x^+ v_i^n & \text{if } c_i^n \geq 0 \\ c_i^n D_x^- v_i^n & \text{if } c_i^n < 0 \end{array}\right.$$

with

$$D_x^+ v_i^n = \frac{v_{i+1}^n - v_i^n}{\Delta x}, \quad D_x^- v_i^n = \frac{v_i^n - v_{i-1}^n}{\Delta x}$$

and c_i^n is the discrete velocity.
We assume the following CFL condition for the local problem

$$\Delta t \leq \frac{\Delta x}{\sup\limits_{i,n} |c_i^n|} \tag{46}$$

For the reader's convenience, we recall some useful results proved in [2, 3]. We first recall a discrete gradient estimate from above whose proof is given in [2].

Lemma 5.1 (Discrete gradient estimate from above)
If for some $B^0 > 0$ we have $\left|\dfrac{v_{i+1}^0 - v_i^0}{\Delta x}\right| \leq B^0$, $\forall\, i\in\mathbb{Z}$ and $B^{n+1} = B^n\left(1 + 2\Delta t \sup\limits_{j\in\mathbb{Z}}\left|\dfrac{c_{j+1}^n - c_j^n}{\Delta x}\right|\right)$ then

$$\left|\frac{v_{i+1}^n - v_i^n}{\Delta x}\right| \leq B^n \quad \forall\, i\in\mathbb{Z},\ \forall\, n\in\mathbb{N}\,.$$

In the following, we also need a discrete gradient estimate from below.

Lemma 5.2 (Discrete gradient estimate from below)
If for some $b^0 > 0$ we have $\dfrac{v_{i+1}^0 - v_i^0}{\Delta x} \geq b^0$, $\forall\, i\in\mathbb{Z}$, and $b^{n+1} = b^n\left(1 - 2\Delta t \sup\limits_{j\in\mathbb{Z}}\left|\dfrac{c_{j+1}^n - c_j^n}{\Delta x}\right|\right)$ with

$$\Delta t < \frac{1}{2\sup\limits_{j\in\mathbb{Z}}\left|\dfrac{c_{j+1}^n - c_j^n}{\Delta x}\right|} \tag{47}$$

then

$$\frac{v_{i+1}^n - v_i^n}{\Delta x} \geq b^n\,,\ \forall\, i\in\mathbb{Z},\ \forall\, n\in\mathbb{N}\,.$$

Proof of Lemma 5.2

First, let us remark that $b^n \geq 0$ because of the condition $\Delta t < \dfrac{1}{2\sup\limits_{j\in\mathbb{Z}}\left|\dfrac{c_{j+1}^n - c_j^n}{\Delta x}\right|}$. Let $w_i^n = v_{i-1}^n + b^n \Delta x$. By assumption, we have $w_i^n \leq v_i^n$, $\forall\, i \in \mathbb{Z}$. In order to show that $w_i^{n+1} \leq v_i^{n+1}$ for all $i \in \mathbb{Z}$, we check that w^n is a discrete subsolution *i.e.* $w_i^{n+1} - (w_i^n + \Delta t H_d(w^n, i)) \leq 0$. Indeed,

$$\begin{aligned}
&w_i^{n+1} - (w_i^n + \Delta t H_d(w^n, i)) \\
&= v_{i-1}^{n+1} + b^{n+1}\Delta x - \left(v_{i-1}^n + b^n \Delta x + \Delta t H_d(w^n, i)\right) \\
&= \left(b^{n+1} - b^n\right)\Delta x + \Delta t\left(H_d(v^n, i-1) - H_d(w^n, i)\right) \\
&= \left(b^{n+1} - b^n\right)\Delta x + \Delta t\left(H_d(v^n, i-1) - H_d(v_{\cdot-1}^n, i)\right)
\end{aligned}$$

If c_i^n and c_{i-1}^n have the same sign, we assume that they are nonnegative (the proof is similar when they are nonpositive), then

$$\begin{aligned}
&w_i^{n+1} - (w_i^n + \Delta t H_d(v^n, i)) \\
&= -2b^n \Delta t\, \Delta x \sup_{j\in\mathbb{Z}}\left|\frac{c_j^n - c_{j-1}^n}{\Delta x}\right| + \Delta t\left(c_{i-1}^n - c_i^n\right) D_x^+ v_i^n \\
&= -2b^n \Delta t \sup_{j\in\mathbb{Z}}\left|c_j^n - c_{j-1}^n\right| - (c_i^n - c_{i-1}^n)\Delta t D_x^+ v_{i-1}^n \\
&\leq -b^n \Delta t\left(2\sup_{j\in\mathbb{Z}}\left|c_j^n - c_{j-1}^n\right| + c_i^n - c_{i-1}^n\right) \\
&\leq 0\,.
\end{aligned}$$

Therefore, w^n is a discrete subsolution and then $w_i^{n+1} \leq v_i^{n+1}$ for all $i \in \mathbb{Z}$. If c_i^n and c_{i-1}^n do not have the same sign (we refer the reader to the end of the proof of Lemma 5.1 in [3]) the conclusion prevails because of the following estimate for $a, b \geq 0$:

$$\begin{aligned}
\left|c_i^n a - c_{i-1}^n b\right| \leq &\quad \max(a,b)\max\left(\left|c_i^n\right|, \left|c_{i-1}^n\right|\right) \\
\leq &\quad \max(a,b)\left|c_i^n - c_{i-1}^n\right| \\
\leq &\quad \max(a,b)\left|\frac{c_i^n - c_{i-1}^n}{\Delta x}\right|\Delta x\,.
\end{aligned}$$

This achieves the proof of Lemma 5.2. □

We introduce the grid $I_d^T = \left\{(i\Delta x, n\Delta t);\ i \in \mathbb{Z},\ n \leq N_T = \dfrac{T}{\Delta t}\right\}$. We recall the following numerical stability result whose proof is given in [3, 2].

Proposition 5.3 (Numerical stability)
We consider $v^{1,n}$ and $v^{2,n}$ two numerical solutions of the following monotone scheme (with the same initial condition)

$$v_i^{l,n+1} = v_i^{l,n} + \Delta t\, H_d\left(v^{l,n}, i\right) \tag{48}$$

where

$$H_d\left(v^{l,n}, i\right) = \begin{cases} c_i^{l,n} D_x^+ v_i^{l,n} & \text{if } c_i^{l,n} \geq 0 \\ c_i^{l,n} D_x^- v_i^{l,n} & \text{if } c_i^{l,n} < 0 \end{cases}, \quad \text{for } l = 1, 2,\ \forall\, i \in \mathbb{Z},\ \forall\, n \in \mathbb{N}.$$

Then there exists a constant $C > 0$, depending on the discrete gradient estimates on v^1 and v^2, such that

$$\sup_{I_d^T}\left|v_i^{1,n+1} - v_i^{2,n+1}\right| \leq CT \sup_{I_d^T}\left|c_i^{1,n} - c^{2,n}\right|\,. \tag{49}$$

6 Preliminary result for the discrete non-local problem

We will prove the analogue of Proposition 3.1 in the framework of discrete solutions. We will use this result in Section 7.

Proposition 6.1 (Estimate of the difference of integer parts in the discrete case) *Consider a discrete function v^1 such that*

$$v^1_{i+K} = v^1_i + P \text{ where } P \in \mathbb{N}\setminus\{0\} \quad \text{and} \quad K = \frac{1}{\Delta x} \in \mathbb{N}\setminus\{0\}\,. \tag{50}$$

Assume that there exists two constants $B \geq b > 0$ such that for every $i \in \mathbb{Z}$ we have

$$b \leq \frac{v^1_{i+1} - v^1_i}{\Delta x} \leq B\,.$$

Then for all discrete function v^2 satisfying (50), we get

$$\sup_{i\in\mathbb{Z}} \sum_{j\in J_i=[i,i+K[} \left|E(v^2_j) - E(v^1_j)\right| \Delta x \leq 2\left(P + \sup_{i\in\mathbb{Z}}\left|v^2_i - v^1_i\right|\right)\left(\frac{1}{b}\sup_{i\in\mathbb{Z}}\left|v^2_i - v^1_i\right| + \Delta x\right)\,.$$

Remark 6.2 *Note that if $\sup_{i\in\mathbb{Z}}\left|v^2_i - v^1_i\right| \leq 1$ then*

$$\sup_{i\in\mathbb{Z}} \sum_{j\in J_i=[i,i+K[} \left|E(v^2_j) - E(v^1_j)\right| \Delta x \leq 4P\left(\frac{1}{b}\sup_{i\in\mathbb{Z}}\left|v^2_i - v^1_i\right| + \Delta x\right)\,.$$

This result is the discrete analogue of Proposition 3.1. This is also the generalization of Lemma 5.5 in [3] to the case of several dislocations where the characteristic function $v^l > 0$ is replaced with the floor part $E(v^l)$. For the proof of Proposition 6.1 we need to introduce the following notations. We denote $\Lambda' = \sup_{j\in\mathbb{Z}}\left|v^2_j - v^1_j\right|$ and we assume that $\Lambda' \in (0,+\infty)$ (other cases are trivial). For $m \in \mathbb{Z}$ and for $l = 1,2$, we denote $E^l_m = \left\{j \in \mathbb{Z} :\ v^l_j < m+1\right\}$. First, we remark that since $\frac{v^1_{i+1} - v^1_i}{\Delta x} \geq b > 0$, there exists the greatest integer $j_0 \in \mathbb{Z}$ such that $v^1_{j_0} < m+1$ and we have $E^1_m = \{j \in \mathbb{Z} :\ j \leq j_0\}$. We will use the following lemma for the proof of Proposition 6.1.

Lemma 6.3 (Estimate for the distance between the sets E^1_m and E^2_m) *Under the notations above and the assumptions of Proposition 6.1, we have*

$$E^1_m - E\left(\frac{\Lambda'}{b\Delta x}\right) - 1 \subset E^2_m \subset E^1_m + E\left(\frac{\Lambda'}{b\Delta x}\right) + 1\,.$$

Proof of Lemma 6.3

The main idea in this proof is to use the discrete gradient estimate from below. We will estimate in two steps the distance between E^1_m and E^2_m.

Step 1 We have $\mathbf{E^1_m - E\left(\frac{\Lambda'}{b\Delta x}\right) - 1 \subset E^2_m}$.

Indeed, let $j \in E^1_m - E\left(\frac{\Lambda'}{b\Delta x}\right) - 1$. Then, $j \leq j_0 - E\left(\frac{\Lambda'}{b\Delta x}\right) - 1$ *i.e.* $j_0 - j \geq E\left(\frac{\Lambda'}{b\Delta x}\right) + 1 > \frac{\Lambda'}{b\Delta x}$ *i.e.* $(j_0 - j)b\Delta x \geq \Lambda'$. Since $\frac{v^1_{j_0} - v^1_j}{(j_0 - j)\Delta x} \geq b$ and $j_0 - j > 0$, we have

$$v^1_{j_0} - v^1_j \geq (j_0 - j)b\Delta x > \Lambda'$$

which implies (by definition of Λ')

$$m+1 > v^1_{j_0} > v^1_j + \Lambda' \geq v^2_j$$

and therefore $v^2_j < m+1$. Thus, $j \in E^2_m$.

Step 2 We have $\mathbf{E^2_m \subset E^1_m + E\left(\dfrac{\Lambda'}{b\Delta x}\right) + 1}$.

Similarly, considering $j \in \left(E^1_m + E\left(\dfrac{\Lambda'}{b\Delta x}\right)+1\right)^c$ we prove that $j \in \left(E^2_m\right)^c$. Indeed, $j > j_0 + E\left(\dfrac{\Lambda'}{b\Delta x}\right)+1$ then $j \geq j_0+1+E\left(\dfrac{\Lambda'}{b\Delta x}\right)+1$ implies $j-(j_0+1) \geq E\left(\dfrac{\Lambda'}{b\Delta x}\right)+1 > \dfrac{\Lambda'}{b\Delta x}$ *i.e.* $(j-(j_0+1))b\Delta x > \Lambda'$. Since $\dfrac{v^1_j - v^1_{j_0+1}}{(j-(j_0+1))\Delta x} \geq b$ and $j-(j_0+1) > 0$, we have

$$v^1_j - v^1_{j_0+1} \geq (j-(j_0+1))b\Delta x > \Lambda'$$

which implies (by definition of Λ')

$$v^2_j \geq v^1_j - \Lambda' > v^1_{j_0+1} \geq m+1\,.$$

Then we have $j \in \left(E^2_m\right)^c$ and therefore $\left(E^1_m + E\left(\dfrac{\Lambda'}{b\Delta x}\right)+1\right)^c \subset \left(E^2_m\right)^c$.

□

Proof of Proposition 6.1

The main idea in this proof is to bound the quantity $\left|E\left(v^2_j\right) - E\left(v^1_j\right)\right|$ by the characteristic functions the sets $E^2_m \vartriangle E^1_m$. We then bound the discrete analogue of its L^1_{unif}-norm.

From the definition of E^l_m, for $l=1,2$, we remark that $E^l_{m-1} \subset E^l_m$. Then $E(v^l_j) = m$ for any $j \in E^l_m \setminus E^l_{m-1}$, for $l=1,2$. We define

$$1_A(j) = \begin{cases} 1 & \text{if } j \in A \subset \mathbb{Z}\,, \\ 0 & \text{if not.} \end{cases}$$

Then we can write

$$E(v^l_j) = \sum_{m\in\mathbb{N}} 1_{(E^l_m)^c}(j) - \sum_{m\in\mathbb{Z}\setminus\mathbb{N}} 1_{E^l_m}(j)\,.$$

Similarly to (22) in the proof of Proposition 3.1, we get

$$\left|E(v^2_j) - E(v^1_j)\right| \leq \sum_{m\in\mathbb{Z}} 1_{E^2_m \vartriangle E^1_m}(j)$$

Let us fix $i\in\mathbb{Z}$ and define $J_i = [i, i+K[$. Then the discrete analogue of L^1_{unif}-norm of $E(v^2)-E(v^1)$ satisfies

$$\begin{aligned}
\sum_{j\in J_i} \left|E(v^2_j) - E(v^1_j)\right| &\leq \sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} 1_{J_i}(j) 1_{(E^2_m \vartriangle E^1_m)}(j) \\
&\leq \sum_{m\in\mathbb{Z}} \left|J_i \cap \left(E^2_m \vartriangle E^1_m\right)\right| \\
&\leq 2N'\left(\frac{\Lambda'}{b\Delta x}+1\right)
\end{aligned}$$

where $N' = Card\left\{m \in \mathbb{Z},\ \left|J_i \cap \left(E^2_m \vartriangle E^1_m\right)\right| \neq 0\right\}$ and where we have used the fact that the measure $\left|E^2_m \vartriangle E^1_m\right| \leq 2\left(E\left(\frac{\Lambda'}{b\Delta x}\right)+1\right) \leq 2\left(\frac{\Lambda'}{b\Delta x}+1\right)$. Similarly to (24) in the proof of Proposition 3.1, we get

$$N' \leq P + \sup_{i\in\mathbb{Z}} \left|v^2_i - v^1_i\right| .$$

We then conclude that:

$$\sup_{i\in\mathbb{Z}} \sum_{j\in J_i} \left|E(v^2_j) - E(v^1_j)\right| \Delta x \leq 2\left(P + \sup_{i\in\mathbb{Z}}\left|v^2_i - v^1_i\right|\right)\left(\frac{1}{b}\sup_{i\in\mathbb{Z}}\left|v^2_i - v^1_i\right| + \Delta x\right).$$

□

7 Proof of Theorem 1.5

In this section, we first recall how to get an error estimate between the continuous solution and the discrete solution for a general non-local transport equation for some $\bar{T} > 0$. We are inspired by the work of [2].

7.1 An abstract error estimate

We consider the continuous viscosity solution u of a general non-local transport equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) &= c[u](x,t)\,\dfrac{\partial u}{\partial x}(x,t) \quad \text{in} \quad \mathbb{R}\times(0,\bar{T}) \\ u(x,0) &= u^0(x) \qquad\qquad\qquad \text{on} \quad \mathbb{R} \end{cases} \tag{51}$$

We recall that the non-local velocity $c[u]$ belongs to $L^\infty((0,\bar{T}), W^{1,\infty}(\mathbb{R}))$ and that the solution u is Lipschitz continuous. We will consider a discrete solution v satisfying

$$v = G^\Delta \circ c^\Delta(v) \tag{52}$$

where this abstract scheme will be precised below.
For $0 < T \leq \bar{T}$, and given a mesh $\Delta = (\Delta x, \Delta t)$, we denote $E^\Delta_T = \mathbb{R}^{\mathbb{Z}\times\{0,\ldots,N_T\}}$, N_T is the floor part of $\dfrac{T}{\Delta t}$, the space of discrete functions defined on $I^T_d = \{(i\Delta x, n\Delta t),\ i\in\mathbb{Z},\ n \leq N_T\}$. We consider two subsets of E^Δ_T

$$U^\Delta_T = \left\{w \in E^\Delta_T \quad \text{such that} \quad \left|\frac{w^n_{i+1} - w^n_i}{\Delta x}\right| \leq B^0 e^{LT} \quad \text{and} \quad \frac{w^n_{i+1} - w^n_i}{\Delta x} \geq b^0 e^{-LT}\right\}$$

and

$$V^\Delta_T = \left\{ c \in E^\Delta_T \quad \text{such that} \quad |c^n_i| \leq K \quad \text{and} \quad \left|\frac{c^n_{i+1} - c^n_i}{\Delta x}\right| \leq L \ \ \forall\, i\in\mathbb{Z},\ \forall\, n \leq N_T \right\}.$$

We also consider two operators:

$$\begin{aligned} G^\Delta : V^\Delta_T &\longrightarrow U^\Delta_T \\ c &\longmapsto v \end{aligned} \qquad \text{and} \qquad \begin{aligned} c^\Delta : U^\Delta_T &\longrightarrow V^\Delta_T \\ w &\longmapsto c^\Delta(w) \end{aligned}$$

where c^Δ is an approximation of the non-local velocity. For $c \in V^\Delta_T$, $v = G^\Delta(c)$ is defined by

$$v^0_i = u^0(x_i), \quad v^{n+1}_i = v^n_i + \Delta t\, c^n_i \times \begin{cases} D^+_x v^n_i & \text{if } c^n_i \geq 0 \\ D^-_x v^n_i & \text{if } c^n_i < 0 \end{cases} \tag{53}$$

We are looking for a solution of (52). Our goal is to give an abstract error estimate between the continuous solution u and the discrete solution v. To this end, we need to introduce a long serie of assumptions. Our error estimate will be given in Theorem 7.1. We make the following assumptions.

(A1) **CFL condition**

$$\Delta t \leq \frac{\Delta x}{\sup\limits_{i\in\mathbb{Z}} |c_i(v^n)|}$$

(A2) $(u)^\Delta \in U_T^\Delta$ where $(u)^\Delta$ is the restriction of the continuous solution u of (51) to I_d^T.

(A3) $(c)^\Delta \in V_T^\Delta$ where $(c)^\Delta$ is the restriction of the non-local velocity $c[u]$ to I_d^T.

(A4) U_T^Δ and V_T^Δ are respectively equi-Lipschitz and equibounded in the sense that there is a constant K such that, for every mesh Δ,

$$\left|D_x^+ w\right| \leq K,\ |c| \leq K, \text{ for every } w \in U_T^\Delta,\ c \in V_T^\Delta\,. \tag{54}$$

(A5)

$$G^\Delta(V_T^\Delta) \subset U_T^\Delta \quad \text{for every } T\,. \tag{55}$$

(A6) **The discrete velocity c^Δ is stationary** *i.e.* there is a map $\bar{c}^\Delta$ such that

$$\bar{c}^\Delta(w(\cdot, t_n)) = c^\Delta(w)(\cdot, t_n)\,. \tag{56}$$

(A7)

$$c^\Delta(U_T^\Delta) \subset V_T^\Delta \quad \text{for every } T\,. \tag{57}$$

(A8) **Stability of the operator G^Δ** (see Proposition 5.3).
There is a constant $K > 0$ such that for every mesh Δ satisfying the CFL condition (A1), for every T and every c_1, $c_2 \in V_T^\Delta$,

$$\sup_{I_d^T} \left|G^\Delta(c_2) - G^\Delta(c_1)\right| \leq KT \sup_{I_d^T} |c_2 - c_1|\,. \tag{58}$$

(A9) **Consistency of the discrete velocity c^Δ.**
There is a constant $K > 0$ such that for every mesh Δ and every T,

$$\sup_{I_d^T} \left|c[u] - c^\Delta(u^\Delta)\right| \leq K\Delta x \tag{59}$$

where u is the solution of (51) and $u^\Delta = (u)^\Delta$ is the restriction of u to I_d^T.

(A10) **Stability of the discrete velocity c^Δ.**
There is a constant $K > 0$ such that for every mesh Δ, for every $T \leq \bar{T}$ and every w_1, $w_2 \in U_T^\Delta$,

$$\sup_{I_d^T} \left|c^\Delta(w_1) - c^\Delta(w_2)\right| \leq K\left(\sup_{I_d^T} |w_2 - w_1| + \Delta x\right)\,. \tag{60}$$

We have the following abstract error estimate (see [3, 2]).

Theorem 7.1 (An abstract error estimate for a short time)
Let us consider $\bar{T} > 0$ and $\Delta x + \Delta t \leq 1$. Let us assume that (A1)-(A10) hold for any $T \leq \bar{T}$ and that there exists a unique continuous solution u of (51) on $[0, \bar{T}]$. There then exists a constant $K' > 0$, depending on $|c^{\text{ext}}|_{L^\infty(\mathbb{R})}$, P, $\left|c^0\right|_{L^1(\mathbb{R})}$, the bound constants of $D_x^+ v_i^n$, $D_x^- v_i^n$ and $\left|(u^0)_x\right|_{L^\infty(\mathbb{R})}$, and there exists a constant $0 < \bar{T}^* \leq \bar{T}$ with $\bar{T}^*$ only depending on $\bar{T}$ and K', such that for every $T \leq \bar{T}^*$, we have

$$\sup_{I_d^T} |u - v| \leq K'\sqrt{\Delta x} \quad \text{if} \quad \Delta x \leq \frac{\bar{T}^*}{K'}\,.$$

7.2 Application of the abstract error estimate: proof of Theorem 1.5

We check successively assumptions (A1) to (A10).

1. We assume the CFL condition (12) which implies (A1) and (47) because

$$\begin{aligned}\sup_{j\in\mathbb{Z}}\left|c^n_{j+1}-c^n_j\right| &\leq \sup_{j\in\mathbb{Z}}\left|c^n_{j+1}\right|+\sup_{j\in\mathbb{Z}}\left|c^n_j\right|\\ &\leq 2\sup_{j\in\mathbb{Z}}\left|c^n_j\right|\\ &\leq 2\left|c\right|_{L^\infty(\mathbb{R}\times(0,+\infty))}\\ &\leq 2\left(\left|c^{\text{ext}}\right|_{L^\infty(\mathbb{R})}+P\left|c^0\right|_{L^1(\mathbb{R})}\right)\end{aligned}$$

   which will allow us to apply Lemma 5.2.

   Here we will apply Theorem 7.1 with $\bar{T}=T_0$ given in (32) and with $T_1=\bar{T}^*$, $C=K'$ given by Theorem 7.1. We recall the following notations (see (28) and (31)):

$$K=\left|c^{\text{ext}}\right|_{L^\infty(\mathbb{R})}+P\left|c^0\right|_{L^1(\mathbb{R})}\quad\text{and}\quad L=\left|c^{\text{ext}}_x\right|_{L^\infty(\mathbb{R})}+P\left|c^0_x\right|_{L^1(\mathbb{R})}\,.$$

2. By Proposition 2.2, we have $(u)^\Delta\in U^\Delta_T$ where u is the solution of (51).

3. It is clear that $(c)^\Delta\in V^\Delta_T$ where $c=c[u]$ given by (2) for the solution u of (51).

4. It is also clear that, by definition, the sets U^Δ_T and V^Δ_T are respectively equi-Lipschitz and equi-bounded.

5. We now check that $G^\Delta(V^\Delta_T)\subset U^\Delta_T$. Let $c\in V^\Delta_T$. By Lemma 5.1 we have

$$\left|\frac{G^\Delta(c)^n_{i+1}-G^\Delta(c)^n_i}{\Delta x}\right|=\left|\frac{v^n_{i+1}-v^n_i}{\Delta x}\right|\leq B^n\,.$$

   Moreover,

$$B^n=B^{n-1}\left(1+\Delta t\sup_{i\in\mathbb{Z}}\left|\frac{c^{n-1}_{i+1}-c^{n-1}_i}{\Delta x}\right|\right)\leq B^{n-1}(1+L\Delta t)\leq B^{n-1}e^{L\Delta t}\,.$$

   We deduce that $B^n\leq B^0e^{L\,n\Delta t}\leq B^0e^{LT}$. Therefore,

$$\left|\frac{G^\Delta(c)^n_{i+1}-G^\Delta(c)^n_i}{\Delta x}\right|\leq B^0e^{LT}\,.$$

   Similarly, by Lemma 5.2, we have

$$\frac{G^\Delta(c)^n_{i+1}-G^\Delta(c)^n_i}{\Delta x}\geq b^0\,e^{-LT}\,.$$

   Thus, $G^\Delta(c)\in U^\Delta_T$, for all $c\in V^\Delta_T$ and then $G^\Delta(V^\Delta_T)\subset U^\Delta_T$ for all T.

6. We now consider the discrete non-local velocity given in (9), (10):

$$c^{\text{int},\Delta}=c^{\text{int},n}_i=\sum_{l\in\mathbb{Z}}c^0_{i-l}\,E(v^n_l)\,\Delta x\,,\quad c^0_i=\frac{1}{\Delta x}\int_{I_i}c^0(x)\,dx$$

with $I_i = (x_i - \frac{\Delta x}{2}, x_i + \frac{\Delta x}{2})$. It is clearly stationary. We recall from [3], that $c_i^{\mathrm{int},n}$ can be written as the continuous convolution

$$c_i^{\mathrm{int},n} = c^0 \star E(v_\#)(x_i)$$

where $v_\#$ is the piecewise constant lifting of v

$$v_\# = \sum_i v_i 1_{I_i}\,. \tag{61}$$

Obviously c_i^{ext} is stationary. Therefore c^Δ is stationary.

7. We now check that $c^\Delta\left(U_T^\Delta\right) \subset V_T^\Delta$. Indeed, for all $v \in U_T^\Delta$, we have

$$\left|c^\Delta(v)\right| \le K, \quad \left|D^+ c^\Delta(v)\right| \le \left|\frac{\partial}{\partial x}\left(c^0 \star E(v_\#)\right)\right|_{L^\infty(\mathbb{R})} \le L\,.$$

Therefore $c^\Delta\left(U_T^\Delta\right) \subset V_T^\Delta$.

8. The assumption (A7) holds by Proposition 5.3.

9. **Consistency of the discrete velocity $c^{\mathrm{int},\Delta}$.** We estimate:

$$\begin{aligned}\sup_{i\in\mathbb{Z}}\left|c_i^{\mathrm{int},\Delta}(u^\Delta)(\cdot,t_n) - c[u](x_i,t_n)\right| &\le \sup_{x\in\mathbb{R}}\left|c^0 \star E(u_\#^\Delta)(\cdot,t_n) - c^0 \star E(u)(\cdot,t_n)\right| \\ &\le \left|c^0\right|_{L^\infty_{\mathrm{int}}(\mathbb{R})}\left|E(u_\#)(\cdot,t_n) - E(u)(\cdot,t_n)\right|_{L^1_{\mathrm{unif}}(\mathbb{R})} \\ &\le \frac{4P}{b}\left|c^0\right|_{L^\infty_{\mathrm{int}}(\mathbb{R})}\left|u_\#(\cdot,t_n) - u(\cdot,t_n)\right|_{L^\infty(\mathbb{R})}\,.\end{aligned}$$

Then $c^{\mathrm{int},\Delta}$ is consistent.

10. **Stability of the discrete velocity $c^{\mathrm{int},\Delta}$.** We estimate

$$\begin{aligned}\left|c_i^{\mathrm{int},\Delta}(w^1) - c_i^{\mathrm{int},\Delta}(w^2)\right| &= \left|c^0 \star E(w_\#^1)(x_i) - c^0 \star E(w_\#^2)(x_i)\right| \\ &\le \left|c^0\right|_{L^\infty_{\mathrm{int}}(\mathbb{R})}\left|E(w_\#^1) - E(w_\#^2)\right|_{L^1_{\mathrm{unif}}(\mathbb{R})} \\ &\le 4P\left|c^0\right|_{L^\infty_{\mathrm{int}}(\mathbb{R})}\left(\frac{1}{b}\sup_{i\in\mathbb{Z}}\left|w_i^1 - w_i^2\right| + \Delta x\right)\end{aligned}$$

where we have used in the last time Proposition 6.1.

Finally, we apply Theorem 7.1 and we obtain Theorem 1.5.

8 Example of a simulation

In this section, we provide some numerical simulations showing the behavior of the solution and the dislocations dynamics through obstacles.
We start by an initial data $u^0(x) = 2x$. The velocity is chosen as

$$c[u](x,t) = A + B\sin(2k\pi x) + c^0 \star E(u(\cdot,t))(x)$$

with $A = 1.2$, $B = 1$, the number of obstacles is $k = 2$, the kernel c^0 is the one of Peierls Nabarro given by (4) with $\dfrac{\mu b^2}{2\pi(1-\nu)} = 1$ and $\zeta = 0.1$. We choose $\Delta x = 0.0099$ and $\Delta t = 0.00263$. Numerically we work on the interval for $x \in \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$.

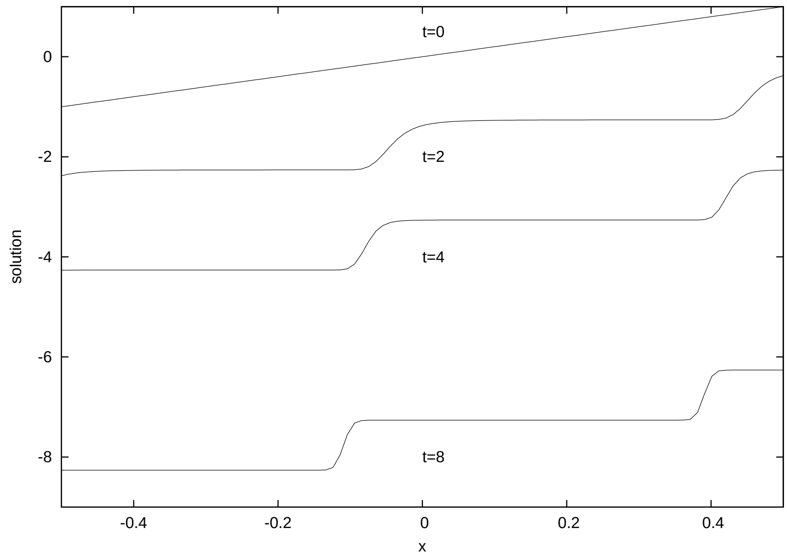

Figure 3: Behavior of the solution in time

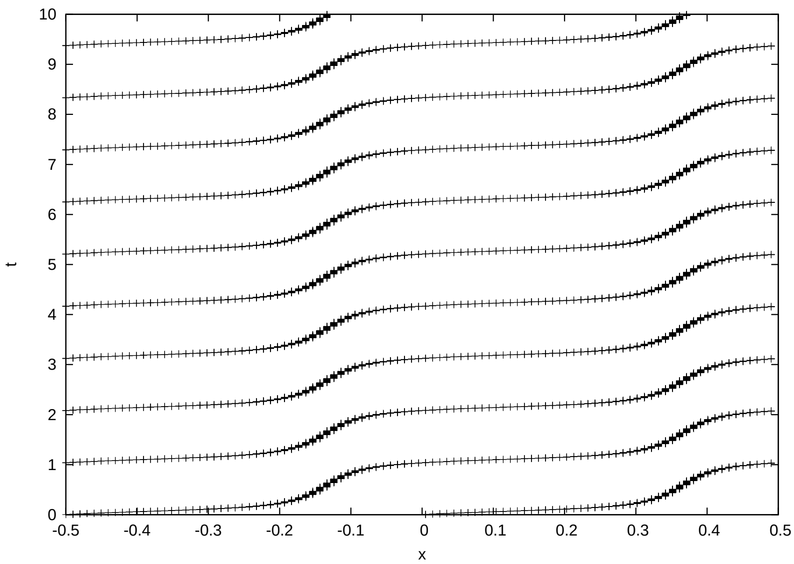

Figure 4: Dislocations dynamics through obstacles

In Figure 3, we represent the solution $u(x,t)$ as a function of $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ for different values of $t = 0, 2, 4, 8$. On this figure we see that the gradient of the solution remains numerically in time bounded from above and from below, even if the lower bound of the gradient is very small. In Figure 4, we represent the trajectories of the dislocations $x(t)$ (here there are 2 dislocations) with the time on the vertical axis and the space on the horizontal one. We recall that the positions of dislocations correspond to the jumps of the floor part of the solution. On Figure 4, we see that the dislocations slow down on the obstacles. Finally, we remark numerically on Figure 3 that the gradient of the solution is far from zero in the regions where we take the floor part of the solution, which is a good behavior for this simulation. We can even say that we can localize the dislocations by the strong variations of the solution.

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