

When and how an error yields a Dirichlet form

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WHEN AND HOW AN ERROR YIELDS A DIRICHLET FORM

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Abstract. We consider a random variable Y and approximations Y_n , $n \in \mathbb{N}$, defined on the same probability space with values in the same measurable space as Y . We are interested in situations where the approximations Y_n allow to define a Dirichlet form in the space $L^2(\mathbb{P}_Y)$ where \mathbb{P}_Y is the law of Y . Our approach consists in studying both biases and variances. The article attempts to propose a general theoretical framework. It is illustrated by several examples.

Keywords : error, approximation, Dirichlet form, square field operator, bias, Wiener space, stochastic differential equation.

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I. Bias operators.

We consider a random variable Y defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with values in a measurable space (E, \mathcal{F}) and approximations Y_n , $n \in \mathbb{N}$, also defined on $(\Omega, \mathcal{A}, \mathbb{P})$ with values in (E, \mathcal{F}) . In the whole study we suppose that there exist an algebra \mathcal{D} of bounded functions from E into \mathbb{R} or \mathbb{C} dense in $L^2(E, \mathcal{F}, \mathbb{P}_Y)$ containing the constants and a sequence $(\alpha_n)_{n \in \mathbb{N}}$ of positive numbers, with which the following hypotheses are considered :

$$(H1) \quad \begin{cases} \forall \varphi \in \mathcal{D}, \text{ there exists } \bar{A}[\varphi] \in L^2(E, \mathcal{F}, \mathbb{P}_Y) \quad s.t. \quad \forall \chi \in \mathcal{D} \\ \lim_{n \rightarrow \infty} \alpha_n \mathbb{E}[(\varphi(Y_n) - \varphi(Y))\chi(Y)] = \mathbb{E}_Y[\bar{A}[\varphi]\chi]. \end{cases}$$

the expectation \mathbb{E}_Y being relative to the law \mathbb{P}_Y .

$$(H2) \quad \begin{cases} \forall \varphi \in \mathcal{D}, \text{ there exists } \underline{A}[\varphi] \in L^2(E, \mathcal{F}, \mathbb{P}_Y) \quad s.t. \quad \forall \chi \in \mathcal{D} \\ \lim_{n \rightarrow \infty} \alpha_n \mathbb{E}[(\varphi(Y) - \varphi(Y_n))\chi(Y_n)] = \mathbb{E}_Y[\underline{A}[\varphi]\chi]. \end{cases}$$

$$(H3) \quad \begin{cases} \forall \varphi \in \mathcal{D}, \text{ there exists } \tilde{A}[\varphi] \in L^2(E, \mathcal{F}, \mathbb{P}_Y) \quad s.t. \quad \forall \chi \in \mathcal{D} \\ \lim_{n \rightarrow \infty} \alpha_n \mathbb{E}[(\varphi(Y_n) - \varphi(Y))(\chi(Y_n) - \chi(Y))] = -2\mathbb{E}_Y[\tilde{A}[\varphi]\chi]. \end{cases}$$

We first note that as soon as two of hypotheses (H1) (H2) (H3) are fulfilled (with the same algebra \mathcal{D} and the same sequence α_n), the third one follows thanks to the relation

$$\tilde{A} = \frac{\bar{A} + \underline{A}}{2}.$$

When defined, the operator \bar{A} which considers the asymptotic error from the point of view of the limit model, will be called *the theoretical bias operator*.

The operator \underline{A} which considers the asymptotic error from the point of view of the approximating model will be called *the practical bias operator*.

Because of the property

$$\langle \tilde{A}[\varphi], \chi \rangle_{L^2(\mathbb{P}_Y)} = \langle \varphi, \tilde{A}[\chi] \rangle_{L^2(\mathbb{P}_Y)}$$

the operator \tilde{A} will be called *the symmetric bias operator*.

Remark 1. Under (H1) the limit of $\alpha_n \mathbb{E}[(\varphi(Y) - \varphi(Y_n))\chi(Y_n)]$ exists and is equal to

$$\mathbb{E}_Y[\bar{A}[\chi]\varphi] - \mathbb{E}_Y[\bar{A}[\varphi]\chi].$$

The operator \bar{A} with dense domain possesses an adjoint \bar{A}^* . If $\mathcal{D} \subset \mathcal{D}(\bar{A}^*)$ then (H2) is satisfied and

$$(1) \quad \underline{A}[\varphi] = \bar{A}^*[\varphi] - \varphi \bar{A}^*[1]. \quad \forall \varphi \in \mathcal{D}$$

Reciprocally, if (H1) and (H2) are supposed and if $1 \in \mathcal{D}(\bar{A}^*)$, the map $\chi \mapsto \mathbb{E}_Y[\underline{A}[\varphi]\chi] = \mathbb{E}_Y[\bar{A}[\chi]\varphi] - \mathbb{E}_Y[\bar{A}[\varphi]\chi]$ is continuous and so is $\chi \mapsto \mathbb{E}[\bar{A}[\chi]\varphi]$ which shows $\mathcal{D} \subset \mathcal{D}\bar{A}^*$

and relation (1) holds. We see that the hypothesis $1 \in \mathcal{D}(\overline{A}^*)$ is rather strong, it will be not fulfilled in general.

Example I.1. Let us take for (E, \mathcal{F}) a metrisable compact set with its Borel σ -field and let (X_t) be a Feller process with values in E and transition semi-group (P_t) (A Feller process on an l.c.d. space reduces to this situation by the Alexandrov compactification see [12] chap. XIII §20-21).

Let (A_P, D_P) be the generator of the strongly continuous contraction semi-group (P_t) on $\mathcal{C}(E)$. Suppose (P_t) be in duality with a strongly continuous semi-group (Q_t) with respect to a probability measure ν and let (A_Q, D_Q) be the generator of (Q_t) on $\mathcal{C}(E)$. Then, if there is an algebra of bounded functions containing constants $\mathcal{D} \subset D_P \cap D_Q$ dense in $L^2(\nu)$, the approximation X_t of X_0 satisfies hypotheses (H1) to (H3) and we have on \mathcal{D} :

$$\begin{aligned}\overline{A}[\varphi] &= A_P[\varphi] \\ \underline{A}[\varphi] &= A_Q[\varphi] - \varphi A_Q[1] \\ \overline{A}^*[\varphi] &= \underline{A}[\varphi] + \varphi A_Q[1].\end{aligned}$$

Indeed, if $\varphi, \chi \in \mathcal{D}$

$$\frac{1}{t} \mathbb{E}_\nu[(\varphi(X_t) - \varphi(X_0))\chi(X_0)] = \frac{1}{t} \langle P_t \varphi - \varphi, \chi \rangle_\nu \rightarrow \langle A_P[\varphi], \chi \rangle_\nu$$

$$\frac{1}{t} \mathbb{E}_\nu[(\varphi(X_0) - \varphi(X_t))\chi(X_t)] = \frac{1}{t} \langle Q_t \varphi - \varphi + \varphi(1 - Q_t 1), \chi \rangle_\nu \rightarrow \langle A_Q[\varphi] - \varphi A_Q[1], \chi \rangle_\nu.$$

Hence (H1) to (H3) are fulfilled and, by theorem 1 below, the limit $\frac{1}{t} \mathbb{E}_\nu[(\varphi(X_t) - \varphi(X_0))^2]$ extends to a symmetric Dirichlet form on $L^2(\nu)$. \diamond

The basis of our study is the following theorem :

Theorem 1. *Under hypothesis (H3)*

a) *the limit*

$$\tilde{\mathcal{E}}[\varphi, \chi] = \lim_n \frac{\alpha_n}{2} \mathbb{E}[(\varphi(Y_n) - \varphi(Y))(\chi(Y_n) - \chi(Y))] \quad \varphi, \chi \in \mathcal{D}$$

defines a closable positive bilinear form whose smallest closed extension is denoted $(\mathcal{E}, \mathbb{D})$.

b) *$(\mathcal{E}, \mathbb{D})$ is a Dirichlet form*

c) *$(\mathcal{E}, \mathbb{D})$ admits a square field operator Γ satisfying $\forall \varphi, \chi \in \mathcal{D}$*

$$\Gamma[\varphi] = \tilde{A}[\varphi^2] - 2\varphi \tilde{A}[\varphi]$$

$$\mathbb{E}_Y[\Gamma[\varphi]\chi] = \lim_n \alpha_n \mathbb{E}[(\varphi(Y_n) - \varphi(Y))^2(\chi(Y_n) + \chi(Y))/2]$$

d) *$(\mathcal{E}, \mathbb{D})$ is local if and only if $\forall \varphi \in \mathcal{D}$*

$$\lim_n \alpha_n \mathbb{E}[(\varphi(Y_n) - \varphi(Y))^4] = 0.$$

Demonstration. a) That $(\tilde{\mathcal{E}}, \mathcal{D})$ be closable comes from the Friedrich construction of the minimal selfadjoint extension of a symmetric operator. Let us recall the argument.

By $\tilde{\mathcal{E}}[\varphi, \chi] = - \langle \tilde{A}[\varphi], \chi \rangle_{L^2(\mathbb{P}_Y)} \quad \forall \varphi, \chi \in \mathcal{D}$ the form $(\tilde{\mathcal{E}}, \mathcal{D})$ satisfies

$$u_n \in \mathcal{D}, \|u_n\| \rightarrow 0 \quad \Rightarrow \quad \mathcal{E}[u_n, v] \rightarrow 0 \quad \forall v \in \mathcal{D}$$

and this property suffices to imply closability (cf. [13] ex. 1.1.2, [9] Chap. I ex. 1.3.4 or [5] lemma III.24).

b) In order to prove that the form $(\mathcal{E}, \mathbb{D})$ is Dirichlet we will use the following elementary property :

If K is a compact subset of \mathbb{R} , $\forall \varepsilon > 0$ there exists a polynomial $p(x)$ such that

$$(i) \quad 0 \leq p(y) - p(x) \leq y - x \quad \forall x < y \in K$$

$$(ii) \quad |p(x) - x| \leq \varepsilon \quad \forall x \in [0, 1] \cap K$$

$$(iii) \quad p(x) \geq -\varepsilon \quad \forall x \in K.$$

Let $(R_\lambda)_{\lambda > 0}$ be the strongly continuous contraction resolvent associated with $(\mathcal{E}, \mathbb{D})$, we have to prove that the operators λR_λ are sub-Markov (cf. [13], [9], [23]). For that, since here $\lambda R_\lambda 1 = 1$, $\forall \lambda > 0$, because $\mathcal{E}[1, u] = 0 \quad \forall u \in \mathbb{D}$, it is enough to show that $0 \leq u \leq 1 \Rightarrow R_\lambda u \geq 0$ and this for $\lambda \geq 1$ since $R_\alpha = \frac{1}{\beta} \sum_{k=1}^{\infty} (\beta R_{\alpha+\beta})^k \quad \forall \alpha, \beta > 0$ by the resolvent equation.

Let u be a measurable function from E into \mathbb{R} s.t. $0 \leq u \leq 1$, denoting as usual \mathcal{E}_λ the form $\mathcal{E} + \lambda \|\cdot\|_{L^2}^2$, the functional

$$\Phi(v) = \mathcal{E}[v] + \lambda \|v - \frac{u}{\lambda}\|^2 \quad v \in \mathbb{D}$$

satisfies

$$\Phi(R_\lambda u) + \mathcal{E}_\lambda[R_\lambda u - v] = \Phi(v) \quad v \in \mathbb{D}.$$

It follows that $R_\lambda u$ is the unique element of \mathbb{D} minimizing Φ on \mathbb{D} , and the sequence v_n converges in \mathbb{D} (equipped with the norm $\sqrt{\mathcal{E}_1}$) to $R_\lambda u$ if and only if $\lim_n \Phi(v_n) = \Phi(R_\lambda u)$.

Since \mathcal{D} is dense in \mathbb{D} , let $v_n \in \mathcal{D}$ be such a sequence converging to $R_\lambda u$. Let p_n be a polynomial fulfilling property (i) to (iii) for $\varepsilon = \frac{1}{n}$ and K containing the range of the bounded function v_n . Let us put $w_n = p_n \circ v_n$. Since \mathcal{D} is an algebra, $w_n \in \mathcal{D}$ and by (i) we have $\tilde{\mathcal{E}}[w_n] \leq \tilde{\mathcal{E}}[v_n]$. Now, by (i) and (ii) $|p_n \circ v_n - y| \leq |v_n - y| + \frac{1}{n} \quad \forall y \in [0, 1]$ so $|w_n - \frac{u}{\lambda}| \leq |v_n - \frac{u}{\lambda}| + \frac{1}{n}$ what gives $\|w_n - \frac{u}{\lambda}\|_{L^2}^2 \leq \|v_n - \frac{u}{\lambda}\|_{L^2}^2 + \frac{2}{n} \|v_n - \frac{u}{\lambda}\|_{L^2} + \frac{1}{n^2}$ and

$$\Phi(w_n) \leq \Phi(v_n) + \frac{2}{n} \|v_n - \frac{u}{\lambda}\| + \frac{1}{n^2}.$$

The sequence v_n being bounded in L^2 , it follows that $\Phi(w_n) \rightarrow \Phi(R_\lambda u)$ and $w_n \rightarrow R_\lambda u$ in \mathbb{D} . Taking eventually a subsequence converging a.s. and using $w_n \geq -\frac{1}{n}$ by (iii), we obtain $R_\lambda u \geq 0$ what proves the property.

c) Let us denote (A, \mathcal{DA}) the selfadjoint operator associated with $(\mathcal{E}, \mathbb{D})$ (Friedrich's extension of (\tilde{A}, \mathcal{D})), since the algebra $\mathcal{D} \subset \mathcal{DA}$ is dense in \mathbb{D} , the theorem 4.2.2

of [9] applies and the form $(\mathcal{E}, \mathbb{D})$ admits a square field operator satisfying $\Gamma[\varphi] = \tilde{A}[\varphi^2] - 2\varphi\tilde{A}[\varphi] \quad \forall \varphi \in \mathcal{D}$ and $\Gamma[\psi] = A[\psi^2] - 2\psi A[\psi]$ if $\psi \in \mathcal{D}A$ and $\psi^2 \in \mathcal{D}A$. The formula of the statement comes from

$$\mathbb{E}_Y[\Gamma[\varphi]\chi] = \mathbb{E}_Y[\tilde{A}[\varphi^2]\chi] - 2\mathbb{E}_Y[\tilde{A}[\varphi]\varphi\chi]$$

coming back to the definition of \tilde{A} .

d) Let us remark that for $f \in \mathcal{D}$ we have

$$4\tilde{\mathcal{E}}[f^3, f] - 3\tilde{\mathcal{E}}[f^2, f^2] = \lim_n \alpha_n \mathbb{E}[(f(Y_n) - f(Y))^4].$$

So, if the assumption of the statement holds, using the fact that one dimensional contractions are continuous on \mathbb{D} (cf. [])

$$4\mathcal{E}[u^3, u] - 3\mathcal{E}[u^2, u^2] = 0 \quad \forall u \in \mathbb{D} \cap L^\infty$$

this is enough to imply that \mathcal{E} is local (cf [9] Chap. I §5). Reciprocally, if \mathcal{E} is local, since \mathcal{E} admits a square field operator, the functional calculus applies (cf. [9] Chap. I §6) and

$$4\mathcal{E}[u^3, u] - 3\mathcal{E}[u^2, u^2] = 2\mathbb{E}[3u^2\Gamma[u]] - \frac{3}{2}\mathbb{E}[4u^2\Gamma[u]] = 0$$

$\forall u \in \mathbb{D} \cap L^\infty$ and the condition of the statement is fulfilled. \diamond

Comment. Considering for Y a Brownian motion B indexed by $[0, 1]$ as random variable with values in $\mathcal{C}([0, 1])$ and taking for Y_ε the approximation $Y_\varepsilon = B + \sqrt{\varepsilon}W$ where W is an independent standard Brownian motion, we may apply the theorem with \mathcal{D} the linear combinations of functions $\varphi(B) = e^{i \int_0^1 f dB}$ with regular f say C_b^1 .

We have with $\chi(B) = e^{i \int_0^1 g dB}$

$$\begin{aligned} \mathbb{E}[(e^{i \int_0^1 f dY_\varepsilon} - e^{i \int_0^1 f dY})(e^{i \int_0^1 g dY_\varepsilon} - e^{i \int_0^1 g dY})] \\ = \mathbb{E}[e^{i \int (f+g) dY}] \mathbb{E}[(e^{i \sqrt{\varepsilon} \int f dW} - 1)(e^{i \sqrt{\varepsilon} \int g dW} - 1)] \end{aligned}$$

so that

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \mathbb{E}[(\varphi(Y_\varepsilon) - \varphi(Y))(\chi(Y_\varepsilon) - \chi(Y))] = (- \int_0^1 fg dt) e^{-\frac{1}{2} \int_0^1 (f+g)^2 dt}$$

what may be written $-2 \langle \tilde{A}[\varphi], \chi \rangle$ with

$$\tilde{A}[\varphi] = e^{i \int f dB} [-\frac{i}{2} \int f dB - \frac{1}{2} \int f^2 dt]$$

as seen by an elementary calculation. Hypothesis (H3) is satisfied. The theorem yields the well known Ornstein-Uhlenbeck structure on the Wiener space (see e.g. [9] or [27]).

We can say that from a pedagogical point of view, in order to introduce the error calculus on the Wiener space (basic Malliavin calculus) and the same would be true for

the Poisson space or the Monte Carlo space (cf. [5]), theorem 1 is a quite convenient tool. It is simpler than the theorem on infinite products of Dirichlet structures used in [9] or [5]. It allows also to construct Dirichlet forms in a variety of situations as will show the examples below.

This being said, when studying mathematical approximations, the most useful part of the theorem is often the easiest one to prove, i.e. part a), because the limit form is often recognized as a classical form whose properties (Dirichlet character, square field operator, locality) are known.

Remark 2. Suppose that instead of (H3) we assume that the limit $\lim_n \alpha_n \mathbb{E}[(\varphi(Y_n) - \varphi(Y))^2]$ exists $\forall \varphi \in \mathcal{D}$. Then, denoting $2\tilde{\mathcal{E}}[\varphi]$ this limit, if the form $(\tilde{\mathcal{E}}, \mathcal{D})$ is closable,

i) the hypothesis (H3) is equivalent to $\mathcal{D} \subset \mathcal{DA}$ where $(\mathcal{A}, \mathcal{DA})$ is the generator of the form $(\mathcal{E}, \mathbb{D})$ smallest closed extension of $(\tilde{\mathcal{E}}, \mathcal{D})$ (Indeed, $u \in \mathcal{DA} \Leftrightarrow f \mapsto \mathcal{E}[f, u]$ is continuous on \mathcal{D} , hence if $\mathcal{D} \subset \mathcal{DA}$ hypothesis (H3) is satisfied with $\tilde{A} = A$ on \mathcal{D} and if (H3) is satisfied $\mathcal{D} \subset \mathcal{DA}$)

ii) parts b) and c) of the demonstration of the theorem apply and show that $(\mathcal{E}, \mathbb{D})$ is Dirichlet with square field operator.

Remark 3. Under (H3) the condition d) of the theorem $\forall \varphi \in \mathcal{D} \quad \lim_n \alpha_n \mathbb{E}[(\varphi(Y_n) - \varphi(Y))^4] = 0$ is equivalent to either of the conditions :

$$(j) \exists \lambda > 2 \quad \lim_n \alpha_n \mathbb{E}[|\varphi(Y_n) - \varphi(Y)|^\lambda] = 0 \quad \forall \varphi \in \mathcal{D}.$$

$$(jj) \forall \lambda > 2 \quad \lim_n \alpha_n \mathbb{E}[|\varphi(Y_n) - \varphi(Y)|^\lambda] = 0 \quad \forall \varphi \in \mathcal{D}.$$

Indeed, it suffices to prove that (j) implies (jj). Let λ be such that (j) is true, then for $\mu > \lambda$

$$\alpha_n \mathbb{E}[|\varphi(Y_n) - \varphi(Y)|^\mu] \leq \alpha_n \mathbb{E}[|\varphi(Y_n) - \varphi(Y)|^\lambda] (2\|\varphi\|_\infty)^{\mu-\lambda} \rightarrow 0$$

and for $\lambda > \mu = 2 + \varepsilon$

$$\alpha_n \mathbb{E}[|\varphi(Y_n) - \varphi(Y)|^\mu] \leq (\alpha_n \mathbb{E}|\varphi(Y_n) - \varphi(Y)|^2)^{1/2} (\alpha_n \mathbb{E}|\varphi(Y_n) - \varphi(Y)|^{2+2\varepsilon})^{1/2}$$

iterating the procedure until $2 + 2^k \varepsilon \geq \lambda$ gives the result. \diamond

Remark 4. As an example where the limit form is non-local, let us consider the case where $(E, \mathcal{F}) = (\Omega, \mathcal{A})$ and let θ be a measurable map from Ω into itself preserving the probability \mathbb{P} and defining a strongly mixing endomorphism $A \in \mathcal{A} \mapsto \theta^{-1}(A)$, i.e. such that

$$\lim_n \mathbb{E}[f \circ \theta^n \cdot g] = \mathbb{E}f \mathbb{E}g \quad \forall f, g \in L^2(\mathbb{P})$$

then taking for Y the identity map and $Y_n = \theta^n$ with $\alpha_n = 1$, hypotheses (H1) to (H3) hold on $\mathcal{D} = L^\infty(\mathbb{P})$ with $\bar{A}[\varphi] = \underline{A}[\varphi] = \tilde{A}[\varphi] = -\varphi + \mathbb{E}[\varphi]$ and $\tilde{\mathcal{E}}[\varphi] = \text{var}\varphi$. \diamond

We introduce now the fourth bias operator \mathbb{A} defined under (H1) and (H2) on \mathcal{D} as

$$\mathbb{A} = \frac{1}{2}(\bar{A} - \underline{A}).$$

By $\mathbb{E}_Y[\mathbb{A}[\varphi]\chi] = \lim_n \mathbb{E}[(\varphi(Y_n) - \varphi(Y))(\chi(Y) + \chi(Y_n))/2]$ we see that \mathbb{A} represent the asymptotic error from the point of view of an exterior observator according the same weight to both the theoretical and the practical models and measuring the error algebraically on the same axis. Because of the properties of \mathbb{A} proved below, \mathbb{A} will be called *the singular bias operator*.

We shall say that an operator B from \mathcal{D} into $L^2(\mathbb{P}_Y)$ is a *first order operator* if it satisfies

$$B[\varphi\chi] = B[\varphi]\chi + \varphi B[\chi] \quad \forall \varphi, \chi \in \mathcal{D}$$

Proposition 1. *Under (H1) to (H3)*

a) *the theoretical variance $\lim_n \alpha_n \mathbb{E}[(\varphi(Y_n) - \varphi(Y))^2 \psi(Y)]$ and the practical variance $\lim_n \alpha_n \mathbb{E}[(\varphi(Y_n) - \varphi(Y))^2 \psi(Y_n)]$ exist and we have $\forall \varphi, \chi, \psi \in \mathcal{D}$*

$$\begin{aligned} \lim_n \alpha_n \mathbb{E}[(\varphi(Y_n) - \varphi(Y))(\chi(Y_n) - \chi(Y))\psi(Y)] &= \mathbb{E}_Y[-\underline{\mathbb{A}}[\varphi\psi]\chi + \underline{\mathbb{A}}[\psi]\varphi\chi - \overline{\mathbb{A}}[\varphi]\chi\psi] \\ \lim_n \alpha_n \mathbb{E}[(\varphi(Y_n) - \varphi(Y))(\chi(Y_n) - \chi(Y))\psi(Y_n)] &= \mathbb{E}_Y[-\overline{\mathbb{A}}[\varphi\psi]\chi + \overline{\mathbb{A}}[\psi]\varphi\chi - \underline{\mathbb{A}}[\varphi]\chi\psi] \end{aligned}$$

b) *These two variances coincide if and only if \mathbb{A} is a first order operator, and then are equal to $\mathbb{E}_Y[\Gamma[\varphi]\psi]$.*

Proof. The part a) comes directly from the definition of $\overline{\mathbb{A}}$ and $\underline{\mathbb{A}}$. The difference between the two expressions in φ, χ, ψ is

$$2\mathbb{E}_Y[(\mathbb{A}[\varphi\psi] - \mathbb{A}[\psi]\varphi - \mathbb{A}[\varphi]\psi)\chi]$$

and vanishes iff \mathbb{A} is first order. ◇

A sufficient condition for the equality of the theoretical and the practical variances is given by

Proposition 2. *Under (H1) to (H3) If there is a real number $p \geq 1$ s.t.*

$$\lim_n \alpha_n \mathbb{E}[(\varphi(Y_n) - \varphi(Y))^2 |\psi(Y_n) - \psi(Y)|^p] = 0 \quad \forall \varphi, \psi \in \mathcal{D}$$

then \mathbb{A} is first order.

Proof. Let be $\lambda \in [0, 2[$. We have $|\alpha_n \mathbb{E}[(\varphi(Y_n) - \varphi(Y))^2 (\psi(Y_n) - \psi(Y))]|$

$$\leq \alpha_n \mathbb{E}[|\varphi(Y_n) - \varphi(Y)|^\lambda |\varphi(Y_n) - \varphi(Y)|^{2-\lambda} |\psi(Y_n) - \psi(Y)|]$$

the case $p = 1$ is obtained taking $\lambda = 0$. If $p > 1$ we go on with $\lambda > 0$

$$\leq (\alpha_n \mathbb{E}[(\varphi(Y_n) - \varphi(Y))^2])^{\lambda/2} (\alpha_n \mathbb{E}[(\varphi(Y_n) - \varphi(Y))^2 |\psi(Y_n) - \psi(Y)|^{2/(2-\lambda)}])^{(2-\lambda)/2}$$

the result follows taking $2/(2 - \lambda) = p$. ◇

In particular under (H1) to (H3), if the locality condition of theorem 1 is fulfilled then \mathfrak{A} is a first order operator.

Remark 5. In the frequent cases where \mathfrak{A} is a first order operator, \tilde{A} captures all the diffusive part of \bar{A} and of \underline{A} .

Similarly, we can remark that for deterministic approximations, the operator \tilde{A} is often nought (see prop. 17 in fine). For example let us consider the ordinary differential equation

$$x_t = x_0 + \int_0^t f(x_s) y_s ds$$

approximated by the Euler scheme

$$x_t^n = x_0 + \int_0^t f(x_{[ns]/n}^n) y_s ds$$

even if we suppose x_0 to be random, errors are of deterministic nature and as soon as f is \mathcal{C}^1 with at most linear growth and $\int_0^1 y_s^2 ds < +\infty$ we have for $\varphi, \chi \in \mathcal{C}_b^1$ (bounded with bounded derivative)

$$\begin{aligned} n\mathbb{E}[(\varphi(x_t^n) - \varphi(x_t))\chi(x_t)] &\rightarrow \mathbb{E}[u_t \varphi'(x_t)\chi(x_t)] \\ n\mathbb{E}[(\varphi(x_t) - \varphi(x_t^n))\chi(x_t^n)] &\rightarrow -\mathbb{E}[u_t \varphi'(x_t)\chi(x_t)] \\ \text{and } \mathbb{E}[(\varphi(x_t^n) - \varphi(x_t))^2] &\rightarrow 0 \end{aligned}$$

where u_t is given by $u_t = -\frac{1}{2} \int_0^t f'(x_s) f(x_s) y_s^2 e^{\int_s^t f'(x_\alpha) y_\alpha d\alpha} ds$. (cf. [17] theorem 1.1). Thus

$$\bar{A}[\varphi](x) = \mathbb{E}[u_t \varphi'(x_t) | x_t = x] = -\underline{A}[\varphi](x)$$

and we have $\mathfrak{A} = \bar{A}$ and $\tilde{A} = 0$. ◇

Let us derive some consequences of the only (H1) hypothesis. Under (H1) we may consider the symmetric *positive* bilinear form

$$e[\varphi, \chi] = -\mathbb{E}[\bar{A}[\varphi]\chi + \varphi\bar{A}[\chi] - \bar{A}[\varphi\chi]]$$

Proposition 3. Under (H1) the following conditions are equivalent

- 1) (H2)
- 2) (H3)
- 3) (e, \mathcal{D}) satisfies the following sufficient closability condition

$$\chi_n \in \mathcal{D} \quad \chi_n \rightarrow 0 \text{ in } L^2 \quad \Rightarrow \quad e[\varphi, \chi_n] \rightarrow 0 \quad \forall \varphi \in \mathcal{D}.$$

Proof. Since $\lim_n \alpha_n \mathbb{E}[(\varphi(Y) - \varphi(Y_n))\chi(Y_n)] = \mathbb{E}_Y[\bar{A}[\chi]\varphi - \bar{A}[\varphi\chi]]$, (H2) is equivalent to

$$\forall \varphi \in \mathcal{D} \quad \chi \mapsto \mathbb{E}_Y[\bar{A}[\chi]\varphi - \bar{A}[\varphi\chi]] \text{ is continuous on } \mathcal{D} \text{ in } L^2$$

which is equivalent to

$$\forall \varphi \in \mathcal{D} \quad \chi \mapsto e[\varphi, \chi] \text{ continuous on } \mathcal{D} \text{ in } L^2,$$

i.e. equivalent to this bilinear form be continuous at 0 which is the condition of the statement. \diamond

Proposition 4. *Under (H1), the conditions of the preceding proposition are fulfilled if $\forall \varphi \in \mathcal{D}$*

$$\chi_p \in \mathcal{D}, \quad \chi_p \rightarrow 0 \text{ in } L^2 \quad \Rightarrow \quad \lim_p \lim_n \alpha_n \mathbb{E}[(\chi_p(Y_n) - \chi_p(Y))\varphi(Y)] = 0.$$

Proof. The condition of the statement means $\chi \rightarrow \mathbb{E}[\bar{A}[\chi]\varphi]$ continuous, i.e. $\mathcal{D} \subset \mathcal{D}(\bar{A}^*)$ hence by remark 1 hypothesis (H2) holds. \diamond

Remark 6. If $\forall \varphi \in \mathcal{D}$ the conditional expectation $\alpha_n \mathbb{E}[\varphi(Y_n) - \varphi(Y)|Y = y]$ converges weakly in $L^2(\mathbb{P}_Y)$ then (H1) is satisfied, because the weak limit is necessarily an element of L^2 .

Proposition 5. *Under (H1), if the law of the pair (Y_n, Y) is asymptotically symmetric in the following sense :*

$$\lim_n \alpha_n \mathbb{E}[\varphi(Y_n)\psi(Y) - \varphi(Y)\psi(Y_n)] = 0 \quad \forall \varphi, \psi \in \mathcal{D}$$

then the conditions of proposition 3 are fulfilled, $\underline{A} = \bar{A} = \tilde{A}$ and $\mathbb{A} = 0$.

Proof. Taking $\psi = 1$ gives $\lim_n \alpha_n \mathbb{E}[\varphi(Y_n)\chi(Y_n) - \varphi(Y)\chi(Y)] = 0$ hence

$$\lim_n \alpha_n \mathbb{E}[(\varphi(Y) - \varphi(Y_n))\chi(Y_n)] = \lim_n \alpha_n \mathbb{E}[(\varphi(Y_n) - \varphi(Y))\chi(Y)]$$

and (H2) holds with $\underline{A} = \bar{A}$. \diamond

We now come back to the situation where the only assumption (H3) is supposed.

Theorem 2. *Under (H3). If the form $(\mathcal{E}, \mathbb{D})$ (cf. theorem 1) is local, then the principle of asymptotic error calculus is valid on*

$$\tilde{\mathcal{D}} = \{F(f_1, \dots, f_p) : f_i \in \mathcal{D}, \quad F \in \mathcal{C}^1(\mathbb{R}^p, \mathbb{R})\}$$

$$\begin{aligned} \text{i.e. } \lim_n \alpha_n \mathbb{E}[(F(f_1(Y_n), \dots, f_p(Y_n)) - F(f_1(Y), \dots, f_p(Y)))^2] \\ = \mathbb{E}_Y[\sum_{i,j=1}^p F'_i(f_1, \dots, f_p)F'_j(f_1, \dots, f_p)\Gamma[f_i, f_j]]. \end{aligned}$$

Demonstration. a) Let us first give the argument in the case $p = 1$. When the form is local, by remark 3, $\lim_n \alpha_n \mathbb{E}[|f(Y_n) - f(Y)|^k] = 0 \quad \forall f \in \mathcal{D}$ for any integer

$k \geq 3$. Let $F \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$, writing the finite increments formula $F(y) - F(x) = (y - x) \int_0^1 F'(x + t(y - x)) dt$ we have

$$\alpha_n \mathbb{E}[(F \circ f(Y_n) - F \circ f(Y))^2] = \alpha_n \mathbb{E}[(f(Y_n) - f(Y))^2 \left(\int_0^1 F'(f(Y) + t(f(Y_n) - f(Y))) dt \right)^2].$$

Let P_k be a polynomial uniformly close to F' on the closed ball $B(0, \|f\|_\infty)$, considering

$$\begin{aligned} (I) &= \alpha_n \mathbb{E}[(f(Y_n) - f(Y))^2 \left(\int_0^1 P_k(f(Y) + t(f(Y_n) - f(Y))) dt \right)^2] \\ &= \alpha_n \mathbb{E}[(f(Y_n) - f(Y))^2 \left((P_k(f(Y)))^2 + \sum_{\ell=0, m=1}^N \lambda_{\ell m} (f(Y))^\ell (f(Y_n) - f(Y))^m \right)] \\ &= \alpha_n \mathbb{E}[(f(Y) - f(Y_n))^2 \left((P_k(f(Y_n)))^2 + \sum_{\ell=0, m=1}^N \lambda_{\ell m} (f(Y_n))^\ell (f(Y) - f(Y_n))^m \right)] \end{aligned}$$

we observe that (I) has same limit as

$$\alpha_n \mathbb{E}[(f(Y_n) - f(Y))^2 ((P_k(f(Y)))^2 + (P_k(f(Y_n)))^2) / 2]$$

which converges to $\mathbb{E}_Y[\Gamma[f] P_k^2(f)]$ by theorem 1.

Now, $|\alpha_n \mathbb{E}[(F \circ f(Y_n) - F \circ f(Y))^2] - \mathbb{E}_Y[F'^2 \circ f \Gamma[f]]|$

$$\begin{aligned} &\leq \alpha_n \mathbb{E} \left[(f(Y_n) - f(Y))^2 \left| \left(\int_0^1 F'(f(Y) + t(f(Y_n) - f(Y))) dt \right)^2 \right. \right. \\ &\quad \left. \left. - \left(\int_0^1 P_k(f(Y) + t(f(Y_n) - f(Y))) dt \right)^2 \right| \right] \\ &\quad + \left| \alpha_n \mathbb{E} \left[(f(Y_n) - f(Y))^2 \left(\int_0^1 P_k(f(Y) + t(f(Y_n) - f(Y))) dt \right)^2 \right] - \mathbb{E}_Y[P_k^2 \circ f \Gamma[f]] \right| \\ &\quad + |\mathbb{E}_Y[P_k^2 \circ f \Gamma[f]] - \mathbb{E}_Y[F'^2 \circ f \Gamma[f]]|. \end{aligned}$$

Since $\sup_n \alpha_n \mathbb{E}[(f(Y_n) - f(Y))^2] < +\infty$, the first and the last terms may be made small uniformly in n by a suitable choice of k , as the second term goes to zero when $n \uparrow +\infty$, the proof is complete in this case.

b) In the general case the finite increments formula writes

$$F(y_1, \dots, y_p) - F(x_1, \dots, x_p) = \sum_{i=1}^p (y_i - x_i) \int_0^1 F'_i(y_1, \dots, y_{i-1}, x_i + t(y_i - x_i), x_{i+1}, \dots, x_p) dt.$$

The local property of the form implies

$$(2) \quad \lim_n \alpha_n \mathbb{E} \left[\prod_{i=1}^k |f_i(Y_n) - f_i(Y)| \right] = 0 \quad \forall f_1, \dots, f_k \in \mathcal{D} \quad \forall k \geq 3$$

by Hölder inequality $\mathbb{E} \left[\prod_{i=1}^k X_i \right] \leq \prod_{i=1}^k (\mathbb{E}[|X_i|^k])^{1/k}$. Then the proof proceeds similarly, approximating the derivatives F'_i by polynomials $P_{k,i}$ on the ball $B(0, \max_i \|f_i\|_\infty)$ of \mathbb{R}^p using (2) and the property $\forall \varphi, \chi \in \mathcal{D}$

$$\lim_n \alpha_n \mathbb{E}[(f_i(Y_n) - f_i(Y))(f_j(Y_n) - f_j(Y)) (\varphi(Y_n)\chi(Y) + \varphi(Y)\chi(Y_n)) / 2] = \mathbb{E}_Y[\Gamma[f_i, f_j]\varphi\chi]$$

which is consequence of theorem 1. ◇

Let us end this part by a remark concerning the transportation of the four bias operators by image (cf. also [8]).

Remark 7. Let Y_n be an approximation of Y satisfying (H1) to (H3) on the same dense algebra \mathcal{D} of bounded functions with constants. Let Φ be a map from (E, \mathcal{F}) to (G, \mathcal{H}) such that the algebra $\mathcal{D}_\Phi = \{u : u \circ \Phi \in \mathcal{D}\}$ be dense in $L^2(\mathbb{P}_{\Phi \circ Y})$. Let us put $Z_n = \Phi \circ Y_n$ and $Z = \Phi \circ Y$. Then Z_n and Z satisfy (H1) to (H3) with \mathcal{D}_Φ and the same sequence α_n :

$$\begin{aligned}\bar{A}_\Phi[u](z) &= \mathbb{E}_Y[\bar{A}[u \circ \Phi] \mid \Phi = z] \\ \underline{A}_\Phi[u](z) &= \mathbb{E}_Y[\underline{A}[u \circ \Phi] \mid \Phi = z]\end{aligned}$$

and similar relations for \tilde{A}_Φ and \mathbb{A}_Φ . The Dirichlet form associated with \tilde{A}_Φ is the image by Φ of the Dirichlet form associated with \tilde{A} , and the properties of images of Dirichlet forms (cf. [9] chapter V) apply (square field operator, locality). If \mathbb{A} is first order, \mathbb{A}_Φ is first order.

II. Examples.

II.0. Preliminary example. This is not strictly speaking an example but a part of Dirichlet forms theory itself. Let $(\mathcal{E}, \mathbb{D})$ be a Dirichlet form on the Hilbert space $L^2(E, \mathcal{F}, m)$ where m is a probability measure and let (P_t) be the strongly continuous contraction semi-group associated with $(\mathcal{E}, \mathbb{D})$.

Let us suppose that the quasi-regularity assumption is fulfilled so that we may construct a Markov process Y_t with P_t as transition semi-group (cf. [23] chapter IV §3), and let us suppose also that the domain \mathcal{DA} of the generator (A, \mathcal{DA}) contains an algebra \mathcal{D} of bounded functions with constants dense in L^2 . Then for $f \in \mathcal{D}$, the approximate forms

$$\mathcal{E}_t[f] = \frac{1}{t} \langle f - P_t f, f \rangle_{L^2(m)} = \frac{1}{2t} \mathbb{E}_m[(f(Y_0) - f(Y_t))^2]$$

do converge (increasingly) when $t \downarrow 0$ to $\mathcal{E}[f] = - \langle Af, f \rangle$ (cf. [13], [9] or [5]). Hence hypothesis (H3) is fulfilled. The form $(\mathcal{E}, \mathbb{D})$ is an extension of that one provided by theorem 1.

Here, as easily seen, we have

$$\bar{A}[f] = \underline{A}[f] = \tilde{A}[f] = A[f] \quad \forall f \in \mathcal{D}$$

and the operator \mathbb{A} vanishes. The above properties of Dirichlet forms hold either for local or non-local forms. Since $\frac{1}{2t} \uparrow +\infty$ we see that the hypothesis (H3) may be satisfied with $\alpha_n \uparrow +\infty$ the limit form being nevertheless non-local (cf. e) of theorem 1).

Note. In the whole article the positive symmetric bilinear forms of real functions $\mathcal{E}[f, g]$ are extended to complex functions, not as Hermitian forms, but as bilinear forms with the same symbol \mathcal{E} , in other words

$$\mathcal{E}[f_1 + if_2, g_1 + ig_2] = \mathcal{E}[f_1, f_2] + i\mathcal{E}[f_1, g_2] + i\mathcal{E}[f_2, g_1] - \mathcal{E}[f_2, g_2].$$

II.1. Error in the Glivenko-Cantelli theorem. Let us begin with a simple one-dimensional example related to the Glivenko-Cantelli theorem. Let X be a real random variable with continuous distribution function F and let X_k be i.i.d. copies of X .

If we put

$$Y_n = \frac{1}{n} \sum_{k=1}^n 1_{\{X_k \leq X\}}$$

and $Y = F(X)$, we have $Y_n \rightarrow Y$ a.s. and we may study Y_n as approximation of Y . Thinking for instance X and the X_k 's to be simulated by the inversion method shows that the pair (Y_n, Y) has the same law as $(\frac{1}{n} \sum_{k=1}^n 1_{U_k \leq U}, U)$ where U is uniformly distributed on $[0, 1]$ and the U_k are i.i.d. copies of U . Since our framework involves only the joint law of Y_n and Y we may work with (U_k, U) instead of (X_k, X) . Choosing $\alpha_n = n$ and $\mathcal{D} = \mathcal{L}\{x \mapsto e^{2i\pi px}, p \in \mathbb{Z}\}$, we obtain that hypotheses (H1) to (H3) are fulfilled with

$$\begin{aligned} \overline{A}[\varphi](y) &= \frac{y-y^2}{2} \varphi''(y) \\ \underline{A}[\varphi](y) &= \frac{y-y^2}{2} \varphi''(y) + (1-2y)\varphi'(y) \\ \widetilde{A}[\varphi](y) &= \frac{y-y^2}{2} \varphi''(y) + \frac{1-2y}{2} \varphi'(y) \\ \widetilde{\mathcal{E}}[\varphi] &= - \langle \underline{A}\varphi, \varphi \rangle = \int_0^1 \frac{y-y^2}{2} \varphi'^2(y) dy \\ \mathbb{A}[\varphi](y) &= (y - \frac{1}{2})\varphi'(y) \end{aligned}$$

the theoretical and practical variances coincide and $\mathbb{D} = \{f \in L^2[0, 1] : f' \text{ in distribution sense } \in L^1_{loc}([0, 1]) \text{ and } y \mapsto \sqrt{y-y^2}f'(y) \in L^2(dy)\}$. Let us give some indication on the proof of hypothesis (H1) for instance. Denoting \mathbb{E}_y the conditional law given $Y = y$, we have to study

$$n \int_0^1 \mathbb{E}_y[(e^{2i\pi p Y_n} - e^{2i\pi p y})e^{2i\pi q y}] dy = n \int_0^1 ((e^{2i\pi p/n} y + 1 - y)^n - e^{2i\pi p y})e^{2i\pi q y} dy$$

which may be expanded as

$$= n \int_0^1 \left(\exp\left\{-\frac{y-y^2}{2n}(2\pi p)^2\right\} (1 + \varepsilon(n, y)) - 1 \right) e^{2i\pi(p+q)y} dy$$

where $\varepsilon(n, y)$ goes to zero uniformly in y when $n \rightarrow \infty$. Using $e^{-\lambda} - 1 = -\lambda \int_0^1 e^{-t\lambda} dt$, the dominated convergence theorem applies and the limit is

$$\int_0^1 -\frac{y-y^2}{2}(2\pi p)^2 e^{2i\pi(p+q)y} dy = \mathbb{E}_Y\left[\frac{Y-Y^2}{2}\varphi''(Y)\chi(Y)\right]$$

for $\varphi(y) = \exp 2i\pi py$ and $\chi(y) = \exp 2i\pi qy$.

II.2. Typical formulae of finite dimensional error calculus.

II.2.a. Let us consider a triplet of real random variables (Y, Z, T) and a real random variable G independent of (Y, Z, T) centered with variance one. We are interested in the approximation Y_ε of Y given by

$$(3) \quad Y_\varepsilon = Y + \varepsilon Z + \sqrt{\varepsilon}TG.$$

In the multidimensional case, Y is with values in \mathbb{R}^p as Z , T is a $p \times q$ -matrix and G is independent of (Y, Z, T) with values in \mathbb{R}^q , centered, square integrable, such that $\mathbb{E}[G_i G_j] = \delta_{ij}$.

Operator \bar{A} .

Proposition 6. *If Z and T are square integrable, if φ is \mathcal{C}^2 bounded with bounded derivatives of first and second orders ($\varphi \in \mathcal{C}_b^2$) and if χ is bounded,*

$$\frac{1}{\varepsilon} \mathbb{E}[(\varphi(Y_\varepsilon) - \varphi(Y))\chi(Y)] \rightarrow \mathbb{E}_Y[\bar{A}[\varphi]\chi]$$

where $\bar{A}[\varphi](y) = \mathbb{E}[Z|Y=y]\varphi'(y) + \frac{1}{2}\mathbb{E}[T^2|Y=y]\varphi''(y)$.

In the multidimensional case

$$\bar{A}[\varphi](y) = \mathbb{E}[Z^t|Y=y]\nabla\varphi(y) + \frac{1}{2} \sum_{ij} \mathbb{E}[(TT^t)_{ij}|Y=y]\varphi''_{ij}(y).$$

Proof. Let us give the argument with the notation of the case $q = p = 1$. The Taylor-Lagrange formula applied up to second order gives

$$\begin{aligned} \frac{1}{\varepsilon} \mathbb{E}[(\varphi(Y_\varepsilon) - \varphi(Y))\chi(Y)] &= \mathbb{E}[Z\varphi'(Y)\chi(Y)] \\ &+ \frac{1}{2} \mathbb{E}[(\varepsilon Z^2 + 2\sqrt{\varepsilon}ZTG + T^2G^2) \int_0^1 \int_0^1 \varphi''(Y + ab(\varepsilon Z + \sqrt{\varepsilon}TG)) 2adadb \chi(Y)] \end{aligned}$$

(note that ZTG and $T^2G^2 \in L^1$ because of the independence) and this converges by dominated Lebesgue theorem to $\mathbb{E}[Z\varphi'(Y)\chi(Y)] + \frac{1}{2}\mathbb{E}[T^2\varphi''(Y)\chi(Y)]$. \diamond

Quadratic form and operator \tilde{A} .

Proposition 7. *If Z and T are square integrable, if φ and χ are \mathcal{C}_b^1*

$$\frac{1}{\varepsilon} \mathbb{E}[(\varphi(Y_\varepsilon) - \varphi(Y))(\chi(Y_\varepsilon) - \chi(Y))] \rightarrow \mathbb{E}[T^2\varphi'(Y)\chi'(Y)]$$

and in the multidimensional case

$$\frac{1}{\varepsilon} \mathbb{E}[(\varphi(Y_\varepsilon) - \varphi(Y))(\chi(Y_\varepsilon) - \chi(Y))] \rightarrow \mathbb{E}[(\nabla\varphi)^t(Y)TT^t\nabla\chi(Y)].$$

Proof. The demonstration is similar with a first order expansion. \diamond

In order to exhibit the operator \tilde{A} , we must examine the conditions of an integration by parts in the preceding limit. Let us put $\theta_{ij}(y) = \mathbb{E}[(TT^t)_{ij}|Y = y]$ so that $\mathbb{E}[(\nabla\varphi)^t(Y)TT^t\nabla\chi(Y)] = \sum_{ij} \mathbb{E}_Y[\varphi'_i\theta_{ij}\chi'_j]$.

Proposition 8. *If Z and T are square integrable, if for $i, j = 1, \dots, p$ the measure $\theta_{ij}\mathbb{P}_Y$ on \mathbb{R}^p possesses a partial derivative in the sense of distributions $\partial_j(\theta_{ij}\mathbb{P}_Y)$ which is a bounded measure absolutely continuous w.r. to \mathbb{P}_Y , say $\rho_{ij}\mathbb{P}_Y$, then as soon as θ_{ij} and $\rho_{ij} \in L^2(\mathbb{P}_Y)$ the form $\tilde{\mathcal{E}}[\varphi, \chi] = \frac{1}{2} \sum_{ij} \mathbb{E}_Y[\varphi'_i\theta_{ij}\chi'_j]$ is closable on the algebra $\mathcal{D} = \mathcal{C}_b^2$, hypotheses (H1) to (H3) are fulfilled and*

$$\tilde{A}[\varphi] = \frac{1}{2} \sum_{ij} \theta_{ij} \varphi''_{ij} + \frac{1}{2} \sum_{ij} \rho_{ij} \varphi'_j.$$

Proof. We have

$$\sum_{ij} \int \theta_{ij} \varphi'_i \chi'_j d\mathbb{P}_Y = \sum_{ij} \int \theta_{ij} (\partial_j(\varphi'_i \chi) - \varphi''_{ij} \chi) d\mathbb{P}_Y$$

and the equality

$$\int \theta_{ij} \partial_j(\varphi'_i \chi) d\mathbb{P}_Y = - \int \varphi'_i \chi \rho_{ij} d\mathbb{P}_Y$$

valid for $\varphi, \chi \in \mathcal{C}_K^\infty$ extends, under the assumptions of the statement, to $\varphi, \chi \in \mathcal{C}_b^2$. This yields

$$\frac{1}{2} \sum_{ij} \mathbb{E}[\varphi'_i \theta_{ij} \chi'_j] = -\frac{1}{2} \int (\sum_{ij} \theta_{ij} \varphi''_{ij} + \sum_{ij} \rho_{ij} \varphi'_j) \chi d\mathbb{P}_Y.$$

Q.E.D. \diamond

The operator \tilde{A} depends only on T , not on Z . We obtain \underline{A} by difference :

$$\underline{A}[\varphi] = \frac{1}{2} \sum_{ij} \theta_{ij} \varphi''_{ij} + \sum_j (\sum_i \rho_{ij} - z_j) \varphi'_j$$

where $z_j(y) = \mathbb{E}[Z_j|Y = y]$. At last, \underline{A} is first order :

$$\underline{A}[\varphi] = \sum_j (z_j - \frac{1}{2} \sum_i \rho_{ij}) \varphi'_j.$$

Remark 9. The results of this section II.2.a) would be identical with an approximation of the form

$$(4) \quad Y_\varepsilon = Y + \varepsilon Z + T.B_\varepsilon$$

where B is a centered Brownian motion vanishing at zero independent of (Y, Z, T) since only the joint law of (Y, Y_ε) is used.

The question is very close to the classical approach of Kolmogorov [20] to study Markov processes starting from the assumptions

$$\lim_{h \downarrow 0} \frac{1}{h} \mathbb{E}[(X_{t+h} - X_t) | \mathcal{F}_t] = b(t, X_t)$$

$$\lim_{h \downarrow 0} \frac{1}{h} \mathbb{E}[(X_{t+h} - X_t)^2 | \mathcal{F}_t] = a(t, X_t)$$

Indeed, it is easy to see that the representation (4) occurs naturally for Ito processes and for diffusion processes given by an Ito equation :

On a filtered probability space $(\Omega, (\mathcal{F}_t), \mathbb{P})$, let B_t be an (\mathcal{F}_t) -Brownian motion centered vanishing at zero and let ξ be an Ito process defined by

$$\xi_t = \xi_0 + \int_0^t \sigma_s dB_s + \int_0^t b_s ds$$

where the processes σ and b are adapted and continuous at zero in $L^2(\mathbb{P})$ and $L^1(\mathbb{P})$ respectively. Then approximating ξ_0 by ξ_t is equivalent to approximating ξ_0 by $\tilde{\xi}_t = \xi_0 + tb_0 + \sigma_0 B_t$ because for $\varphi \in \mathcal{C}_b^2$

$$\lim_{t \rightarrow 0} \frac{1}{t} \mathbb{E}[(\varphi(\xi_t) - \varphi(\tilde{\xi}_t))^2 | \mathcal{F}_0] = 0$$

$$\lim_{t \rightarrow 0} \frac{1}{t} \mathbb{E}[(\varphi(\xi_t) - \varphi(\tilde{\xi}_t)) | \mathcal{F}_0] = 0$$

as soon as for instance $\mathbb{E}\sigma_t^4$ is bounded in a neighborhood of zero, as seen by application of Ito formula and standard inequalities.

II.2.b. Series with independent increments.

Let be

$$S = \sum_{n=1}^{\infty} \frac{X_n}{n^2} + \frac{Z_n}{n}$$

where $X_n, Z_n \in L^{2+\varepsilon}$, Z_n centered, (X_n, Z_n) i.i.d., we approximate S by its partial sum $S_n = \sum_{k=1}^n \frac{X_k}{k^2} + \frac{Z_k}{k}$.

Using Burkholder inequality, we observe that $n\mathbb{E}[|S - S_n|^{2+\varepsilon}] \rightarrow 0$ as $n \rightarrow \infty$. Thus, taking $\mathcal{D} = \mathcal{C}_K^\infty$, we have for $\varphi, \chi \in \mathcal{D}$

$$\lim_n n\mathbb{E}[(\varphi(S) - \varphi(S_n))^2] = \lim_n n\mathbb{E}[(S - S_n)^2 \varphi'^2(S_n)] = \mathbb{E}[Z_1^2] \mathbb{E}[\varphi'^2(S)]$$

$$\begin{aligned} \lim_n n\mathbb{E}[(\varphi(S) - \varphi(S_n))\chi(S_n)] &= \lim_n n\mathbb{E}[(S - S_n)\varphi'(S_n)\chi(S_n) + \frac{1}{2}(S - S_n)^2 \varphi''(S_n)\chi(S_n)] \\ &= \frac{1}{2} \mathbb{E}[Z_1^2] \mathbb{E}[\varphi''(S)\chi(S)] + \mathbb{E}[X_1] \mathbb{E}[\varphi'(S)\chi(S)]. \end{aligned}$$

We can conclude that hypothesis (H2) is satisfied and

$$\underline{A}[\varphi] = \frac{\mathbb{E}[Z_1^2]}{2} \varphi'' + \mathbb{E}[X_1] \varphi'.$$

Assumption (H3) holds as soon as the law of S satisfies the Hamza condition ([13] p.105) and then the Dirichlet form is local.

II.2.c. Tails of martingales.

Let us first consider the classical case of Polya's urn in its simplest configuration with two colors, one ball added each time, and an initial composition of one white ball and one black ball.

The ratio X_n of white balls after the n -th drawing satisfies

$$X_{n+1}(n+3) = X_n(n+1) + 1_{U_{n+1} \leq X_n}$$

where U_{n+1} is a random variable uniformly distributed on $[0, 1]$ independent of $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$, i.e.

$$X_{n+1} = X_n + \frac{1}{n+3}(1_{U_{n+1} \leq X_n} - X_n).$$

Let X be the (a.s. and L^p , $1 \leq p < +\infty$) limit of the bounded martingale X_n , we study the approximation of X by X_n .

We note that $\lim_n n\mathbb{E}[|X - X_n|^3] = 0$ as easily seen using Burkholder inequality. Then, taking for \mathcal{D} the functions of class \mathcal{C}^3 on $[0, 1]$ vanishing at 0 and 1, we have $\forall \varphi, \chi \in \mathcal{D}$

$$\lim_n n\mathbb{E}[(\varphi(X) - \varphi(X_n))\chi(X_n)] = \lim_n n\mathbb{E}[(X - X_n)\varphi'(X_n)\chi(X_n) + \frac{1}{2}(X - X_n)^2\varphi''(X_n)\chi(X_n)]$$

and

$$\begin{aligned} \mathbb{E}(X - X_n)^2 &= \mathbb{E} \sum_{k=n}^{\infty} \frac{\mathbb{E}[(1_{U_{k+1} \leq X_k} - X_k)^2 | \mathcal{F}_k]}{(k+3)^2} \\ &= \sum_{k=n}^{\infty} \frac{1}{(k+3)^2} \mathbb{E}[X_k(1 - X_k)] \\ &\sim \frac{1}{6n} \end{aligned}$$

because $X_n \rightarrow X$ and X is uniformly distributed on $[0, 1]$, as easily verified.

We obtain

$$\lim_n n\mathbb{E}[(\varphi(X) - \varphi(X_n))\chi(X_n)] = \frac{1}{12}\mathbb{E}[\varphi''(X)\chi(X)]$$

$$\lim_n n\mathbb{E}[(\varphi(X) - \varphi(X_n))^2] = \frac{1}{6}\mathbb{E}[\varphi'^2(X)].$$

Hence (H1) to (H3) are fulfilled $\underline{A}[\varphi] = \frac{1}{12}\varphi''$ and $\tilde{A}[\varphi] = \frac{1}{12}\varphi''$ so that $\underline{A} = \bar{A}$ and $\mathbb{A} = 0$. The limit error structure is the uniform error structure on $[0, 1]$. This analysis could be easily extended to any configuration of Polya's urn, mutatis mutandis.

More generally, this kind of asymptotic behavior appears, under regularity assumptions, for the approximation between a martingale and its limit.

Let $M_n = \sum_{i=1}^n X_i$ be a martingale w.r. to the filtration \mathcal{F}_n . Let us suppose M_n centered, square integrable s.t. $\sum_{i=1}^{\infty} \mathbb{E}X_i^2 < +\infty$ and let us put $\sigma_n^2 = \sum_{i=n+1}^{\infty} \mathbb{E}X_i^2$.

Proposition 9. *Supposing $\frac{1}{\sigma_n} \sup_{i>n} |X_i| \rightarrow 0$ in probability, $\mathbb{E}[\frac{1}{\sigma_n^2} \sup_{i>n} X_i^2]$ bounded in n , $\mathbb{E}[(\frac{1}{\sigma_n^2} (\sum_{i>n} X_i)^2)^p]$ bounded in n for some $p > 1$, and $\frac{1}{\sigma_n^2} \sum_{i>n} X_i^2 \rightarrow \zeta^2$ in probability, then*

a) $\frac{1}{\sigma_n} \sum_{i>n} X_i \xrightarrow{d} Z$, where Z has for characteristic function $\mathbb{E}[e^{-\frac{1}{2}\zeta^2 t^2}]$

b) hypothesis (H2) is satisfied and $\forall \varphi, \chi \in \mathcal{C}_K^{\infty}$,

$\lim_n \frac{1}{\sigma_n^2} \mathbb{E}[(\varphi(M_{\infty}) - \varphi(M_n))\chi(M_n)] = \frac{1}{2} \mathbb{E}Z^2 \mathbb{E}[\varphi''(M_{\infty})\chi(M_{\infty})]$ what gives

$\underline{A}[\varphi] = \frac{1}{2} \mathbb{E}[Z^2] \varphi''$.

c) $\lim_n \frac{1}{\sigma_n^2} \mathbb{E}[(\varphi(M_{\infty}) - \varphi(M_n))^2] = \mathbb{E}[Z^2] \mathbb{E}[\varphi'^2(M_{\infty})]$, hypothesis (H3) is satisfied as soon as the law of M_{∞} satisfies the Hamza condition, then $\underline{A} = \bar{A} = \tilde{A}$ and $\mathfrak{A} = 0$.

Proof. The proposition is a direct consequence of a result of Hall and Heyde ([14] §3.5 p.76 et seq.).

II.3. Conditionally Gaussian case

Let us begin with the finite dimensional case before applying the approach to processes.

II.3.a. Let Y be a r.v. with values in \mathbb{R}^d , $V = (V_{ij})_{i,j=1,\dots,d}$ be an application from \mathbb{R}^d into symmetric positive $d \times d$ -matrices, ξ_j be r.v. with values in \mathbb{R}^d which conditionally given $Y = y$ are i.i.d. Gaussian with common law $\mathcal{N}_d(y, V(y))$. We consider

$$Y_n = \frac{1}{n} \sum_{j=1}^n \xi_j$$

as approximation of Y . We take $\mathcal{D} = \mathcal{L}\{x \mapsto e^{i\langle u, x \rangle}, u \in \mathbb{R}^d\}$ and $\alpha_n = n$.

Lemma 1. *Let us suppose $\mathbb{E}[\text{trace}(V(Y))] < +\infty$, then*

1) $\lim_n n \mathbb{E}[(\varphi(Y_n) - \varphi(Y))^2] = \mathbb{E}_Y[\sum_{i,j=1}^d \varphi'_i \varphi'_j V_{ij}] \quad \forall \varphi \in \mathcal{D}$,

2) the hypothesis (H1) is satisfied and $\bar{A}[\varphi](y) = \frac{1}{2} \sum_{i,j=1}^d V_{ij}(y) \varphi''_{ij}(y) \quad \forall \varphi \in \mathcal{D}$.

The proof proceeds without difficulties as in the preceding example by finite expansions of the exponential functions.

Lemma 2. *Let us suppose $\mathbb{E}[\text{trace}(V(Y))] < +\infty$ and the following regularity condition: $\forall i, j$, the measure $V_{ij} \mathbb{P}_Y$ has a partial derivative $\partial_j(V_{ij} \mathbb{P}_Y)$ in the sense of distributions which is a bounded measure absolutely continuous w.r. to \mathbb{P}_Y , say $\rho_{ij} \mathbb{P}_Y$,*

then as soon as V_{ij} and $\rho_{ij} \in L^2(\mathbb{P}_Y)$, (H1) to (H3) are fulfilled and

$$\begin{aligned}\underline{A}[\varphi] &= \frac{1}{2} \sum_{ij} V_{ij} \varphi''_{ij} + \sum_{ij} \rho_{ij} \varphi'_j \\ \widetilde{A}[\varphi] &= \frac{1}{2} \sum_{ij} V_{ij} \varphi''_{ij} + \frac{1}{2} \sum_{ij} \rho_{ij} \varphi'_j \\ \mathbb{A}[\varphi] &= -\frac{1}{2} \sum_{ij} \rho_{ij} \varphi'_j\end{aligned}$$

Proof. The condition of the statement allows to perform an integration by parts in the limit obtained in lemma 2. That gives (H3) hence (H2) as well. \diamond

Remark 9. There are several sufficient conditions in order that the form $\hat{\mathcal{E}}[u, v] = \mathbb{E}_Y[\sum_{ij} u'_i v'_j V_{ij}]$ be closable on $\mathcal{C}_K^\infty(\mathbb{R}^d)$ (cf. [13] chapter 3 §3.1 and [23] chapter II §2). Suppose such a condition holds, then by the argument of remark 2 the hypothesis (H3) is equivalent to $\mathcal{D} \subset \mathcal{DA}$ where $(\mathcal{A}, \mathcal{DA})$ is the generator of the smallest closed extension of $(\hat{\mathcal{E}}, \mathcal{C}_K^\infty(\mathbb{R}^d))$.

II.3.b. Let us apply this to the approximation of processes. Let Y be a real process indexed by a set T . Let us consider Y as a measurable map from $(\Omega, \mathcal{A}, \mathbb{P})$ into $(E, \mathcal{F}) = (\mathbb{R}^T, (\mathcal{B}(\mathbb{R}))^{\otimes T})$. Let y_t be the coordinate mappings from E on \mathbb{R} . We consider the algebra

$$\mathcal{D} = \mathcal{L}\{e^{iu_1 y_{t_1} + \dots + iu_k y_{t_k}}, u_j \in \mathbb{R}, t_j \in T\}.$$

Thanks to the monotone class theorem, \mathcal{D} is a dense algebra in $L^2(E, \mathcal{F}, \mathbb{P}_Y)$. We put $Y_t = y_t \circ Y$.

Let $\xi^j = (\xi_t^j)_{t \in T}$ be a sequence of real processes such that, conditionally given Y the ξ^j are independent with the same Gaussian law with

$$\mathbb{E}[\xi_t^1 | Y] = Y_t$$

$$\mathbb{E}[(\xi_s^1 - Y_s)(\xi_t^1 - Y_t) | Y] = C_{s,t}(Y_s, Y_t)$$

the function $C_{s,t}(x_1, x_2)$ and the process Y being such that $\mathbb{E}[C_{s,t}(Y_s, Y_t)] < +\infty \quad \forall s, t \in T$.

We approximate Y by the process $Y_n = \frac{1}{n} \sum_{j=1}^n \xi^j$. The results obtained in the finite dimensional case give the following proposition :

Proposition 10. *If the marginal laws of Y $\mathbb{P}_{(Y_{t_1}, \dots, Y_{t_k})}(dy_{t_1} \dots dy_{t_k})$ possess partial derivatives in the sense of distributions $\frac{\partial}{\partial y_{t_i}}$ which are bounded measures absolutely continuous w.r. to $\mathbb{P}_{(Y_{t_1}, \dots, Y_{t_k})}$, say $\lambda_{ij}(y_{t_1}, \dots, y_{t_k}) \mathbb{P}_Y$, then hypotheses (H1) to (H3) are verified and for $\varphi \in \mathcal{D}$ we have*

$$\begin{aligned}\overline{A}[\varphi](y_{t_1}, \dots, y_{t_k}) &= \frac{1}{2} \sum_{i,j=1}^k C_{t_i, t_j}(y_{t_i}, y_{t_j}) \frac{\partial^2 \varphi}{\partial y_{t_i} \partial y_{t_j}} \\ \widetilde{A}[\varphi](y_{t_1}, \dots, y_{t_k}) &= \frac{1}{2} \sum_{i,j=1}^k C_{t_i, t_j}(y_{t_i}, y_{t_j}) \frac{\partial^2 \varphi}{\partial y_{t_i} \partial y_{t_j}} + \sum_{i,j} \lambda_{ij}(y_{t_1}, \dots, y_{t_k}) \frac{\partial \varphi}{\partial y_{t_j}} \\ \Gamma[\varphi] &= \sum_{i,j=1}^k C_{t_i, t_j} \frac{\partial \varphi}{\partial y_{t_i}} \frac{\partial \varphi}{\partial y_{t_j}}.\end{aligned}$$

Under the hypotheses of proposition 10, the form $(\mathcal{E}, \mathbb{D})$ is local and theorem 2 on asymptotic error calculus applies. Let us also remark that if we define the operator $\#$ on \mathcal{D} by

$$\varphi^\# = \sum_{j=1}^k \frac{\partial \varphi}{\partial y_{t_j}} \circ Y.(\xi_{t_j}^1 - Y_{t_j})$$

when φ depends only on y_{t_1}, \dots, y_{t_k} , we have

$$\mathbb{E}[(\varphi^\#)^2 | Y] = \Gamma[\varphi](Y)$$

and $\int (\varphi^\#)^2 d\mathbb{P}_Y(y) = 2\mathcal{E}[\varphi]$. It follows that the operator $\#$ extends uniquely to \mathbb{D} in a closed operator satisfying for $F \in \mathcal{C}^1 \cap Lip(\mathbb{R}^p, \mathbb{R})$

$$(F(f_1, \dots, f_p))^\# = \sum_{j=1}^p F'_j(f_1, \dots, f_p) f_j^\# \quad \forall f_1, \dots, f_p \in \mathbb{D}$$

In other words $\#$ plays the role of a gradient w.r. to the Dirichlet form $(\mathcal{E}, \mathbb{D})$.

Special case 1. Y is a real process indexed by \mathbb{R}_+ , and $\xi_t^j = Y_t + W_t^j$ where W_t^j are independent standard Brownian motions independent of Y . If the marginal laws of Y have densities f_{t_1, \dots, t_k} s.t. $\frac{\partial f_{t_1, \dots, t_k}}{\partial y_{t_i}} = \lambda_i(y_{t_1}, \dots, y_{t_k}) f_{t_1, \dots, t_k}$ with $\lambda_i \in L^1(\mathbb{R}^k)$, the construction applies and yields a Dirichlet form with square field operator Γ s.t.

$$\Gamma[\varphi](Y) = - \sum_{ij} u_i u_j t_i \wedge t_j \varphi^2(Y) \quad \text{for } \varphi(Y) = e^{i \sum_k u_k Y_{t_k}}.$$

Suppose Y possesses second order moments, then the linear forms $\ell = \sum_{p=1}^k a_p (Y_{t_{p+1}} - Y_{t_p})$ are in \mathbb{D} and $\Gamma[\ell] = \sum_{p=1}^k a_p^2 (t_{p+1} - t_p)$, so that on step functions f

$$\Gamma\left[\int f dy\right] = \int f^2(s) ds.$$

Thus this error structure may be called *the Ornstein-Uhlenbeck structure on the process Y* .

Special case 2. With the above notation, let us assume that

$$\xi_t^j = Y_t - h(t) W_t^j$$

where h is a deterministic function. With the same hypotheses as above, we have on step functions f

$$\Gamma\left[\int f dy\right] = \mathbb{E}\left[\left(\int f(t) d(h(t) W_t)\right)^2\right]$$

Special case 3. Suppose eventually

$$\xi_t^j = Y_t + \int_0^t h(s) dW_s^j$$

then $C_{t_i, t_j} = \int_0^{t_i \wedge t_j} h^2(s) ds$ and $\Gamma[\int f dy] = \int f^2(s) h^2(s) ds$. We obtain a structure which may be called *the Ornstein-Uhlenbeck structure with weight h on the process Y*.

II.4. Natural inaccuracy of the Brownian motion simulated by the Donsker theorem.

We begin with the simplest case of one dimensional marginal laws which is here nothing else than the central limit theorem.

II.4.a. Natural inaccuracy in the central limit theorem.

Let be $S_p = \sum_{i=1}^p V_i$ where the random variables V_i are i.i.d. centered with variance σ^2 . We consider two indices m and n linked by the relation

$$n = n(m) = m + k(m) \quad \text{with} \quad \theta\sqrt{m} \leq k(m) \leq \frac{1}{\theta}\sqrt{m}$$

for a $\theta \in]0, 1[$.

Let us observe some evident properties : $\lim_{m \rightarrow \infty} \frac{m}{n} = 1$; $\frac{n-m}{\sqrt{m}} \in [\theta, \frac{1}{\theta}]$; $\sqrt{n} \geq \sqrt{m}\sqrt{1+\theta} \geq \theta\sqrt{1+\theta}(n-m)$; $\frac{1}{\sqrt{m}} - \frac{1}{\sqrt{n}} \leq \frac{1}{2\theta m}$.

Writing $\frac{1}{\sqrt{n}}S_n - \frac{1}{\sqrt{m}}S_m = \frac{1}{\sqrt{n}}(S_n - S_m) + (\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{m}})S_m$ and using these properties, shows that $\frac{1}{\sqrt{n}}S_n - \frac{1}{\sqrt{m}}S_m \rightarrow 0$ a.s.

We consider the mutual approximation of $\frac{1}{\sqrt{m}}S_m$ and $\frac{1}{\sqrt{n}}S_n$ (which is an obvious extension of the framework of part I). That is for \bar{A} we study

$$(5) \quad \alpha(m)\mathbb{E}[(\varphi(\frac{1}{\sqrt{m}}S_m) - \varphi(\frac{1}{\sqrt{n}}S_n))\chi(\frac{1}{\sqrt{n}}S_n)]$$

and for \underline{A} we study

$$(6) \quad \alpha(m)\mathbb{E}[(\varphi(\frac{1}{\sqrt{n}}S_n) - \varphi(\frac{1}{\sqrt{m}}S_m))\chi(\frac{1}{\sqrt{m}}S_m)]$$

with $\alpha(m) = \frac{m}{k(m)}$ (so that $\theta\sqrt{m} \leq \alpha(m) \leq \frac{1}{\theta}\sqrt{m}$). For the algebra \mathcal{D} we take the linear combinations of imaginary exponentials.

Proposition 11. *Suppose the V_i 's possess a third order moment, then hypotheses (H1) to (H3) are fulfilled and for $\varphi \in \mathcal{D}$*

$$\bar{A}[\varphi](x) = \underline{A}[\varphi](x) = \tilde{A}[\varphi](x) = \frac{1}{2}\sigma^2\varphi'' - \frac{1}{2}x\varphi'.$$

The Dirichlet form is the Ornstein-Uhlenbeck form on \mathbb{R} (endowed with the normal law $\mathcal{N}(0, \sigma^2)$).

Proof. For \bar{A} , taking $\varphi(x) = e^{iux}$, $\chi(x) = e^{ivx}$ we have to look at

$$J_m = \alpha(m) \mathbb{E}[e^{i(\frac{u}{\sqrt{m}} + \frac{v}{\sqrt{m}})S_m + \frac{iv}{\sqrt{n}}(S_n - S_m)} - e^{i\frac{u+v}{\sqrt{n}}S_n}].$$

Let $\xi(t) = \mathbb{E}[e^{itV_1}]$ be the characteristic function of the V_i 's

$$J_m = \alpha(m) [\xi(\frac{u}{\sqrt{m}} + \frac{v}{\sqrt{n}})^m (\xi(\frac{v}{\sqrt{n}}))^{n-m} - (\xi(\frac{u+v}{\sqrt{n}}))^n]$$

using the existence of a third moment we have

$$(7) \quad \log \xi(t) = -\frac{t^2}{2} \sigma^2 (1 + at + t o(1))$$

and this allows to write $J_m = \alpha(m) [e^{-\frac{\sigma^2}{2} K_m} - e^{-\frac{\sigma^2}{2} L_m}]$ with

$$K_m = (u+v)^2 + \frac{a}{\sqrt{m}}(u+v)^3 + 2uv(\frac{\sqrt{m}}{\sqrt{n}} - 1) + \frac{1}{\sqrt{m}} o(1)$$

and $L_m = (u+v)^2 + \frac{a}{\sqrt{m}}(u+v)^3 + \frac{1}{\sqrt{m}} o(1)$. This yields

$$J_m = \alpha(m) e^{-\frac{\sigma^2}{2}(u+v)^2} [(-\frac{\sigma^2}{2})(2uv(\sqrt{\frac{m}{n}} - 1) + \frac{1}{\sqrt{m}} o(1))]$$

hence

$$\lim_m J_m = \frac{1}{2} \sigma^2 uv e^{-\frac{\sigma^2}{2}(u+v)^2} = -\frac{\sigma^2}{2} \int_{\mathbb{R}} \varphi' \chi' d\mathcal{N}(0, \sigma^2)$$

what gives easily the proposition. ◇

II.4.b. The Donsker case.

Let the V_i 's be as before and

$$X_n(t) = \frac{1}{\sqrt{n}} \left(\sum_{k=1}^{[nt]} V_k + (nt - [nt]) V_{[nt]+1} \right)$$

for $t \in [0, 1]$, $[nt]$ denoting the entire part of nt .

The laws of the variables X_n are probability measures on $\mathcal{C}([0, 1])$ as well as their limit in law which is a Brownian motion by Donsker theorem.

For the algebra \mathcal{D} we take the linear combinations of exponential of the form $\varphi(X) = \exp\{iX(f)\}$ where $X(f) = \int_0^1 f(s) dX(s)$ and with $f \in \mathcal{C}^1$ in order that $\int_0^1 f(s) dX(s)$ may be defined as $X(1)f(1) - X(0)f(0) - \int_0^1 X(s)df(s)$ for the general coordinate process $X(s)$ on $\mathcal{C}[0, 1]$. As easily seen the algebra \mathcal{D} is dense in $L^2(\mathcal{C}([0, 1]), \mu)$ μ being the Wiener measure.

Thus we have $X_n(f) = \sqrt{n} \sum_{k=0}^{n-1} \int_{k/n}^{(k+1)/n} f(s) ds V_{k+1}$ and for studying the operator \bar{A} we have to look at

$$\begin{aligned} M_m &= \alpha(m) \mathbb{E}[(\varphi(X_m) - \varphi(X_n))\chi(X_n)] = \alpha(m) \mathbb{E}[(e^{iX_m(f)} - e^{iX_n(f)})e^{iX_n(g)}] \\ &= \alpha(m) \mathbb{E} \left[\left(e^{i\sqrt{m} \sum_{k=0}^{m-1} \int_{\frac{k}{m}}^{\frac{k+1}{m}} f(s) ds V_{k+1}} - e^{i\sqrt{n} \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(s) ds V_{k+1}} \right) e^{i\sqrt{n} \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} g(s) ds V_{k+1}} \right]. \end{aligned}$$

We take as before $\alpha(m) = m/k(m)$.

Proposition 12. *Suppose the V_i 's possess a third order moment, then hypotheses (H1) to (H3) are fulfilled. We have $\bar{A} = \underline{A} = \tilde{A}$ on \mathcal{D} . The Dirichlet form is the Ornstein-Uhlenbeck form on the Wiener space (with a Brownian motion s.t. $\langle B \rangle_t = \sigma^2 t$) normalized so that the square field operator satisfies $\Gamma[\int_0^1 h(s) dB_s] = \int_0^1 h^2(s) \sigma^2 ds \quad \forall h \in L^2([0, 1])$.*

Since the Dirichlet form is local, some limits are automatically obtained (theorem 2). Since $\mathbb{A} = 0$, the theoretical and practical variances coincide (prop. 1).

Proof. For studying \bar{A} we consider the quantity M_m defined above. By the third moment assumption, the characteristic function $\xi(t)$ of the V_i 's satisfies (7) and we can write

$$\begin{aligned} M_m &= \alpha(m) \left[\prod_{j=m}^{n-1} \xi\left(\sqrt{n} \int_{\frac{j}{n}}^{\frac{j+1}{n}} g\right) \prod_{k=0}^{m-1} \xi\left(\sqrt{m} \int_{\frac{k}{m}}^{\frac{k+1}{m}} f + \sqrt{n} \int_{\frac{k}{n}}^{\frac{k+1}{n}} g\right) - \prod_{j=m}^{n-1} \xi\left(\sqrt{n} \int_{\frac{j}{n}}^{\frac{j+1}{n}} (f+g)\right) \right] \\ (8) \qquad &= \alpha(m) \left[e^{-\frac{\sigma^2}{2} N_m} - e^{-\frac{\sigma^2}{2} P_m} \right] \end{aligned}$$

with

$$\begin{aligned} N_m &= \sum_{j=m}^{n-1} n \left(\int_{\frac{j}{n}}^{\frac{j+1}{n}} g \right)^2 + an\sqrt{n} \left(\int_{\frac{j}{n}}^{\frac{j+1}{n}} g \right)^3 (1 + o(1)) \\ &\quad + \sum_{k=0}^{m-1} \left(\sqrt{m} \int_{\frac{k}{m}}^{\frac{k+1}{m}} f + \sqrt{n} \int_{\frac{k}{n}}^{\frac{k+1}{n}} g \right)^2 + a \left(\sqrt{m} \int_{\frac{k}{m}}^{\frac{k+1}{m}} f + \sqrt{n} \int_{\frac{k}{n}}^{\frac{k+1}{n}} g \right)^3 (1 + o(1)) \\ P_m &= \sum_{k=0}^{n-1} n \left(\int_{\frac{k}{n}}^{\frac{k+1}{n}} (f+g) \right)^2 + an\sqrt{n} \left(\int_{\frac{k}{n}}^{\frac{k+1}{n}} (f+g) \right)^3 (1 + o(1)). \end{aligned}$$

Using $m \sum_{j=0}^{m-1} \left(\int_{\frac{j}{m}}^{\frac{j+1}{m}} f \right)^2 = \int_0^1 f^2 + \frac{1}{m} O(1)$ and $\sup_j \left| \sqrt{n} \int_{\frac{j}{n}}^{\frac{j+1}{n}} g \right| \leq \frac{1}{\sqrt{n}} \|g\|_\infty$ we obtain

$$N_m = \int_0^1 g^2 + \int_0^1 f^2 + 2 \sum_{k=0}^{m-1} \sqrt{m} \sqrt{n} \int_{\frac{k}{m}}^{\frac{k+1}{m}} f \int_{\frac{k}{n}}^{\frac{k+1}{n}} g + \frac{a}{\sqrt{m}} \int_0^1 (f+g)^3 (1 + o(1)) + \frac{1}{n} O(1)$$

and

$$P_m = \int_0^1 (f + g)^2 + \frac{a}{\sqrt{m}} \int_0^1 (f + g)^3 (1 + o(1)) + \frac{1}{n} O(1)$$

Putting these expressions in (8) leads to

$$M_m = \alpha(m) \left(\exp\left[-\frac{\sigma^2}{2} \int_0^1 (f + g)^2\right] \right) \left[(-\sigma^2) \left(\sqrt{\frac{m}{n}} - 1\right) \int_0^1 fg + \frac{1}{\sqrt{m}} o(1) \right]$$

Eventually, for $\varphi(X) = \exp[i \int_0^1 f dX]$ and $\chi(X) = \exp[i \int_0^1 g dX]$ we get

$$\alpha(m) \mathbb{E}[(\varphi(X_m) - \varphi(X_n))\chi(X_n)] \rightarrow \frac{\sigma^2}{2} \exp\left[-\frac{\sigma^2}{2} \int_0^1 (f + g)^2 ds\right] \int_0^1 fg ds.$$

In order to recognize the obtained limit, let Γ_{ou} be the Ornstein-Uhlenbeck square field operator on the standard Wiener space (s.t. $\Gamma_{ou}[\int h dB] = \int h^2 ds$). We have by the functional calculus

$$\mathbb{E}\Gamma_{ou}[e^{i \int_0^1 f dB}, e^{i \int_0^1 g dB}] = - \int fg ds \mathbb{E}[e^{i \int (f+g) dB}] = - \int fg ds \exp\left[-\frac{1}{2} \int (f + g)^2 ds\right].$$

It follows that for a Wiener measure s.t. $\langle B \rangle_t = \sigma^2 t$ and an Ornstein-Uhlenbeck structure s.t. $\Gamma[\int_0^1 h dB] = \int h^2 d \langle B \rangle$ whose generator will be denoted (A, \mathcal{DA}) , we have $\mathcal{D} \subset \mathcal{DA}$ and

$$\begin{aligned} \langle A\varphi, \chi \rangle &= -\mathcal{E}[\varphi, \chi] = -\frac{1}{2} \mathbb{E}[\Gamma[\varphi, \chi]] \\ &= \frac{\sigma^2}{2} \int_0^1 fg ds \exp\left[-\frac{\sigma^2}{2} \int_0^1 (f + g)^2 ds\right] = \langle \bar{A}\varphi, \chi \rangle. \end{aligned}$$

The operator \bar{A} is therefore symmetric on \mathcal{D} , which implies $\bar{A} = \underline{A} = \tilde{A}$ and the proposition is proved. \diamond

Comment. As noted already by Louis Bachelier, assets quoted on the organized markets *look like* Brownian paths. This is displayed in any course in mathematical finance in order to introduce modelling by diffusion processes and stochastic calculus. Nevertheless some concrete facts prevent this resemblance from being accurate at microscopic scale. First because the spot is only defined at discrete instants, second because a Brownian path possesses ideal properties (like the fact that it cuts uncountably many times every level that it reaches) that cannot be verified by material recordings.

So that in order to be completely pragmatic, we might replace in financial models any Brownian motion by a random walk by application of Donsker theorem with n sufficiently large and consider the Brownian motion of the model *is nothing else* than a class of such sufficiently fine random walks. An infinite precision for stochastic calculus in finance is therefore a priori absurd and, by the results of this section, we may represent the intrinsic fuzzyness of these computations by the Ornstein-Uhlenbeck form on the Wiener space. This is a justification of the approach proposed in [4].

II.5. Empirical laws and natural inaccuracy of the Brownian bridge

If (V_n) are i.i.d. real random variables $0 \leq V_n \leq 1$ with distribution function F and if $F_n(x) = \frac{1}{n} \sum_{i=1}^n 1_{V_i \leq x}$ is the empirical distribution function, then $\sqrt{n}(F_n - F)$ converges in law on the Skorohod space to a transformed Brownian bridge $B_{F(x)} - F(x)B_1$ (see for instance [3] p. 141).

Considering the V_i 's are simulated by the inversion method shows that this result is a consequence of the special case where the V_i 's are uniformly distributed on $[0, 1]$. From now on, we restrict to this case. Putting $Z_n(x) = \sqrt{n}(F_n(x) - x)$ we are interested in the limit $R_m = \alpha(m) \mathbb{E}[(\varphi(Z_m) - \varphi(Z_n))\chi(Z_n)]$ for m and n linked as in the preceding example, with also $\alpha(m) = m/k(m)$ and for $\varphi, \chi \in \mathcal{D}$ where \mathcal{D} is the algebra of linear combinations of imaginary exponentials of the form $\varphi(Z) = \exp\{i \int_0^1 f(s) dZ(s)\} = \exp\{-i \int_0^1 Z(s) df(s)\}$ for $f \in \mathcal{C}^1([0, 1])$.

Thus

$$R_m = \alpha(m) \mathbb{E} \left[\left(e^{i \frac{1}{\sqrt{m}} \sum_{k=1}^m (f(V_k) - f)} - e^{i \frac{1}{\sqrt{n}} \sum_{k=1}^n (f(V_k) - f)} \right) e^{i \frac{1}{\sqrt{n}} \sum_{k=1}^n (g(V_k) - f)} \right].$$

Putting $\tilde{f} = f - \int_0^1 f ds$ and $\tilde{g} = g - \int_0^1 g ds$ and denoting η and ζ the characteristic functions of \tilde{f} and $\tilde{f} + \tilde{g}$ gives

$$R_m = \alpha(m) \left[(\rho(m))^m \left(\left(\eta\left(\frac{1}{\sqrt{n}}\right) \right)^{n-m} - \left(\zeta\left(\frac{1}{\sqrt{n}}\right) \right)^n \right) \right]$$

with $\rho(m) = \mathbb{E}[e^{i \frac{1}{\sqrt{m}} \tilde{f} + i \frac{1}{\sqrt{n}} \tilde{g}}]$.

The estimates

$$\begin{aligned} \rho(m) &= 1 - \frac{1}{2m} \mathbb{E}[(\tilde{f} + \sqrt{\frac{m}{n}} \tilde{g})^2] + \left(\frac{i}{\sqrt{m}}\right)^3 \frac{1}{6} \mathbb{E}[(\tilde{f} + \sqrt{\frac{m}{n}} \tilde{g})^3] + \frac{1}{m^2} O(1) \\ \zeta\left(\frac{1}{\sqrt{n}}\right) &= 1 - \frac{1}{2n} \mathbb{E}[(\tilde{f} + \tilde{g})^2] + \left(\frac{i}{\sqrt{n}}\right)^3 \frac{1}{6} \mathbb{E}[(\tilde{f} + \tilde{g})^3] + \frac{1}{m^2} O(1) \\ \eta\left(\frac{1}{\sqrt{n}}\right) &= 1 - \frac{1}{2n} \mathbb{E}[\tilde{g}^2] + \left(\frac{i}{\sqrt{n}}\right)^3 \frac{1}{6} \mathbb{E}[\tilde{g}^3] + \frac{1}{m^2} O(1) \end{aligned}$$

allow to obtain

$$\begin{aligned} \lim_m R_m &= e^{-\frac{1}{2} \mathbb{E}[(\tilde{f} + \tilde{g})^2]} \left(-\frac{1}{2} \mathbb{E}[\tilde{g}^2] + \frac{1}{2} \text{cov}(g, f + g) \right) \\ &= \frac{1}{2} e^{-\frac{1}{2} \mathbb{E}[(\tilde{f} + \tilde{g})^2]} \text{cov}(f, g). \end{aligned}$$

In order to recognize the limit, let Γ_{ou} be as before the Ornstein-Uhlenbeck square field operator on the Wiener space and $Z_t = B_t - tB_1$. We have

$$\Gamma_{ou}[e^{i \int f dZ}, e^{i \int g dZ}] = -e^{i \int (f+g) dZ} \Gamma_{ou} \left[\int f dZ, \int g dZ \right] = -e^{i \int (f+g) dZ} \left(\int f g ds - \int f ds \int g ds \right)$$

and consequently

$$\begin{aligned} \mathcal{E}_{ou}[\varphi, \chi] &= \frac{1}{2} \mathbb{E}[\Gamma_{ou}[\varphi, \chi]] = -\frac{1}{2} e^{-\frac{1}{2} \int (f+g)^2 ds - (\int (f+g) ds)^2} \left(\int f g ds - \int f ds \int g ds \right) \\ &= \lim_m R_m. \end{aligned}$$

As before denoting \underline{A} the Ornstein-Uhlenbeck operator, we see that $\overline{A} = A$ on \mathcal{D} and therefore $\overline{A} = \underline{A} = \widetilde{A}$, the Dirichlet form is the image of the Ornstein-Uhlenbeck form on the Wiener space.

II.6. Erroneous empirical laws and generalized Mehler type structures on the Brownian bridge

We still consider a sequence $V = (V)_{i \in \mathbb{N}}$ of i.i.d. random variables uniformly distributed on $[0, 1]$ and the empirical distribution function

$$F_n(x) = \frac{1}{n} \sum_{i=1}^n 1_{V \leq x}$$

but the problem that we tackle is different. We suppose that the law of V^1 is not perfectly known. We assume that there is a sequence of r.v. U_m^1 approximating V^1 and copies (U_m^i, V^i) of (U_m^1, V^1) such that the sequence $(U_m^i, V^i)_{i \in \mathbb{N}}$ be i.i.d. and we suppose that the law of U_m^1 has support in $[0, 1]$ with distribution function F^m . We define the empirical distribution function

$$F_n^m(x) = \frac{1}{n} \sum_{j=1}^n 1_{U_m^j \leq x}.$$

We are interested in the approximation

$$Z_n^m = \frac{1}{\sqrt{n}} \sum_{j=1}^n (1_{U_m^j \leq x} - F^m(x))$$

of the process

$$Z_n = \frac{1}{\sqrt{n}} \sum_{j=1}^n (1_{V^j \leq x} - x).$$

We take $\mathcal{D} = \mathcal{L}\{\varphi(z) = \exp[-i \int_0^1 z(t) df(t)], \quad f \in \mathcal{C}^1([0, 1])\}$ and we study

$$Q_{m,n} = \beta_m \mathbb{E}[(\varphi(Z_n^m) - \varphi(Z_n))^2] = \beta_m \mathbb{E}[(\exp[i \frac{1}{\sqrt{n}} \sum_{i=1}^n \widetilde{f(U_m^i)}] - \exp[i \frac{1}{\sqrt{n}} \sum_{i=1}^n \widetilde{f(V^i)}])^2]$$

in which the symbol $\widetilde{}$ represents the centering operation.

Denoting η [resp. η_m] the characteristic function of $\widetilde{f(V^1)}$ [resp. $\widetilde{f(U_m^i)}$], θ_m the characteristic function of $\widetilde{f(U_m^1) + f(V^1)}$, σ^2 [resp. σ_m^2] the variance of f [resp. of $\widetilde{f(U_m^1)}$], we have

$$Q_{m,n} = \beta_m [(\eta_m(\frac{2}{\sqrt{n}}))^n - 2(\theta_m(\frac{1}{\sqrt{n}}))^n + (\eta(\frac{2}{\sqrt{n}}))^n]$$

The estimates

$$\begin{aligned}\eta_m(t) &= 1 - \frac{t^2}{2}\sigma_m^2 + t^2o(1) \\ \theta_m(t) &= 1 - \frac{t^2}{2}\mathbb{E}[(\widetilde{f(U_m^1)} + \widetilde{f(V^1)})^2] + t^2o(1) \\ \eta(t) &= 1 - \frac{t^2}{2}\sigma^2 + t^2o(1)\end{aligned}$$

give

$$Q_{m,n} = \beta_m e^{-2\sigma^2} \left[e^{-2(\sigma_m - \sigma^2) + o(1)} - 2e^{2\sigma^2 - \frac{1}{2}\mathbb{E}[(\widetilde{f(U_m^1)} + \widetilde{f(V^1)})^2] + o(1)} + \exp o(1) \right].$$

Remarking that $2\sigma^2 - \frac{1}{2}\mathbb{E}[(\widetilde{f(U_m^1)} + \widetilde{f(V^1)})^2] = \frac{1}{2}\mathbb{E}[(\widetilde{f(U_m^1)} + \widetilde{f(V^1)})^2] + \sigma^2 - \sigma_m^2$, if we assume $\mathbb{E}[(\widetilde{f(U_m^1)} - \widetilde{f(V^1)})^2] \rightarrow_{m \rightarrow \infty} 0$ and $\sigma^2 - \sigma_m^2 \rightarrow 0$, we obtain

$$Q_{m,n} = \beta_m e^{-2\sigma^2} (-\mathbb{E}[(\widetilde{f(U_m^1)} - \widetilde{f(V^1)})^2] + o(1))$$

we can state

Proposition 13. *If there is a sequence $\beta_m \rightarrow +\infty$ s.t.*

$$\beta_m \mathbb{E}[(\widetilde{f(U_m^1)} - \widetilde{f(V^1)})^2] \rightarrow \frac{1}{2}e[f]$$

where $e[\cdot]$ is a quadratic form defined on $\mathcal{C}^1([0, 1])$ closable in $L^2([0, 1])$ (with a Dirichlet extension non necessarily local, cf. remark 2) and supposing $\mathbb{E}[(\widetilde{f(U_m^1)})^2] \rightarrow \mathbb{E}[(\widetilde{f(V^1)})^2]$ then

$$\lim_{m, n \uparrow \infty} \beta_m \mathbb{E}[(e^{i \int f dZ_n^m} - e^{i \int f dZ_n})^2] = \frac{1}{2}e^{-2\sigma^2} e[f],$$

hypothesis (H3) is fulfilled and the limit Dirichlet form is the image by the Brownian bridge of the generalized Mehler type form on the Wiener space associated with the form $e[\cdot]$ (cf. [5] chapter VI §2.5 p113 et seq).

Proof. The hypotheses of the proposition imply what we needed during the above computation. It suffices therefore, as before, to recognize the limit as a closable form. But that comes from the functional calculus and the fact that the generalized Mehler type structure associated with the form $e[\cdot]$ satisfies $\Gamma[\int_0^1 f(s) dB_s] = e[f]$. \diamond

For example if $U_m^1 = \frac{1}{m} \sum_{p=1}^m (V^1 + a(V^1)G_p)$ where the G_p are i.i.d. reduced normal variables independent of V^1 and where a is continuous, then by the lemma 1 of the conditionally Gaussian case and the Hamza condition, the form

$$e[f] = \lim_n n \mathbb{E}[(\widetilde{f(U_m^1)} - \widetilde{f(V^1)})^2] = \int_0^1 f'^2(x) a^2(x) dx$$

is closable and the proposition applies. This generalized Mehler type structure satisfies

$$(9) \quad \Gamma\left[\int_0^1 h dB\right] = \int_0^1 f'^2(s) ds \quad \forall f \in H^1([0, 1])$$

this structure may be constructed elementarily as in [5] or by the white noise theory.

Remark 10. There exists an extension of Donsker theorem to the case where the variables V_k (notation of section II.4.b) are erroneous with a functional convergence in the sense of Dirichlet forms (see [6] and [10]). The limit structure obtained is the Ornstein-Uhlenbeck structure. This is related to the fact that the perturbation involved in this approach is a *transversal* perturbation of the random walk hence at the limit a transversal perturbation of the Brownian path (we will display this result in terms of an approximation procedure in section II.7 below).

Here instead, we change the law of the starting random variables : In the expression of Z_n^m

$$Z_n^m = \frac{1}{\sqrt{n}} \sum_{j=1}^n (1_{U_m^j \leq x} - F^m(x))$$

if we simulate U_m^j by the inversion method we see that Z_n^m has same law as

$$\frac{1}{\sqrt{n}} \sum_{j=1}^n (1_{(F^m)^{-1}(X^j) \leq x} - F^m(x)) = \frac{1}{\sqrt{n}} \sum_{j=1}^n (1_{X^j \leq F^m(x)} - F^m(x))$$

where $(X^j)_{j \in \mathbb{N}}$ is a copy of $(V^j)_{j \in \mathbb{N}}$. We see that when F^m changes, the path is *longitudinally* perturbed and so is the limit Brownian bridge. This explains a formula like (9).

II.7. Erroneous random walk and Donsker theorem.

This example displays many similarities with examples II.4, II.5 and II.6. We give only the framework and the results.

Let U^1 be a centered square integrable r.v. approximated by U_m^1 also centered and square integrable. We suppose

$$\alpha_m \mathbb{E}[(U_m - U)^2] \rightarrow \lambda.$$

Considering i.i.d. copies (U_m^i, U^i) of (U_m^1, U^1) we look at

$$X_m^n(t) = \frac{1}{\sqrt{n}} \left(\sum_{i=1}^{[nt]} U_m^i + (nt - [nt]) U_m^{[nt]+1} \right)$$

$$X^n(t) = \frac{1}{\sqrt{n}} \left(\sum_{i=1}^{[nt]} U^i + (nt - [nt]) U^{[nt]+1} \right)$$

and we study $T_{m,n} = \alpha_m \mathbb{E}[(\varphi(X_m^n) - \varphi(X^n))^2]$ for φ belonging to

$$\mathcal{D} = \mathcal{L} \left\{ \exp \left[i \int_0^1 f(s) dX_s \right], \quad f \in \mathcal{C}^\infty \text{ with support in }]0, 1[\right\}.$$

Putting $\sigma^2 = \text{var}(U^1)$ we find that

$$\lim_{m,n \uparrow \infty} T_{m,n} = -\lambda \left(\int f^2 ds \right) e^{-2\sigma^2 \int f^2 ds}$$

The limit Dirichlet form is once more the Ornstein-Uhlenbeck form on the Wiener space s.t. $\langle B \rangle_t = \sigma^2 t$ and $\Gamma[\int_0^1 f dB] = \lambda \int_0^1 f^2(s) d \langle B \rangle_s$.

Comment. The Dirichlet-version of the Donsker theorem proved in [6] supposes the r.v. U^1 has a regular law allowing to carry a (non zero) Dirichlet form. This excludes the case of a discrete law. Here instead, we do not need such a restriction and the present construction applies for instance to the Cox-Ross-Rubinstein model approximating the Black-Scholes model. But the convergence here is weaker than the one used in [6].

II.8. Approximation of the Brownian motion defined through the Wiener integral, i.e. as centered orthogonal measure.

Let X_n^1 be a centered square integrable real r.v. approximating the variable X^1 which is reduced Gaussian. Let $(X_n^i, X^i)_{i \in \mathbb{N}^*}$ be i.i.d. copies of (X_n^1, X^1) .

We assume X_n^1 and X^1 satisfy the hypothesis (H3) with α_n and an algebra \mathcal{D}_0 dense in $L^2(\mathcal{N}(0, 1))$ of bounded functions containing the constants and the imaginary exponentials: $\forall \varphi, \chi \in \mathcal{D}_0$

$$(10) \quad \alpha_n \mathbb{E}[(\varphi(X_n^1) - \varphi(X^1))(\chi(X_n^1) - \chi(X^1))] \rightarrow \lambda \int_{\mathbb{R}} \varphi' \chi' d\mathcal{N}(0, 1).$$

Let ξ_k be an orthonormal basis of $L^2(E_1, \mathcal{F}_1, \mu_1)$ where $(E_1, \mathcal{F}_1, \mu_1)$ is a σ -finite measured space and let us consider the mappings

$$\begin{aligned} f \in L^2(E_1, \mathcal{F}_1, \mu_1) &\xrightarrow{J_n} \sum_{k=1}^{\infty} \langle f, \xi_k \rangle X_n^k \\ f \in L^2(E_1, \mathcal{F}_1, \mu_1) &\xrightarrow{J} \sum_{k=1}^{\infty} \langle f, \xi_k \rangle X^k. \end{aligned}$$

We consider J_n as an approximation of J and for the algebra \mathcal{D} we choose

$$\mathcal{D} = \mathcal{L}\{\Phi : \theta \in \text{CORM} \mapsto \Phi(\theta) = e^{i\langle f, \theta \rangle} \text{ where } f \in L^2(E_1, \mathcal{F}_1, \mu_1) \text{ has a finite expansion on the basis } (\xi_k)\}$$

here CORM denotes the set of centered orthogonal random measures on $L^2(E_1, \mathcal{F}_1, \mu_1)$.

We study the limit of $\Delta_n = \alpha_n \mathbb{E}[\Phi(J_n) - \Phi(J)]^2$ for $\Phi(\theta) = e^{i\langle f, \theta \rangle}$ with $f = \sum_{q=1}^Q f_q \xi_q$.

We have

$$\begin{aligned}
\Delta_n &= \alpha_n \left(\prod_{k=1}^Q \mathbb{E} e^{2if_k X_n^1} - 2 \prod_{k=1}^Q \mathbb{E} e^{if_k(X_n^1 + X^1)} + \prod_{k=1}^Q \mathbb{E} e^{2if_k X^1} \right) \\
&= \alpha_n \left(\prod_{k=1}^Q \mathbb{E} e^{2if_k X_n^1} - \prod_{k=1}^Q \mathbb{E} e^{if_k(X_n^1 + X^1)} + \prod_{k=1}^Q \mathbb{E} e^{2if_k X^1} - \prod_{k=1}^Q \mathbb{E} e^{if_k(X_n^1 + X^1)} \right) \\
&= \alpha_n (A_n + B_n)
\end{aligned}$$

and we may write

$$A_n = \sum_{q=1}^Q \prod_{k=1}^{q-1} \mathbb{E} e^{2if_k X_n^1} \left(\mathbb{E} e^{2if_q X_n^1} - \mathbb{E} e^{2if_q(X_n^1 + X^1)} \right) \prod_{k=q+1}^Q \mathbb{E} e^{2if_k(X_n^1 + X^1)}$$

because the intermediate terms cancel and remain only the first and the last ones.

Let us assume now in addition that the pair (X_n^1, X^1) converges in law to (X^1, X^1) . Then

$$\begin{aligned}
\lim_n \alpha_n (A_n + B_n) &= \sum_{q=1}^Q \prod_{k=1}^{q-1} \mathbb{E} e^{2if_k X^1} \lim_n \mathbb{E} [(e^{if_q X_n^1} - e^{if_q X^1})^2] \prod_{k=q+1}^Q \mathbb{E} e^{2if_k X^1} \\
&= \sum_{q=1}^Q \mathbb{E} [e^{2i \sum_{k=1}^Q f_k X^k}] \lim_n \alpha_n \frac{\mathbb{E} [(e^{if_q X_n^1} - e^{if_q X^1})^2]}{\mathbb{E} [e^{2if_q X^1}]}
\end{aligned}$$

and by the assumption (10) this is nothing else than

$$\lim_n \Delta_n = \mathbb{E} [e^{2i \sum_{k=1}^Q f_k X^k}] \sum_{q=1}^Q (-\lambda f_q^2) = \mathbb{E} [e^{2iJ(f)}] \|f\|^2$$

We recognize once more the Ornstein-Uhlenbeck structure on the abstract Wiener space defined by J . We can state

Proposition 14. *If X_n^1 is an approximation of X^1 satisfying (10) and if $(X_n^1, X^1) \xrightarrow{d} (X^1, X^1)$ then the approximation J_n of the centered orthogonal random measure J satisfies (H3) on \mathcal{D} and yields the Ornstein-Uhlenbeck form.*

We would prove easily following the same lines that if the construction is done with different speeds for the different approximations of X^k by X_n^k , for example replacing X_n^k by \tilde{X}_n^k defined by

$$\tilde{X}_n^k = X_{\ell_k(n)}^k \text{ with } \frac{\alpha_n}{\alpha_{\ell_k(n)}} \rightarrow a_k,$$

we would have

$$\alpha_n \mathbb{E} [(\varphi(\tilde{X}_n^k) - \varphi(X^k))^2] \rightarrow a_k \int_{\mathbb{R}} \varphi'^2 d\mathcal{N}(0, 1)$$

and we would obtain the generalized Mehler type error structure on the abstract Wiener space defined by J associated with the quadratic form

$$e[f] = \sum_q a_q f_q^2$$

i.e. associated with the semi-group

$$p_t f = \sum_q \langle f, \xi_q \rangle e^{-a_q t} \xi_q$$

(cf. [5] p113 et seq).

II.9. Approximation of a Poisson point process.

Let X be a r.v. with values in a metric space E endowed with its Borel σ -field \mathcal{F} .

Let X_n be an approximation of X satisfying hypothesis (H3) with the sequence α_n and an algebra \mathcal{D}_0 composed of bounded continuous functions (containing the constants and dense in $L^2(\mathbb{P}_X)$). We suppose that the Dirichlet form defined by

$$\lim_n \alpha_n \mathbb{E}[(\varphi(X_n) - \varphi(X))^2] \quad \varphi \in \mathcal{D}_0$$

is *local*. We denote by $\gamma[\cdot]$ its square field operator.

Let μ be the law of X on (E, \mathcal{F}) . Let (X_n^j, X^j) be i.i.d. copies of (X_n, X) and let J be an integer valued r.v. with Poisson law of parameter 1 independent of the sequence (X_n^j, X^j) .

We consider the Poisson point processes

$$N_n = \sum_{j=1}^J \delta_{X_n^j} \quad N = \sum_{j=1}^J \delta_{X^j}$$

(\sum_1^0 meaning zero). N_n and N are r.v. with values in the space of point measures \mathcal{M}_p on (E, \mathcal{F}) equipped with the smallest σ -field making all maps $A \mapsto \nu(A)$ $A \in \mathcal{F}$ measurable for $\nu \in \mathcal{M}_p$.

We consider the algebra

$$\mathcal{D} = \mathcal{L}\{\nu \in \mathcal{M}_p \mapsto e^{\int \varphi d\nu} \quad \varphi \in \mathcal{D}_0\}$$

Lemma 3. \mathcal{D} is dense in $L^2(\mathbb{P}_N)$.

Proof. By the chaos decomposition, it is enough to prove that the constants and the elements of $L^2(\mathbb{P}_N)$ of the form $\int f_1 d\tilde{N} \cdots \int f_k d\tilde{N}$ where the functions f_1, \dots, f_k are measurable bounded on (E, \mathcal{F}) and where \tilde{N} denotes $N - \mu$, may be approached by elements of \mathcal{D} . Since the constants are in \mathcal{D} , it suffices to reach $\int f_1 dN \cdots \int f_k dN$. Now $\lim_{\lambda \rightarrow 0} (e^{\lambda \int \varphi dN} - 1)/\lambda = \int \varphi dN$ and this gives easily the lemma. \diamond

We study the approximation of N by N_n by looking at $E_n = \alpha_n \mathbb{E}[(\Phi(N_n) - \Phi(N))^2]$

for $\Phi(\nu) = \exp[i \int \varphi d\nu]$. We may write

$$\begin{aligned}
E_n &= \alpha_n \mathbb{E}[\exp(2i \sum_{j=1}^J \varphi(X_n^j)) - 2 \exp(i \sum_{j=1}^J (\varphi(X_n^j) + \varphi(X^j))) + \exp(2i \sum_{j=1}^J \varphi(X^j))] \\
&= \alpha_n \sum_{p=0}^{\infty} \frac{e^{-1}}{p!} [(\mathbb{E}e^{2i\varphi(X_n^1)})^p - 2(\mathbb{E}e^{i(\varphi(X_n^1) + \varphi(X^1))})^p + (\mathbb{E}e^{2i\varphi(X^1)})^p] \\
&= \alpha_n \sum_{p=0}^{\infty} \frac{e^{-1}}{p!} [(\mathbb{E}e^{2i\varphi(X_n^1)})^p - (\mathbb{E}e^{i(\varphi(X_n^1) + \varphi(X^1))})^p + (\mathbb{E}e^{2i\varphi(X^1)})^p - (\mathbb{E}e^{i(\varphi(X_n^1) + \varphi(X^1))})^p] \\
&= \alpha_n \sum_{p=0}^{\infty} \frac{e^{-1}}{p!} [A_p + B_p]
\end{aligned}$$

We apply a similar idea to what we have done in section II.7 writing A_p under the form

$$A_p = \mathbb{E} \left[e^{i\varphi(X_n^1)} (e^{i\varphi(X_n^1)} - e^{i\varphi(X^1)}) \right] \sum_{k=1}^p (\mathbb{E}e^{2i\varphi(X_n^1)})^{p-k} (\mathbb{E}e^{i(\varphi(X_n^1) + \varphi(X^1))})^{k-1}$$

Making now the additional assumption that $(X_n^1, X^1) \xrightarrow{d} (X^1, X^1)$ and using the fact that $\varphi \in \mathcal{D}_0$ is continuous and bounded, we see that $\alpha_n(A_p + B_p)$ has the same limit as

$$\alpha_n \mathbb{E}[(e^{i\varphi(X_n^1)} - e^{i\varphi(X^1)})^2] p (\mathbb{E}e^{2i\varphi(X^1)})^{p-1}$$

but, since the form $\lim_n \alpha_n \mathbb{E}[(\varphi(X_n) - \varphi(X))^2]$ is local, theorem 2 on asymptotic error calculus applies and gives

$$\lim_n \alpha_n \mathbb{E}[(e^{i\varphi(X_n^1)} - e^{i\varphi(X^1)})^2] = -\mathbb{E}[e^{2i\varphi(X^1)} \gamma[\varphi](X^1)]$$

and we obtain

$$\begin{aligned}
\lim_n E_n &= -\mathbb{E}[e^{2i\varphi(X)} \gamma[\varphi](X)] \sum_{p=0}^{\infty} \frac{e^{-1}}{p!} p (\mathbb{E}e^{2i\varphi(X)})^{p-1} \\
&= -e^{-1} e^{\mathbb{E}[2i\varphi(X)]} \mathbb{E}[e^{2i\varphi(X)} \gamma[\varphi](X)] \\
&= -\exp\left[-\int (1 - e^{2i\varphi}) d\mu\right] \int e^{2i\varphi} \gamma[\varphi] d\mu.
\end{aligned}$$

In order to recognize the limit, let us consider what we have called the white structure on N associated with the structure $(E, \mathcal{F}, \mu, \mathcal{D}_0, \gamma)$ (cf. [5] chapter VI §3, cf. also [7]), its square field operator Γ and its Dirichlet form \mathcal{E} satisfy

$$\begin{aligned}
\Gamma[e^{i \int \varphi dN}] &= -e^{2i \int \varphi dN} \int \gamma[\varphi] dN \\
\mathcal{E}[e^{i \int \varphi dN}] &= -\frac{1}{2} \mathbb{E}[e^{2i \int \varphi dN} \int \gamma[\varphi] dN]
\end{aligned}$$

which using the Laplace characteristic functional $\mathbb{E}e^{i \int \varphi dN} = e^{-\int (1 - e^{i\varphi}) d\mu}$, may easily be seen to be equal to

$$-\frac{1}{2} \exp\left[-\int (1 - e^{2i\varphi}) d\mu\right] \int \gamma[\varphi] e^{2i\varphi} d\mu$$

what we obtained up to the factor 1/2. In conclusion

Proposition 15. *Let X_n be an approximation of X satisfying (H3) on an algebra of continuous functions with a local asymptotic Dirichlet form and square field operator γ . Assuming in addition the weak convergence $(X_n, X) \xrightarrow{d} (X, X)$ then the approximation N_n of the Poisson point process N constructed above satisfies (H3) with the same α_n and with asymptotic Dirichlet form the so-called white form characterized by its square field operator*

$$\Gamma\left[\int f dN\right] = \int \gamma[f] dN.$$

II.9. Stochastic integral.

We now consider a stochastic integral

$$Y = \int_0^1 H_s dB_s$$

approximated by the sum

$$Y_n = \sum_{k=0}^{n-1} H_{\frac{k}{n}} (B_{\frac{k+1}{n}} - B_{\frac{k}{n}})$$

(B_t) is a standard Brownian motion defined as the coordinate process of $\mathcal{C}([0, 1])$ equipped with the Wiener measure, and $H_s = H_0 + \int_0^s \xi_u dB_u + \int_0^s \eta_u du$ is an Ito process defined on the same space, processes ξ and η are adapted and regular in Malliavin sense. We suppose they satisfy $\sup_t \mathbb{E}[|\xi_t|^p + |\eta_t|^p] < +\infty$ for some $p > 2$ and we will state their other regularity properties along the calculation.

In order to obtain the limit expressions we are looking for, we will use several times the integration by part formula

$$\mathbb{E}[u\delta U] = \mathbb{E}[\langle Du, U \rangle_{\mathcal{H}}]$$

(cf. for the notation [5] formula (15) p81). This technique has been already used with success by Clement, Kohatsu-Higa and Lambertson [11] to compute, for s.d.e. possibly with delay, an estimate of $\mathbb{E}[\varphi(Y_n) - \varphi(Y)]$, i.e. with our notation, an estimate of $\langle \bar{A}[\varphi], 1 \rangle$. Let us note that this expression which is always equal to $\langle \mathbb{A}[\varphi], 1 \rangle$ erases the diffusive part of the bias and, since \mathbb{A} is here a first order operator, as we will see in a moment, this expression writes $\langle \bar{A}[\varphi], 1 \rangle = \langle \mathbb{A}[\varphi], 1 \rangle = \mathbb{E}_Y[F\varphi']$ and reduces, when regularity allows an integration by parts, to the form $\mathbb{E}[G\varphi]$. In the case of Ito type s.d.e. under rather general hypotheses, $\mathbb{E}[\varphi(Y_n) - \varphi(Y)]$ keeps the same order of magnitude for the speed of convergence even when φ is only bounded and measurable [2].

In this section we attempt to explicit the four bias operators for the above approximation problem. They occur with the sequence $\alpha_n = n$.

a) **The local property is satisfied.**

Lemma 4. *If ξ and η satisfy $\sup_t \mathbb{E}[|\xi_t|^p + |\eta_t|^p] < +\infty$ for some $p > 2$ then*

$$n\mathbb{E}[|Y_n - Y|^{2+\alpha}] \rightarrow 0 \quad \forall \alpha : 2 < 2 + \alpha \leq p$$

Proof. Let α be s.t. $2 < 2 + \alpha \leq p$, by Burkholder-Davis-Gundy inequality

$$\mathbb{E}[|Y_n - Y|^{2+\alpha}] \leq C_1 \mathbb{E} \int_0^1 |H_s - H_{\frac{[ns]}{n}}|^{2+\alpha} ds.$$

Now

$$\begin{aligned} \|H_s - H_{\frac{[ns]}{n}}\|_{2+\alpha} &\leq (C_1 \mathbb{E}[\int_{\frac{[ns]}{n}}^s \xi_u^2 du]^{\frac{p}{2}})^{\frac{1}{p}} + (\mathbb{E}[(\int_{\frac{[ns]}{n}}^s |\eta_u| du)^p])^{\frac{1}{p}} \\ &\leq (s - \frac{[ns]}{n})^{\frac{1}{2}} (C_1 \sup_t \mathbb{E}|\xi_t|^p)^{\frac{1}{p}} + (s - \frac{[ns]}{n}) (\sup_t \mathbb{E}|\eta_t|^p)^{\frac{1}{p}} \\ \mathbb{E}[|H_s - H_{\frac{[ns]}{n}}|^{2+\alpha}] &\leq (s - \frac{[ns]}{n})^{\frac{2+\alpha}{2}} (C_2 + o(1)). \end{aligned}$$

Hence $n\mathbb{E}[|Y_n - Y|^{2+\alpha}] \leq \frac{1}{n^{\alpha/2}} (C_3 + o(1))$. Q.E.D. \diamond

It follows that if our test functions algebra \mathcal{D} consists of bounded \mathcal{C}^2 -functions with bounded derivatives, we have $\lim_n n\mathbb{E}[|\varphi(Y_n) - \varphi(Y)|^{2+\alpha}] = 0$ so that by remark 3 if we succeed in proving assumption (H3) with $\alpha_n = n$, the asymptotic Dirichlet form will be local.

b) It follows also if we assume a little bit more for instance that the functions in \mathcal{D} are \mathcal{C}^3 bounded with bounded derivatives, that in the study of \bar{A} , the expression

$$n\mathbb{E}[(\varphi(Y_n) - \varphi(Y))\chi(Y)]$$

has the same limit as

$$n\mathbb{E}[(Y_n - Y)\varphi'(Y)\chi(Y) + \frac{1}{2}(Y_n - Y)^2\varphi''(Y)\chi(Y)].$$

Similarly, in the study of \tilde{A} , the expression $n\mathbb{E}[(\varphi(Y_n) - \varphi(Y))^2]$ has the same limit as $n\mathbb{E}[(Y_n - Y)^2\varphi'^2(Y)]$.

For simplicity we shall suppose that the functions in \mathcal{D} are \mathcal{C}^∞ bounded with bounded derivatives.

c) **Study of the symmetric bias operator.**

Let us remark first that in the study of $n\mathbb{E}[(Y_n - Y)^2\varphi'^2(Y)]$ we may suppose $\eta \equiv 0$ and that H be of the form $H_t = H_0 + \int_0^t \xi_s dB_s$.

Indeed, putting $K_s = \int_0^s \eta_u du$ we have

$$n\mathbb{E}[(\int_0^1 (K_s - K_{\frac{[ns]}{n}}) dB_s)^2 \varphi'^2(Y)] \leq Cn \int_0^1 \mathbb{E}[(\int_{\frac{[ns]}{n}}^s \eta_u du)^2] ds = O(\frac{1}{n}).$$

Now for studying $n\mathbb{E}[(Y_n - Y)^2\varphi'^2(Y)]$ we apply Ito's formula to the continuous martingale $Y_n - Y$:

$$\begin{aligned}(Y_n - Y)^2 &= \int_0^1 2 \int_0^t (H_s - H_{\lfloor ns \rfloor}) dB_s (H_t - H_{\lfloor nt \rfloor}) dB_t + \int_0^1 (H_s - H_{\lfloor ns \rfloor})^2 ds \\ &= (1) + (2)\end{aligned}$$

i) Contribution due to the second term.

Let us apply once more Ito's formula

$$\begin{aligned}\left(\int_{\lfloor ns \rfloor}^s \xi_u dB_u\right)^2 &= 2 \int_{\lfloor ns \rfloor}^s \int_{\lfloor nt \rfloor}^t \xi_u dB_u \xi_t dB_t + \int_{\lfloor ns \rfloor}^s \xi_u^2 du \\ &= (2, 1) + (2, 2).\end{aligned}$$

The contribution of the term (2,2) is $n\mathbb{E}[\int_0^1 \int_{\lfloor ns \rfloor}^s \xi_u^2 du ds \varphi'^2(Y)]$ which tends to $\frac{1}{2}\mathbb{E}[\int_0^1 \xi_s^2 ds \varphi'^2(Y)]$.

The contribution of the term (2,1) is zero. Indeed by integration by parts it is the limit of

$$2n \int_0^1 \mathbb{E} \int_{\lfloor ns \rfloor}^s \int_{\lfloor nt \rfloor}^t \xi_u dB_u \xi_t D_t[\varphi'^2(Y)] dt ds$$

which by an other integration by part in order to get rid of the stochastic integral, gives

$$2n \int_0^1 \int_{\lfloor ns \rfloor}^s \int_{\lfloor nt \rfloor}^t \mathbb{E}[\xi_u D_u[\xi_t D_t[\varphi'^2(Y)]]] du dt ds$$

and is $O(\frac{1}{n})$ as soon as $\mathbb{E}[\xi_u D_u[\xi_t D_t[\varphi'^2(Y)]]]$ remains bounded.

ii) Contribution of the first term.

We shall show by several integration by parts that this contribution is zero. We are concerned by the limit of

$$\begin{aligned}2n\mathbb{E} \int_0^1 \int_0^t (H_s - H_{\lfloor ns \rfloor}) dB_s (H_t - H_{\lfloor nt \rfloor}) D_t[\varphi'^2(Y)] dt \\ = 2n\mathbb{E} \int_0^1 \int_0^t (H_s - H_{\lfloor ns \rfloor}) (H_t - H_{\lfloor nt \rfloor}) D_s D_t[\varphi'^2(Y)] ds dt \\ + 2n\mathbb{E} \int_0^1 \int_0^t (H_s - H_{\lfloor ns \rfloor}) D_s [H_t - H_{\lfloor nt \rfloor}] D_t[\varphi'^2(Y)] ds dt \\ = (1, 1) + (1, 2).\end{aligned}$$

The term (1,1) decomposes in (1,11)+(1,12) with

$$\begin{aligned}(1, 11) &= 2n \int_0^1 \int_0^t \int_{\lfloor ns \rfloor}^s \mathbb{E}[\xi_u (H_t - H_{\lfloor nt \rfloor}) D_u D_s D_t[\varphi'^2(Y)]] du ds dt \\ (1, 12) &= 2n \int_0^1 \int_0^t \int_{\lfloor ns \rfloor}^s \mathbb{E}[\xi_u D_u [H_t - H_{\lfloor nt \rfloor}] D_s D_t[\varphi'^2(Y)]] du ds dt\end{aligned}$$

we have

$$(1, 11) = 2n \int_0^1 \int_0^t \int_{\lfloor ns \rfloor}^s \int_{\lfloor nt \rfloor}^t \mathbb{E}[\xi_v D_v [\xi_u D_u D_s D_t[\varphi'^2(Y)]]] dv du ds dt$$

Thus (1,11) is $O(\frac{1}{n})$ as soon as the expectation inside is bounded. About (1,12) we get still two terms

$$(1, 12) = 2n \int_0^1 \int_0^t \int_{\frac{[ns]}{n}}^s \mathbb{E}[\xi_u \left(\int_{\frac{[nt]}{n}}^t D_u[\xi_\alpha] dB_\alpha + \xi_u 1_{(\frac{[nt]}{n}, t)}(u) D_s D_t[\varphi'^2(Y)] \right)] dudsd t$$

the second one requires $\frac{[ns]}{n} = \frac{[nt]}{n}$ and tends to zero. The first one may be written

$$2n \int_0^1 \int_0^t \int_{\frac{[ns]}{n}}^s \int_{\frac{[nt]}{n}}^t D_u[\xi_\alpha] D_\alpha D_s D_t[\varphi'^2(Y)] d\alpha dudsd t = O\left(\frac{1}{n}\right)$$

Let us come to the term (1,2) which may be written

$$2n \int_0^1 \int_0^t \mathbb{E} \left[\left(H_t - H_{\frac{[nt]}{n}} \right) \left[\int_{\frac{[nt]}{n}}^t D_s[\xi_\alpha] dB_\alpha + \xi_s 1_{(\frac{[nt]}{n}, t)}(s) D_t[\varphi'^2(Y)] \right] \right] dsdt$$

still two terms, the second one requires $\frac{[ns]}{n} = \frac{[nt]}{n}$ and tends to zero. The first one is

$$2n \int_0^1 \int_0^t \mathbb{E} \left[\int_{\frac{[ns]}{n}}^s \xi_\beta dB_\beta \int_{\frac{[nt]}{n}}^t D_s[\xi_\alpha] dB_\alpha D_t[\varphi'^2(Y)] \right] dsdt$$

which may be handled as term (1,1).

Eventually, supposing ξ and η are bounded with bounded Malliavin derivatives up to order four, then

$$n\mathbb{E}[(\varphi(Y_n) - \varphi(Y))^2] \rightarrow \frac{1}{2}\mathbb{E}\left[\int_0^1 \xi_s^2 ds \varphi'^2(Y)\right].$$

d) Study of the theoretical bias \bar{A} .

By the remark done in b) above, it is sufficient to study the limit of

$$n\mathbb{E}[(Y_n - Y)\varphi'(Y)\chi(Y) + \frac{1}{2}(Y_n - Y)^2\varphi''(Y)\chi(Y)].$$

The work is already done for the second term, it suffices to replace $\varphi'^2(Y)$ by $\varphi''(Y)\chi(Y)$ in point c).

The first term may be written

$$\begin{aligned} n\mathbb{E}\left[\int_0^1 (H_{\frac{[ns]}{n}} - H_s) dB_s \varphi'(Y)\chi(Y)\right] &= n\mathbb{E}\int_0^1 (H_{\frac{[ns]}{n}} - H_s) D_s[\varphi'(Y)\chi(Y)] ds \\ &= n\mathbb{E}\int_0^1 \left(\int_{\frac{[ns]}{n}}^s \xi_u dB_u + \int_{\frac{[ns]}{n}}^s \eta_u du \right) D_s[\varphi'(Y)\chi(Y)] ds \\ &= (a) + (b) \end{aligned}$$

$$(a) = n \int_0^1 \int_{\frac{[ns]}{n}}^s \mathbb{E}[\xi_u D_u D_s[\varphi'(Y)\chi(Y)]] dud s \text{ which tends to } \frac{1}{2} \int_0^1 \mathbb{E}[\xi_s D_s D_s[\varphi'(Y)\chi(Y)]] ds.$$

$$(b) = \int_0^1 \int_{\frac{[ns]}{n}}^s \mathbb{E}[\eta_s D_s [\varphi'(Y)\chi(Y)]] du ds \rightarrow \frac{1}{2} \int_0^1 \mathbb{E}[\eta_s D_s [\varphi'(Y)\chi(Y)]] ds$$

therefore we have

$$(11) \quad \lim_n \mathbb{E}[(Y_n - Y)\varphi'(Y)\chi(Y)] \\ = \frac{1}{2} \int_0^1 \mathbb{E}[\xi_s D_s D_s [\varphi'(Y)\chi(Y)]] ds + \frac{1}{2} \int_0^1 \mathbb{E}[\eta_s D_s [\varphi'(Y)\chi(Y)]] ds.$$

Hence, with the same hypotheses as for c), the conclusion is :

$$\lim_n n \mathbb{E}[(\varphi(Y_n) - \varphi(Y))\chi(Y)] \\ = \frac{1}{2} \int_0^1 \mathbb{E}[\xi_s D_s D_s [\varphi'(Y)\chi(Y)] + \eta_s D_s [\varphi'(Y)\chi(Y)]] ds + \frac{1}{4} \int_0^1 \mathbb{E}[\xi_s^2 \varphi''(Y)\chi(Y)] ds.$$

e) Interpretation of the results.

If we put $\frac{1}{2}\mathbb{E}[\int_0^1 \xi_s^2 ds | Y = y] = \rho(y)$ the form $\mathcal{E}[\varphi] = \frac{1}{2} \int \rho \varphi'^2 d\mathbb{P}_Y$ is closable iff the measure $\frac{1}{2}\rho \mathbb{P}_Y$ has a density satisfying the Hamza condition (cf. [13] p105). In that case the operator \tilde{A} exists and is uniquely defined by

$$\langle \tilde{A}[\varphi], \chi \rangle = -\frac{1}{4} \mathbb{E}[\int_0^1 \xi_s^2 ds \varphi'(Y)\chi'(Y)].$$

The operator \bar{A} would be defined by

$$\langle \bar{A}[\varphi], \chi \rangle = \lim_n n \mathbb{E}[(\varphi(Y_n) - \varphi(Y))\chi(Y)] \\ = \frac{1}{2} \mathbb{E}[\int_0^1 \xi_s D_s D_s [\varphi'(Y)\chi(Y)] ds] + \frac{1}{2} \mathbb{E}[\int_0^1 \eta_s D_s [\varphi'(Y)\chi(Y)] ds] \\ + \frac{1}{4} \mathbb{E}[\int_0^1 \xi_s^2 (\varphi'\chi)'(Y) ds] - \frac{1}{4} \mathbb{E}[\int_0^1 \xi_s^2 \varphi'(Y)\chi'(Y) ds]$$

provided that the righthand side may be put in the form of the lefthand side.

Sufficient conditions are easy to be listed.

(j) By the functional calculus the first term may be written

$$\frac{1}{2} \mathbb{E} \int_0^1 \xi_s [(D_s[Y])^2 (\varphi'\chi)''(Y) + D_s D_s[Y] (\varphi'\chi)'(Y)] ds$$

it will have the desired form as soon as the measures $\int_0^1 \xi_s (D_s[Y])^2 ds \cdot \mathbb{P}_Y$ and $\int_0^1 \xi_s D_s D_s[Y] ds \cdot \mathbb{P}_Y$ will be sufficiently regular to allow an integration by parts.

(jj) The second term is equal to

$$\frac{1}{2} \mathbb{E} \left[\int_0^1 \eta_s dB_s \varphi'(Y)\chi(Y) \right].$$

(jjj) The third term requires the conditions of an integration by part.

(jv) The fourth term is $\langle \tilde{A}[\varphi], \chi \rangle$.

We see that the operator \tilde{A} exists under a quite simple condition involving only the law of the pair $(\int_0^1 \xi_s^2 ds, Y)$. The regularity conditions insuring the existence of the operators \bar{A} or \underline{A} are more intricate. When they hold, putting

$$(12) \quad \langle \mathbb{A}[\varphi], \chi \rangle = \frac{1}{2} \mathbb{E} \int_0^1 (\xi_s D_s D_s + \eta_s D_s) [\varphi'(Y) \chi(Y)] ds + \frac{1}{4} \mathbb{E} \left[\int_0^1 \xi_s^2 ds (\varphi' \chi)'(Y) \right]$$

we have $\bar{A} = \mathbb{A} + \tilde{A}$ and $\underline{A} = -\mathbb{A} + \tilde{A}$ and by the general theory, hypotheses (H1) to (H3) being fulfilled and the Dirichlet form being local, \mathbb{A} is a first order operator, as may be seen also on the obtained form (12).

Remark 11. Our approach is direct. But the heaviest part of the proof i.e. the proof of

$$n \mathbb{E} [(\varphi(Y_n) - \varphi(Y))^2] \rightarrow \frac{1}{2} \mathbb{E} \int_0^1 \xi_s^2 ds \varphi'^2(Y),$$

may be considerably shortened if we use a result of weak convergence like

$$(\sqrt{n}(Y_n - Y), Y) \xrightarrow{d} \left(\frac{1}{\sqrt{2}} \int_0^1 \xi_s dW_s, Y \right)$$

with an “extra” Brownian motion W independent of Y and ξ .

This gives our results thanks to the uniform integrability of $n(Y_n - Y)^2$ which is a consequence of the inequality $\mathbb{E}[|Y_n - Y|^{2+\alpha}] \leq n^{-\frac{2+\alpha}{2}} (C_3 + o(1))$ established in the proof of lemma 4.

Such a weak convergence result has been obtained long time ago by Rootzen [28] for the case where the process H has the form $H_s = f(B_s, s)$. This kind of weak convergence results for stochastic integrals have been now considerably extended, see especially [16], [18], [29], [19], [21], [21], [17], [15]. Our approach to s.d.e. in the next section is based on such results.

II.10. Stochastic differential equations and Euler scheme.

As we have just explained we will base our approach on results on convergence in law, in particular on the article of Jacod and Protter [17]. We consider only the case of a continuous semi-martingale in which the main ideas already appear.

Let $X = (X^i)_{i=1, \dots, d}$ be a continuous semi-martingale with values in \mathbb{R}^d vanishing at zero defined on the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. For $t \in [0, 1]$ we consider the q -dimensional s.d.e.

$$(13) \quad dY_t = f(Y_t) dX_t \quad Y_0 = y_0$$

where $y_0 \in \mathbb{R}^q$, f is \mathcal{C}^1 from \mathbb{R}^q into $\mathbb{R}^{q \times d}$ with at most linear growth ($|f(x)| \leq K(1+|x|)$) denoting $|\cdot|$ the norms on \mathbb{R}^k . It is known that (13) has a unique strong solution. We study the resolution of (13) by the Euler scheme :

$$dY_t^n = f\left(Y_{\frac{[nt]}{n}}^n\right) dX_t \quad Y_0^n = y_0$$

where $[nt]$ is the entire part of nt .

We denote $U_t^n = Y_t^n - Y_t$ the error process. U^n as process with values in $\mathcal{C}([0, 1])$ tends to zero in probability (as soon as f is locally Lipschitz with at most linear growth [17]).

It is supposed that $X = M + A$ where M is a continuous local martingale vanishing at zero with values in \mathbb{R}^d and A is a continuous finite variation adapted process vanishing at zero satisfying

$$\begin{aligned} A_t^i &= \int_0^t a_s^i ds \quad \text{with} \quad \int_0^1 (a_s^i)^2 ds < +\infty \text{ a.s.} \\ < M^i, M^j >_t &= \int_0^t c_s^{ij} ds \quad \text{with} \quad \int_0^1 (c_s^{ij})^2 ds < +\infty \text{ a.s.} \end{aligned}$$

then for every starting point y_0 and for all function $f \in \mathcal{C}^1$ with at most linear growth, the process $\sqrt{n}U^n$ converges in law on $\mathcal{C}([0, 1])$ to the solution to

$$dU_t^i = \sum_{j=1}^d \sum_{k=1}^q \frac{\partial f^{ij}}{\partial x_k}(Y_t) \left[U_t^k dX_t^j - \sum_{\ell=1}^d f^{k\ell}(Y_t) dZ_t^{\ell j} \right], \quad U_0^i = 0,$$

Z being given by

$$Z_t^{ij} = \frac{1}{\sqrt{2}} \sum_{k, \ell=1}^q \int_0^t \sigma_s^{ik} \sigma_s^{j\ell} dW_s^{k\ell}$$

where W is a standard q^2 -dimensional Brownian motion defined on an extension of the space independent of X and σ is a matrix of processes s.t. $(\sigma\sigma^t)^{ij} = c^{ij}$ which exists as soon as $q \geq d$ case to which the question may be always reduced.

The proof consists of the conjunction of theorems 3.3, 5.1 and 5.5 of [17] and their proofs.

In order to study the hypotheses (H1) to (H3) we consider the algebra \mathcal{D} of the linear combinations of functions φ defined on $\mathcal{C}([0, 1])$ by

$$\varphi(Y) = e^{i\langle u_1, Y_{t_1} \rangle + \dots + i\langle u_r, Y_{t_r} \rangle} \quad u_\ell \in \mathbb{R}^q \quad t_\ell \in [0, 1] \quad \ell = 1, \dots, r$$

and the sequence $\alpha_n = n$.

a) Symmetric bias operator.

We study $n\mathbb{E}[(\varphi(Y^n) - \varphi(Y))^2]$.

Lemma 5. *If for fixed t the sequence $n|Y_t^n - Y_t|^2 = |\sqrt{n}U_t^n|^2$ is uniformly integrable,*

$$(14) \quad n\mathbb{E}[(\varphi(Y^n) - \varphi(Y))^2] \rightarrow \mathbb{E} \left[\left(\sum_{j=1}^q \sum_{\ell=1}^r U_{t_\ell}^j \frac{\partial \varphi}{\partial y_{t_\ell}^j}(Y) \right)^2 \right].$$

Proof. Let us argue in the case $q = r = 1$, the general case being similar.

$$\begin{aligned} n\mathbb{E}[(\varphi(Y^n) - \varphi(Y))^2] &= \mathbb{E}[n(Y_t^n - Y_t)^2 (\int_0^1 \varphi'(Y_t + \lambda(Y_t^n - Y_t)) d\lambda)^2] \\ &\leq \mathbb{E}[\{n(Y_t^n - Y_t)^2 - (n(Y_t^n - Y_t)^2) \wedge a\} \|\varphi'\|_\infty^2] \\ &\quad + \mathbb{E}[\{(n(Y_t^n - Y_t)^2) \wedge a\} (\int_0^1 \varphi'(Y_t + \lambda(Y_t^n - Y_t)) d\lambda)^2] \end{aligned}$$

By the uniform integrability the first term may be made smaller than $\varepsilon > 0$ uniformly in n by suitable choice of a , then the second term goes to zero by the weak convergence of $n(Y^n - Y)$ and the convergence in probability of $Y_t^n - Y_t$ to zero. \diamond

Remark 12. In the classical case of an s.d.e. defining a diffusion process from a Brownian motion, if the coefficients are regular, for instance \mathcal{C}^∞ with bounded derivatives, it is known that $\sqrt{n}\|Y_t^n - Y_t\|_p$ is bounded for any $p \in [1, +\infty[$, the uniform integrability of $n|Y_t^n - Y_t|^2$ follows.

Considering that X and W are defined on a product space whose samples are denoted ω and $\hat{\omega}$, formula (14) shows that if hypothesis (H3) is verified and if $n|U_t^n|^2$ is uniformly integrable, the limit Dirichlet form satisfies $Y_t \in \mathbb{D}$ and its square field operator satisfies

$$\Gamma[Y_t^j] = \hat{\mathbb{E}}[(U_t^j)^2].$$

In other words, the limit process $U(\omega, \hat{\omega})$ appears to be a gradient in the sense of Dirichlet forms of the process Y : we may write

$$(15) \quad (Y_t)^\#(\omega, \hat{\omega}) = U_t(\omega, \hat{\omega})$$

and formula (14) follows by the chain rule.

The remaining question is whether the form defined on \mathcal{D} by (14) is closable in $L^2(\mathcal{C}([0, 1]), \mathbb{P}_Y)$. To this question we have yet only an answer in the simplest case where $q = 1$. When

$$dY_t = a(Y_t, t)dB_t + b(Y_t, t)dt$$

with $a, b \in \mathcal{C}^1$ with at most linear growth, the process U is given by

$$U_t = N_t \int_0^t \frac{a(Y_s, s)a'_y(Y_s, s)}{\sqrt{2}N_s} dW_s$$

with

$$N_t = \exp\left\{\int_0^t a'_y(Y_s, s)dB_s - \frac{1}{2}\int_0^t a_y^2(Y_s, s)ds + \int_0^t b'_y(Y_s, s)ds\right\}.$$

Let us denote $(\mathcal{E}_{ou}^\theta, \mathbb{D}_{ou}^\theta)$ the Dirichlet form on the Wiener space of type Ornstein-Uhlenbeck with deterministic weight θ , and let us denote D_{ou}^θ its gradient operator defined with the auxiliary Hilbert space $L^2([0, 1], dt)$. We have

Proposition 16. *If the coefficient a satisfies $\mathbb{E} \int_0^1 a_y'^2(Y_s, s) ds < +\infty$ and if $a_y'^2(Y_s, s) \geq \theta(s) > 0$, hypothesis (H3) is fulfilled. The asymptotic Dirichlet form is the image by Y of the form $(\mathcal{E}_w, \mathbb{D}_w)$ defined on the Wiener space by*

$$\mathbb{D}_w = \left\{ F \in \mathbb{D}_{ou}^\theta : \int_0^1 \mathbb{E}[(D_{ou}^\theta[F](t))^2 \frac{a_y'^2(Y_t, t)}{\theta(t)}] dt < +\infty \right\}$$

$$\mathcal{E}_w[F] = \frac{1}{4} \int_0^1 \mathbb{E}[(D_{ou}^\theta[F](t))^2 \frac{a_y'^2(Y_t, t)}{\theta(t)}] dt.$$

The proof has been exposed at the Fifth Seminar on Stochastic Analysis, Random Fields and Application at Ascona in 2005 and will appear in the proceedings.

The form $(\mathcal{E}_w, \mathbb{D}_w)$ admits the square field operator

$$\Gamma_w[F] = \frac{1}{2} \int_0^1 (D_{ou}^\theta[F](t))^2 \frac{a_y'^2(Y_t, t)}{\theta(t)} dt.$$

Putting $\xi_t = \frac{1}{2} a_y'^2(Y_t, t)$ the operator \tilde{A} is given by

$$\tilde{A}[\varphi](y) = \mathbb{E}[A_w[\varphi(Y)] | Y = y]$$

where $A_w[\varphi(Y)] = -\frac{1}{2} \delta_{ou}^\theta[\frac{\xi}{\theta} D_{ou}^\theta[F]]$, and δ_{ou}^θ being the Skorokod stochastic integral operator associated with $(\mathcal{E}_{ou}^\theta, \mathbb{D}_{ou}^\theta)$.

From the concrete point of view of error calculus, the relation

$$(16) \quad Y_t^\# = U_t$$

is the most important. It allows to propagate errors by the chain rule and using also, in order to manage limit objects, the fact that the operator $\#$ is closed what is a consequence of the closedness of the form.

b) The theoretical bias operator.

As in the case of the approximation of a stochastic integral (cf. section II.9) the operator \bar{A} involves an iterated gradient.

The main part of the calculation has been performed by Malliavin and Thalmaier ([24] and [25]) and we adopt their hypotheses : Y is solution of the s.d.e.

$$dY_t = a(Y_t)dB_t + b(Y_t)dt$$

where B is a $(d - 1)$ -dimensional Brownian motion and where the matrix a and the function b are \mathcal{C}^∞ with bounded derivatives.

The operator \bar{A} is given by $\lim_n n\mathbb{E}[(\varphi(Y^n) - \varphi(Y))\chi(Y)]$. Since \mathcal{D} consists of functions of finite dimensional marginals, we restrict for simplicity to marginals of order one and to the case where B and Y are scalar, we have (cf. lemma 4) :

$$\lim_n n\mathbb{E}[(\varphi(Y_t^n) - \varphi(Y_t))\chi(Y_t)] = \lim_n n\mathbb{E}[(Y_t^n - Y_t)\varphi'(Y_t)\chi(Y_t) + \frac{1}{2}(Y_t^n - Y_t)^2\varphi''(Y_t)\chi(Y_t)]$$

P. Malliavin and A. Thalmaier have computed the first term which may be pulled back on the Wiener space

$$(17) \lim_n n \mathbb{E}[(Y_t^n - Y_t) \varphi'(Y_t) \chi(Y_t)] = \int_0^1 \mathbb{E}[a_{11}(Y_s) D_s D_s F + b_1(Y_s) D_s F + c_1(Y_s) F] ds$$

where $F = \varphi'(Y_t) \chi(Y_t)$ and where a_{11}, b_1, c_1 are functions of the coefficients a, b and of their four first derivatives. It should be noted the similarity between the above formula (17) and formula (11) obtained for a stochastic integral ((17) reduces to (11) when $b = 0$).

The second term is consequence of the preceding results on convergence in law.

$$(18) \quad \frac{n}{2} \mathbb{E}[(Y_t^n - Y_t)^2 \varphi''(Y_t) \chi(Y_t)] \rightarrow \frac{1}{2} \mathbb{E}[U_t^2 \varphi''(Y_t) \chi(Y_t)].$$

We see, by formulae (17) and (18) that the operator \bar{A} is the image by Y of a singular distribution operator on the Wiener space.

We have to conclude that, up to now, the study of the approximation of the solution of an s.d.e. by the Euler scheme is far from being achieved : the operator \bar{A} is yielded by the quoted recent works but the existence of the operator \tilde{A} (hence of the Dirichlet form) is only shown in a very particular case.

III. Conclusive comments.

We focus in this conclusion on remarks concerning the comparison between deterministic and stochastic approximation.

The hypothesis of uniqueness of the approximation of order n .

Let us consider a situation where given Y the approximation Y_n is completely determined, i.e. for \mathbb{P}_Y -a.e. y , the conditional law of Y_n given $Y = y$, has the form $\delta_{\eta_n(y)}$. We call this assumption of uniqueness hypothesis (U).

Example. Such a hypothesis is often *implicitly* supposed when numerical results are given under the form

$$(19) \quad \begin{aligned} Y_3 &= 2.3769 \pm 10^{-4} \\ Y_5 &= 2.376985 \pm 10^{-6} \\ Y_7 &= 2.37698534 \pm 10^{-8} \\ &\dots \end{aligned}$$

and it is underlying the concept of *number of significant digits*.

Indeed, let us take the decimal representation of real numbers in $[0, 1]$:

$$y = \sum_{n=0}^{\infty} \frac{a_n}{10^{n+1}} \quad \text{with} \quad a_n \in \{0, 1, \dots, 9\}$$

If the a_n 's are drawn independently uniformly on $\{0, 1, \dots, 9\}$ the random variable $Y = \sum_{n=0}^{\infty} \frac{a_n}{10^{n+1}}$ is uniformly distributed on $[0, 1[$ and as soon as y is not decimal, which is a negligible set, the expansion of y is unique, so that the above hypothesis (U) is fulfilled for the approximation $Y_n = \sum_{k=0}^n \frac{a_k}{10^{k+1}}$.

Some martingales satisfy hypothesis (U). Let us consider $(\Omega, \mathcal{A}, \mathbb{P})$ with an increasing sequence of sub- σ -fields \mathcal{B}_n generated by countable partitions \mathcal{P}_n of Ω . Then for $Y \in L^1$, $Y_n = \mathbb{E}[Y|\mathcal{B}_n]$ satisfies (U) with

$$\eta_n(y) = \sum_{A \in \mathcal{P}_n} 1_A(y) \frac{\mathbb{E}[Y 1_A]}{\mathbb{P}(A)}.$$

This happens in particular for Haar systems (cf. [26] chap. III §3).

Proposition 17. *Suppose hypothesis (U). If for $\alpha_n \rightarrow +\infty$ and an algebra \mathcal{D} ,*

$$\alpha_n \mathbb{E}[(\varphi(Y_n) - \varphi(Y))\chi(Y)] \rightarrow \langle \bar{A}[\varphi], \chi \rangle_{\mathbb{P}_Y} \quad \forall \varphi \in \mathcal{D}, \quad \forall \chi \in L^2(\mathbb{P}_Y),$$

then (H1) to (H3) hold, $\bar{A} = -\underline{A} = \mathbb{A}$ are first order operators and $\tilde{A} = 0$.

Proof. The sequence $\alpha_n(\mathbb{E}[\varphi(Y_n)|Y = y] - \varphi(y))$ is weakly bounded in $L^2(\mathbb{P}_Y)$ hence strongly bounded, i.e.

$$\alpha_n^2 \int (\mathbb{E}[\varphi(Y_n)|Y = y] - \varphi(y))^2 \mathbb{P}_Y(dy) \leq K.$$

Now $\mathbb{E}[\varphi(Y_n)|Y = y] = \varphi(\eta_n(y))$, hence

$$\alpha_n \mathbb{E}[(\varphi(Y_n) - \varphi(Y))^2] = \alpha_n \int (\varphi(\eta_n(y)) - \varphi(y))^2 \mathbb{P}_Y(dy) \leq \frac{K}{\alpha_n} \rightarrow 0.$$

The Dirichlet form is zero, hence it is local and \mathbb{A} is a first order operator. \diamond

Examples in part II show that in many probabilistic approximations, hypothesis (U) does not hold. The law of Y_n given $Y = y$ has a non zero variance. Polya's urn is a generic example showing that information at each step cannot be resumed by boxes of size $\pm 10^{-k}$ but by standard deviation of laws whose support doesn't go in general to zero.

When we are interested in computing a sample of a random quantity, we have to display the result with specifications adapted to the stochastic case.

The interest of such specifications is particularly clear in infinite dimension when we have to compute ω by ω a path of a process. For example in the GPS or GALILEO systems when modelling the ionosphere by a spatio-temporal process, computing a sample is necessary to obtain the shift in the signals coming from several satellites. The accuracy of this sample is important to get the accuracy of the whole positioning system.

For such numerical computations of sample paths, we suggest that, as much as possible, the following specifications be displayed : (i) the law \mathbb{P}_Y of Y , (ii) the sequence α_n , (iii) the theoretical and practical bias operators \tilde{A} and \underline{A} .

Then the operator \tilde{A} , the Dirichlet form and the square field operator follow and the approximation $Y_n(\omega)$ may be (if the Dirichlet form is local) the starting point of an error calculus for the studied model. Non locality of the form, when it happens, is also a precious warning to be particularly carefull in the sensitivity analysis.

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