# A convergent scheme for a non-local coupled system modelling dislocations densities dynamics

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**Abstract**

In this paper, we study a non-local coupled system that arises in the theory of dislocations densities dynamics. Within the framework of viscosity solutions, we prove a long time existence and uniqueness result for the solution of this model. We also propose a convergent numerical scheme and we prove a Crandall-Lions error estimate between the continuous solution and the numerical one. As far as we know, this is the first error estimate of Crandall-Lions type for Hamilton-Jacobi systems. We also provide some numerical simulations.

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**Keywords :** Hamilton Jacobi equations, viscosity solutions, dislocations densities dynamics, numerical scheme, error estimate, system.

# 1 Introduction

## 1.1 Presentation and physical motivations

A dislocation is a crystal defect which corresponds to a discontinuity in the crystalline structure organisation. This concept has been introduced by Polanyi, Taylor and Orowan in 1934 as the main explanation at the microscopic scale of plastic deformation. A dislocation creates around it a perturbation that can be seen as an elastic field. Under an exterior strain, a dislocation moves according to its Burgers vector which caracterize the intensity and the direction of the defect displacement (see Hirth and Lothe [17] for an introduction to dislocations).

Here, we are interested by dislocations densities dynamics. More precisely, we consider edge dislocations, *i.e* the Burgers vectors and dislocations are in the same plane. These dislocations are moving with the Burgers vectors $\pm\vec{b}$ (see figure 1). This model has been introduced by Groma, Balogh as a coupled system, namely a transport problem where the velocity is given by the elasticity equations in the 2-D case (see [16]).

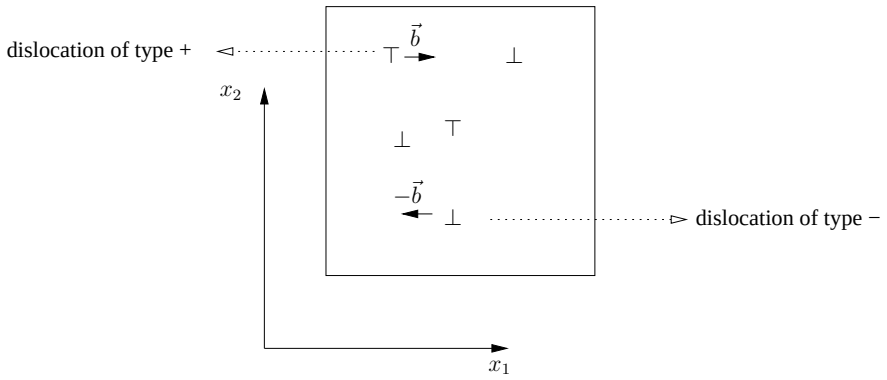


FIG. 1 – The cross-section of the dislocations lines.

If the 2-D domain is 1-periodic in $x_1$ and $x_2$, and if the dislocations densities depends only on the variable $x = x_1 + x_2$ (where $(x_1, x_2)$ are the coordinates of a point in $\mathbb{R}^2$), when $\vec{b} = (1, 0)$ the 2-D

model of [16] reduces to the 1-D non local Hamilton-Jacobi system (see Section 3)

$$\begin{cases} (\rho_+)_t = -\left(\rho_+ - \rho_- + \displaystyle\int_0^1 (\rho_+(x,t) - \rho_-(x,t))\,dx + L(t)\right)|D\rho_+| & \text{on} \quad \mathbb{R}\times(0,T) \\ \\ (\rho_-)_t = \left(\rho_+ - \rho_- + \displaystyle\int_0^1 (\rho_+(x,t) - \rho_-(x,t))\,dx + L(t)\right)|D\rho_-| & \text{on} \quad \mathbb{R}\times(0,T) \end{cases} \tag{1.1}$$

where $\rho_+, \rho_-$ are the unknown scalars such that $(\rho_+ - \rho_-)$ represents the plastic deformation, there space derivatives $D\rho_\pm := \frac{\partial \rho_\pm}{\partial x}$ are the dislocations densities and $L(t)$ represents the exterior shear stress field. From a physical viewpoint, $D\rho_\pm \geq 0$, however, here we don't make this assumption to remain on a more general case. The initial conditions for the system (1.1) are defined as follows :

$$\rho_\pm(x,0) = \rho_\pm^0(x) = P_\pm^0(x) + L_0 x \text{ on } \mathbb{R} \tag{1.2}$$

where $P_\pm^0$ are periodic of period 1 and Lipschitz. In particular, $\rho_+^0 - \rho_-^0$ is a 1-periodic function. $L_0$ is a given constant which is the total densities of type $\pm$, *i.e.* we suppose that initially, we have the same total density of type + and -.

## 1.2 Main Results

The first goal of our paper is to prove the existence and uniqueness for the solution of the non-local system (1.1)-(1.2). A natural framework for our study is viscosity solutions. We refer to Barles [7], Bardi, Capuzzo-dolcetta [6] and Crandall, Ishii, Lions [10] for a good introduction to this theory in the scalar case. We also refer to Ishii, Koike [18], [19] for the vectorial case and to Engler, Lenhart [12], Lenhart [22], Lenhart, Belbas [23], Lenhart, Yamada [24] and Yamada [27] for some applications.

We have the following existence and uniqueness result for the non local system

**Theorem 1.1 (Existence and uniqueness for the non-local problem)** *For all $T > 0$, for all $L_0 \in \mathbb{R}$, suppose that $\rho_\pm^0 \in \mathrm{Lip}(\mathbb{R})$ satisfy (1.2) and $L \in W^{1,\infty}(\mathbb{R}^+)$. Then, the system (1.1)-(1.2) admits a unique viscosity solution, $\rho = (\rho_+, \rho_-)$. Moreover, this solution is uniformly Lipschitz continuous in space and time.*

**Remark 1.2** *If at initial time, we have $D\rho_\pm^0 \geq 0$, then this remains true for $t \geq 0$, i.e., $D\rho_\pm(x,t) \geq 0$ for all $(x,t) \in \mathbb{R}\times[0,T]$. This allows to treat the physical case where $D\rho_\pm \geq 0$.*

The main difficulty comes from the fact that the comparison principle does not hold because of the non-local term. In order to overcome this problem, we introduce a fixed point method by freezing the non-local term. In a first time, we give an existence and uniqueness result for the local problem (this is a simple adaptation of [18]). Then, we use Lipschitz estimates on the solution to prove the short time existence and uniqueness for the non-local system. In the third step, we obtained the result for all time by iterating the process.

Here, we are interested by the dislocations densities dynamics. Some others models have studied the dynamics of dislocations lines. We recall some recent results. A non-local Hamilton-Jacobi equation have been proposed by Alvarez, Hoch, Le Bouar and Monneau [5] [4] for modelling dislocation dynamics. They also proved a short time existence and uniqueness result for this model. We also refer to Alvarez, Cardaliaguet, Monneau [1] and Barles, Ley [8] for a long time result under certain monotony assumptions and to Forcadel [13] for a short time result for dislocations dynamics with a mean curvature term.

The second result is a numerical analysis of the non-local system (1.1). We propose a numerical scheme for our non-local system. Then, we give an error estimate between the continuous solution and the numerical one.

We want to approximate the solution of (1.1)-(1.2). Given a mesh size $\Delta x,\ \Delta t$ we define

$$\Xi = \{i\Delta x,\ i \in \mathbb{Z}\} \quad \Xi_T = \Xi \times \{0, ..., (\Delta t)N_T\}$$

where $N_T$ is the integer part of $T/\Delta t$. We refer generically to the lattice by $\Delta$ in the sequel. The discrete running point is $(x_i, t_n)$ with $x_i = i(\Delta x),\ t_n = n(\Delta t)$. We assume that $\Delta x + \Delta t \leq 1$. The approximation of the solution $\rho_k$ at the node $(x_i, t_n)$ is written indifferently as $v_k(x_i, t_n)$ or $v^n_{k,i}$ according to whether we view it as a function defined on the lattice or as a sequence.

Now, we will introduce the numerical scheme. The main difficulty is due to the non-local term, which requires the availability of the solution we are intending to approximate. To solves this problem, we fix the solution $v^n_i = (v^n_{+,i}, v^n_{-,i})$ at each time step on the interval $[t_n, t_{n+1}]$ and we apply the following monotone scheme,

$$v^0_i = (v^0_{+,i}, v^0_{-,i}) = \tilde{\rho}^0(x_i) = (\tilde{\rho}^0_+, \tilde{\rho}^0_-), \tag{1.3}$$

where $\tilde{\rho}^0_\pm(x_i)$ is an approximation of $\rho^0_\pm(x_i)$

$$v^{n+1}_{k,i} = v^n_{k,i} + \Delta t C^\Delta_k[v](x_i, t_n) \begin{cases} E^+\left(D^+v^n_{k,i}, D^+v^n_{k,i}\right) & \text{if } C^\Delta_k[v](x_i, t_n) \geq 0 \\ E^-\left(D^+v^n_{k,i}, D^+v^n_{k,i}\right) & \text{else} \end{cases} \quad \forall k \in \{+, -\} \tag{1.4}$$

where

$$C^\Delta_k[v](x_i, t_n) = -k\left(v^n_{+,i} - v^n_{-,i} + a^\Delta[v](t_n)\right)$$

and the non-local term $a^\Delta[v](t_n)$ is given by

$$a^\Delta[v](t_n) = \sum_{i=0}^{N_x - 1} \Delta x \left(v_+(x_i, t_n) - v_-(x_i, t_n)\right) + L(t_n) \tag{1.5}$$

where $N_x$ is the integer part of $1/\Delta x$. $E^\pm$ are the approximation of the Euclidean norm proposed by Osher and Sethian [25] :

$$E^+(P, Q) = \left(\max(P, 0)^2 + \min(Q, 0)^2\right)^{\frac{1}{2}}, \tag{1.6}$$

$$E^-(P, Q) = \left(\min(P, 0)^2 + \max(Q, 0)^2\right)^{\frac{1}{2}}$$

and $D^+v^n_{k,i},\ D^-v^n_{k,i}$ are the discrete gradient for all $n \in \{0, ..., N_T\},\ i \in \mathbb{Z}$ and $k \in \{+, -\}$ :

$$D^+v^n_{k,i} = \frac{v^n_{k,i+1} - v^n_{k,i}}{\Delta x}, \tag{1.7}$$

$$D^-v^n_{k,i} = \frac{v^n_{k,i} - v^n_{k,i-1}}{\Delta x}.$$

Finally, we assume the following uniform CFL condition (see the beginning of Section 5.2 for more details)

$$\Delta t \leq \frac{1}{2L_2}\Delta x \tag{1.8}$$

where

$$L_2 = (2B_0T + 1)\left(2B_0(2B_0 + \|L\|_{L^\infty(0,T)})T + M + \|L\|_{L^\infty(0,T)}\right) + M + 2$$

with $B_0 = \max_{k \in \{+,-\}} \|D\rho^0_k\|_{L^\infty(\mathbb{R})}$ and $M = \|P^0_+ - P^0_-\|_{L^\infty(\mathbb{R})}$.

We then have the following error estimate

**Theorem 1.3 (Discrete-continuous error estimate)** *Let $T \geq 0$. Assume that $\Delta x + \Delta t \leq 1$, $L \in W^{1,\infty}(\mathbb{R} \times [0,T))$ and that the CFL condition (1.8) holds.*

*Then there exists a constant $K > 0$ depending only on $\|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})}$, $\max_{k\in\{+,-\}} \|D\rho_k^0\|_{L^\infty(\mathbb{R})}$ and $\|L\|_{W^{1,\infty}(0,T)}$ such that the error estimate between the solution $\rho$ of the continuous system (1.1)-(1.2) and the discrete solution $v$ of the finite difference scheme (1.3)-(1.4) is given by*

$$\max_{k\in\{+,-\}} \sup_{\Xi_T} |\rho_k - v_k| \leq K \left( (T + \sqrt{T}) \, (\Delta x + \Delta t)^{1/2} + \max_{k\in\{+,-\}} \sup_{\Xi} |\rho_k^0 - v_k^0| \right)$$

*provided* $K \left( (T + \sqrt{T})(\Delta x + \Delta t)^{\frac{1}{2}} + \max\limits_{k\in\{+,-\}} \sup\limits_{\Xi} (\rho_k^0 - v_k^0) \right) \leq 1.$

**Remark 1.4** *In the condition* $K \left( (T + \sqrt{T})(\Delta x + \Delta t)^{\frac{1}{2}} + \max\limits_{k\in\{+,-\}} \sup\limits_{\Xi} (\rho_k^0 - v_k^0) \right) \leq 1$, *we can replace the* 1 *by any positive constant.*

In fact in the proof of this theorem, we mimic the continuous problem by considering the approximate solution of (1.1) as a fixed point of a local system. We are inspired by [3] to prove a Crandall-Lions [11] rate of convergence between the continuous solution of (1.1) and the numerical one. As far as we know, this is the first error estimate of Crandall-Lions type for Hamilton-Jacobi systems. We also refer to Jakobsen, Karlsen [20] and Jakobsen, Karlsen, Risebro [21] where they proved an error estimate for a weakly coupled system of the form

$$(u_i)_t + H_i(t, x, u_i, Du_i) = G_i(t, x, u) \quad \text{in } \mathbb{R}^N \times (0, T) \tag{1.9}$$

for $i = 1, ..., M$. Their error estimate is in $O(\Delta t)$ for a semi-discrete splitting algorithm that they propose to approach the solution of (1.9). However, we obtain an error estimate in $O(\sqrt{\Delta t + \Delta x})$ because we also discretize in space.

In the dynamics of dislocations lines case, the model have also been numerically studied by Alvarez, Carlini, Monneau and Rouy [2] [3]. In their paper, they proposed a numerical scheme for the non-local Hamilton-Jacobi equation and they proved a Crandall-Lions rate of convergence.

Let us now explain how the paper is organised. In Section 2, we fix some notations. We present the formal derivation of the model in Section 3. Then, in Section 4, we study the continuous problem. First in Subsection 4.1, we give an existence and uniqueness result for a local system. Then, in Subsection 4.2, we prove Theorem 1.1 using a fixed point method. In Section 5, we prove a Crandall-Lions error estimate for the local problem and then we prove Theorem 1.3 on the non-local problem. Some numerical examples are displayed in Section 6 where we show some tests illustrating our error estimate and then an evolution approximation of dislocation densities.

# 2 Notation

For simplicity of presentation, we fix some notations :

1. *Order relation* : for $r = (r_1,\ r_2)\,,\ s = (s_1,\ s_2) \in \mathbb{R}^2$, we say that $r \leq s$ if $r_k \leq s_k$ for $k \in \{1, 2\}$.
2. *Addition vector-scalar* : for $r = (r_1,\ r_2) \in \mathbb{R}^2$, $\lambda \in \mathbb{R}$, we denote by $r + \lambda$ the vector $(r_1 + \lambda,\ r_2 + \lambda)$.
3. *$P$-periodic plus $L_0$-linear function* : we say that $\rho$ is $P-$periodic plus $L_0-$linear if there exists a vectorial periodic in space function $P^\rho = (P_+^\rho, P_-^\rho)$ of period $P$ and a constant $L_0$ such that $\rho(x,t) = P^\rho(x,t) + L_0 x = \big(P_+^\rho(x,t) + L_0 x,\ P_-^\rho(x,t) + L_0 x\big)$.

# 3 Modelling

We denote by $\mathbf{X}$ the vector $\mathbf{X} = (x_1, x_2)$. We consider a crystal with periodic deformation, namely the case where the total displacement of the crystal $U = (U_1, U_2) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^2$ can be decomposed in a 1-periodic displacement $u = (u_1, u_2)$ and a linear displacement $A(t)^t\mathbf{X}$ with $A(t)$ a given $2 \times 2$ matrix which represents the shear stress

$$A(t) = \begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{pmatrix}.$$

The displacement $U$ is then given by

$$U(\mathbf{X}, t) = u(\mathbf{X}, t) + A(t)^t\mathbf{X}$$

and we define the total strain by

$$\varepsilon(U) = \frac{1}{2}(\nabla U + {}^t\nabla U) = \frac{1}{2}\left(\nabla u + {}^t\nabla u + A(t) + {}^tA(t)\right).$$

where the coefficients of $\nabla U$ are $(\nabla U)_{ij} = \dfrac{\partial U_i}{\partial x_j}$, $i, j \in \{1, 2\}$.
This total strain is decomposed in the form

$$\varepsilon(U) = \varepsilon^e(U) + \varepsilon^p,$$

where $\varepsilon^e(U)$ is the elastic deformation and $\varepsilon^p$ the plastic deformation which is connected to the densities of dislocations by

$$\varepsilon^p = \varepsilon^0 \left(\rho_+ - \rho_-\right), \tag{3.10}$$

where $\rho_\pm$ represent the edge dislocation of type $\pm$, such that $\vec{b}.\nabla\rho_\pm \geq 0$ is the density of dislocation of type $\pm$, $\vec{b} = (b_1, b_2)$ is the Burgers vector and

$$\varepsilon^0 = \frac{1}{2}\left(\vec{b} \otimes \vec{b}^\perp + \vec{b}^\perp \otimes \vec{b}\right) \tag{3.11}$$

where $\vec{b}^\perp$ is the orthogonal vector to $\vec{b}$ and $\left(\vec{b} \otimes \vec{b}^\perp\right)_{ij} = b_i b_j^\perp$.

The stress is then given by

$$\sigma = \Lambda : \varepsilon^e(U), \tag{3.12}$$

*i.e* the coefficients of the matrix $\sigma$ are :

$$\sigma_{ij} = \sum_{k,l \in \{1,2\}} \Lambda_{ijkl}\varepsilon^e_{kl}(U) \quad i, j \in \{1, 2\}$$

with $\Lambda = (\Lambda_{ijkl})_{ijkl}$, $i, j, k, l = 1, 2$, $\Lambda_{ijkl}$ are the elastic constant coefficients of the material, satisfying for $m > 0$ :

$$\sum_{ijkl=1,2} \Lambda_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \geq m \sum_{ij=1,2} \varepsilon_{ij}^2 \tag{3.13}$$

for all symmetric matrix $\varepsilon = (\varepsilon_{ij})_{ij}$ *i.e.* such that $\varepsilon_{ij} = \varepsilon_{ji}$

The functions $\rho_\pm$ and $u$ are then solutions of the coupled system (see Groma, Balogh [16], [15] and Groma [14]) :

$$\left\{\begin{array}{lll} div\ \sigma & = 0 & \text{in } \mathbb{R}^2 \times (0\ \ T), \\ \sigma & = \Lambda : (\varepsilon(U) - \varepsilon^p) & \text{in } \mathbb{R}^2 \times (0\ \ T), \\ \varepsilon(U) & = \frac{1}{2}\left(\nabla u + {}^t\nabla u + A(t) + {}^tA(t)\right) & \text{in } \mathbb{R}^2 \times (0\ \ T), \\ \varepsilon^p & = \varepsilon^0\left(\rho_+ - \rho_-\right) & \text{in } \mathbb{R}^2 \times (0\ \ T), \\ & & \\ (\rho_\pm)_t & = \pm(\sigma : \varepsilon^0)\vec{b}.\nabla\rho_\pm & \text{in } \mathbb{R}^2 \times (0\ \ T), \end{array}\right. \tag{3.14}$$

*i.e* in the coordinates

$$\left\{\begin{array}{lll} \displaystyle\sum_{j=1,2} \frac{\partial \sigma_{ij}}{\partial x_j} & = 0 & \text{in } \mathbb{R}^2 \times (0\ \ T), \\ \sigma_{ij} & = \displaystyle\sum_{k,l\in\{1,2\}} \Lambda_{ijkl}\left(\varepsilon_{kl}(U) - \varepsilon^p_{kl}\right) & \text{in } \mathbb{R}^2 \times (0\ \ T), \\ \varepsilon_{ij}(U) & = \frac{1}{2}\Big((\nabla u)_{ij} + (\nabla u)_{ij} + A_{ij}(t) + A_{ji}(t)\Big) & \text{in } \mathbb{R}^2 \times (0\ \ T), \\ \varepsilon^p_{ij} & = \varepsilon^0_{ij}\left(\rho_+ - \rho_-\right) & \text{in } \mathbb{R}^2 \times (0\ \ T), \\ & & \\ (\rho_\pm)_t & = \pm\left(\displaystyle\sum_{i,j\in\{1,2\}} \sigma_{ij}\varepsilon^0_{ij}\right)\vec{b}.\nabla\rho_\pm & \text{in } \mathbb{R}^2 \times (0\ \ T), \end{array}\right. \tag{3.15}$$

where the unknowns of the system are $\rho_\pm$ and the displacement $u = (u_1, u_2)$ and with $\varepsilon^0$ defined by (3.11). The sign $\pm$ comes from $\pm$ in $\pm\vec{b}$.

To simplify, we consider the homogeneous case. The coefficients $\Lambda_{ijkl}$ are such that

$$\sigma = 2\mu\varepsilon^e(U) + \lambda tr(\varepsilon^e(U))I_d, \tag{3.16}$$

where $\mu > 0$ and $\lambda + \mu > 0$ (consequence of (3.13)) are the Lamé coefficients and $I_d$ the identity matrix. Then, the following *lemma* holds

***Lemma* 3.1 (Equivalence between 2-D and 1-D models)** *If we assume that the Burger vector is $\vec{b} = (1, 0)$, and that the densities of dislocations only depends on one variable $x = x_1 + x_2$ (as shown in Figure 2), the 2-D problem (3.14), with $\Lambda$ defined by (3.16), becomes equivalent to the 1-D problem*

$$\left\{\begin{array}{lll} (\rho_+)_t & = -C_1\left(\left(\rho_+ - \rho_-\right) + C_2\displaystyle\int_0^1 (\rho_+ - \rho_-) + L(t)\right) D\rho_+ & \textit{in } \mathbb{R} \times (0\ \ T) \\ & & \\ (\rho_-)_t & = C_1\left(\left(\rho_+ - \rho_-\right) + C_2\displaystyle\int_0^1 (\rho_+ - \rho_-) + L(t)\right) D\rho_- & \textit{in } \mathbb{R} \times (0\ \ T) \end{array}\right. \tag{3.17}$$

*where $L(t) = -\frac{(\lambda+2\mu)}{(\lambda+\mu)}(A_{12}(t) + A_{21}(t))$, $C_1 = \frac{(\lambda+\mu)\mu}{\lambda+2\mu}$, and $C_2 = \frac{\mu}{(\lambda+\mu)}$.*

**Proof of lemma 3.1**
We can rewrite the first equation of (3.14) and (3.16) as

$$div\left(2\mu\varepsilon(U) + \lambda tr(\varepsilon(U))I_d\right) = div\left(2\mu\varepsilon^p + \lambda tr(\varepsilon^p)I_d\right).$$

This implies by (3.10)

$$\mu\Delta u + (\lambda + \mu)\nabla(div\ u) = \mu\begin{pmatrix} \dfrac{\partial(\rho_+ - \rho_-)}{\partial x_2} \\ \\ \dfrac{\partial(\rho_+ - \rho_-)}{\partial x_1} \end{pmatrix}.$$

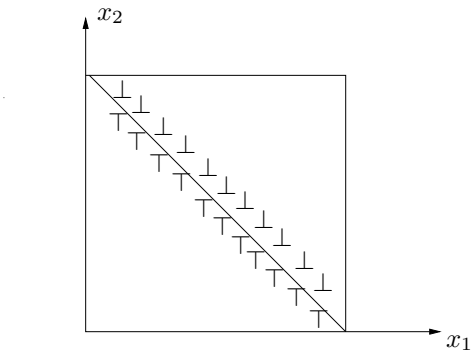


FIG. 2 – 1D sub-model for invariance by translation in the direction(-1,1)

Using the fact that $x = x_1 + x_2$, yields

$$2\mu \begin{pmatrix} \dfrac{\partial^2 u_1}{\partial x^2} \\ \\ \dfrac{\partial^2 u_2}{\partial x^2} \end{pmatrix} + (\lambda+\mu) \begin{pmatrix} \dfrac{\partial^2 (u_1+u_2)}{\partial x^2} \\ \\ \dfrac{\partial^2 (u_1+u_2)}{\partial x^2} \end{pmatrix} = \mu \begin{pmatrix} \dfrac{\partial (\rho_+ - \rho_-)}{\partial x} \\ \\ \dfrac{\partial (\rho_+ - \rho_-)}{\partial x} \end{pmatrix}.$$

Now, by adding the two above equations, we obtain

$$\frac{\partial^2 (u_1+u_2)}{\partial x^2} = \frac{\mu}{\lambda+2\mu} \left( \frac{\partial (\rho_+ - \rho_-)}{\partial x} \right).$$

Integrating the above equation yields since $u$ is 1-periodic

$$\frac{\partial (u_1+u_2)}{\partial x} = \frac{\mu}{\lambda+2\mu} \left( (\rho_+ - \rho_-) - \int_0^1 (\rho_+ - \rho_-) \right). \tag{3.18}$$

Using the fact that

$$(\sigma : \varepsilon^0) = \sigma_{12} = 2\mu(\varepsilon^e(U))_{12} = \mu \left( \frac{\partial (u_1+u_2)}{\partial x} + A_{12}(t) + A_{21}(t) - (\rho_+ - \rho_-) \right)$$

and (3.18) yields

$$(\sigma : \varepsilon^0) = -\frac{(\lambda+\mu)\mu}{\lambda+2\mu} \left( (\rho_+ - \rho_-) + \frac{\mu}{2(\lambda+\mu)} \int_0^1 (\rho_+ - \rho_-) + L(t) \right)$$

where $L(t) = -\dfrac{(\lambda+2\mu)}{(\lambda+\mu)}(A_{12}(t) + A_{21}(t))$. We then deduce, if $\vec{b} = (1,0)$, that the system (3.14) can be rewritten as (3.17). As the constant $C_1$, $C_2$ are positive, to simplify the notations, we can put them to 1 in the following without lost of generality on the system (1.1).

# 4 The continuous problem

To prove the existence and uniqueness result for the non-local problem, we use a fixed point method. In order to do that we freeze the non-local term and we study the following local problem

$$\begin{cases} (\rho_+)_t = -\left(\rho_+ - \rho_- + a(t)\right) |D\rho_+| & \text{on} \quad \mathbb{R} \times (0,T) \\ (\rho_-)_t = \left(\rho_+ - \rho_- + a(t)\right) |D\rho_-| & \text{on} \quad \mathbb{R} \times (0,T) \\ \rho_+(\cdot,0) = \rho_+^0 & \text{on} \quad \mathbb{R} \\ \rho_-(\cdot,0) = \rho_-^0 & \text{on} \quad \mathbb{R}. \end{cases} \tag{4.19}$$

The assumptions are the following :

(H1) $a \in W^{1,\infty}(\mathbb{R}^+)$,

(H2) $\rho^0 = (\rho^0_+, \rho^0_-)$ is $1-$periodic plus $L_0-$linear, *i.e.*, $\rho^0_\pm(x) = P^0_\pm(x) + L_0 x$ where $P^0_\pm$ are periodic of period 1 and $L_0$ is a constant.

(H3) $P^0_\pm \in \mathrm{Lip}(\mathbb{R})$.

## 4.1 The local problem

We denote by USC (resp. LSC) the set of locally bounded upper (resp. lower) semi-continuous functions. The key point is that our system is quasi monotone in the sense of Ishii, Koike [18] (A.1) (see *Lemma* 4.1) and so we can extend their results to our system in unbounded domain and with unbounded initial condition using the well-known arguments of the scalar case.

***Lemma* 4.1 (Quasi-monotony of the Hamiltonian)** *The Hamiltonian $H_k(t,\rho,p) = k(\rho_+ - \rho_- + a(t))|p|$ is quasi-monotone, i.e., for all vectors $r$ and $s$ such that*

$$r_j - s_j = \max_{k\in\{+,-\}}(r_k - s_k) \geq 0$$

*then*

$$H_j(t,r,p) - H_j(t,s,p) \geq 0. \tag{4.20}$$

We then have the following theorem

**Theorem 4.2 (The local problem)** *Let $T \geq 0$. Assume (H1)-(H2)-(H3). We set $M = \|P^0_+ - P^0_-\|_{L^\infty(\mathbb{R})}$ and $B_0 = \max_{k\in\{+,-\}} \|D\rho^0_k\|_{L^\infty(\mathbb{R})}$. Then, the following holds*

*(i)* **Comparison principle.** *Let $\rho \in USC(\Omega\times]0,T[)$ and $v \in LSC(\Omega\times]0,T[)$ be respectively sub and super-solution of (1.1)-(1.2). We assume that there exists a constant $C > 0$ such that*

$$\rho^0(x) - Ct \leq \rho, v \leq \rho^0(x) + Ct. \tag{4.21}$$

*Then if $\rho(\cdot,0) \leq v(\cdot,0)$ in $\mathbb{R}$ then $\rho \leq v$ in $\mathbb{R}\times[0,T]$.*

*(ii)* **Existence.** *There exists a unique viscosity solution $\rho$ of problem (4.19) satisfying*

$$\rho^0(x) - (M + \|a\|_{L^\infty(0,T)})B_0 t \leq \rho(x,t) \leq \rho^0(x) + (M + \|a\|_{L^\infty(0,T)})B_0 t. \tag{4.22}$$

*Moreover, the solution is $1-$periodic plus $L_0-$linear.*

*(iii)* **Regularity.** *The solution $\rho$ of (4.19) is Lipschitz continuous in space and time and satisfies*

$$\max_{k\in\{+,-\}} \|D\rho_k\|_{L^\infty(\mathbb{R}\times(0,T))} \leq B_0$$

$$\max_{k\in\{+,-\}} \|(\rho_k)_t\|_{L^\infty(\mathbb{R}\times(0,T))} \leq B_0(2B_0(M + \|a\|_{L^\infty(0,T)})T + M + \|a\|_{L^\infty(0,T)}).$$

*(iv)* **Estimate on the solution.** *The solution $\rho$ satisfies*

$$\|\rho_+ - \rho_-\|_{L^\infty(\mathbb{R}\times(0,T))} \leq 2B_0\|a\|_{L^\infty(0,T)}T + \|\rho^0_+ - \rho^0_-\|_{L^\infty(\mathbb{R})}.$$

**Proof of Theorem 4.2**
The comparison principle is just an extension of the one of Ishii, Koike [18] Theorem 4.7 for quasi-monotone Hamiltonian. For the existence, it suffices to use Perron's method by remarking that $\rho \pm (M + \|a\|_{L^\infty(0,T)})B_0 t$ are resp. super and sub-solution of (4.19). The fact that $\rho$ is $1-$periodic plus $L_0-$linear comes from the fact that $\rho(x+1,t) + L_0$ is also solution of (4.19).

The Lipschitz estimate in space comes from the fact that Problem (4.19) is invariant by space translation. To obtain the Lipschitz estimate in time, it is sufficient to bound the velocity using (4.22).

We now prove $(iv)$. We set $w = \rho_+ - \rho_-$. We then have

$$w_t = -(w + a(t))(|D\rho_+| + |D\rho_-|).$$

We set $m(t) = \sup_{x\in\mathbb{R}} w(x,t)$ (this sup is reached thanks to $(ii)$) and we get for $m \geq 0$

$$\begin{aligned} m_t =& -(m + a(t))(|D\rho_+| + |D\rho_-|) \\ =& -m(|D\rho_+| + |D\rho_-|) - a(t)(|D\rho_+| + |D\rho_-|) \\ \leq& 2B_0\|a\|_{L^\infty(0,T)} \end{aligned}$$

which implies the upper bound of $(iv)$. The lower bound is proved similarly. This ends the proof of the theorem.

## 4.2 The non-local problem

Before to prove Theorem 1.1, we need the following *lemma*

***Lemma* 4.3 (Stability of the solution with respect to the velocity)** *Let $T \geq 0$. We consider for $i = 1, 2$ two different equations*

$$\begin{cases} (\rho_k^i)_t = -k\left(\rho_+^i - \rho_-^i + a_i(t)\right)|D\rho_k^i| & \text{for} \quad k \in \{+,-\} \\ \rho_k^i(\cdot,0) = \rho_k^0 & \text{for} \quad k \in \{+,-\} \end{cases} \tag{4.23}$$

*where the coefficients $a_i$ satisfy $(H1)$ and the initial conditions $\rho^0 = (\rho_+^0, \rho_-^0)$ satisfies (H2)-(H3). Then, we have*

$$\max_{k\in\{+,-\}} \|\rho_k^2 - \rho_k^1\|_{L^\infty(\mathbb{R}\times(0,T))} \leq B_0 T\|a_2 - a_1\|_{L^\infty(0,T)},$$

*where $\rho^i$ for $i = 1, 2$ are the solutions of (4.23) given by Theorem 4.2.*

**Proof of *Lemma* 4.3**
We set $K = \|a_2 - a_1\|_{L^\infty(0,T)}$. We remark that $\rho^2$ is a sub-solution of

$$(\rho_k)_t + k\left(\rho_+ - \rho_- + a_1(t)\right)|D\rho_k| - KB_0 = 0.$$

Moreover $\rho^1 + KB_0 t$ is solution of the same problem. By comparison principle, we then deduce

$$\max_{k\in\{+,-\}} \|\rho_k^2 - \rho_k^1\|_{L^\infty(\mathbb{R}\times(0,T))} \leq KB_0 T.$$

This is the estimate we want.

We have the following *lemma* which proof is trivial

***Lemma* 4.4 (Stability of the velocity a)** *Let $\rho^1$, $\rho^2$ be $1-$periodic plus $L_0-$linear. We set $a[\rho^i](t) = \int_0^1 \rho_+^i(x,t) - \rho_-^i(x,t)dx + L(t)$. Then the following holds*

$$\|a[\rho^2] - a[\rho^1]\|_{L^\infty(0,T)} \leq 2 \max_{k\in\{+,-\}} \|\rho_k^2 - \rho_k^1\|_{L^\infty(\mathbb{R}\times(0,T))}.$$

We now prove Theorem 1.1

**Proof of Theorem 1.1**
We define the set :

$$U_T = \left\{ \rho = \begin{pmatrix} \rho_+ \\ \rho_- \end{pmatrix} \in (L^\infty_{\text{Loc}})^2, s.t. \left| \begin{array}{l} \|\rho_+ - \rho_-\|_{L^\infty} \le 2M + \|L\|_{L^\infty(0,T)} \\ \rho \text{ is } 1-\text{periodic plus } L_0-\text{linear} \\ \max\limits_{k\in\{+,-\}} \|D\rho_k\|_{L^\infty} \le B_0 \\ \max\limits_{k\in\{+,-\}} \|(\rho_k)_t\|_{L^\infty} \le B_0(4M + 3\|L\|_{L^\infty(0,T)}) \end{array} \right. \right\},$$

where $L_0$ is defined in $(H2)$, $B_0 = \max\limits_{k\in\{+,-\}} \|D\rho_k^0\|_{L^\infty(\mathbb{R})}$ and $M = \|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})}$. For $\rho \in U_T$, we set

$$a[\rho](t) = \int_0^1 \rho_+(x,t) - \rho_-(x,t)dx + L(t). \tag{4.24}$$

We see that $a[\rho]$ satisfies $(H1)$ with $\|a[\rho]\|_{L^\infty(0,T)} \le 2(M + \|L\|_{L^\infty(0,T)})$.

For $\rho \in U_T$, we then define $v = G(\rho) = (G_+(\rho), G_-(\rho))$ as the unique viscosity solution for $k = 1, 2$ (see Theorem 4.2) of

$$\begin{cases} (v_k)_t = -k\left(v_+ - v_- + a[\rho](t)\right)|Dv_k| & \text{on} \quad (0,T)\times\mathbb{R} \\ v_k(\cdot,0) = \rho_k^0 & \text{on} \quad \mathbb{R}. \end{cases} \tag{4.25}$$

We will show that $G : U_T \to U_T$ is a strict contraction for $T$ small enough. First, we will prove that $G$ is well defined. By Theorem 4.2, we know that $v$ is $1-$ periodic plus $L_0-$linear. Moreover, we have

$$\max_{k\in\{+,-\}} \|Dv_k\|_{L^\infty(\mathbb{R}\times(0,T))} \le B_0$$

and

$$\begin{aligned} \max_{k\in\{+,-\}} \|(v_\pm)_t\|_{L^\infty(\mathbb{R}\times(0,T))} \le & B_0(2B_0(M + \|a\|_{L^\infty(0,T)})T + \|a\|_{L^\infty(0,T)} + M) \\ \le & B_0(2B_0(3M + 2\|L\|_{L^\infty(0,T)})T + 3M + 2\|L\|_{L^\infty(0,T)}) \\ \le & B_0(4M + 3\|L\|_{L^\infty(0,T)}) \end{aligned}$$

for $T \le T^* = \dfrac{1}{6B_0}$. It thus remains to be shown that $\|v_+ - v_-\|_{L^\infty(\mathbb{R}\times(0,T))} \le 2M + \|L\|_{L^\infty(0,T)}$. By Theorem 4.2, we have

$$\begin{aligned} \|v_+ - v_-\|_{L^\infty(\mathbb{R}\times(0,T))} \le & 2B_0(M + \|a\|_{L^\infty(0,T)})T + M \\ \le & 2B_0(3M + 2\|L\|_{L^\infty(0,T)})T + M \le 2M + \|L\|_{L^\infty(0,T)} \end{aligned}$$

for $T \le T^* = \dfrac{1}{6B_0}$ and so $v \in U_T$ for $T \le T^*$.

It thus remains to show that $G$ is a contraction. For $v^i = G(\rho^i)$, according to *Lemma* 4.3 and *Lemma* 4.4, we have

$$\begin{aligned} \|v^2 - v^1\|_{L^\infty} = & \sup_{\{k\in\{+,-\}\}} \|v_k^2 - v_k^1\|_{L^\infty} \le B_0 T\|a[\rho^2] - a[\rho^1]\|_{L^\infty} \\ \le & 2B_0T\|\rho^1 - \rho^2\|_{L^\infty} \le \frac{1}{3}\|\rho^1 - \rho^2\|_{L^\infty} \end{aligned}$$

for $T \le T^* = \dfrac{1}{6B_0}$. And so $G$ is a contraction on $U_T$ which is a closed set. So, there exists a unique viscosity solution of (1.1)-(1.2) in $U_T$ on $(0, T^*)$ where $T^* = \dfrac{1}{6B_0}$. By iterating this process, one can construct a solution for all $T > 0$. Indeed, $T^*$ depends only on $B_0$ which do not change during time.

**Proposition 4.5 (Estimate for the non-local solution)** *Let $T \geq 0$. The solution $\rho$ of (1.1)-(1.2) satisfies*

$$\|\rho_+ - \rho_-\|_{L^\infty(\mathbb{R}\times(0,T))} \leq 2B_0(2B_0 + \|L\|_{L^\infty(0,T)})T + M$$

*where* $B_0 = \max_{k\in\{+,-\}} \|D\rho_k^0\|_{L^\infty(\mathbb{R})}$ *and* $M = \|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})}$.

**Proof of Proposition 4.5**
The proof is very similar to the one of the local case Theorem 4.2 $(iv)$. We denote by $w = \rho_+ - \rho_-$. We have

$$\begin{aligned} w_t =& -(w + \int_0^1 w + L(t))(|D\rho_+| + |D\rho_-|) \\ =& -2w(|D\rho_+| + |D\rho_-|) + (w - \int_0^1 w - L(t))(|D\rho_+| + |D\rho_-|). \end{aligned}$$

Using the Poincaré-Wirtingerle inequality, we deduce that the second term is bounded by $2B_0(2B_0 + \|L\|_{L^\infty(0,T)})$. Using the same argument of the one of Theorem 4.2, we deduce the result.

# 5 Numerical scheme

## 5.1 Approximation of the local system

In this subsection, we propose a finite difference scheme for the local system (4.19). Given a discrete velocity $a^\Delta$, we consider the discrete solution $v$ that approximates the solution of (4.19), given by the following explicit scheme

$$v_{k,i}^0 = \tilde{\rho}_k^0(x_i), \tag{5.26}$$

$$v_{k,i}^{n+1} = v_{k,i}^n + \Delta t \left(C_k^{\Delta,\mathrm{Loc}}[v](x_i,t_n)\right) \begin{cases} E^+\left(D^+v_{k,i}^n, D^+v_{k,i}^n\right) & \text{if } C_k^{\Delta,\mathrm{Loc}}[v](x_i,t_n) \geq 0 \\ E^-\left(D^+v_{k,i}^n, D^+v_{k,i}^n\right) & \text{else} \end{cases} \quad \forall k \in \{+,-\} \tag{5.27}$$

where $\tilde{\rho}^0(x_i)$ are defined in (1.3), $E^\pm$ are the approximation of the Euclidean norm proposed by Osher and Sethian [25] defined in (1.6) (we also can use the one proposed by Rouy, Tourin [26]), $D^+v_k^n$, $D^-v_k^n$ are the discrete gradient defined in (1.7) and

$$C_k^{\Delta,\mathrm{Loc}}[w](x_i,t_n) = -k(w_+(x_i,t_n) - w_-(x_i,t_n) + a^\Delta(t_n)) \tag{5.28}$$

where $a^\Delta$ is an approximation of $a$ satisfying

$$a^\Delta(t_n) = a(t_n). \tag{5.29}$$

In particular, the functions $E^\pm$ are Lipschitz continuous with respect to the discrete gradients, *i.e.*

$$\left|E^\pm(P,Q) - E^\pm(P',Q')\right| \leq (|P-P'| + |Q-Q'|) \tag{5.30}$$

are consistent with the Euclidean norm

$$E^\pm(P,P) = |P| \tag{5.31}$$

and enjoy suitable monotonicity with respect to each variable

$$\frac{\partial E^+}{\partial P^+} \geq 0, \quad \frac{\partial E^+}{\partial P^-} \leq 0, \quad \frac{\partial E^-}{\partial P^+} \geq 0, \quad \frac{\partial E^-}{\partial P^-} \leq 0. \tag{5.32}$$

Denoting by $S^k$ the operator on the right-hand side of (5.27), we can rewrite the scheme more compactly as

$$v^0_{k,i} = \tilde{\rho}^0_k(x_i), \quad v^{n+1}_k = S^k v^n.$$

Finally, we also assume that the mesh satisfies the following CFL condition (cf Remark 5.2)

$$\Delta t \leq \frac{1}{2L_1}\Delta x \tag{5.33}$$

where

$$L_1 = (2B_0T+1)\|a\|_{L^\infty(0,T)} + M + 2.$$

**Theorem 5.1 (Crandall-Lions rate of convergence)** *Let $T \leq 1$. Assume that $\Delta x + \Delta t \leq 1$. Assume that $a \in W^{1,\infty}(\mathbb{R}\times[0,T))$ and that the CFL condition (5.33) holds.*

*Then there exists a constant $K > 0$ depending only on $\|P^0_+ - P^0_-\|_{L^\infty(\mathbb{R})}$, $\max_{k\in\{+,-\}}\|D\rho^0_k\|_{L^\infty(\mathbb{R})}$ and $\|a\|_{W^{1,\infty}(0,T)}$ such that the error estimate between the solution $\rho$ of the continuous system (4.19) and the discrete solution $v$ of the finite difference scheme (5.26)-(5.27) is given by*

$$\max_{k\in\{+,-\}}\sup_{\Xi_T}|\rho_k(x_i,t_n) - v^n_{k,i}| \leq K\sqrt{T}\,(\Delta x + \Delta t)^{1/2} + \max_{k\in\{+,-\}}\sup_{\Xi}|\rho^0_k(x_i) - v^0_{k,i}|$$

*provided* $K(\Delta x+\Delta t)^{\frac{1}{2}} + \max_{k\in\{+,-\}}\sup_{\Xi}(\rho^0_k(x_i) - v^0_{k,i}) \leq 1$.

**Remark 5.2 (Monotony of the scheme)** *Under the assumptions of Theorem 5.1, we have*

$$\begin{aligned}|v^n_{+,i} - v^n_{-,i}| \leq& |v^n_{+,i} - \rho_+(x_i,t_n)| + |\rho_+(x_i,t_n) - \rho_-(x_i,t_n)| + |\rho_-(x_i,t_n) - v^n_{-,i}|\\ \leq& 2 + 2B_0\|a\|_{L^\infty(0,T)}T + M\end{aligned}$$

*where we have used Theorem 4.2 $(iv)$ for the second term. We then deduce that the discrete velocity*

$$C^{\Delta,\mathrm{Loc}}_k[v] \leq (2B_0T+1)\|a\|_{L^\infty(0,T)} + M + 2 = L_1.$$

*Then, one can show that the scheme is monotone in the following sense : Let $v$ and $w$ be two discrete functions such that $v^n_i \leq w^n_i$ then*

$$S^k(v^n)(x_i) \leq S^k(w^n)(x_i), \quad \text{for } k \in \{+,-\}.$$

For the proof of Theorem 5.1, we need the following *lemma*

***Lemma* 5.3** *If $v^n_i$ is the numerical solution of (5.26)-(5.27), then*

$$-Kt_n - \mu^0 \leq \rho^0(x_i) - v(x_i,t_n) \leq Kt_n + \mu^0. \tag{5.34}$$

*where* $K = 2(\|P^0_+ - P^0_-\|_{L^\infty(\mathbb{R})} + \|a\|_{L^\infty(0,T)})\max_{k\in\{+,-\}}\|D\rho^0_k\|_{L^\infty(\mathbb{R})}$ *and*

$$\mu^0 = \max_{k\in\{+,-\}}\sup_{\Xi}|\rho^0_k(x_i) - v^0_{k,i}| \geq 0. \tag{5.35}$$

**Proof of Lemma 5.3**
To prove this, we set $w_\pm(x_i,t_n) = \rho^0_\pm(x_i) - Kt_n - \mu^0$ and we show that for $K$ large enough $w$ is a discrete sub-solution. Indeed, we have

$$\begin{aligned}&w^{n+1}_{\pm,i} - (S^\pm w^n)_i\\ =& -K\Delta t - \Delta t C^{\Delta,\mathrm{Loc}}_\pm[\rho^0](x_i,t_n)E^{Sgn(C^{\Delta,\mathrm{Loc}}_\pm[\rho^0](x_i,t_n))}(D^+\rho^0_\pm(x_i), D^-\rho^0_\pm(x_i))\\ =& -\Delta t\left(K \mp (\rho^0_+(x_i) - \rho^0_-(x_i) + a^\Delta(t_n))E^{Sgn(C^{\Delta,\mathrm{Loc}}_\pm[\rho^0](x_i,t_n))}(D^+\rho^0_\pm(x_i), D^-\rho^0_\pm(x_i))\right)\\ \leq& -\Delta t\left(K - 2(\|P^0_+ - P^0_-\|_{L^\infty(\mathbb{R})} + \|a\|_{L^\infty(0,T)})\max_{k\in\{+,-\}}\|D\rho^0_k\|_{L^\infty(\mathbb{R})}\right).\end{aligned}$$

where $C_k^{\Delta,\mathrm{Loc}}[w](x_i,t_n)$ is defined in (5.28) and $sgn(f)$ is the sign of $f$.

So, for every $K \geq 2(\|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})} + \|a\|_{L^\infty(0,T)}) \max_{k\in\{+,-\}} \|D\rho_k^0\|_{L^\infty(\mathbb{R})}$, $w$ is a discrete subsolution. Moreover

$$w_{\pm,i}^0(x_i) = \rho_\pm^0(x_i) - \mu^0 \leq v_\pm^0(x_i).$$

Using the monotony of the scheme, we deduce $w_i^n \leq v_i^n$ and so

$$\rho^0(x_i) - v_i^n \leq Kt_n + \mu_0.$$

The lower bound is proved similarly.

We now give the proof of Theorem 5.1

**Proof of Theorem 5.1**

The proof is an adaptation for systems of the one of Crandall Lions [11], revisited by Alvarez *et al.* [2]. The proof splits into three steps. We denote throughout by $K$ various constant depending only on $\|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})}$, $\max_{k\in\{+,-\}} \|D\rho_k^0\|_{L^\infty(\mathbb{R})}$ and $\|a\|_{W^{1,\infty}(0,T)}$.

We first assume that

$$\rho^0(x_i) \geq v_i^0 \tag{5.36}$$

and we set

$$\mu^0 = \max_{k\in\{+,-\}} \sup_{\Xi} |\rho_k^0(x_i) - v_{k,i}^0| \geq 0. \tag{5.37}$$

We set a few notations. We put

$$\mu = \max_{k\in\{+,-\}} \sup_{\Xi_T} (\rho_k(x_i,t_n) - v_{k,i}^n).$$

For every $0 < \alpha \leq 1$, $0 < \varepsilon \leq 1$ and $\sigma > 0$, we set

$$M_\sigma^{\alpha,\varepsilon} = \sup_{\mathbb{R}\times[0,T]\times\Xi_T\times\{+,-\}} \Psi_\sigma^{\alpha,\varepsilon}(x,t,x_i,t_n,k),$$

with

$$\Psi_\sigma^{\alpha,\varepsilon}(x,t,x_i,t_n,k) = \rho_k(x,t) - v_k(x_i,t_n) - \frac{|x-x_i|^2}{2\varepsilon} - \frac{|t-t_n|^2}{2\varepsilon} - \sigma t - \alpha|x|^2 - \alpha|x_i|^2.$$

We shall drop the super and subscripts on $\Psi$ when no ambiguity arises as concern the value of the parameter.

Since $\rho^0$ is Lipschitz, we have by (4.22)

$$|\rho_\pm(x,t)| \leq K(1+|x|). \tag{5.38}$$

Moreover by *Lemma* 5.3 we have

$$\begin{aligned} |v_\pm(x_i,t_n)| \leq & |v_\pm(x_i,t_n) - \rho_\pm^0(x_i)| + |\rho_\pm^0(x_i)| \\ \leq & Kt_n + K(1+|x_i|) \\ \leq & K(1+|x_i|). \end{aligned}$$

We then deduce that $\Psi$ achieves its maximum at some point that we denote by $(x^*,t^*,x_i^*,t_n^*,k^*)$.

**Step 1 : Estimates for the maximum point of $\Psi$**

The maximum point of $\Psi$ enjoys the following estimates

$$\alpha|x^*| + \alpha|x_i^*| \leq K, \tag{5.39}$$

and

$$|x^* - x_i^*| \leq K\varepsilon, \quad |t^* - t_n^*| \leq (K + 2\sigma)\varepsilon. \tag{5.40}$$

Indeed, by inequality $\Psi(x^*, t^*, x_i^*, t_n^*, k^*) \geq \Psi(0, 0, 0, 0, k^*) \geq 0$, we obtain

$$\alpha|x^*|^2 + \alpha|x_i^*|^2 \leq \rho_{k^*}(x^*, t^*) - v_{k^*}(x_i^*, t_n^*) \leq K(1 + |x^*| + |x_i^*|) \leq K + \frac{K^2}{\alpha} + \frac{\alpha}{2}|x^*|^2 + \frac{\alpha}{2}|x_i^*|^2.$$

This implies (5.39), since $\alpha \leq 1$.

The first bound of (5.40) follows from the Lipschitz in space regularity of $\rho$ (see Theorem 4.2 $(iii)$), from the inequality $\Psi(x^*, t^*, x_i^*, t_n^*, k^*) \geq \Psi(x_i^*, t^*, x_i^*, t_n^*, k^*)$ and from (5.39). Indeed, this implies

$$\begin{aligned}\frac{|x^* - x_i^*|^2}{2\varepsilon} \leq& \rho_{k^*}(x^*, t^*) - \rho_{k^*}(x_i^*, t^*) - \alpha|x^*|^2 + \alpha|x_i^*|^2 \\ \leq& K|x^* - x_i^*| + \alpha|x^* - x_i^*|(|x^*| + |x_i^*|) \leq K|x^* - x_i^*|.\end{aligned}$$

The second bound of (5.40) is obtained in the same way, using the inequality $\Psi(x^*, t^*, x_i^*, t_n^*, k^*) \geq \Psi(x^*, t_n^*, x_i^*, t_n^*, k^*)$ and Theorem 4.2 $(iii)$.

**Step 2 : A better estimate for the maximum point of $\Psi$**

Inequality (5.39) can be strengthened to

$$\alpha|x^*|^2 + \alpha|x_i^*|^2 \leq K. \tag{5.41}$$

Indeed, using the Lipschitz regularity of $\rho$, the inequality $\Psi(x^*, t^*, x_i^*, t_n^*, k^*) \geq \Psi(0, 0, 0, 0, k^*)$ and equations (4.22), (5.34) and (5.40), yields

$$\begin{aligned}\alpha|x^*|^2 + \alpha|x_i^*|^2 \leq& \rho_{k^*}(x^*, t^*) - v_{k^*}(x_i^*, t_n^*) + \rho(x_i^*, 0) - \rho(x_i^*, 0) \\ \leq& K(|x^* - x_i^*| + t^*) + Kt_n^* + \mu^0 \leq K.\end{aligned}$$

**Step 3 : Upper bound of $\mu$**

We have the bound $\mu \leq K\sqrt{T}\,(\Delta x + \Delta t)^{\frac{1}{2}} + \mu^0$ if $\Delta x + \Delta t \leq \frac{1}{K^2}$.

First, we claim that for $\sigma$ large enough, we have either $t^* = 0$ or $t_n^* = 0$. Suppose the contrary. Then the function $(x, t) \mapsto \Psi(x, t, x_i^*, t_n^*, k^*)$ achieves its maximum at a point of $\mathbb{R} \times (0, T]$. Using the fact that $\rho$ is a sub-solution of the continuous problem, we obtain the inequality

$$\sigma + p_t^* \leq -k^*(\rho_+ - \rho_- + a(t^*))|p_x^* + 2\alpha x^*| \tag{5.42}$$

with $p_t^* = \frac{t^* - t_n^*}{\varepsilon}$, $p_x^* = \frac{x^* - x_i^*}{\varepsilon}$.

Since $t_n^* > 0$, we also have $\Psi(x^*, t^*, x_i^*, t_n^*, k^*) \geq \Psi(x^*, t^*, x_i, t_n^* - \Delta t, k^*)$. This implies

$$v_{k^*}(\cdot, t_n^* - \Delta t) \geq \varphi(\cdot, t_n^* - \Delta t) + v_{k^*}(x_i^*, t_n^*) - \varphi(x_i^*, t_n^*)$$

for $\varphi(x_i, t_n) = -\frac{|x^* - x_i|^2}{2\varepsilon} - \frac{|t^* - t_n|^2}{2\varepsilon} - \alpha|x_i|^2$. Using the fact that the scheme is monotone and commutes with the addition of constants, yields

$$\begin{aligned}& v_{k^*}(x_i^*, t_n^*) \\ =& S^{k^*}\left(v_{k^*}(\cdot, t_n^* - \Delta t)\right)(x_i^*) \\ \geq& \varphi(x_i^*, t_n^* - \Delta t) + v_{k^*}(x_i^*, t_n^*) - \varphi(x_i^*, t_n^*) \\ &+ \Delta t\left(c_{k^*}^{\Delta,\mathrm{Loc}}[v](x_i^*, t_n^*)\right) E^{l^*}\left(D^+\varphi(x_i^*, t_n^* - \Delta t), D^-\varphi(x_i^*, t_n^* - \Delta t)\right)\end{aligned}$$

where $l^* = Sgn\left(c_{k^*}^{\Delta,\mathrm{Loc}}[v](x_i^*, t_n^*)\right)$. We set

$$c[v] = -c_{k^*}^{\Delta,\mathrm{Loc}}[v](x_i^*, t_n^*), \quad c[\rho] = k^* \left(\rho_+(x^*, t^*) - \rho_-(x^*, t^*) + a(t^*)\right).$$

We then obtain the super-solution inequality :

$$\frac{\varphi(x_i^*, t_n^*) - \varphi(x_i^*, t_n^* - \Delta t)}{\Delta t} \geq -c[v] E^{l^*}(D^+\varphi(x_i^*, t_n^* - \Delta t), D^-\varphi(x_i^*, t_n^* - \Delta t)).$$

Straightforward computations of the discrete derivative of $\varphi$ yield

$$p_t^* + \frac{\Delta t}{2\varepsilon} \geq -c[v] E^{l^*}\left(p_x^* - \frac{\Delta x}{2\varepsilon} - \alpha(2x_i^* + \Delta x), p_x^* + \frac{\Delta x}{2\varepsilon} - \alpha(2x_i^* - \Delta x)\right).$$

Subtracting the above inequality to (5.42), we deduce

$$\begin{aligned}
\sigma \leq & \frac{\Delta t}{2\varepsilon} - c[\rho]|p_x^* + 2\alpha x^*| + c[v] E^{l^*}\left(p_x^* - \frac{\Delta x}{2\varepsilon} - \alpha(2x_i^* + \Delta x), p_x^* + \frac{\Delta x}{2\varepsilon} - \alpha(2x_i^* - \Delta x)\right) \\
\leq & \frac{\Delta t}{2\varepsilon} - (c[\rho] - c[v])\,|p_x^*| + \alpha K|x^*| \\
& + |c[v]|\left|E^{l^*}\left(p_x^* - \frac{\Delta x}{2\varepsilon} - \alpha(2x_i^* + \Delta x), p_x^* + \frac{\Delta x}{2\varepsilon} - \alpha(2x_i^* - \Delta x)\right) - E^{l^*}\left(p_x^*, p_x^*\right)\right| \\
\leq & \frac{\Delta t}{2\varepsilon} - (c[\rho] - c[v])\,|p_x^*| + K\alpha|x^*| + K\frac{\Delta x}{\varepsilon} + 2\alpha K|x_i^*| + 2\alpha K\Delta x
\end{aligned}$$

where we have used, for the second line, the fact that

$$c[\rho] \leq M + 2B_0(M + \|a\|_{L^\infty(0,T)})T \leq K$$

with $M = \|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})}$ and $B_0 = \max_{k\in\{+,-\}} \|D\rho_k^0\|_{L^\infty(\mathbb{R})}$ (see Theorem 4.2). Now, since $\rho_{k^*}(x^*, t^*) - v_{k^*}(x_i^*, t_n^*) = \max_{k\in\{+,-\}} (\rho_k(x^*, t^*) - v_k(x_i^*, t_n^*)) \geq 0$, by *Lemma* 4.1, we obtain

$$\begin{aligned}
-(c[\rho] - c[v])|p_x^*| = & -k^*\left(\rho_+(x^*, t^*) - \rho_-(x^*, t^*) + a(t^*)\right)|p_x^*| \\
& + k^*\left(v_+(x_i^*, t_n^*) - v_-(x_i^*, t_n^*) + a(t^*)\right)|p_x^*| + k^*\left(a^\Delta(t_n^*) - a(t^*)\right)|p_x^*| \\
\leq & \left|a^\Delta(t_n^*) - a(t^*)\right|\,|p_x^*| \leq K|t_n^* - t^*||p_x^*|
\end{aligned}$$

where we have used (5.29). This implies

$$\begin{aligned}
\sigma \leq & \frac{\Delta t}{2\varepsilon} + K|t^* - t_n^*||p_x^*| + K\alpha|x^*| + K\frac{\Delta x}{\varepsilon} + 2\alpha K|x_i^*| + 2\alpha K\Delta x \\
\leq & K\frac{\Delta x + \Delta t}{\varepsilon} + K\alpha^{1/2} + K\varepsilon.
\end{aligned}$$

Putting

$$\sigma^*(\Delta x + \Delta t, \varepsilon, \alpha) = K\frac{\Delta x + \Delta t}{\varepsilon} + K(\alpha^{1/2} + \varepsilon),$$

we therefore conclude that we must have $t^* = 0$ or $t_n^* = 0$ provided $\sigma \geq \sigma^*$. Whenever $t^* = 0$, we deduce from *Lemma* 5.3 and from (5.40) that

$$\begin{aligned}
M_\sigma^{\alpha,\varepsilon} = & \Psi(x^*, 0, x_i^*, t_n^*, k^*) \leq \rho_{k^*}^0(x^*) - v_{k^*}(x_i^*, t_n^*) \\
\leq & \rho_{k^*}^0(x^*) - \rho_{k^*}^0(x_i^*) + Kt_n^* + \mu^0 \\
\leq & K\left(|x^* - x_i^*| + t_n^*\right) + \mu^0 \leq K(1 + \sigma)\varepsilon + \mu^0.
\end{aligned}$$

Similarly, whenever $t_n^* = 0$, we deduce from the Lipschitz regularity of $\rho$ and from (5.40) that

$$\begin{aligned} M_\sigma^{\alpha,\varepsilon} =& \Psi(x^*, t^*, x_i^*, 0, k^*) \leq \rho_{k^*}(x^*, t^*) - v_{k^*}(x_i^*, 0) \\ \leq& K\left(|x^* - x_i^*| + t^*\right) + \mu^0 \leq K(1+\sigma)\varepsilon + \mu^0. \end{aligned}$$

To sum up, we have shown that

$$M_\sigma^{\alpha,\varepsilon} \leq K(1+\sigma)\varepsilon + \mu^0 \leq K\varepsilon + \mu^0$$

provided $K\frac{\Delta x + \Delta t}{\varepsilon} + K(\alpha^{1/2} + \varepsilon) \leq \sigma \leq 1$.We then deduce that, for every $(x_i, t_n)$ and for every $k$, we have

$$\rho_k(x_i, t_n) - v_k(x_i, t_n) - \left(K\frac{\Delta x + \Delta t}{\varepsilon} + K(\alpha^{1/2} + \varepsilon)\right) T - 2\alpha|x_i|^2 \leq M_\sigma^{\alpha,\varepsilon} \leq K\varepsilon + \mu^0.$$

Sending $\alpha \to 0$, taking the supremum over $(x_i, t_n)$, the maximum over $k$ and choosing $\varepsilon = T^{1/2}\left(\Delta x + \Delta t\right)^{1/2}$, we conclude that

$$\max_{k\in\{+,-\}} \sup_{\Xi_T}(\rho_k(x_i, t_n) - v_{k,i}^n) = \mu \leq K\left(\Delta x + \Delta t\right)^{1/2}\sqrt{T} + \max_{k\in\{+,-\}} \sup_{\Xi}(\rho_k^0(x_i) - v_{k,i}^0), \quad (5.43)$$

provided that $\Delta x,\ \Delta t$ are small enough $T \leq 1$, $\mu_0 \leq 1$ and (5.36) is assumed.

In the general case, we consider $\overline{\rho} = \rho + \mu^1$ with $\mu^1 = \max_{k\in\{+,-\}} \sup_{\Xi}(v_{k,i}^0 - \rho_k^0(x_i))$. We remark that $\overline{\rho}$ is solution of (4.19) and satisfies $\overline{\rho}^0(x_i) \geq v_i^0$. Then (5.43) is true with $\overline{\rho}$ in place of $\rho$, *i.e.*

$$\max_{k\in\{+,-\}} \sup_{\Xi_T}(\rho_k(x_i, t_n) + \mu^1 - v_{k,i}^n) \leq K\left(\Delta x + \Delta t\right)^{1/2}\sqrt{T} + \max_{k\in\{+,-\}} \sup_{\Xi}(\rho_k^0(x_i) + \mu^1 - v_{k,i}^0),$$

which still implies (5.43) with $\max_{k\in\{+,-\}} \sup_{\Xi} |\rho_k^0(x_i) - v_{k,i}^0|$.

The lower bound for the error estimate is obtained by exchanging $\rho$ and $v$. As the proof is similar to the above, we omit it.

## 5.2 Approximation of the non-local system

To solve numerically the non-local system (1.1)-(1.2), we use the finite difference scheme (1.3)-(1.4)-(1.5). We also assume the CFL condition (1.8). In particular, using Proposition 4.5, we deduce that the CFL condition (5.33) is satisfied uniformly for all $a$ defined by (1.5) because

$$\|a[\rho]\|_{L^\infty(0,T)} \leq 2B_0(2B_0 + \|L\|_{L^\infty(0,T)})T + M + \|L\|_{L^\infty(0,T)}$$

and so $L_1 \leq L_2$.

Let $\bar{T} \geq 0$ which will be precised later. To prove our convergence result, we mimic the continuous case. Before to prove Theorem 1.3 we need to introduce some notations and *lemmata*. Defining $X_T^{1,\Delta} = \mathbb{R}^{\{0,\dots,N_T\}}$ and $X_T^{2,\Delta} = (\mathbb{R}^2)^{\mathbb{Z}\times\{0,\dots,N_T\}}$, the set of discrete functions defined on $\{0,...,N_T\}$ and on the mesh $\Xi_T$ respectively, we denote by $G^\Delta : X_{\bar{T}}^{1,\Delta} \to X_{\bar{T}}^{2,\Delta}$ the operator that gives the discrete solution $v$ of the local Problem (5.27) for a given velocity $a^\Delta \in X_{\bar{T}}^{1,\Delta}$, *i.e.*

$$(G_+^\Delta(a^\Delta), G_-^\Delta(a^\Delta)) = G^\Delta(a^\Delta) = v.$$

In particular, the scheme can be rewritten

$$v = G^\Delta(a^\Delta[v])$$

with $a^\Delta[\cdot]$ defined in (1.5). We set, for all $T \le \bar{T}$ :

$$U_T^\Delta = \left\{ w \in X_T^{2,\Delta} : \left| \begin{array}{l} \sup_{\Xi_T} |D_x^+ w_\pm| \le B_0, \\ \sup_{\Xi_T} |D_t^+ w_\pm| \le 2B_0(4M + 3\|L\|_{L^\infty(0,T)} + 6), \\ \sup_{\Xi_T} |w_+ - w_-| \le 2M + \|L\|_{L^\infty(0,T)} + 3 \end{array} \right. \right\}$$

and

$$V_T^\Delta = \left\{ a^\Delta \in X_T^{1,\Delta} : \left| \begin{array}{l} \sup_{\{0,\dots,N_T\Delta t\}} |a^\Delta| \le 2(M + \|L\|_{L^\infty}) + 3, \\ \sup_{\{0,\dots,N_T\Delta t\}} |D_t^+ a^\Delta| \le 4B_0(4M + 3\|L\|_{L^\infty(0,T)} + 6) + \|L\|_{W^{1,\infty}(0,T)} \end{array} \right. \right\}$$

where $M = \|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})}$. One can easily check that

$$\{(\rho)^\Delta \backslash \rho \in U_T\} \subset U_T^\Delta$$

and

$$\left\{ (a)^\Delta : \left| \begin{array}{l} \|a\|_{L^\infty(0,T)} \le 2(M + \|L\|_{L^\infty(0,T)}), \\ \|a_t\|_{L^\infty(0,T)} \le 4B_0(4M + 3\|L\|_{L^\infty(0,T)}) + \|L\|_{W^{1,\infty}(0,T)} \end{array} \right. \right\} \subset V_T^\Delta$$

where $(f)^\Delta$ is the restriction to $\Xi_T$ of the continuous function $f$. We have the following *Lemma* :

***Lemma* 5.4** *Assume that (1.8) holds, then for all $T \le \bar{T}$, the following inclusion hold*

*(i)* $a^\Delta[U_T^\Delta] \subset V_T^\Delta$,

*(ii)* $G^\Delta(V_T^\Delta) \subset U_T^\Delta$.

**Proof of *Lemma* 5.4**

The proof of $(i)$ is just a simple computation. We prove $(ii)$.

Let $a^\Delta \in V_T^\Delta$ and $v = G^\Delta(a^\Delta)$. We set $w(x_i, t_n) = v(x_{i+1}, t_n) - \Delta x B_0$. Then $w$ is still solution of the discrete scheme (5.27) and satisfies $w^0 \le v^0$. Using the monotony of the scheme, yields

$$\frac{v_\pm(x_{i+1}, t_n) - v_\pm(x_i, t_n)}{\Delta x} \le B_0.$$

Using Lemma 5.3, we have (since $\mu_0 \le 1$)

$$\begin{aligned} |v_+ - v_-| \le & M + 4B_0(M + \sup_{\{0,\dots,N_T\Delta t\}} |a^\Delta|)T + 2 \le M + 12B_0(M + \|L\|_{L^\infty(0,T)} + 1)T + 2 \\ \le & 2M + \|L\|_{L^\infty(0,T)} + 3 \end{aligned}$$

for $T \le \bar{T} = \dfrac{1}{12B_0}$.

For the estimate in time, we have using *Lemma* 5.3

$$\begin{aligned} \left| \frac{v_i^{n+1} - v_i^n}{\Delta t} \right| \le & 2B_0 |C_k^{\Delta,\mathrm{Loc}}[v](x_i, t_n)| \\ \le & 2B_0(4B_0(M + \sup_{\{0,\dots,N_T\Delta t\}} |a^\Delta|)T + M + 2 + \sup_{\{0,\dots,N_T\Delta t\}} |a^\Delta|) \\ \le & 2B_0(12B_0(M + \|L\|_{L^\infty(0,T)} + 1)T + 3M + 2\|L\|_{L^\infty(0,T)} + 5) \\ \le & 2B_0(4M + 3\|L\|_{L^\infty(0,T)} + 6) \end{aligned}$$

for $T \le \bar{T} = \dfrac{1}{12B_0}$. So $G^\Delta(V_T^\Delta) \subset U_T^\Delta$. This ends the proof of the *lemma*.

We now have to prove some consistency and stability results for the velocity $a^\Delta$ and for the operator $G^\Delta$.

***Lemma* 5.5 (Consistency for the discrete velocity $a^\Delta$)** *There is a constant $K = 2B_0+2M+\|L\|_\infty$ such that, for every mesh $\Delta$, for every $0 \le T \le \bar{T}$ and for $\rho \in U_T$, we have*

$$\sup_{\{0,\dots,N_T\Delta t\}} |\,(a[\rho])^\Delta - a^\Delta[(\rho)^\Delta]| \le K\Delta x$$

*where $(\rho)^\Delta$ is the restriction to $\Xi_T$ of the continuous function $\rho$ and $a[\cdot]$ is defined in (4.24).*

**Proof of *Lemma* 5.5**
We set $\tilde{\rho}(x,t) = \rho_+(x,t) - \rho_-(x,t)$. The following holds :

$$\begin{aligned}
\left|a[\rho](t_n) - a^\Delta[(\rho)^\Delta](t_n)\right| &= \left|\int_0^1 \tilde{\rho}(x,t_n)dx - \sum_{i=0}^{N_x-1} \Delta x \tilde{\rho}(x_i,t_n)\right| \\
&\le \sum_{i=0}^{N_x-1} \left|\int_{i\Delta x}^{(i+1)\Delta x} \tilde{\rho}(x,t_n)dx - \Delta x\tilde{\rho}(x_i,t_n)\right| + \int_{N_x\Delta x}^1 \tilde{\rho}(x,t_n)dx \\
&\le \Delta x \sum_{i=0}^{N_x-1} \sup_{[i\Delta x,(i+1)\Delta x]} |\tilde{\rho}(\cdot,t_n) - \tilde{\rho}(x_i,t_n)| + (2M+\|L\|_\infty)\Delta x \\
&\le \Delta x(2B_0 + 2M + \|L\|_\infty).
\end{aligned}$$

We have the following *lemma* which proof is just a simple computation

***Lemma* 5.6 (Stability property of the velocity $a^\Delta$)** *For every mesh $\Delta$, for every $0 \le T \le \bar{T}$ and every $v_1, v_2 \in U_T^\Delta$, the following holds*

$$\sup_{\{0,\dots,N_T\Delta t\}} |a^\Delta[v_2] - a^\Delta[v_1]| \le 2 \max_{k\in\{+,-\}} \sup_{\Xi_T} |v_2 - v_1|.$$

***Lemma* 5.7 (Stability property of the operator $G^\Delta$)** *There is a constant $K = 2B_0$ so that, for every mesh $\Delta$ satisfying the uniform CFL condition (1.8), for all $0 \le T \le \bar{T}$ and all $a_1^\Delta,\ a_2^\Delta \in V_T^\Delta$*

$$\max_{k\in\{+,-\}} \sup_{\Xi_T} |G_k^\Delta(a_2^\Delta) - G_k^\Delta(a_1^\Delta)| \le KT \sup_{\{0,\dots,N_T\Delta t\}} |a_2^\Delta - a_1^\Delta|.$$

**Proof of *Lemma* 5.7**
We set $v_i = G^\Delta\left(a_i^\Delta\right)$. Using the fact that

$$c_1E^{Sgn(c_1)} - c_2E^{Sgn(c_2)} \le |c_1 - c_2| \max(E^+, E^-)$$

yields

$$\begin{aligned}
&v_{2,k}^{n+1} - v_{2,k}^n + k\Delta t\left(v_{2,+}^n - v_{2,-}^n + a_1^\Delta(t_n)\right) E^{sgn(v_{2,+}^n - v_{2,-}^n + a_1^\Delta(t_n))}(D^+v_2^n, D^-v_2^n) \\
\le &\Delta t|a_2^\Delta(t_n) - a_1^\Delta(t_n)| \max(E^+(D^+v_2^n, D^-v_2^n), E^-(D^+v_2^n, D^-v_2^n)) \\
\le &2B_0\Delta t \sup_{\{0,\dots,N_T\Delta t\}} |a_1^\Delta - a_2^\Delta|.
\end{aligned}$$

Moreover $\tilde{v}_1(x_i,t_n) = v_1(x_i,t_n) + 2B_0 \sup_{\{0,\dots,N_T\Delta t\}} |a_1^\Delta - a_2^\Delta|t_n$ is solution of the same discrete equation. Since the scheme is monotone, one deduces that

$$\max_{k\in\{+,-\}} \sup_{\Xi_T} |G_k^\Delta(a_2^\Delta) - G_k^\Delta(a_1^\Delta)| \le 2B_0T \sup_{\{0,\dots,N_T\Delta t\}} |a_2^\Delta - a_1^\Delta|.$$

This achieves the proof.

We now prove Theorem 1.3.

**Proof of Theorem 1.3**
We use the main idea of Alvarez *et al.* [2].
We first assume that $T \geq \bar{T}$ and we set, for every $l \geq 1$

$$Q_l^\Delta = \Delta x \mathbb{Z} \times \{\Delta t N_l, ..., \Delta t N_{l+1}\}$$

where $N_l$ is the integer part of $\dfrac{l\bar{T}}{\Delta t}$. As in the continuous case, on each interval $(l\bar{T}, (l+1)\bar{T})$, we can iterate the process (since $\bar{T}$ depends only on $B_0$ which does not change during time) and construct, using a fix point method (denoting by $G$ and $G^\Delta$), $\rho$ and $v$ respectively solution of (1.1)-(1.2) and (1.3)-(1.4)-(1.5). We then have the inequality

$$\begin{aligned}
\max_{k\in\{+,-\}} \sup_{Q_l^\Delta} |\rho_k - v_k| \leq & \max_{k\in\{+,-\}} \sup_{Q_l^\Delta} |G_{k,l}(a[\rho]) - G_{k,l}^\Delta(a^\Delta[v])| \\
\leq & \max_{k\in\{+,-\}} \sup_{Q_l^\Delta} \left|G_{k,l}(a[\rho]) - G_{k,l}^\Delta\left((a[\rho])^\Delta\right)\right| \\
& + \max_{k\in\{+,-\}} \sup_{Q_l^\Delta} \left|G_{k,l}^\Delta\left((a[\rho])^\Delta\right) - G_{k,l}^\Delta\left(a^\Delta[v]\right)\right|,
\end{aligned}$$

where the function $G_l^\Delta\left((a[\rho])^\Delta\right) = \left(G_{+,l}^\Delta\left((a[\rho])^\Delta\right), G_{-,l}^\Delta\left((a[\rho])^\Delta\right)\right)$ (resp. $G_l(a[\rho])$) is simply the discrete solution of (5.26) (resp. the continuous solution of (1.1)) with the velocity $a[\rho]$ and initial condition $v^{N_l}$ (resp. $\rho(\cdot, N_l\mathcal{D}t)$). From Theorem 5.1, we then deduce

$$\begin{aligned}
\max_{k\in\{+,-\}} \sup_{Q_l^\Delta} \left|G_{k,l}(a[\rho]) - G_{k,l}^\Delta\left((a[\rho])^\Delta\right)\right| & \leq K\sqrt{\bar{T}\Delta x} + \max_{k\in\{+,-\}} \sup_{\Delta x\mathbb{Z}\times N_l\Delta t} |\rho_k - v_k| \\
& \leq lK\sqrt{\bar{T}\Delta x} + \max_{k\in\{+,-\}} \sup_{\Xi} |\rho_k^0 - v_k^0|.
\end{aligned} \tag{5.44}$$

For the second term, we use *Lemma* 5.5, *Lemma* 5.6 and *Lemma* 5.7 to obtain

$$\begin{aligned}
& \max_{k\in\{+,-\}} \sup_{Q_l^\Delta} \left|G_{k,l}^\Delta\left((a[\rho])^\Delta\right) - G_{k,l}^\Delta\left(a^\Delta[v]\right)\right| \\
\leq & K\bar{T} \sup_{\{N_l\Delta t,...,N_{l+1}\Delta t\}} \left|(a[\rho])^\Delta - a^\Delta[v]\right| \\
\leq & K\bar{T} \sup_{\{N_l\Delta t,...,N_{l+1\Delta t}\}} \left(\left|\left(a[\rho]\right)^\Delta - a^\Delta\left[(\rho)^\Delta\right]\right| + \left|a^\Delta\left[(\rho)^\Delta\right] - a^\Delta[v]\right|\right) \\
\leq & K\bar{T}\left(\Delta x + \max_{k\in\{+,-\}} \sup_{Q_l^\Delta} |\rho_k - v_k|\right).
\end{aligned}$$

This implies, for $\bar{T}\Delta x \leq 1$ and $K\bar{T} < 1$

$$\max_{k\in\{+,-\}} \sup_{Q_l^\Delta} |\rho_k - v_k| \leq \frac{lK}{1-K\bar{T}}\sqrt{\bar{T}\Delta x} + \left(\max_{k\in\{+,-\}} \sup_{\Xi} |\rho_k^0 - v_k^0|\right)\frac{1}{1-K\bar{T}}.$$

We now take $\bar{l} \geq 1$ such that

$$\bar{l}\bar{T} \leq T \leq (\bar{l}+1)\bar{T}.$$

Then the following holds

$$\begin{aligned}
\max_{k\in\{+,-\}} \sup_{\Xi_T} |\rho_k - v_k| \leq & \frac{\bar{l}K}{1-K\bar{T}}\sqrt{\bar{T}\Delta x} + \left(\max_{k\in\{+,-\}} \sup_{\Xi} |\rho_k^0 - v_k^0|\right)\frac{1}{1-K\bar{T}} \\
\leq & KT\sqrt{\Delta x} + K\left(\max_{k\in\{+,-\}} \sup_{\Xi} |\rho_k^0 - v_k^0|\right), \text{ if } T \geq \bar{T}.
\end{aligned}$$

where we have used the fact that $\bar{T}$ depends only on $B_0$.

Notice that, in the case where $T \leq \bar{T}$, from Theorem 5.1, (5.44) is replaced by

$$\max_{k\in\{+,-\}} \sup_{\Xi_T} \left|G_k(a[\rho]) - G_k^{\Delta}\left((a[\rho])^{\Delta}\right)\right| \leq K\sqrt{T\Delta x} + \max_{k\in\{+,-\}} \sup_{\Xi} |\rho_k^0 - v_k^0|$$

and so we obtain

$$\max_{k\in\{+,-\}} \sup_{\Xi_T} |\rho_k - v_k| \leq K\sqrt{T\Delta x} + K\left(\max_{k\in\{+,-\}} \sup_{\Xi} |\rho_k^0 - v_k^0|\right), \text{ if } T \leq \bar{T}.$$

This ends the proof of the theorem.

# 6 Numerical results

In this Section, we present some numerical simulations of the 1-D Groma-Balogh problem (1.1)-(1.2) discretized by the numerical scheme (1.3)-(1.4)-(1.5).

## 6.1 Numerical error estimate

Here, we show a numerical test in order to confirm our error estimate for local system. Let us fix $L(t) = 0$ even if we become quite from the physical case, let us choose the following initial conditions : $\rho_+^0(x) = -|x - 1/2| + 1/2$, and $\rho_-^0(x) = -|2x - 1| + 1$ on $[0, 1[$ (and extend it by periodicity on $\mathbb{R}$).

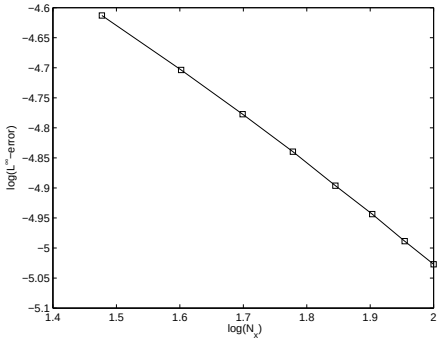


FIG. 3 – $\log(L^\infty - error)$ of $|u_{N_T} - u_{N_T-1}|$ versus $\log(N_x)$ at $T = \frac{1}{2}$

Figure (3) show the behaviour of the $L^\infty$-error versus the discretization parameter $\Delta x$. The regression slope is close to 0.7 and the ideal regression is $\frac{1}{2}$. Hence, the behaviour of this errors confirms that our error seems optimal.

## 6.2 dislocations density dynamics

In this paragraph, we are interested by the evolution of dislocations densities for the 1-D Groma-Balogh model (1.1)-(1.2) under the uniformly applied shear stress $L(t) = 3t$.

In this simulations, we choose an example of concentrated dislocations densities, *i.e* where dislocations densities are initially periodic, and equal to zero on some sub-intervals of $[0, 1[$ (see figure 4).

This initial condition means that there exists some regions without dislocations , and some regions where the dislocations are concentrated.

Intuitively, dislocations are intend to be distributed uniformly in the whole crystal as shown in Figure 6 where finally a uniform distribution in all the crystal is observed, *i.e* the density of dislocation becomes a constant. We can remark that when $L(t)$ involves such as a diffusion equation (see [9] for more precision) but evidently when $L(t) = 0$ with the same initial condition, the system doesn't involve.

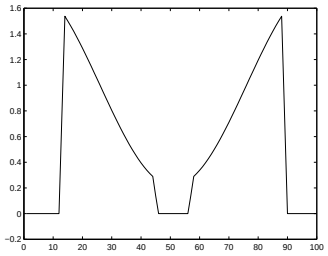

FIG. 4 – dislocations density $(D\rho_+^0(.) = D\rho_-^0(.))$

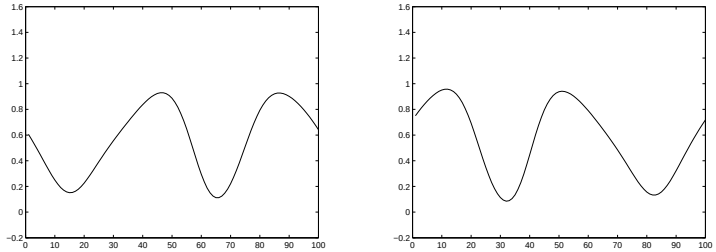

FIG. 5 – On the left : density $(D\rho_+(., \frac{1}{2}))$ ; on the right : dislocations density $(D\rho_-(., \frac{1}{2}))$

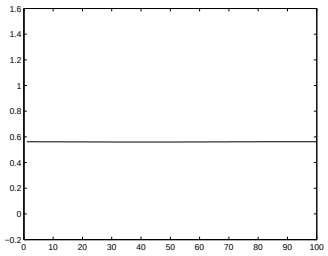

FIG. 6 – dislocations density $(D\rho_+(., 3) = D\rho_-(., 3))$

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