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Abstract

In this paper, we are interested in a nonlinear parabolic evolution equation occuring in rheology. We give a probabilistic interpretation to this equation by associating a nonlinear martingale problem with it. We prove existence of a unique solution P to this martingale problem. For any t, the time-marginal at time t of P admits a density $\rho(t,x)$ with respect to the Lebesgue measure and the function ρ is the unique weak solution of the evolution equation in a well-chosen energy space. Next, we introduce a simulable system of n interacting particles and prove that the empirical measure of this system converges to P as n tends to ∞. This propagation of chaos result ensures that the solution of the equation of interest can be approximated by a Monte-Carlo method. Last, we illustrate the convergence by some numerical experiments.

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Introduction

In rheology, modeling the flow of complex fluids is a very intricate problem which is far from being solved up to now. Hébraux and Lequeux (see[4]) present a model which aims at describing the behavior of very concentrated suspensions of soft particles, known as soft glassy materials, under a simple shear flow. This model is obtained by dividing the material into a large number of mesoscopic elements ("bocks") with a given shear stress. From a mathematical point of view, the probability density $p(t,x)$ for a block to undergo stress x at time t is supposed to satisfy the following evolution equation : $\forall (t,x) \in [0,T] \times \mathbb{R}$,

$$\begin{cases} \frac{\partial p}{\partial t}(t,x) = -b(t)\frac{\partial p}{\partial x}(t,x) + D(p(t))\frac{\partial^2 p}{\partial x^2}(t,x) - \mathbf{1}_{[-1,1]^c}(x)p(t,x) + \frac{2}{\sigma^2}D(p(t))\delta_0(x) \\ p \geq 0 \\ p(0,x) = \rho_0(x), \end{cases} \tag{1}$$

where for $f \in L^1(\mathbb{R})$, we define

$$D(f) := \frac{\sigma^2}{2}\int_{|x|>1} f(x)dx, \quad \sigma > 0.$$

Also, $\mathbf{1}_{[-1,1]^c}$ denotes the characteristic function of the open set $[-1,1]^c =]-\infty,-1[\cup]1,+\infty[$, δ_0 the Dirac delta distribution on $\mathbb{R}$. Last, ρ_0 is a probability density on the line. Let us precise the physical interpretation of the above equation. When a block is sheared, the stress of this block evolves with a variation rate $b(t)$ proportional to the shear rate. In our study, the function b are assumed to be in $L^2([0,T])$. When the modulus of the stress overcomes the critical value of the stress chosen equal to one here, the block becomes unstable and may relax into a state with zero stress after a characteristic relaxation time also chosen equal to one. This phenomenon induces a rearrangement of the blocks modelled through the diffusion term $D(p(t))\frac{\partial^2 p}{\partial x^2}(t,x)$.

Motivated by the physical interest of this model, Cancès, Catto and Gati (see [2]) have studied existence and uniqueness for equation (1). From an analytic point of view, the difficulty of this study comes from the possibility for the coefficient $D(p(t))$ multiplying the second order spatial derivative to vanish. In case the initial density ρ_0 satisfies $D(\rho_0) > 0$ (and under regularity assumptions made precise in Theorem 1 below), Cancès, Catto and Gati were able to control the time evolution of this multiplicative coefficient and prove that (1) admits a unique weak solution ρ in $L^\infty_t([0,T], L^1_x \cap L^2_x) \cap L^2_t([0,T], H^1_x)$, this solution being such that

$$\inf_{t\in[0,T]} D(\rho(t)) > 0. \tag{2}$$

By a weak solution, we mean an integrable function $p : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ such that for any $C^{1,2}$ function ψ with compact support on $[0,T] \times \mathbb{R}$, $\forall t \in [0,T]$,

$$\begin{aligned} \int_{\mathbb{R}} \psi(t,x)p(t,x)dx = \int_{\mathbb{R}} \psi(0,x)\rho_0(x)dx + \int_{[0,t]\times\mathbb{R}} \left(p\frac{\partial \psi}{\partial s} + bp\frac{\partial \psi}{\partial x} + D(p)p\frac{\partial^2 \psi}{\partial x^2}\right)(s,x)dsdx \\ + \int_{[0,t]\times\mathbb{R}} \mathbf{1}_{\{|x|>1\}}p(s,x)\left(\psi(s,0) - \psi(s,x)\right)dsdx. \end{aligned} \tag{3}$$

In this paper, we are interested in constructing and proving the convergence of some Monte-Carlo approximations of the solution p. For this purpose, we first associate a nonlinear martingale problem with (1). Let $D([0,T],\mathbb{R})$ be the space of functions on $[0,T]$ that are right-continuous and have left-hand limits. We denote by X the canonical process on $D([0,T],\mathbb{R})$.

Definition 1 *We say that a probability measure P on $D([0,T],\mathbb{R})$ with time-marginals $(P_t)_{0\leq t\leq T}$ solves the nonlinear martingale problem (MP) if $P_0(dx)=\rho_0(x)dx$ and $\forall\phi\in C_b^2(\mathbb{R})$,*

$$\phi(X_t)-\phi(X_0)-\int_0^t\left(b(s)\phi'(X_s)+\frac{\sigma^2}{2}P_s([-1,1]^c)\phi''(X_s)\right)ds-\int_0^t\left(\phi(0)-\phi(X_s)\right)\mathbf{1}_{\{|X_s|>1\}}ds, \tag{4}$$

is a $P-$martingale on the time interval $[0,T]$.

This problem is nonlinear since the diffusion coefficient $\frac{\sigma^2}{2}P_s([-1,1]^c)$ at time s involves the time-marginal P_s of the solution.
If P solves problem (MP) then according to Lemma 2 (1) below, $\forall\psi\in C_b^{1,2}([0,T]\times\mathbb{R})$,

$$\begin{aligned}\psi(t,X_t)-\psi(0,X_0)-\int_0^t\left(\frac{\partial\psi}{\partial s}(s,X_s)+b(s)\frac{\partial\psi}{\partial x}(s,X_s)+\frac{\sigma^2}{2}P_s([-1,1]^c))\frac{\partial^2\psi}{\partial x^2}(s,X_s)\right)ds\\ -\int_0^t\left(\psi(s,0)-\psi(s,X_s)\right)\mathbf{1}_{\{|X_s|>1\}}ds,\end{aligned} \tag{5}$$

is a $P-$martingale on the time interval $[0,T]$. Writing the constancy of the expectation of this martingale, one deduces the following link between problem (MP) and equation (1) :

Lemma 1 *If P is a solution of the nonlinear martingale problem (MP) then $t\to P_t$ is a weak solution of the partial differential equation (1).*

In the first section of the paper we prove that problem (MP) admits a unique solution P and that for any $t\in[0,T]$, $P_t(dx)=\rho(t,x)dx$ where ρ is the solution of equation (1) obtained by Cancès, Catto and Gati [2].
Then in the second section, we introduce the following system of n interacting particles obtained by replacing the nonlinearity by interaction in the stochastic dynamics associated to the nonlinear martingale problem :

$$Y_t^{i,n}=Y_0^i+\sigma\int_0^t\sqrt{\frac{1}{n}\sum_{j=1}^n\mathbf{1}_{\{|Y_s^{j,n}|>1\}}\vee\frac{1}{n}}dW_s^i+\int_0^t b(s)ds-\int_0^t Y_{s^-}^{i,n}\mathbf{1}_{\{|Y_{s^-}^{i,n}|>1\}}dN_s^i,\ 1\leq i\leq n.$$

Here $(W^i)_{1\leq i\leq n}$ are n independent Brownian motions, $(N^i)_{1\leq i\leq n}$ n independent Poisson processes with intensity one and $(X_0^i)_{1\leq i\leq n}$ n independent random variables with density $\rho_0(dx)$. Also, we assume that $(W^i)_{1\leq i\leq n}$, $(N^i)_{1\leq i\leq n}$ and $(X_0^i)_{1\leq i\leq n}$ are independent. We now face the probabilistic counterpart of the possibility for $D(p(t))$ to vanish : the empirical probability $\frac{1}{n}\sum_{j=1}^n\mathbf{1}_{\{|Y_s^{j,n}|>1\}}$ of the set $[-1,1]^c$ may be equal to 0. That is why we take the supremum of this empirical probability with $1/n$ in the diffusion coefficient of each particle in order to ensure existence of a unique weak solution to this n-dimensional stochastic differential equation. We prove a propagation of chaos result which ensures that the solution $\rho(t,.)$ of (1) can be approximated by $\frac{1}{n}\sum_{i=1}^n\delta_{Y_t^{i,n}}$: indeed we prove that the $\mathcal{P}(D([0,T],\mathbb{R}))$-valued empirical measure $\frac{1}{n}\sum_{i=1}^n\delta_{Y^{i,n}}$ converges in probability to the unique solution P of problem (MP). In the mathematical analysis of the convergence, the main difficulty is that the lower bound $1/n$ of the diffusion coefficient in the system with n particles vanishes as $n\to+\infty$. To overcome this

difficulty, we first prove convergence on a small time interval. Then, to iterate the argument, we take advantage of (2) which holds for the solution of (1) as soon as $D(\rho_0) > 0$.
In the third section, we present some numerical results obtained by simulation of the system with n particles.

Notation :

- For $\tau > 0$ let $L_t^\infty([0,\tau], L_x^1 \cap L_x^2)$ denote the space of real-valued functions f defined on $[0,\tau] \times \mathbb{R}$ and satisfying

$$\sup_{t\in[0,\tau]} \left(\int_{\mathbb{R}} |f(t,x)|dx \right) < \infty \quad \text{and} \quad \sup_{t\in[0,\tau]} \left(\int_{\mathbb{R}} |f(t,x)|^2 dx \right) < \infty.$$

- By $L_t^2([0,\tau], H_x^1)$, we denote the space of functions f on $[0,\tau]\times\mathbb{R}$ such that the distribution derivative $\frac{\partial f}{\partial x}$ is a function and

$$\int_0^\tau \int_{\mathbb{R}} |f(t,x)|^2 + |\frac{\partial f}{\partial x}(t,x)|^2 dx < \infty.$$

- We say that a probability density ρ_0 satisfies the condition (H) if

$$\rho_0 \in L^\infty(\mathbb{R}), \quad \int_{\mathbb{R}} |x|\rho_0(x)dx < +\infty, \quad \text{and} \quad D(\rho_0) > 0.$$

1 Existence and Uniqueness of the Martingale Problem

1.1 On the equation (1)

We are going to recall existence and uniqueness results for equation (1) established in Theorem 1.1 of Cancès, Catto and Gati [2].

Theorem 1 *Let the initial density ρ_0 satisfy the condition (H). Then for every $T > 0$, there exists a unique weak solution ρ to the system (1) in $L_t^\infty([0,T], L_x^1 \cap L_x^2) \cap L_t^2([0,T], H_x^1)$. Moreover, for all $t \in [0,T]$, $\int_{\mathbb{R}} \rho(t,x)dx = 1$ and there exists a positive constant ν such that*

$$\frac{2}{\sigma^2} D(\rho(t)) > \nu, \qquad \forall t \in [0,T]. \tag{6}$$

In addition,

$$\sup_{t\in[0,T]} \int_{\mathbb{R}} |x|\rho(t,x)dx < \infty. \tag{7}$$

As for all $t \in [0,T]$, $\displaystyle\int_{|x|>1} \rho(t,x)dx \geq \nu$ and $\rho \in L_t^\infty([0,T], L_x^2)$, we obtain the following result.

Corollary 1 *There is an $\alpha > 1$ satisfying*

$$\int_{|x|>\alpha} \rho(t,x)dx \geq \frac{\nu}{2}, \quad \forall t \in [0,T].$$

Proof : In fact, for $\alpha > 1$, we have

$$\int_{[-\alpha,-1]\cup[1,\alpha]} \rho(t,x)dx \leq 2\sqrt{\alpha-1}||\rho(t,.)||_{L^2_x},$$

and we choose α so that the upper bound is less than $\frac{\nu}{2}$.
□

1.2 Main results

Theorem 2 *Assume that ρ_0 satisfies condition (H). The nonlinear martingale problem (MP) admits a unique solution P. In addition, $\forall t \in [0,T]$, $\rho(t,.)$ is a density of the time-marginal P_t with respect to the Lebesgue measure on $\mathbb{R}$.*

For the reader convenience, the rather technical proof of the following proposition, which ensures that the last statement holds, is postponed to section 1.3.

Proposition 1 *Assume that ρ_0 satisfies condition (H). If P solves the martingale problem (MP), then, $\forall t \in [0,T]$, P_t admits $\rho(t,.)$ as a density with respect to the Lebesgue measure.*

In order to deduce Theorem 2 from Proposition 1, we need to introduce a linear martingale problem :

Definition 2 *Let a be a nonnegative function. We say that a probability measure P on $D([0,T],\mathbb{R})$ solves the linear martingale problem (LMP) starting at $\lambda \in \mathcal{P}(\mathbb{R})$ if $P_0 = \lambda$ and $\forall \phi \in C_b^2(\mathbb{R})$,*

$$\phi(X_t) - \phi(X_0) - \int_0^t \left(b(s)\phi'(X_s) + a(s)\phi''(X_s)\right) ds - \int_0^t \left(\phi(0) - \phi(X_s)\right) \mathbf{1}_{\{|X_s|>1\}} ds, \quad (8)$$

is a $P-$martingale on $[0,T]$.

On a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, let $(W_t)_{t\geq 0}$ be a Brownian motion and $(N_t)_{t\geq 0}$ an independent Poisson process with intensity one. The stochastic differential equation associated the linear martingale problem (LMP) starting at λ is

$$Y_t = Y_0 + \int_0^t \gamma(s)dW_s + \int_0^t b(s)ds - \int_0^t Y_{s^-}\mathbf{1}_{\{|Y_{s^-}|>1\}}dN_s \quad (9)$$

where $\gamma(s) = \sqrt{2a(s)}$, Y_0 is a λ distributed random variable and $Y_0, (W_t)_{t\geq 0}, (N_t)_{t\geq 0}$ are independent. It is clear that existence and trajectorial uniqueness hold for this stochastic differential equation.

By Lepeltier and Marchal [7], theorems II_9, II_{13} and corollary II_{13}, we deduce the first assertion in the following lemma.

Lemma 2 **(1)** *For any $\lambda \in \mathcal{P}(\mathbb{R})$, the distribution of the unique solution of (9) is the unique solution of the linear martingale problem (LMP) starting at λ, say P.*

(2) *If in addition, $\lambda(dx) = f(x)dx$ with $f \in L^2(\mathbb{R})$ and there exists an interval $[0,\tau]$, $\tau > 0$, such that the function a is bounded from below on $[0,\tau]$ by a positive constant. Then for all $t \in [0,\tau]$, P_t has a density $p(t,x)$ with respect to the Lebesgue measure and the function p belongs to $L_t^\infty([0,\tau], L_x^1 \cap L_x^2) \cap L_t^2([0,\tau], H_x^1)$.*

The proof of the remaining assertion is postponed to section 1.3.

Proof of theorem 2 : Let us suppose that Proposition 1 holds. If P and Q denote two solutions of the nonlinear martingale problem (MP) then both P and Q solve the linear martingale problem (LMP) with diffusion coefficient $a(s) = D(\rho(s))$ starting at $\lambda(dx) = \rho_0(x)dx$. Since uniqueness holds for this linear martingale problem, $P = Q$, and uniqueness holds for the nonlinear martingale problem (MP).

We still have to prove existence for the nonlinear martingale problem (MP). Let P be the solution of the linear martingale problem introduced above. By (6) and lemma 2 (2) above, for all t in $[0,T]$, the probability distribution P_t admits a density $p(t,.)$ with respect to the Lebesgue measure and the function p belongs to $L_t^\infty([0,T], L_x^1 \cap L_x^2) \cap L_t^2([0,T], H_x^1)$. Moreover, reasoning like in the proof of Lemma 1, we obtain that p is a weak solution of the linear partial differential equation

$$\left\{ \begin{array}{l} \frac{\partial p}{\partial t}(t,x) = -b(t)\frac{\partial p}{\partial x}(t,x) + a(t)\frac{\partial^2 p}{\partial x^2}(t,x) - \mathbf{1}_{[-1,1]^c}(x)p(t,x) + \frac{2}{\sigma^2}D(p(t))\delta_0(x) \\ p(0,x) = \rho_0(x), \end{array} \right. \tag{10}$$

As ρ satisfies equation (1) and $a(t) = D(\rho(t))$, ρ also satisfies the above linear partial differential equation. Now, by adapting the ideas of Cancès, Catto and Gati in the proof of uniqueness for (1), we shall prove that $p = \rho$. By subtracting the equations satisfied by p and ρ respectively, we obtain that $q = p - \rho$ satisfies

$$\left\{ \begin{array}{l} \dfrac{\partial q}{\partial t}(t,x) = -b(t)\dfrac{\partial q}{\partial x}(t,x) + a(t)\dfrac{\partial^2 q}{\partial x^2}(t,x) - \mathbf{1}_{[-1,1]^c}(x)q(t,x) + \dfrac{2}{\sigma^2}D(q(t))\delta_0(x) \\ q(0,x) = 0, \end{array} \right.$$

Multiplying equation (11) by q and integrating over $\mathbb{R}$ with respect to x, one obtains formally

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} q^2(t,x)dx + a(t)\int_{\mathbb{R}}\left(\frac{\partial q}{\partial x}(t,x)\right)^2 dx + \int_{|x|>1} q^2(t,x)dx = \frac{2}{\sigma^2}D(q(t))q(t,0). \tag{11}$$

Because of the regularity of functions p and ρ, this formal computation is rigorous. We next remark that since $\int_{\mathbb{R}} p(t,x)dx = \int_{\mathbb{R}} \rho(t,x)dx = 1$, we get

$$|\frac{2}{\sigma^2}D(q(t))| = |\int_{|x|\le 1} q(t,x)dx| \le \sqrt{2}||q(t,.)||_{L_x^2},$$

thanks to the Cauchy-Schwarz inequality. Using moreover the Sobolev embedding of $H^1(\mathbb{R})$ into the space of continuous and bounded functions on $\mathbb{R}$ endowed with the sup norm, we bound from above the term on the right-hand side of (11) in the following way

$$\begin{aligned} |\frac{2}{\sigma^2}D(q(t))q(t,0)| &\le C||q(t,.)||_{L_x^2}||q(t,.)||_{H_x^1} \\ &\le \frac{C^2||q(t,.)||_{L_x^2}^2}{2\varepsilon} + \frac{\varepsilon}{2}||q(t,.)||_{L_x^2}^2 + \frac{\varepsilon}{2}||\frac{\partial}{\partial x}q(t,.)||_{L_x^2}^2, \end{aligned}$$

for any positive constant ε. Since by theorem 1, $\inf_{0\le t\le T} a(t) > 0$, one may choose $\dfrac{\varepsilon}{2} \le \inf\limits_{0\le t\le T} a(t)$ and deduce from equation (11) that

$$\frac{1}{2}\frac{d}{dt}||q(t,.)||^2_{L^2_x} \le (\frac{C^2}{2\varepsilon} + \frac{\varepsilon}{2})||q(t,.)||^2_{L^2_x}.$$

Finally, by applying Gronwall's lemma, we obtain that $\forall t \in [0,T]$ $||q(t,.)||^2_{L^2_x} = 0$, thus $q = 0$. This ensures that $a(t) = D(p(t))$. Therefore, P solves the nonlinear martingale problem (MP). $\square$

1.3 Proofs of technical results

Proof of lemma 2 (2) : By lemma 2 (1), it is enough to deal with the stochastic differential equation (9). For $n \in \mathbb{N}^*$, let $T_n = \inf\{t > 0 : N_t = n\}$. The conditional distribution of $(T_1, \cdots, T_n)$ given $\{N_t = n\}$ is uniform on the $n-$dimensional simplex $\Delta_n = \{0 < t_1 < \cdots < t_n < t\}$. Let $Q_{s,t}$ be the density of the random variable $\int_s^t \gamma(r)dW_r + \int_s^t b(r)dr$. Since N is independent from (Y_0, W), for $n \in \mathbb{N}$, the conditional density $p_n(t,y)$ of Y_t given $\{N_t = n\}$ may be computed by induction on n. For $t > 0$ and $y \in \mathbb{R}$, we have

$$p_0(t,y) = f * Q_{0,t}(y).$$

$$p_1(t,y) = \frac{1}{t}\int_0^t \int_{\mathbb{R}} p_0(t_1,x_1)\left[\mathbf{1}_{\{|x_1|\le 1\}}Q_{t_1,t}(y-x_1) + \mathbf{1}_{\{|x_1|>1\}}Q_{t_1,t}(y)\right] dx_1 dt_1.$$

In general, for $n \ge 1$

$$\begin{aligned} p_n(t,y) &= \frac{n!}{t^n}\int_{\Delta_n}\int_{\mathbb{R}} p_{n-1}(t_n,x)\left[\mathbf{1}_{\{|x|\le 1\}}Q_{t_n,t}(y-x) + \mathbf{1}_{\{|x|>1\}}Q_{t_n,t}(y)\right] dx dt_1\cdots dt_n \\ &= \int_{\mathbb{R}} \frac{ns^{n-1}}{t^n} p_{n-1}(s,x)\left[\mathbf{1}_{\{|x|\le 1\}}Q_{s,t}(y-x) + \mathbf{1}_{\{|x|>1\}}Q_{s,t}(y)\right] dx ds. \end{aligned}$$

Now we give the Fourier transform $\widehat{p}_n(t,\zeta) = \int_{\mathbb{R}} e^{i\zeta y}p_n(t,y)dy$ of $p_n(t,.)$:

$$\widehat{p}_0(t,\zeta) = \widehat{f}(\zeta)\widehat{Q}_{0,t}(\zeta).$$

and

$$\widehat{p}_n(t,\zeta) = \int_0^t\int_{\mathbb{R}} \frac{ns^{n-1}}{t^n}p_{n-1}(s,x)\left[\mathbf{1}_{\{|x|\le 1\}}e^{i\zeta x}\widehat{Q}_{s,t}(\zeta) + \mathbf{1}_{\{|x|>1\}}\widehat{Q}_{s,t}(\zeta)\right] dx ds.$$

Assume that the function γ^2 is bounded from below by $\varepsilon > 0$. The modulus of the Fourier transform $|\widehat{p}_n(t,.)|$ is bounded as follows.

$$|\widehat{p}_0(t,\zeta)| = |\widehat{f}(\zeta)||Q_{0,t}(\zeta)| = |\widehat{f}(\zeta)|\exp\{-\frac{\zeta^2}{2}\int_0^t \gamma^2(s)ds\} \le |\widehat{f}(\zeta)|\exp\{-\frac{\zeta^2}{2}\varepsilon t\}$$

and for $n \geq 1$

$$\begin{aligned}
|\widehat{p}_n(t,\zeta)| &\leq \frac{n}{t^n}\int_0^t \int_{\mathbb{R}} s^{n-1} p_{n-1}(s,x)|\widehat{Q}_{s,t}(\zeta)|dxds \\
&\leq \frac{n}{t^n}\int_0^t s^{n-1}\exp\{-\frac{\zeta^2}{2}\int_s^t \gamma^2(u)du\}ds \\
&\leq \frac{n}{t}\int_0^t \exp\{-\frac{\zeta^2}{2}\varepsilon(t-s)\}ds = \frac{2n(1-\exp\{-\frac{\zeta^2}{2}\varepsilon t\})}{t\zeta^2\varepsilon}.
\end{aligned}$$

The density of Y_t is

$$p(t,y) = \sum_{n=0}^{\infty} e^{-t}\frac{t^n}{n!}p_n(t,y).$$

To check that p belongs to $L_t^\infty([0,\tau],L_x^2)$, we combine Parseval-Plancherel theorem with the bound on the modulus of the Fourier transform given before :

$$\begin{aligned}
2\pi\int_{\mathbb{R}} p^2(t,y)dy &= \int_{\mathbb{R}} |\widehat{p}(t,\zeta)|^2 d\zeta \\
&\leq \sum_{n=0}^{\infty} e^{-t}\frac{t^n}{n!}\int_{\mathbb{R}} |\widehat{p}_n(t,\zeta)|^2 d\zeta \\
&\leq e^{-t}\int_{\mathbb{R}} |\widehat{f}(\zeta)|^2 d\zeta + \sum_{n=1}^{\infty} e^{-t}\frac{t^n}{n!}\int_{\mathbb{R}} \frac{4n^2\left(1-\exp\{-\frac{\zeta^2}{2}\varepsilon t\}\right)^2}{t^2\zeta^4\varepsilon^2}d\zeta.
\end{aligned}$$

The change of variable $x = \zeta\sqrt{\varepsilon t}$ allows us to write

$$2\pi\int_{\mathbb{R}} p^2(t,y)dy \leq e^{-t}||f||_{L^2}^2 + \sum_{n=1}^{\infty} e^{-t}\frac{t^n}{n!}\frac{4n^2}{\sqrt{\varepsilon t}}\int_{\mathbb{R}} \frac{\left(1-\exp\{-\frac{x^2}{2}\}\right)^2}{x^4}dx.$$

As the right hand side is bounded uniformly when t belongs to $[0,\tau]$, $p \in L_t^\infty([0,\tau],L_x^2)$. To check that p belongs to $L_t^2([0,\tau],H_x^1)$, we note that $\widehat{\frac{\partial p}{\partial y}}(t,\zeta) = i\zeta\widehat{p}(t,\zeta)$ and we write

$$\begin{aligned}
2\pi\int_0^\tau \int_{\mathbb{R}} |\frac{\partial p}{\partial y}(t,y)|^2 dtdy &= \int_0^\tau \int_{\mathbb{R}} |\widehat{\frac{\partial p}{\partial y}}(t,\zeta)|^2 dtd\zeta = \int_0^\tau \int_{\mathbb{R}} \zeta^2|\widehat{p}(t,\zeta)|^2 dtd\zeta \\
&\leq \sum_{n=0}^{\infty}\int_0^\tau \int_{\mathbb{R}} e^{-t}\frac{t^n}{n!}\zeta^2|\widehat{p}_n(t,\zeta)|^2 dtd\zeta \\
&\leq \int_0^\tau \int_{\mathbb{R}} \zeta^2 e^{-t}\exp\{-\zeta^2\varepsilon t\}|\widehat{f}(\zeta)|^2 dtd\zeta \\
&\quad + \sum_{n=1}^{\infty}\int_0^\tau \int_{\mathbb{R}} e^{-t}\frac{t^n}{n!}\frac{4n^2}{t^2}\frac{\left(1-\exp\{-\frac{\zeta^2}{2}\varepsilon t\}\right)^2}{\zeta^2\varepsilon^2}dtd\zeta.
\end{aligned}$$

The change of variable $x = \zeta\sqrt{\varepsilon t}$ yields

$$2\pi\int_0^\tau\int_{\mathbb{R}}|\frac{\partial p}{\partial y}(t,y)|^2 dtdy \le \int_{\mathbb{R}}\left(\int_0^\tau \zeta^2\exp\{-\zeta^2\varepsilon t\}dt\right)|\widehat{f}(\zeta)|^2 d\zeta + \sum_{n=1}^{\infty}\frac{4Cn}{\varepsilon^{\frac{3}{2}}(n-1)!}\int_0^\tau e^{-t}t^{n-\frac{3}{2}}dt$$

with $C = \int_{\mathbb{R}}\frac{\left(1-\exp\{-\frac{x^2}{2}\}\right)^2}{x^2}dx$. Finally,

$$2\pi\int_0^\tau\int_{\mathbb{R}}|\frac{\partial p}{\partial y}(t,y)|^2 dtdy \le \frac{1}{\varepsilon}||f||_{L^2}^2 + \frac{4C}{\varepsilon^{\frac{3}{2}}}\sum_{n=1}^{\infty}\frac{n\tau^{n-\frac{1}{2}}}{(n-\frac{1}{2})(n-1)!}.$$

As the right hand side is finite we have $p \in L_t^2([0,\tau], H_x^1)$.

□

We are now ready to prove proposition 1.

Proof of proposition 1 : To obtain this result we proceed by inductive reasoning. The idea is to build a positive increasing sequence $0 \le t_1 \le \ldots \le t_K = T$ such that for $k \in \{1,\cdots,K\}$ we are able to prove the following property : for all $t \in [0,t_k]$, the marginal distribution P_t has a probability density $p(t,.)$ and $(p(t,.))_{0\le t\le t_k}$ belongs to $L_t^\infty([0,t_k], L_x^1\cap L_x^2)\cap L_t^2([0,t_k],H_x^1)$. Since by Lemma 1, p is a weak solution of (1), by the uniqueness result in theorem 1, $(p(t,.))_{0\le t\le t_k}$ can then be identified with the restriction of ρ to the time interval $[0,t_k]$.

Let α be such that the conclusion of corollary 1 holds and $K \in \mathbb{N}^*$ be such that $\frac{T}{K} \le \left(\frac{\alpha-1}{2||b||_{L^2}}\right)^2$. We set $t_k = k\frac{T}{K}$, $k \in \{1,\cdots,K\}$.

- At the first step, we use that, by lemma 2 (1), P is the distribution of the solution of the stochastic differential equation

$$Y_t = Y_0 + \int_0^t \sigma\sqrt{P_s([-1,1]^c)}dW_s + \int_0^t b(s)ds - \int_0^t Y_{s-}\mathbf{1}_{\{|Y_{s-}|>1\}}dN_s.$$

with Y_0 distributed according to the density ρ_0.

Let $t \in [0,t_1]$. Since $t_1 \le \left(\frac{\alpha-1}{2||b||_{L^2}}\right)^2$, $\int_0^t|b(s)|ds \le \|b\|_{L^2}\sqrt{t} \le \frac{\alpha-1}{2}$. Therefore

$$\begin{aligned}
P_t([-1,1]^c) &\ge \mathbb{P}(|Y_0|>\alpha, N_t=0, |Y_t|>1)\\
&= \mathbb{P}(|Y_0|>\alpha, N_t=0)\\
&- \mathbb{P}(|Y_0|>\alpha, N_t=0, |Y_0+\sigma\int_0^t\sqrt{P_s([-1,1]^c)}dW_s + \int_0^t b(s)ds| \le 1)\\
&\ge e^{-t}\int_{|x|>\alpha}\rho_0(x)dx\left(1-\mathbb{P}\left(|\sigma\int_0^t\sqrt{P_s([-1,1]^c)}dW_s| \ge \alpha-1-\int_0^t|b(s)|ds\right)\right)\\
&\ge \frac{\nu}{2}e^{-t}\left(1-\frac{2}{\sqrt{2\pi}}\int_{\frac{\alpha-1}{2\sigma\sqrt{t}}}^{\infty}\exp\{-\frac{x^2}{2}\}dx\right) \qquad (12)
\end{aligned}$$

by corollary 1.
Therefore the diffusion coefficient $a(t) = \frac{\sigma^2}{2} P_t([-1,1]^c)$ of the martingale problem satisfied by P is bounded from below by a positive constant on the time interval $[0, t_1]$. By lemma 2 (2), we deduce that for all $t \in [0, t_1]$, P_t has a density $p(t,.)$ with respect to the Lebesgue measure on $\mathbb{R}$ and that the function p belongs to $L^\infty_t([0,t_1], L^1_x \cap L^2_x) \cap L^2_t([0,t_1], H^1_x)$. On the other hand, by lemma 1, p is a weak solution of equation (1). From theorem 1, we deduce that for $t \in [0, t_1]$, $p(t,.) = \rho(t,.)$.

- Now, assume that the inductive assumption is true at order k, $k \in \{1, \cdots, N-1\}$ and let us show that this property remains true at oder $k+1$. The image $\tilde{P}$ of P by the mapping $x \in D([0,T], \mathbb{R}) \to (x_{t+t_k})_{t\in[0,t_1]}$ solves the nonlinear martingale problem on the time interval $[0, t_1]$ with the initial probability distribution $\tilde{P}_0 = P_{t_k}$.
  Now $\tilde{P}_0([-1,1]^c) = \int_{|x|>\alpha} p(t_k, x)dx = \int_{|x|>\alpha} \rho(t_k,x)dx \geq \frac{\nu}{2}$ by Corollary 1. By computations similar to the ones made at the first step, we obtain that for $t \in [0,t_1]$, $\tilde{P}_t([-1,1]^c)$ is greater than the right-hand-side of (12). Again, we deduce from lemma 2 (2) that for $t \in [0,t_1]$, $\tilde{P}_t$ has a density $\tilde{p}(t,.)$ and the function $\tilde{p}$ belongs to $L^\infty_t([0,t_1], L^1_x \cap L^2_x) \cap L^2_t([0,t_1], H^1_x)$. Since $t_{k+1} = t_k + t_1$, putting all this material together, we conclude that for all $t \in [0, t_{k+1}]$, P_t has a density $p(t,.)$, and $(p(t,.))_{0\leq t\leq t_{k+1}}$ belongs to $L^\infty_t([0,t_{k+1}], L^1_x \cap L^2_x) \cap L^2_t([0,t_{k+1}], H^1_x)$. Moreover, $(p(t,.))_{0\leq t \leq t_{k+1}}$ can be identified to the restriction of ρ to the interval $[0, t_{k+1}]$.

This concludes the proof. $\square$

2 Propagation of Chaos

We define a system of n interacting particles by the following stochastic differential equation :

$$Y^{i,n}_t = Y^i_0 + \sigma \int_0^t \sqrt{\frac{1}{n}\sum_{j=1}^n \mathbf{1}_{\{|Y^{j,n}_s|>1\}} \vee \frac{1}{n}} dW^i_s + \int_0^t b(s)ds - \int_0^t Y^{i,n}_{s^-} \mathbf{1}_{\{|Y^{i,n}_{s^-}|>1\}} dN^i_s, \ 1 \leq i \leq n. \tag{13}$$

Here $(W^i)_{1\leq i\leq n}$ are independent Brownian motions, $(N^i)_{1\leq i\leq n}$ independent Poisson processes with intensity one and $(Y^i_0)_{1\leq i\leq n}$ independent $\rho_0(x)dx$ distributed random variables. Also, we assume that $(W^i)_{1\leq i\leq n}$, $(N^i)_{1\leq i\leq n}$ and $(Y^i_0)_{1\leq i\leq n}$ are independent.
Between the jump times of the Poisson processes, $(Y^{1,n}, \ldots, Y^{n,n})$ evolves as a n-dimensional diffusion process with a piecewise constant (in the n-dimensional spatial variable) and non-degenerated diffusion matrix. Hence, by Bass and Pardoux [1], and exercise 7.3.2 p. 191 in Stroock and Varadhan [6], existence and uniqueness in law hold for equation (13).
Let $\mu_n = \frac{1}{n}\sum_{i=1}^n \delta_{Y^{i,n}}$ denote the empirical measure of the particle system. We are going to prove the following law of large numbers

Theorem 3 *Assume that ρ_0 satisfies condition (H). As n tends to infinity, the $\mathcal{P}(D([0,T],\mathbb{R}))$−valued random variables μ^n converge in probability to the unique solution P of the nonlinear martingale problem (MP).*

Since the particles $Y^{i,n}, 1 \le i \le n$ are exchangeable, according to Sznitman proposition 2.2 p. 177 in [9], this result is equivalent to propagation of chaos : for any fixed $k \in \mathbb{N}^*$, when n goes to infinity, the joint distribution of the processes $(Y_t^{1,n}, \cdots, Y_t^{k,n})_{t\in[0,T]}$ converges to $P^{\otimes k}$.
In order to establish the theorem we need to control the possibility for the diffusion coefficient to vanish. That is why, for $\varepsilon > 0$, we introduce the stopping time

$$\tau_n^\varepsilon := \inf\left\{ t > 0 : \ \frac{1}{n}\sum_{j=1}^{n} \mathbf{1}_{\{|Y_t^{i,n}|>1\}} < \varepsilon \right\}.$$

Let π^n be the probability distribution of the empirical measure μ^n. We will denote by Q the canonical variable on $\mathcal{P}\left(D\left([0,T],\mathbb{R}\right)\right)$. The next Lemma implies that as soon as $\mathbb{P}(\tau_n^\varepsilon \le t)$ converges to 0 as n tends to ∞, then any weak limit π^∞ of the sequence $(\pi^n)_n$ has the following regularity property which is desirable to take the limit in the martingale problem formulation : $\pi^\infty(dQ)$ a.e., dr a.e. on $[0,t]$, Q_r does not weight the set of discontinuity points $\{-1,1\}$ of the characteristic function $x \to 1_{\{|x|>1\}}$ which appears in the nonlinear diffusion coefficient $\frac{\sigma^2}{2}P_s([-1,1]^c) = \frac{\sigma^2}{2}\mathbb{E}^P(1_{\{|X_s|>1\}})$ in problem (MP).

Lemma 3 *There is a constant $C > 0$ such that for all $t \in [0,T]$ and for all bounded functions f in $L^2(\mathbb{R})$,*

$$\left|\mathbb{E}^{\pi^n}\left(\int_0^t \langle Q_s, f\rangle\, ds\right)\right| \le t||f||_\infty \mathbb{P}(\tau_n^\varepsilon \le t) + C||f||_{L^2}. \tag{14}$$

The second technical Lemma prepares an inductive reasoning to prove that $\mathbb{P}(\tau_n^\varepsilon \le T)$ tends to 0 as n tends to ∞.

Lemma 4 *For all $\alpha > 1$ and for all $\kappa > 0$, there are $\varepsilon > 0$ and $K \in \mathbb{N}^*$ such that*

$$\limsup_{n\to+\infty} \mathbb{P}(\tau_n^\varepsilon \le k\frac{T}{N}) \le \sum_{\ell=0}^{k-1} \limsup_{n\to+\infty} \mathbb{P}\left(\mu^n_{\ell\frac{T}{K}}([-\alpha,\alpha]^c) \le \kappa\right). \tag{15}$$

for all $k \in 1, \cdots, K$.

For the reader convenience the proof of the above technical lemmas is postponed after the proof of the theorem.

Proof of theorem 3 : By exchangeability of the particles, the tightness of the sequence $(\pi^n)_{n\ge1}$ is equivalent to the tightness of the laws of the random variables $(Y^{1,n})_{n\ge1}$ (see again proposition 2.2 page 177 in [9]). As the diffusion coefficient and the drift coefficient are uniformly bounded in n and the intensity of jumps remains smaller than one, the tightness of the sequence $(Y^{1,n})_{n\ge1}$ holds (Aldous criterion for instance).

Let π^∞ be the limit of a convergent subsequence that we still index by n for notational simplicity. We are going to check that π^∞ *a.s.*, Q solves the martingale problem (MP). To do so, for $p \in \mathbb{N}^*$, $\phi \in C_b^2(\mathbb{R})$, $g \in C_b(\mathbb{R}^p)$ and $T \geq t \geq s \geq s_1 \geq \cdots \geq s_p \geq 0$, we define a mapping F on $\mathcal{P}\left(D\left([0,T],\mathbb{R}\right)\right)$ by

$$F(Q) = \Big\langle Q, \Big(\phi(X_t) - \phi(X_s) - \int_s^t b(r)\phi'(X_r) + \frac{\sigma^2}{2} Q_r([-1,1]^c)\phi''(X_r)dr \\ - \int_s^t (\phi(0) - \phi(X_r))\mathbf{1}_{\{|X_r|>1\}}dr\Big) g(X_{s_1}, \cdots, X_{s_p})\Big\rangle,$$

and we want to prove that $\mathbb{E}^{\pi^\infty}\left(|F(Q)|\right) = 0$. By Itô's formula,

$$\begin{aligned} F(\mu^n) &= \frac{\sigma}{n}\sum_{i=1}^n g(Y_{s_1}^{i,n}, \cdots, Y_{s_p}^{i,n}) \int_s^t \phi'(Y_r^{i,n})(\mu_r^n([-1,1]^c) \vee \frac{1}{n})dW_r^i \\ &+ \frac{1}{n}\sum_{i=1}^n g(Y_{s_1}^{i,n}, \cdots, Y_{s_p}^{i,n}) \int_s^t (\phi(0) - \phi(Y_{r^-}^{i,n}))\mathbf{1}_{\{|Y_{r^-}^{i,n}|>1\}}d(N_r^i - r) \\ &+ \frac{\sigma^2}{2n}\sum_{i=1}^n g(Y_{s_1}^{i,n}, \cdots, Y_{s_p}^{i,n}) \int_s^t \phi''(Y_r^{i,n})(\mu_r^n([-1,1]^c) \vee \frac{1}{n} - \mu_r^n([-1,1]^c))dr. \end{aligned}$$

Next, using the inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$ for $(a,b,c) \in \mathbb{R}^3$, the independence between the Brownian motions and between the Poisson processes and the boundedness of the function g, we get

$$\begin{aligned} \mathbb{E}(F(\mu^n)^2) &\leq \frac{3\sigma^2||g||_\infty^2}{n^2}\sum_{i=1}^n \mathbb{E}\left(\int_s^t (\phi'(Y_r^{i,n}))^2(\mu_r^n([-1,1]^c) \vee \frac{1}{n})^2 dr\right) \\ &+ \frac{3||g||_\infty^2}{n^2}\sum_{i=1}^n \mathbb{E}\left(\int_s^t (\phi(0) - \phi(Y_r^{i,n}))^2\mathbf{1}_{\{|Y_{r^-}^{i,n}|>1\}}dr\right) \\ &+ \frac{3\sigma^4||g||_\infty^2}{4n^2}\mathbb{E}\left(\sum_{i=1}^n \left|\int_s^t \phi''(Y_r^{i,n})(\mu_r^n([-1,1]^c) \vee \frac{1}{n} - \mu_r^n([-1,1]^c))dr\right|\right)^2. \end{aligned}$$

Now, the boundedness of functions ϕ, ϕ', ϕ'' and the inequalities

$$0 \leq \mu_r^n([-1,1])^c \leq 1 \quad \text{and} \quad 0 \leq \mu_r^n([-1,1]^c) \vee \frac{1}{n} - \mu_r^n([-1,1]^c) \leq \frac{1}{n},$$

yield

$$\mathbb{E}^{\pi^n}\left(|F(Q)|\right) = \mathbb{E}\left(|F(\mu^n)|\right) \leq \sqrt{\mathbb{E}\left(F_k(\mu^n)^2\right)} \leq \frac{C}{\sqrt{n}}, \tag{16}$$

where C is a positive constant. Hence $\mathbb{E}^{\pi^n}\left(|F(Q)|\right)$ converges to 0 as n tends to infinity. Unfortunately the mapping F is not continuous and we cannot deduce that $\mathbb{E}^{\pi^\infty}\left(|F(Q)|\right) = 0$. Nevertheless F is continuous at any Q such that dr *p.p.*, $Q_r(\{-1,1\}) = 0$. So we should first prove that π^∞ gives full weight to such probability measures. To do so, we need to bound the diffusion coefficient of the particle system from below. We are only able to obtain such a control on a small time interval. That is why we will first pass to the limit on this time interval. Then, to iterate the reasoning, we take advantage of the bound

$$\forall t \in [0,T],\ P_t([-\alpha,\alpha]^c) \geq \frac{\nu}{2}, \tag{17}$$

which holds for some $\alpha > 1$ according to corollary 1 and theorem 2. Applying lemma 4 with this α and with $\kappa = \dfrac{\nu}{4}$, we deduce that we can choose $\varepsilon > 0$ and $K \in \mathbb{N}^*$ such that

$$\limsup_{n\to+\infty} \mathbb{P}(\tau_n^\varepsilon \leq k\frac{T}{K}) \leq \sum_{\ell=0}^{k-1} \limsup_{n\to+\infty} \mathbb{P}\left(\mu^n_{\ell\frac{T}{N}}([-\alpha,\alpha]^c) \leq \frac{\nu}{4}\right). \tag{18}$$

Let $\pi^{\infty,k}$ be the law of the image of Q by the restriction mapping $(Y_t)_{t\leq T} \in D([0,T],\mathbb{R}) \to (Y_t)_{t\leq k\frac{T}{N}} \in D([0,k\frac{T}{K}],\mathbb{R})$ under π^∞ and P^k be the image of P by this mapping. We are going to prove by induction on $k \in \{0,\dots,K\}$ that $\pi^{\infty,k} = \delta_{P^k}$. Since the initial variables Y_0^i are independent and identically distributed according to $\rho_0(x)dx$, the inductive property holds for $k=0$. Then we assume that it holds at order $k-1$ and show that it remains true at order k.

From the recurrence assumption at order $k-1$, since under P the canonical process is quasi-left-continuous, we can deduce that $\forall t \in [0,(k-1)\dfrac{T}{K}]$, μ_t^n converges weakly to P_t (see lemma 4.8. p.71 in [8]). Let $(m_n)_{n\geq1}$ and m be probability measures on $\mathbb{R}$, it's well-known that the weak convergence of $(m_n)_{n\geq1}$ to m entails $\liminf_{n\to\infty} m_n(O) \geq m(O)$ for all open sets O of $\mathbb{R}$. This proves that $\{m \in \mathcal{P}(\mathbb{R}), \quad m([-\alpha,\alpha]^c) > \frac{\nu}{4}\}$ is an open set for the topology of weak convergence. Thus, by (17),

$$\liminf_{n\to\infty} \mathbb{P}(\mu^n_{\ell\frac{T}{N}}([-\alpha,\alpha]^c) > \frac{\nu}{4}) \geq P_{\ell\frac{T}{N}}([-\alpha,\alpha]^c) > \frac{\nu}{4}) = 1, \forall \ell \in \{0,\cdots,k-1\}.$$

Then, by (18), $\limsup_{n\to+\infty} \mathbb{P}(\tau_n^\varepsilon \leq k\dfrac{T}{K}) = 0$. With lemma 3, we deduce that for any continuous and bounded function $f \in L^2(\mathbb{R})$,

$$\left|\mathbb{E}^{\pi^{\infty,k}}\left(\int_0^{k\frac{T}{K}} \langle Q_t, f\rangle\, dt\right)\right| \leq C||f||_{L^2}.$$

Now, let $f_\gamma(x) := \max\left(0, 1-\frac{|1-|x||}{\gamma}\right)$ for $0<\gamma<1$. As $||f_\gamma||_{L^2} = \sqrt{\dfrac{4\gamma}{3}}$, if we replace f by f_γ in the equation above and we let γ go to zero, we deduce that $\pi^{\infty,k}$ *a.s.*, dr *p.p.*, $Q_r(\{-1,1\}) = 0$.

Finally, since when the parameter t in the definition of F is smaller than $k\frac{T}{K}$, then this function is continuous at all points $Q \in \mathcal{P}\left(D\left([0,k\dfrac{T}{K}],\mathbb{R}\right)\right)$ satisfying dr *p.p.* $Q_r(\{-1,1\}) = 0$, we deduce from equation (16) that

$$\mathbb{E}^{\pi^{\infty,k}}(|F(Q)|) = \lim_{n\to+\infty} \mathbb{E}^{\pi^n}(|F(Q)|) = 0.$$

Hence $\pi^{\infty,k} = \delta_{P^k}$, which concludes the proof.

□

Let us prove now lemma 3.

Proof of lemma 3: Let f be a bounded function on the real line and $t \in [0,T]$,

$$\begin{aligned}\left|\mathbb{E}^{\pi^n}\left(\int_0^t \langle Q_s, f\rangle\, ds\right)\right| &= \left|\mathbb{E}\left(\int_0^t \langle \mu_s^n, f\rangle\, ds\right)\right| = \left|\frac{1}{n}\sum_{i=1}^n \mathbb{E}\left(\int_0^t f(Y_s^{i,n})ds\right)\right| \\ &\leq t||f||_\infty \mathbb{P}(\tau_n^\varepsilon \leq t) + \left|\frac{1}{n}\sum_{i=1}^n \mathbb{E}\left(\mathbf{1}_{\{\tau_n^\varepsilon > t\}}\int_0^t f(Y_s^{i,n})ds\right)\right|. \qquad (19)\end{aligned}$$

In order to use estimations due to Krylov [5] and stated in Lemma 5 below, we introduce the following stochastic differential equation

$$\begin{cases} Y_0^{i,n,\varepsilon} = Y_0^i \\ dY_t^{i,n,\varepsilon} = \mathbf{1}_{\{\tau_n^\varepsilon > t\}} dY_t^{i,n} + \mathbf{1}_{\{\tau_n^\varepsilon \leq t\}}\left(\sigma\sqrt{\varepsilon}dW_t^i + b(t)dt - Y_{t^-}^{i,n,\varepsilon}\mathbf{1}_{\{|Y_{t^-}^{i,n,\varepsilon}|>1\}}dN_t^i\right) \end{cases}, \ 1 \leq i \leq n.$$

Up to time τ_n^ε the processes $(Y_t^{i,n,\varepsilon}, 1 \leq i \leq n)$ and $(Y_t^{i,n}, 1 \leq i \leq n)$ coincide. This result with the exchangeability of $(Y^{i,n,\varepsilon})_{1\leq i\leq n}$ enable us to replace $Y_t^{i,n}$ by $Y_t^{1,n,\varepsilon}$ in the inequality (19) above. We obtain

$$\left|\mathbb{E}^{\pi^n}\left(\int_0^t \langle Q_s, f\rangle\, ds\right)\right| \leq t||f||_\infty \mathbb{P}(\tau_n^\varepsilon \leq t) + \left|\mathbb{E}\left(\int_0^t f(Y_s^{1,n,\varepsilon})ds\right)\right|. \qquad (20)$$

Moreover the process $Y_t^{1,n,\varepsilon}$ satisfies the following differential equation.

$$Y_t^{1,n,\varepsilon} = Y_0^1 + \int_0^t \sigma_s^{1,n,\varepsilon} dW_s^1 + \int_0^t b(s)ds - \int_0^t Y_{s^-}^{1,n,\varepsilon}\mathbf{1}_{\{|Y_{s^-}^{1,n,\varepsilon}|>1\}}dN_s^1.$$

where $\sigma_s^{1,n,\varepsilon} := \mathbf{1}_{\{\tau_n^\varepsilon > s\}}\sigma\sqrt{\frac{1}{n}\sum_{j=1}^n \mathbf{1}_{\{|Y_s^{j,n,\varepsilon}|>1\}} \vee \frac{1}{n}} + \mathbf{1}_{\{\tau_n^\varepsilon \leq s\}}\sigma\sqrt{\varepsilon}.$

Now, we are ready to apply the following estimation which is a consequence of [5] Theorem 2 p. 238.

Lemma 5 *Let $t \leq T$, $(\xi_s)_{s\geq 0}$ be an $(\mathcal{F}_s)$-standard real Brownian motion and*

$$x_s = x + \int_0^s \sigma_r d\xi_r + \int_0^s \beta(r)dr,\ s \in [0,t]$$

with $x \in \mathbb{R}$, β a deterministic function integrable on $[0,t]$ and σ_r an $\mathcal{F}_r-$adapted process . Let us assume that there are constants $0 < \underline{\sigma} \leq \overline{\sigma}$ such that $\underline{\sigma} \leq \sigma_r \leq \overline{\sigma}$ for all $r \in [0,t]$. Then for all $f \in L^2(\mathbb{R})$

$$\left|\mathbb{E}\left(\int_0^t f(x_s)ds\right)\right| \leq C||f||_{L^2},$$

where the constant C depends only on $\underline{\sigma}$ and $\overline{\sigma}$ and T.

Coming back to our process $(Y_s^{1,n,\varepsilon})_{0\le s\le t}$, a simple decomposition of $Y_s^{1,n,\varepsilon}$ on the subsets $\{N_s^1 = k\}, k \in \mathbb{N}$ with the use of the conditional distribution of the jump times of N^1 given $\{N_s^1 = k\}$ yields

$$\begin{aligned}
\mathbb{E}\left(\int_0^t f(Y_s^{1,n,\varepsilon})ds\right) &= \sum_{k=0}^{\infty} \int_0^t \mathbb{E}\left(f(Y_s^{1,n,\varepsilon})|N_s^1 = k\right)\mathbb{P}(N_s^1 = k)ds \\
&= \mathbb{E}\left(\int_0^t e^{-s} f(x_s^{1,n,\varepsilon,0})ds\right) \\
&\quad + \sum_{k=1}^{\infty} \int_{0<s_1<\cdots<s_k<t} \mathbb{E}\left(\int_{s_k}^t e^{-s} f(x_s^{1,n,\varepsilon,k})ds\right) ds_1 \cdots ds_k,
\end{aligned}$$

where

$$\begin{cases} x_s^{1,n,\varepsilon,0} = Y_0^1 + \int_0^s \sigma_r^{1,n,\varepsilon} dW_r^1 + \int_0^s b(r)dr, \\ x_s^{1,n,\varepsilon,k} = Y_{s_k}^{1,n,\varepsilon} + \int_{s_k}^s \sigma_r^{1,n,\varepsilon} dW_r^1 + \int_{s_k}^s b(r)dr, \quad \forall k \ge 1. \end{cases}$$

Noticing that $\sigma\sqrt{\varepsilon} \le \sigma_r^{1,n,\varepsilon} \le \sigma(1+\varepsilon)$ and applying lemma 5, we deduce that there exists a positive constant C such that, for all $f \in L^2$ and for all $k \in \mathbb{N}$,

$$\left|\mathbb{E}\left(\int_{s_k}^t e^{-s} f(x_s^{1,n,\varepsilon,k})ds\right)\right| \le C||f||_{L^2}.$$

Therefore, for all $f \in L^2(\mathbb{R})$,

$$\left|\mathbb{E}\left(\int_0^t f(Y_t^{1,n,\varepsilon})dt\right)\right| \le Ce^t||f||_{L^2}. \tag{21}$$

Equations (20) and (21) together conclude the proof.

□

Let us prove now lemma 4.

Proof of lemma 4: Let $\alpha > 1$ and $\kappa > 0$. Like in the proof of Proposition 1, we introduce $K \in \mathbb{N}^*$ such that $\dfrac{T}{K} \le \left(\dfrac{\alpha - 1}{2||b||_{L^2}}\right)^2$ and set $t_1 = \frac{T}{K}$. Let $\varepsilon = \dfrac{\kappa\beta(t_1)}{2}$, with

$$\beta(t_1) = \mathbb{P}(\sup_{s\le t_1} |W_s^i| \le \frac{\alpha - 1}{2\sigma}, N_{t_1}^i = 0).$$

Let I denote the set of indexes $\{i \le n : |Y_0^i| > \alpha\}$. If we decompose the event $\{\tau_n^\varepsilon \le t_1\}$ on the event $\{card(I) < \kappa n\}$ and its complementary we obtain

$$\mathbb{P}\left(\tau_n^\varepsilon \le t_1\right) \le \mathbb{P}\left(\mu_0^n([-\alpha, \alpha]^c) < \kappa\right) + \mathbb{P}\left(card(I) \ge \kappa n, \tau_n^\varepsilon \le t_1\right). \tag{22}$$

We are going to prove that the limit as $n \to +\infty$ of the second term at the right hand side of equation (22) is 0. Since $\int_0^{t_1} |b(r)|dr \le \|b\|_{L^2}\sqrt{t_1} = \frac{\alpha-1}{2}$, for $j \in I$, the existence of $s \in [0, t_1]$ such that $|Y_s^{j,n}| \le 1$ entails either $N_{t_1}^j \ne 0$ or $\sup_{s\le t_1} |\int_0^s \sigma_r^n dW_r^j| > \dfrac{\alpha - 1}{2}$, where

$\sigma_r^n := \sigma\sqrt{\frac{1}{n}\sum_{j=1}^n \mathbf{1}_{\{|Y_s^{j,n}|>1\}} \vee \frac{1}{n}}$.Therefore the second term of the right hand side of equation (22) is bounded from above by

$$\mathbb{P}\left(card(I) \geq \kappa n, \sum_{j\in I} \mathbf{1}_{\{N^j_{t_1}\neq 0 \text{ or } \sup_{s\leq t_1} |\int_0^s \sigma_r^n dW_r^j| > \frac{\alpha-1}{2}\}} > card(I) - n\varepsilon\right).$$

On the other hand, considering the filtration $\mathcal{G}_t := \sigma\left(Y_0^i, (N_s^i)_{s\leq T, 1\leq i\leq n}, (W_s = (W_s^1, \ldots, W_s^n))_{s\leq t}\right)$, $t \in [0,T]$, and the $\mathcal{G}_t$ martingale $M_t := \int_0^t \sigma_r^n dW_r$, letting $A_t := \int_0^t (\sigma_r^n)^2 dr$ and $\tau_t := \inf\{s, \quad A_s \geq t\}$, by the Dambis and Dubins-Schwarz theorem $B_t := M_{\tau_t} = \int_0^{\tau_t} \sigma_r^n dW_r$ is an $\mathbb{R}^n$ valued $\mathcal{G}_{\tau_t}$ Brownian motion and $\int_0^t \sigma_r^n dW_r = B_{A_t}$. This enables us to write that $\mathbb{P}\left(card(I) \geq \kappa n, \tau_n^\varepsilon \leq t_1\right)$ is smaller than

$$\mathbb{P}\left(card(I) \geq \kappa n, \sum_{j\in I} \mathbf{1}_{\{N^j_{t_1}\neq 0 \text{ or } \sup_{s\leq t_1} |B^j_{A_s}| > \frac{\alpha-1}{2}\}} > card(I) - n\varepsilon\right).$$

Noticing that $A_s \leq \sigma^2 s$ and using the definition of ε, we can substitute the last upper bound by

$$\mathbb{P}\left(card(I) \geq \kappa n, \frac{1}{card(I)}\sum_{j\in I} \mathbf{1}_{\{N^j_{t_1}=0, \sup_{s\leq \sigma^2 t_1} |B^j_s| \leq \frac{\alpha-1}{2}\}} \leq \frac{\beta(t_1)}{2}\right).$$

Now, as $\sigma\left(Y_0^i, (N_s^i)_{s\leq T}, 1\leq i\leq n\right) = \mathcal{G}_0 \subset \mathcal{G}_{\tau_t}$ we deduce that $(N_s^i, s\leq T, 1\leq i\leq n)$, $(B_s^i, s\leq T, 1\leq i\leq n)$ and $(Y_0^i, 1\leq i\leq n)$ are independent. Letting $\mathcal{F}_0 := \sigma\left(Y_0^i, 1\leq i\leq n\right)$, this probability reads

$$\mathbb{E}\left(\mathbf{1}_{\{card(I)\geq\kappa n\}}\mathbb{P}\left(\frac{1}{card(I)}\sum_{j\in I} \mathbf{1}_{\{N^j_{t_1}=0, \sup_{s\leq t_1} |B^j_s| \leq \frac{\alpha-1}{2\sigma}\}} \leq \frac{\beta(t_1)}{2}|\mathcal{F}_0\right)\right).$$

Using Bienaymé Chebychev inequality, we obtain

$$\mathbb{P}\left(\frac{1}{card(I)}\sum_{j\in I} \mathbf{1}_{\{N^j_{t_1}=0, \sup_{s\leq t_1} |B^j_s| \leq \frac{\alpha-1}{2\sigma}\}} \leq \frac{\beta(t_1)}{2}|\mathcal{F}_0\right) \leq \frac{4}{\beta(t_1)card(I)}.$$

Finally the second term of the right hand side of equation (22) is smaller then $\frac{4}{\kappa\beta(t_1)n}$ and converges to 0.

Next, we use an inductive reasoning on $k \in \{1, \cdots, N\}$ to establish equation (15). Noticing that we have just shown the inductive property for $k = 1$, we assume that the recurrence assumption is true at order $k-1$ and we show that it remains true at order k. We have

$$\begin{aligned}\mathbb{P}\left(\tau_n^\varepsilon \leq kt_1\right) \leq \mathbb{P}\left(\tau_n^\varepsilon \leq (k-1)t_1\right) + \mathbb{P}\left(\mu^n_{(k-1)t_1}([-\alpha,\alpha]^c) \leq \kappa\right)& \\ +\mathbb{P}\left(\mu^n_{(k-1)t_1}([-\alpha,\alpha]^c) > \kappa, (k-1)t_1 < \tau_n^\varepsilon \leq kt_1\right)&.\end{aligned}$$

By the inductive assumption,

$$\limsup_{n\to+\infty}\mathbb{P}(\tau_n^\varepsilon\leq kt_1)\leq\sum_{\ell=0}^{k-1}\limsup_{n\to+\infty}\mathbb{P}\left(\mu^n_{\ell t_1}([-\alpha,\alpha]^c)\leq\kappa\right)$$
$$+\limsup_{n\to+\infty}\mathbb{P}\left(\mu^n_{(k-1)t_1}([-\alpha,\alpha]^c)>\kappa,(k-1)t_1<\tau_n^\varepsilon\leq kt_1\right).$$

Setting $\tilde{I}=\{i\leq n,|Y^i_{(k-1)t_1}|>\alpha\}$ and by a reasoning similar to the one made on the time interval $[0,t_1]$ one obtains

$$\mathbb{P}\left(\mu^n_{(k-1)t_1}([-\alpha,\alpha]^c)>\kappa,(k-1)t_1<\tau_n^\varepsilon\leq kt_1\right)\leq\mathbb{E}\left(\mathbf{1}_{\{card(\tilde{I})\geq\kappa n\}}\frac{4}{\beta(t_1)card(\tilde{I})}\right)\leq\frac{4}{\kappa\beta(t_1)n},$$

which vanishes when n goes to infinity. $\square$

From a physical point of view, the average stress $\int_{\mathbb{R}}x\rho(t,x)dx$ is of particular interest. One can deduce from Theorem 3, the convergence of the particle approximation $\frac{1}{n}\sum_{i=1}^n Y_t^{i,n}$ to this quantity as n tends to infinity.

Corollary 2 *Assume that ρ_0 satisfies condition (H). We have*

$$\lim_{n\to+\infty}\mathbb{E}\left|\frac{1}{n}\sum_{i=1}^n Y_t^{i,n}-\int_{\mathbb{R}}x\rho(t,x)dx\right|=0.$$

Proof : By Theorem 3, since under P, the canonical process is quasi left continuous, for any $t\in[0,T]$, μ^n_t converges in probability to $P_t=\rho(t,x)dx$ as n tends to infinity. One has

$$\left|Y_t^{1,n}\right|\leq\left|Y_0^1\right|+\int_0^T|b(s)|\,ds+2\sigma\sup_{s\leq T}\left|\int_0^s\sqrt{\frac{1}{n}\sum_{j=1}^n\mathbf{1}_{\{|Y_s^{j,n}|>1\}}\vee\frac{1}{n}}dW_s^1\right|.$$

Since the diffusion coefficient $\sqrt{\frac{1}{n}\sum_{j=1}^n\mathbf{1}_{\{|Y_s^{j,n}|>1\}}\vee\frac{1}{n}}$ is bounded by 1 the random variables $\sup_{s\leq T}\left|\int_0^s\sqrt{\frac{1}{n}\sum_{j=1}^n\mathbf{1}_{\{|Y_s^{j,n}|>1\}}\vee\frac{1}{n}}dW_s^1\right|$ are uniformely integrable. By hypothesis (H) and the assumption made on the function b, the random variable $\left|Y_0^1\right|+\int_0^T|b(s)|\,ds$ is integrable. One deduces that the random variables $(\left|Y_t^{1,n}\right|)_{n\geq1}$ are uniformely integrable. Now, for $C>0$

$$\begin{aligned}\mathbb{E}\left|\frac{1}{n}\sum_{i=1}^n Y_t^{i,n}-\int_{\mathbb{R}}x\rho(t,x)dx\right| &\leq \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^n\left|Y_t^{i,n}\right|\mathbf{1}_{\{|Y_t^{i,n}|>C\}}\right)+\int_{\mathbb{R}}|x|\,\mathbf{1}_{\{|x|>C\}}\rho(t,x)dx\\ &+\mathbb{E}\left|\frac{1}{n}\sum_{i=1}^n Y_t^{i,n}\mathbf{1}_{\{|Y_t^{i,n}|\leq C\}}-\int_{\mathbb{R}}x\mathbf{1}_{\{|x|\leq C\}}\rho(t,x)dx\right|.\end{aligned}$$

According to equation (7) and the above uniform integrability result the sum of the two first terms of the right hand side is arbitrarily small uniformly in n for C large enough.

Finally, since μ_t^n converges in probability to $P_t = \rho(t,x)dx$, for found C, the third term tends to 0 as n tends to infinity. This concludes the proof. $\square$

3 Numerical results

To check the validity of the results obtained in the previous section with computer simulations, we consider the example of steady states given in [2]. According to proposition 5.1 of [2], when the function $b(t) = b$ constant, equation (1) admits a unique stationary solution in the following two cases.

- If $b = 0$ and $\sigma^2 > 1$, $p(x) = \dfrac{1-|x|+\sqrt{D}}{\sigma^2}\mathbf{1}_{\{x\in[-1,1]\}} + \dfrac{\sqrt{D}}{\sigma^2}e^{\frac{1-|x|}{\sqrt{D}}}\mathbf{1}_{\{x\notin[-1,1]\}}$, with $D = D(p) > 0$ given by $D + \sqrt{D} = \dfrac{\sigma^2-1}{2}$.
- If $b \neq 0$ and $\sigma^2 \neq 0$, $p(x) = a_1 e^{\beta_{sg(x)}x}\mathbf{1}_{\{x\notin[-1,1]\}} + \left(a_2(1+e^{\frac{b}{D}x}) - \dfrac{2D}{b\sigma^2}e^{\frac{b}{D}x^+}\right)\mathbf{1}_{\{x\in[-1,1]\}}$, with $\beta_\pm = \dfrac{b}{2D} \mp \dfrac{1}{2}\sqrt{\dfrac{b^2+4D}{D^2}}$ where $sg(x)$ denotes the sign of x, $x^+ = \sup(0,x)$, $a_1 = \dfrac{2e^{\frac{1}{2}\sqrt{\frac{b^2+4D}{D^2}}}}{\sigma^2(\beta_- e^{\frac{b}{2D}} - \beta_+ e^{-\frac{b}{2D}})}$, $a_2 = \dfrac{2D\beta_- e^{\frac{b}{2D}}}{\sigma^2 b(\beta_- e^{\frac{b}{2D}} - \beta_+ e^{-\frac{b}{2D}})}$. This function always fulfills $D = D(p) > 0$ and the normalization condition $\int_{\mathbb{R}} p(x)dx = 1$ reads

$$\frac{D}{b}\frac{(1+\beta_-) + (\beta_+ - 1)e^{-\frac{b}{D}}}{\beta_- - \beta_+ e^{-\frac{b}{D}}} + D = \frac{\sigma^2}{2}.$$

For fixed n, we want to simulate n interacting particles given by the stochastic differential equation (13). In order to discretize time, we assign n particles positions $(\hat{Y}^{i,n}_{k\frac{T}{K}})_{1\le i\le n}$ to each time $k\frac{T}{K}$, $0 \le k \le K$, where K is a given integer. Let $\{G_k^i, 1 \le i \le n, 1 \le k \le K\}$ and $\{U_k^i, 1 \le i \le n, 1 \le k \le K\}$ be two independent sequences of independent and identically distributed random variables respectively distributed according the normal law and the uniform law on $[0,1]$. At $k=0$ we simulate n independent particles with the initial density $\rho_0(x) = p(x)$ where p is the above stationary solution. For $k \in \{1,\cdots,K\}$, the discretized particles evolve as follows : $\forall i \in \{1,\cdots,n\}$

$$\begin{array}{llll} \text{if} & |\hat{Y}^{i,n}_{(k-1)\frac{T}{K}}| > 1 \text{ and } U_k^i \le \frac{T}{K} & \text{then} & \hat{Y}^{i,n}_{k\frac{T}{K}} = 0 \\ & & \text{else} & \hat{Y}^{i,n}_{k\frac{T}{K}} = \hat{Y}^{i,n}_{(k-1)\frac{T}{K}} + \sigma D_{(k-1)\frac{T}{K}}\sqrt{\frac{T}{K}}G_k^i + b\frac{T}{K}, \end{array}$$

with $D_{(k-1)\frac{T}{K}} = \sqrt{\frac{1}{n}\sum_{i=1}^n \mathbf{1}_{\{|\hat{Y}^{i,n}_{(k-1)\frac{T}{K}}|>1\}}}$. The average stress in the physical model is given by $\tau(t) = \displaystyle\int_{\mathbb{R}} xp(t,x)dx$ and it is approximated at the points $k\frac{T}{K}$, $k \in \{0,\cdots,K\}$, by the empirical mean $\tau^n_{k\frac{T}{K}} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} \hat{Y}^{i,n}_{k\frac{T}{K}}$. The simulation of $\tau^n_{k\frac{T}{K}}$ for $k \in \{0, \cdots, K\}$ must therefore confirm the convergence toward $\displaystyle\int_{\mathbb{R}} xp(x)dx$ when K and n tend to infinity.

Example : We consider the second example of steady states, we take $T = 1$ first with $K = 100$ then with $K = 1000$. The table 1 represents the convergence of the sequence τ^n_1. The

K	100				1000			
n	1000	5000	10000	20000	1000	5000	10000	20000
τ^n_1	1.1221	1.1126	1.1463	1.1363	1.0737	1.1541	1.1384	1.1256
n	40000	60000	80000	100000	40000	60000	80000	100000
τ^n_1	1.1324	1.1260	1.1286	1.1304	1.1312	1.1350	1.1310	1.1281

Table 1: Convergence of τ^n_1 with $K = 100$ and $K = 1000$.

graphical representation in figure 1 illustrates the convergence of the empirical measure of the n interacting particles. We compare their histogram with the stationary solution.

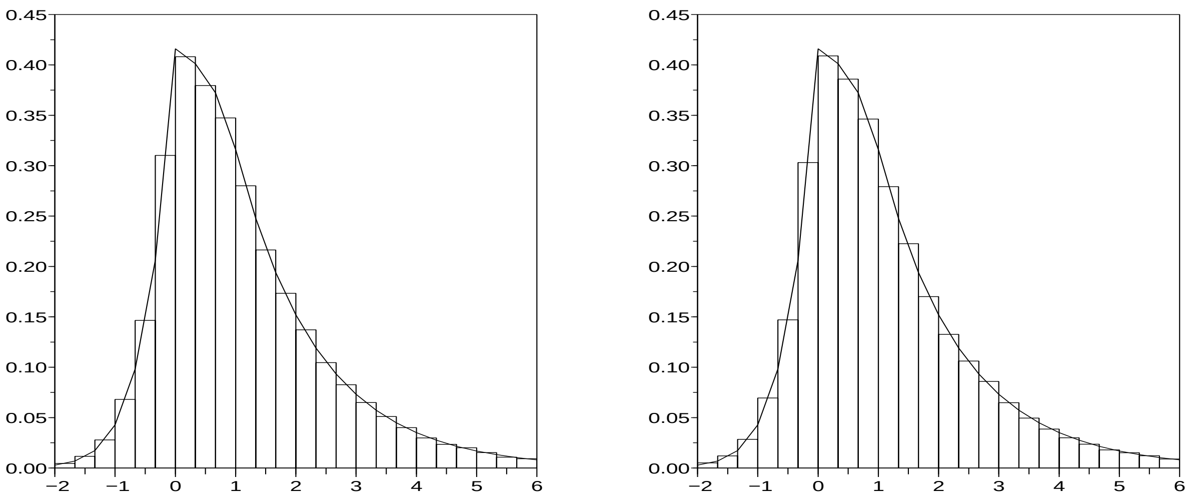

Figure 1: Convergence in distribution of $(Y^{i,n}_1)_{1\leq i\leq n}$, with $n = 100000$ and different K, from the left to the right, we have $K = 100$ and $K = 1000$.

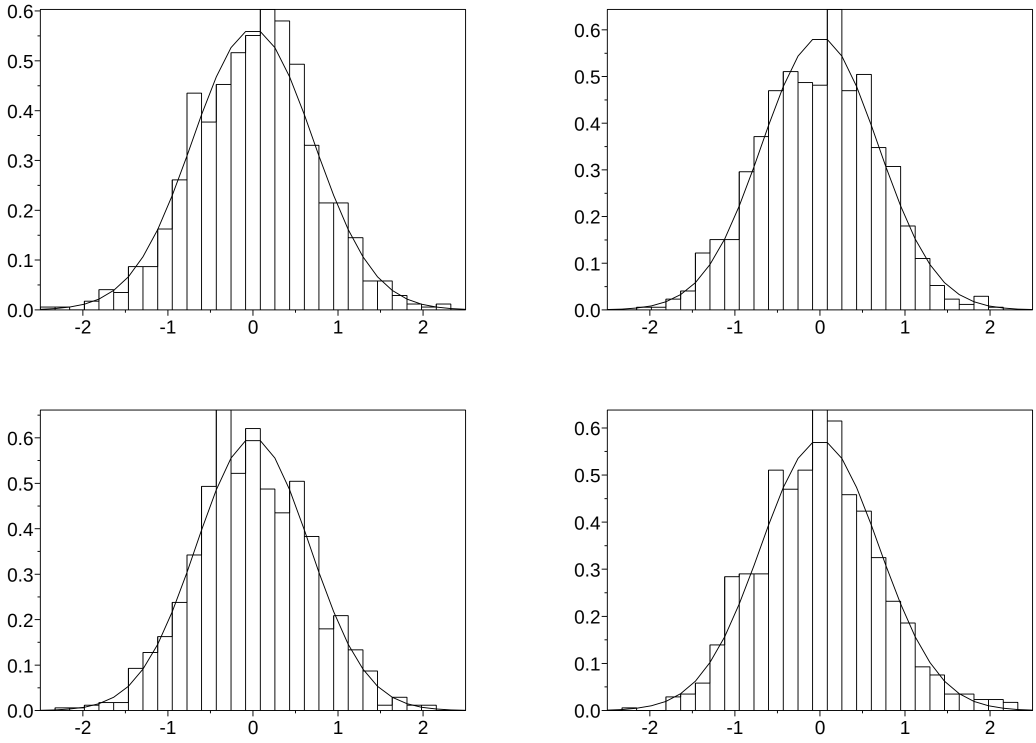

Figure 2: Convergence in distribution of the stress, $\sqrt{n}\tau_1^n$, with different n, from the left to the right, we have $n = 1000$, $n = 5000$, $n = 20000$ and $n = 100000$.

3.1 Rate of convergence

In this section, we investigate the rate of convergence in the number n of particles. We consider the first example of steady states where $\displaystyle\int_{\mathbb{R}} xp(x)dx = 0$.

It seems natural to try to check experimentally if the central limit theorem is satisfied in the number n of particles. In order to do that, we simulate $M = 1000$ independent trajectories of the process τ_1^n, with different values of n and for $K = 100$. We note them by $(\hat{\tau}_1^{j,n})_{1\le j\le M}$ and we plot the histogram of $\sqrt{n}\hat{\tau}_1^{j,n}$, $1 \le j \le M$ on the interval $[-2.5S, 2.5S]$, where S^2 is an estimate of $n\ \mathrm{var}(\tau_1^n)$, $S^2 = \dfrac{n}{M-1}\displaystyle\sum_{j=1}^{M}(\hat{\tau}_1^{j,n} - \bar{\tau}_1^n)^2$ and $\bar{\tau}_1^n = \dfrac{1}{M}\displaystyle\sum_{j=1}^{M}\hat{\tau}_1^{j,n}$. We compare this histogram with the centered Gaussian density with variance S^2. We have $n\ \mathrm{var}(\tau_1^n) = \dfrac{1}{n}\ \mathrm{var}(\displaystyle\sum_{i=1}^{n}\hat{Y}_1^{i,n})$ and table 2 shows numerical convergence of this quantity as $n \to +\infty$, despite the lack of theoretical proof. The graphical representation in figure 2 illustrates the convergence in law of the sequence $\sqrt{n}\tau_1^n$ towards the Gaussian distribution.

n	1000	5000	10000	20000
n var(τ_1^n)	0.5022943	0.4662847	0.4844257	0.4435595
n	40000	60000	80000	100000
n var(τ_1^n)	0.4628567	0.4513587	0.4543330	0.484027

Table 2: Convergence of n var(τ_1^n).

4 Conclusion

The propagation of chaos theorem proved in the present paper provides a theoretical basis to the practical simulation of the average stress which is of interest in physics. Some first tests processed on two examples of steady states are completely conclusive on the convergence and seem promising on the rate of convergence. From a theoretical point of view, the next question is now to investigate the latter subject.

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