A kinetic formulation of moving fronts and application to dislocations dynamics

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Abstract

In this article, we consider hypersurfaces moving with normal velocity depending on the time-space coordinates and on the normal to the hypersurface. We naturally define a measure associated to this hypersurface. This measure is defined on a suitable space/unit normal/curvature configuration space. We show that, while the hypersurface stays smooth, then the measure is a solution to a linear transport equation, that we call a kinetic formulation. In the particular case of curves moving in the plane, we get a simple kinetic formulation. With this kinetic formulation in hands, it is then easy to complete the models of dislocations densities that were proposed in the 60's. As a consequence, we therefore propose a closed mean field model for the dynamics of dislocations densities.

1 Introduction

1.1 Motivation

We are interested in the motion of a smooth bounded connected and oriented hypersurface $\Gamma_t \subset \mathbb{R}^N$ with the first order geometric motion given by the normal velocity

$$(1.1) \qquad V = c(t, y, n(t, y))$$

where $t \in [0, T)$ and $y \in \mathbb{R}^N$ denote respectively the time and space coordinates, and $n(t, y) \in \mathbb{S}^{N-1}$ denotes the unit normal to Γ_t at the point y (for a given choice of orientation). We denote by $K(t, y) \in \mathbb{R}^{N \times N}_{sym}$ the curvature of Γ_t at the point y. This curvature is a symmetric $N \times N$ matrix, and will be defined precisely later. For a given $T > 0$, we define

$$\Gamma = \bigcup_{t \in [0,T)} \{t\} \times \Gamma_t \quad \subset \quad [0, T) \times \mathbb{R}^n$$

To avoid any problem of regularity in this article, we will assume that

$$(1.2) \qquad \begin{cases} \Gamma \in C^3 \quad \text{and} \quad c \in C^3 \left([0, T) \times \mathbb{R}^N \times \mathbb{S}^{N-1} \right) \\ \\ \forall t \in [0, T), \quad \Gamma_t \text{ is } C^3 \text{ bounded, oriented and connected} \end{cases}$$

It is well-known that we can not expect in general existence of smooth solutions Γ_t for all time and that singularities may happen in finite time.

Our goal is to show that it is possible to provide a kinetic formulation of the motion of such fronts. Our motivation comes from the modelling of dislocations dynamics, i.e. in the dynamics of curves moving in their slip planes in crystals. Physically, it is interesting to be able to sum the evolution of several lines to deduce statistical and mean properties of this dynamics. The challenge behind this question is the possibility to describe the dynamics of densities of such curves. We refer to the work of Sedláček, Kratochvil, Werner [23], which was a source of inspiration of the present article.

In this paper, we show that our first goal is achieved, at least while the solution Γ_t stays smooth. Indeed we prove (see Theorem 4.1) that the "density"

$$g(t, y, n, K) = \delta_{\Gamma_t}(y)\delta_0(n - n(t, y))\delta_0(K - K(t, y))$$

which is a measure for $(t, y, n, K) \in [0, T) \times \mathbb{R}^N \times \mathbb{S}^{N-1} \times \mathbb{R}^{N \times N}_{sym}$, satisfies the following kinetic equation

$$g_t + \text{div } (ag) + a_0 g = 0$$

which is a linear transport equation for some function a_0 and some suitable vector field a (which is related to characteristics of Hamilton-Jacobi equations). The precise meaning and the details of these expressions will be given later (see Subsection 1.3 and Theorem 4.1). Let us mention that the vector field a has a quadratic growth at infinity, as a function of the curvature K. This naturally creates some mathematical difficulties (that will not be addressed in the present paper) to get long time existence of solutions. This is obviously related to the fact that geometrically, the curvature of Γ_t can become infinite in finite time. After our work was achieved, we were aware of the developement of results by Hochrainer, Zaiser [13] (see also Hochrainer [12] and Zaiser, Hochrainer [27]) that seem similar in the special case of dimension $N = 2$ for velocities $c(t, y)$ independent on the normal n, these results beeing based on Lie-derivative of differential forms.

Our work focuses on kinetic formulations of hypersurfaces. In the Appendix, we only give some indications for the way to get other kinetic formulations associated to the wavefront of the evolution of submanifolds of codimension higher than 1, in particular for the case of the transport of curves in $\mathbb{R}^N$.

In the particular case of curves moving in the plane, it is possible to use a simplified description. We can describe the normal $n = (\cos\theta, \sin\theta)$ by its angle $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$ and choose a scalar curvature $\kappa \in \mathbb{R}$. In this framework a kinetic formulation (see Theorem 2.1) is proposed for the "density"

$$g(t, y, \theta, \kappa) = \delta_{\Gamma_t}(y)\delta_0(\theta - \theta(t, y))\delta_0(\kappa - \kappa(t, y))$$

Finally, let us mention that our analysis do not cover the case of velocities depending on the curvature itself. This would be an interesting extension in connection with random process (see for instance Buckdahn, Cardaliaguet, Quincampoix [3], Soner, Touzi [24, 25, 26]). We plan to study this problem in a future work.

1.2   Organization of the paper

In Section 2, we present our main result in dimension $N = 2$, namely Theorem 2.1 for the simplified description. We also propose as an application, a model for the dynamics of dislocations densities (see Subsection 2.2). In Section 3, we give the proof of Theorem 2.1 and of Proposition 2.2.

For sake of completeness, we state in Section 4 our main result in any dimensions $N \geq 2$, namely Theorem 4.1. The proof is basically similar to the one of Theorem 2.1, but technically more involved. This is the reason why we chosed to present the result in general dimension *after* the result in dimension $N = 2$. The proof of Theorem 4.1 is done in Section 5. In the Appendix, for sake of completeness, we give in Subsection 6.1 the proof of Lemma 5.1, in Subsection 6.2, we give some indications about a kinetic formulation of the motion of curves in $\mathbb{R}^N$ in the case of pure transport, and in Subsection 6.3, we propose an alternative kinetic

equation for the wavefront of curves moving in the plane which is well-posed for long time existence of solutions.

1.3 Notation

For a smooth oriented hypersurface Γ_t in $\mathbb{R}^N$, we denote by $n(t, y) \in \mathbb{S}^{N-1}$ the unit normal to Γ_t at the point $y \in \Gamma_t$, and by $K(t, y) \in \mathbb{R}^{N \times N}_{sym}$ its curvature, where $\mathbb{R}^{N \times N}_{sym}$ is the set of symmetric $N \times N$ matrices. This matrix $K(t, y)$ is given by

$$K(t, y) = \sum_{i=1}^{N-1} K_i f_i \otimes f_i$$

where the K_i are the principal curvatures and the f_i are the principal directions of curvature of the surface Γ_t at the point y. Recall that the $f_i, i = 1, ..., N - 1$ generate the tangent hyperplane to Γ_t at the point y. We use here the convention that for a sphere, if the normal is pointing out of the ball, then the curvatures K_i are negative.
In dimension $N = 2$, we set

$$n = (\cos \theta, \sin \theta), \quad \tau = (\sin \theta, -\cos \theta)$$

where (τ, n) is a direct orthonormal basis. Depending on the context (but without ambiguity) we will consider either general τ, n depending on the general variable $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$ (and sometimes denoted by $\tau(\theta), n(\theta)$ to clearly specify the dependence on θ), or depending on the particular value $\theta(t, y) \in \mathbb{R}/(2\pi\mathbb{Z})$ which is the angle associated to $n(t, y)$ for $y \in \Gamma_t$. We will also define the scalar curvature $\kappa(t, y)$ by

$$K(t, y) = \kappa(t, y)\tau \otimes \tau$$

We denote by ∂_θ and ∂_κ the derivatives respectively with respect to θ and to κ. With these notation, we have in particular

$$\partial_\theta \tau = n, \quad \partial_\theta n = -\tau$$

In any dimension N, considering a direct orthonormal basis $(e_1, ..., e_N)$, and for tensors $\mathcal{T} = \sum_{i_1,...,i_T=1,...,N} \mathcal{T}_{i_1,...,i_T} e_{i_1} \otimes ... \otimes e_{i_T}$, $\mathcal{S} = \sum_{j_1,...,j_S=1,...,N} \mathcal{S}_{j_1,...,j_S} e_{j_1} \otimes ... \otimes e_{j_S}$, we set the simple contraction of tensors

$$\mathcal{T} \cdot \mathcal{S} = \sum_{i_1,...,i_{T-1}=1,...,N} \sum_{j_2,...,j_S=1,...,N} \left(\sum_{k=1,...,N} \mathcal{T}_{i_1,...,i_{T-1},k} \, \mathcal{S}_{k,j_2,...,j_S} \right) e_{i_1} \otimes ... \otimes e_{i_{T-1}} \otimes e_{j_2} \otimes ... e_{j_S}$$

and the double contraction of tensors

$$\mathcal{T} : \mathcal{S} = \sum_{i_1,...,i_{T-2}=1,...,N} \sum_{j_3,...,j_S=1,...,N} \left(\sum_{k,l=1,...,N} \mathcal{T}_{i_1,...,i_{T-2},l,k} \, \mathcal{S}_{k,l,j_3,...,j_S} \right) e_{i_1} \otimes ... \otimes e_{i_{T-2}} \otimes e_{j_3} \otimes ... e_{j_S}$$

Assuming that the tensor $\mathcal{T}$ depends on $K \in \mathbb{R}^{N \times N}_{sym}$, we define

$$\partial_K \mathcal{T} = \sum_{p,q=1,...,N} \frac{1}{2}(1 + \delta_{pq}) \sum_{i_1,...,i_T=1,...,N} \left(\frac{\partial}{\partial K_{pq}} \mathcal{T}_{i_1,...,i_T} \right) e_p \otimes e_q \otimes e_{i_1} \otimes ... \otimes e_{i_T}$$

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With this definition, we have for instance $\mathcal{T} : \partial_K K = \mathcal{T}$ if $\mathcal{T}$ is symmetric on its two last indices. Similarly, for a tensor $\mathcal{T}$ depending on $y \in \mathbb{R}^N$, we define

$$\partial_y \mathcal{T} = \sum_{p=1,\ldots,N} \sum_{i_1,\ldots,i_T=1,\ldots,N} \left(\frac{\partial}{\partial y_p} \mathcal{T}_{i_1,\ldots,i_T} \right) e_p \otimes e_{i_1} \otimes \ldots \otimes e_{i_T}$$

Similarly, if the tensor $\mathcal{T}(n)$ depends on $n \in \mathbb{S}^{N-1}$ (among other possible variables), we consider $\tilde{n}$ in a neighborhood of $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$ and then define $\widetilde{\mathcal{T}}_{i_1,\ldots,i_T}(\tilde{n}) = \mathcal{T}_{i_1,\ldots,i_T}(\tilde{n}/|\tilde{n}|)$. We set

$$\partial_n \mathcal{T}(n) = \sum_{p=1,\ldots,N} \sum_{i_1,\ldots,i_T=1,\ldots,N} \left(\frac{\partial}{\partial \tilde{n}_p} \left(\widetilde{\mathcal{T}}_{i_1,\ldots,i_T}(\tilde{n}) \right) \right)_{|\tilde{n}=n} e_p \otimes e_{i_1} \otimes \ldots \otimes e_{i_T}$$

We have in particular $n \cdot \partial_n = 0$ and $\partial_n n = I - n \otimes n$ with the identity matrix $I = \sum_{i=1,\ldots,N} e_i \otimes e_i$. We also define

$$\partial_{yy}^2 \mathcal{T} = \partial_y \left(\partial_y \mathcal{T} \right), \quad \partial_{yn}^2 \mathcal{T} = \partial_y \left(\partial_n \mathcal{T} \right), \quad \partial_{ny}^2 \mathcal{T} = \partial_n \left(\partial_y \mathcal{T} \right), \quad \partial_{nn}^2 \mathcal{T} = \partial_n \left(\partial_n \mathcal{T} \right) - n \otimes \partial_n \mathcal{T}$$

where we can check that $\partial_{nn}^2 \mathcal{T}$ is symmetric in its two first indices. We also call $\partial_t \mathcal{T}$ the tensor whose components are time derivatives of the components of $\mathcal{T}$. We set

$$D_t = \partial_t + cn \cdot \partial_y, \quad D_\tau = -(I - n \otimes n) \cdot \partial_y + K \cdot \partial_n$$

Finally in dimension $N = 2$, we keep the same notation for defining

$$D_\tau = \tau \cdot \partial_y + \kappa \partial_\theta$$

and define

$$\partial_{yy}^2 \cdot = \partial_y (\partial_y \cdot), \quad \partial_{y\theta}^2 \cdot = \partial_y (\partial_\theta \cdot), \quad \partial_{\theta y}^2 \cdot = \partial_\theta (\partial_y \cdot), \quad \partial_{\theta\theta}^2 \cdot = \partial_\theta (\partial_\theta \cdot)$$

For a function f, we also set

$$\partial_y^\perp f = -\left(\frac{\partial f}{\partial y_2} \right) e_1 + \left(\frac{\partial f}{\partial y_1} \right) e_2$$

2 Result in dimension $N = 2$

2.1 Main result

In dimension $N = 2$, let us consider a closed connected and oriented curve Γ_t for $t \in [0, T)$ for some fixed $T > 0$, with the normal pointing out the bounded set whose boundary is the curve. At a point y of the curve, we recall that we can write the unit normal $n(t, y) = (\cos \theta(t, y), \sin \theta(t, y))$ with $\theta(t, y) \in \mathbb{R}/(2\pi\mathbb{Z})$, and call $\kappa(t, y) \in \mathbb{R}$ the curvature (negative for a circle).
We set

$$\bar{c}(t, y, \theta) = c(t, y, n(\theta))$$

that up to an abuse of notation, we will continue to call it $c(t, y, \theta)$.

Then for any function $\varphi \in C_c^\infty\left([0,T) \times \mathbb{R}^2 \times (\mathbb{R}/(2\pi\mathbb{Z})) \times \mathbb{R}\right)$, we define the distribution $g_\Gamma(t,y,\theta,\kappa)$ by

$$(2.3) \qquad < g_\Gamma, \varphi >= \int_0^T dt \int_{\Gamma_t} \varphi(t,y,\theta(t,y),\kappa(t,y))$$

Given any distribution g (with compact support in the variable $\kappa \in \mathbb{R}$), we also define formally the distribution $\hat{g}(t,y,\theta)$ by

$$" \; \hat{g} := \int_{\mathbb{R}} d\kappa \; D_\tau g \; " \quad \text{with} \quad D_\tau = \tau \cdot \partial_y + \kappa \partial_\theta$$

i.e. rigorously, for any $\psi \in C_c^\infty\left([0,T) \times \mathbb{R}^2 \times (\mathbb{R}/(2\pi\mathbb{Z}))\right)$

$$(2.4) \qquad < \hat{g}, \psi >:=< D_\tau g, \overline{\psi} > \quad \text{with} \quad \overline{\psi}(t,y,\theta,\kappa) = \psi(t,y,\theta)$$

Then we have the following result

Theorem 2.1 (Equivalence geometric motion/ linear transport, $N = 2$)
Under the regularity assumption (1.2), if $(\Gamma_t)_t$ solves equation (1.1) on the time interval $[0,T)$, then the distribution $g_\Gamma(t,y,\theta,\kappa)$ defined by (2.3) solves the following equation

$$(2.5) \qquad g_t + div \; (ag) + a_0 g = 0 \quad in \quad \mathcal{D}'\left((0,T) \times \mathbb{R}^2 \times (\mathbb{R}/(2\pi\mathbb{Z})) \times \mathbb{R}\right)$$

with

$$(2.6) \qquad div \; (ag) = \partial_y \cdot (a_y g) + \partial_\theta (a_\theta g) + \partial_\kappa (a_\kappa g) \qquad for \qquad a = (a_y, a_\theta, a_\kappa)$$

and

$$(2.7) \qquad \begin{cases} a_0 = \kappa(c + \partial_{\theta\theta}^2 c) + \tau \cdot \partial_{y\theta}^2 c, \quad a_y = cn - \tau\partial_\theta c, \quad a_\theta = \tau \cdot \partial_y c, \\ a_\kappa = \kappa^2(c + \partial_{\theta\theta}^2 c) + \kappa \; (n \cdot \partial_y c + 2\tau \cdot \partial_{y\theta}^2 c) + \tau \otimes \tau : \partial_{yy}^2 c \end{cases}$$

Moreover, if g (with compact support in the variable $\kappa \in \mathbb{R}$) satisfies equation (2.5), then $\hat{g}$ defined by (2.4), satisfies the following equation

$$(2.8) \qquad \hat{g}_t + div \; (a'\hat{g}) = 0 \quad in \quad \mathcal{D}'\left((0,T) \times \mathbb{R}^2 \times (\mathbb{R}/(2\pi\mathbb{Z}))\right)$$

with

$$(2.9) \qquad div \; (a'\hat{g}) = \partial_y \cdot (a_y\hat{g}) + \partial_\theta (a_\theta\hat{g}) \qquad for \qquad a' = (a_y, a_\theta)$$

Finally, for g_Γ defined by (2.3), $\hat{g}_\Gamma$ defined in (2.4) satisfies $\hat{g}_\Gamma = 0$.

Let us note that a single planar curve is now represented as a measure on a space of dimension 4, and this measure satisfies the linear transport equation (2.5). Note also that for a curve we necessarily have $\hat{g}_\Gamma = 0$, which can be interpreted as a kind of compatibility condition. Moreover this compatibility condition is preserved by the equation on g, because $\hat{g}$ satisfies equation (2.8).
Here, general distributions g can be interpreted as the density of curves in the generalized space of space/angle/curvature coordinates. We do not know, if in some sense, any distribution g solving equation (2.5)) and satisfying $\hat{g} = 0$, can be written as a linear combination

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of measures g_Γ for a possibly infinite number of evolutions Γ.

We easily see by an integration in θ, that $\hat{g} = 0$ implies (at least formally) that

$$(2.10) \qquad \partial_y \cdot \left(\int_{\mathbb{R}/(2\pi\mathbb{Z})\times\mathbb{R}} \tau(\theta)g(t,y,\theta,\kappa)\,d\theta d\kappa \right) = 0$$

which can be interpretated as a conservation equation, namely the conservation of the Burgers vector along the dislocatons lines, in the terminology of dislocations dynamics (see Lardner [18]).

More precisely, we have the following result

Proposition 2.2 (Kinetic equation for the vectorial density)
If g is a solution of (2.5) on the time interval $(0,T)$, then the vectorial distribution τg satisfies in $\left(\mathcal{D}'\left((0,T)\times\mathbb{R}^2\times(\mathbb{R}/(2\pi\mathbb{Z}))\times\mathbb{R}\right)\right)^2$

$$(2.11) \qquad 0 = (\tau g)_t - \partial_y^\perp(cg) + (D_\tau g)a_y + \partial_\theta\left((D_\tau c)\tau g - c\kappa n g\right) + \partial_\kappa\left(a_\kappa \tau g\right)$$

Equation (2.11) joint to assumption $\hat{g} = 0$, shows in particular by an integration in θ and κ, that we have (at least formally) the following evolution equation for the reduced vectorial density

$$(2.12) \quad \frac{\partial}{\partial t}\left(\int_{\mathbb{R}/(2\pi\mathbb{Z})\times\mathbb{R}} \tau(\theta)g(t,y,\theta,\kappa)\,d\theta d\kappa \right) = \partial_y^\perp\left(\int_{\mathbb{R}/(2\pi\mathbb{Z})\times\mathbb{R}} c(t,y,\theta)g(t,y,\theta,\kappa)\,d\theta d\kappa \right)$$

which clearly preserves the divergence free property (2.10).

Let us mention the mathematical difficulty due to the fact that the vecor field a has a quadratic growth in the curvature κ, which only allows to expect short time existence of solutions to the kinetic equation in general. See the Appendix (Subsection 6.3) for a different possible kinetic equation which overcomes this difficulty, and well-describes the wavefront solution.

2.2 Application to dislocations dynamics

As a matter of application, let us give a natural model for the dynamics of dislocations densities, using our kinetic formulation.

To simplify the presentation, let us consider only one slip system in a tridimensional crystal with orthonormal basis (e_1, e_2, e_3), and with dislocations curves moving in planes perpendicular to e_3, and with Burgers vector $b \in \mathbb{R}^3$ (with $b \cdot e_3 = 0$ for mobile dislocations without climb). We assume that the density of such dislocations is represented by the quantity

$$g(t,x,\theta,\kappa) \quad \text{with} \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3, \quad \theta \in \mathbb{R}/(2\pi\mathbb{Z}), \quad \kappa \in \mathbb{R}$$

The strain $e(t,x) \in \mathbb{R}^{3\times3}_{sym}$ solves on $\mathbb{R}^3$ (see for instance Alvarez et al. [2])

$$(2.13) \quad \begin{cases} \operatorname{div}(\Lambda : e) = 0 \\[2mm] \operatorname{inc} e = \left(\operatorname{curl}_{row}(b \otimes \beta)\right)_{sym} \quad \text{with} \quad \beta(t,x) = \int_{\mathbb{R}/(2\pi\mathbb{Z})\times\mathbb{R}} d\theta d\kappa\, \tau g(t,x,\theta,\kappa) \end{cases}$$

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where $\Lambda = (\Lambda_{ijkl})_{i,j,k,l=1,2,3}$ is the fourth order tensor of elastic coefficients, and the operator inc e is obtained, taking first the curl of the column vectors of the matrix e, and then the curl of the row vectors of the new matrix. The curl_{row} is the curl of the row vectors of the matrix, and the index $(\quad)_{sym}$ means that we consider the symmetric part of the matrix. The quantity $b \otimes \beta$ is called the Nye tensor of dislocations densities. Here we set

$$(2.14) \qquad \tau = (\sin\theta)e_1 - (\cos\theta)e_2, \quad n = (\cos\theta)e_1 + (\sin\theta)e_2$$

The normal velocity of the dislocations is proportional to the resolved Peach-Koehler force up to a drag coeffient. Even if is easy (using equation (2.5)) to write the model for an anisotropic drag coefficient, for simplicity let us restrict ourself to the case where this coefficient is equal to 1. Then the velocity is simply given by

$$(2.15) \qquad c(t,x) = (b \otimes e_3) : e(t,x)$$

and for $x = (y, x_3)$ with $y = (x_1, x_2)$, for each $x_3 \in \mathbb{R}$, $g^{x_3}(t, y, \theta, \kappa) := g(t, y, x_3, \theta, \kappa)$ solves equation (2.5) with a normal velocity independent on θ defined by $c^{x_3}(t, y, \theta) = c(t, y, x_3)$, i.e.

(2.16)
$$0 = g_t^{x_3} + \kappa c^{x_3} g^{x_3} + n \cdot \partial_y(c^{x_3} g^{x_3}) + \partial_y c^{x_3} \cdot \partial_\theta(\tau g^{x_3}) + \partial_\kappa \left(g^{x_3} \left(c^{x_3}\kappa^2 + \kappa n \cdot \partial_y c^{x_3} + \tau \otimes \tau : \partial_{yy}^2 c^{x_3} \right) \right)$$

i.e.

$$(2.17) \qquad 0 = g_t + \kappa c g + n \cdot \partial_x(cg) + \partial_x c \cdot \partial_\theta(\tau g) + \partial_\kappa \left(g \left(c\kappa^2 + \kappa n \cdot \partial_x c + \tau \otimes \tau : \partial_{xx}^2 c \right) \right)$$

The complete system of equations satisfied by g is then (2.13)-(2.14)-(2.15)-(2.17), with a choice of the initial data satisfying the compatibility condition $\hat{g} = 0$ with the notation (2.4), i.e.

$$\int_{\mathbb{R}} d\kappa \ \{\tau \cdot \partial_x g + \kappa \partial_\theta g\} = 0$$

This system is a generalization of the model of Groma, Balogh [10, 11] that was restricted to the motion of straight lines dislocations with curvature $\kappa = 0$ and only two possible angles $\theta = 0$ or π. See for instance El Hajj, Forcadel [8] for a mathematical analysis of the Groma, Balogh model in a particular geometry. In equation (2.17), the term $\kappa c g$ can be interpretated as a source term created by the curvatures of the dislocations. Our model is also a natural generalization of the model of Sedláček, Kratochvil, Werner [23] whose kinetic equation was written for $\beta(t, y, x_3)$, namely

$$\beta_t = \partial_y^\perp \left(c|\beta| \right)$$

This equation has to be compared to our equation (2.12) which containes more degrees of freedom, or even to (2.11) or (2.17). Let us also underline that equation (2.17) is a natural kinetic equation that was missing for instance in the theory of Kröner [16, 17] or that was under investigation in the theories of Mura [21] or Kosevich [15].

In the case where there are several slip systems, the contribution of each slip system must be summed on the right hand side of the equation giving the inc e, and each density solves an evolution equation similar to (2.5) in its own slip plane direction with its corresponding velocity. The complete system will be studied in a future work.

Let us say that our model (2.13)-(2.14)-(2.15)-(2.17) only describes pure transport of dislocations lines in a quite rough mean field model. For instance we do not treat self-annihilation of dislocation lines, contrarily to the eikonal equation. We really describe only a kind of wavefront propagation (like in Osher et al. [22], see also the Appendix of the present article). Moreover our mean field model is really a zero-order approximation. A more realistic model would also contain some short distances corrections similar to the homogeneization problem studied in Imbert, Monneau [14]. In a more realistic model, other source or collisions terms should be added to describe Frank-Read sources, annihilations of dislocations, cross-slip, etc. See for instance the proposition of El-Azab [7, 6].

3 Proofs in dimension $N = 2$

Let us start with the following result

Lemma 3.1 *Let $\psi \in C_c^1\left((0, T) \times \mathbb{R}^2\right)$ and $(\Gamma_t)_t$ be a smooth evolution with normal velocity $c(t, y)$. Then we have*

(3.18) $$\frac{d}{dt}\left(\int_{\Gamma_t} \psi\right) = \int_{\Gamma_t} D_t\psi - c\kappa\psi \qquad for \qquad D_t\psi := \psi_t + cn \cdot \partial_y\psi$$

where ψ, n, κ and c are evaluated at the current point (t, y).

Proof of Lemma 3.1
Let us fix the time t, and consider a parametrization γ by the curvilinear abscissa s of the connected curve Γ_t, and set $\gamma(s)$ the point associated to $s \in \mathbb{R}/(L\mathbb{Z})$, with L the length of Γ_t and $\dfrac{d\gamma}{ds} = \tau$. Then we can parametrize the curve Γ_{t+h} by γ_{t+h} for h small (even if the parametrization is not yet by curvilinear abscissa)

$$\gamma_{t+h}(s) = \gamma(s) + r(h, s)n(s)$$

where $n(s)$ is the normal to Γ_t at the point $\gamma(s)$. Moreover $r(0, s) = 0$ and $r_h(0, s) = c(t, \gamma(s))$ (where r_h stands for $\dfrac{\partial r}{\partial h}$ and r_s stands for $\dfrac{\partial r}{\partial s}$). We compute

$$\int_{\Gamma_{t+h}} \psi(t+h, \cdot) = \int_{\mathbb{R}/(L\mathbb{Z})} ds \, \psi(t+h, \gamma_{t+h}(s)) \left|\frac{d}{ds}\gamma_{t+h}(s)\right|$$

$$= \int_{\mathbb{R}/(L\mathbb{Z})} ds \, \psi(t+h, \gamma(s) + r(h, s)n) \left|(1 - r(h, s)\,\kappa)\tau + r_s(h, s)\,n\right|$$

Therefore

$$\frac{d}{dh}\left(\int_{\Gamma_{t+h}} \psi(t+h, \cdot)\right)_{|h=0} = \int_{\mathbb{R}/(L\mathbb{Z})} ds \, \{\psi_t + (r_h(0, s))\, n \cdot \partial_y\psi + \psi\,(-r_h(0, s)\kappa)\}$$

$$= \int_{\Gamma_t} \psi_t + cn \cdot \partial_y\psi - c\kappa\psi$$

This ends the proof of the Lemma.

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Lemma 3.2 *For any $\varphi \in C_c^\infty\left((0,T) \times \mathbb{R}^2 \times (\mathbb{R}/(2\pi\mathbb{Z})) \times \mathbb{R}\right)$, we have for $g = g_\Gamma$*

(3.19) $\qquad < g_t, \varphi > = < g, -c\kappa\varphi + +cn \cdot \partial_y\varphi + (D_t\tilde\theta)\,\partial_\theta\varphi + (D_t\tilde\kappa)\,\partial_\kappa\varphi >$

for any C^1 extension $\tilde\theta$ (resp. $\tilde\kappa$), which restricted to Γ is equal to θ (resp. κ).

Proof of Lemma 3.2
We have

$$
\begin{aligned}
< g_t, \varphi > &= -< g, \varphi_t > \\
&= -\int_0^T dt \int_{\Gamma_t} \varphi_t \\
&= \int_0^T dt \left\{ \frac{d}{dt}\left(\int_{\Gamma_t} \varphi\right) - \int_{\Gamma_t} \varphi_t \right\}
\end{aligned}
$$

We now compute using (3.18) with $\psi(t,y) = \varphi(t, y, \tilde\theta(t,y), \tilde\kappa(t,y))$, and the velocity $c(t, y, \tilde\theta(t,y))$

$$
\begin{aligned}
\frac{d}{dt}\left(\int_{\Gamma_t} \varphi(t,y,\theta(t,y),\kappa(t,y))\right) &= \int_{\Gamma_t} \varphi_t + \tilde\theta_t\partial_\theta\varphi + \tilde\kappa_t\partial_\kappa\varphi - c\kappa\varphi \\
&\quad + \int_{\Gamma_t} cn \cdot \left\{ \partial_y\varphi + (\partial_y\tilde\theta)\partial_\theta\varphi + (\partial_y\tilde\kappa)\partial_\kappa\varphi \right\}
\end{aligned}
$$

And then

$$
< g_t, \varphi > = \int_0^T dt \int_{\Gamma_t} -c\kappa\varphi + cn \cdot \partial_y\varphi + \left\{\tilde\theta_t + cn \cdot \partial_y\tilde\theta\right\}\partial_\theta\varphi + \left\{\tilde\kappa_t + cn \cdot \partial_y\tilde\kappa\right\}\partial_\kappa\varphi
$$

which gives the result.

Lemma 3.3 *With the notation of Lemma 3.2, we have on Γ*

$$
\begin{cases}
D_t\tilde\theta &= D_\tau c \\[2mm]
D_t\tilde\kappa &= a_\kappa + (\partial_\theta c)\,(\tau \cdot \partial_y\tilde\kappa)
\end{cases}
$$

with a_κ is given in (2.7).

Proof of Lemma 3.3
Let us remark that the vector field D_t is tangent to the hypersurface Γ. This means that $D_t\tilde\theta$ and $D_t\tilde\kappa$ are intrinsec quantities only depending respectively on the values of θ and κ on the hypersurface Γ. For this reason, it is possible to compute these quantities, considering a particular parametrization of the hypersurface Γ.
Let us consider $(t_0, y_0) \in \Gamma$. Up to a translation and rotations of the coordinates, we can assume that $t_0 = 0 = y_0$ and consider a local representation of Γ as

$$
z = u(t,x) \quad \text{for} \quad y = (x,z) \quad \text{with} \quad u_x(0,0) = 0
$$

where u_x denotes $\dfrac{\partial u}{\partial x}$ and the normal at $(0,0)$ is $(0,1)$ in the (x,y) coordinates. We set $u_{xx} = \dfrac{\partial^2 u}{\partial x^2}$, $u_{xt} = \dfrac{\partial^2 u}{\partial x \partial t}$, and $u_{xxt} = \dfrac{\partial^3 u}{\partial x^2 \partial t}$. In these coordinates, we have

$$
\theta(t,x,u) = \arctan u_x, \quad \kappa(t,x,u) = \frac{u_{xx}}{(1+u_x^2)^{\frac{3}{2}}} \quad \text{and} \quad n(t,x,u) = \frac{1}{\sqrt{1+u_x^2}}(-u_x, 1) \in \mathbb{R}^2
$$

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Let us recall that locally in a neighborhood of $(0,0)$, the function u satisfies

$$(3.20) \qquad u_t = c(t, x, u, \arctan u_x) \sqrt{1 + u_x^2}$$

For t in a neighborhood of zero, let us define the curve γ contained in Γ by:

$$\gamma(t) = (t, 0, u(t, 0))$$

for which we have

$$\frac{d\gamma}{dt}(0) = (1, 0, u_t(0, 0)) = (1, 0, c(0, 0, 0))$$

which is exactly the vector field D_t evaluated at the origin, because we assume that u_x vanishes at the origin. Therefore

$$D_t \tilde{\theta}(0, 0, 0) = \frac{d}{dt}(\tilde{\theta} \circ \gamma)(0) = \frac{d}{dt} \left(\arctan u_x(t, 0) \right)_{|t=0} = u_{xt}(0, 0)$$

Similarly

$$D_t \tilde{\kappa}(0, 0, 0) = \frac{d}{dt}(\tilde{\kappa} \circ \gamma)(0) = \frac{d}{dt} \left(\frac{u_{xx}(t, 0)}{(1 + u_x^2(t, 0))^{\frac{3}{2}}} \right)_{|t=0} = u_{xxt}(0, 0)$$

Derivating (3.20) with respect to x, we get

$$(3.21) \qquad u_{xt} = (1, u_x) \cdot \partial_y c \sqrt{1 + u_x^2} + \partial_\theta c \, \frac{u_{xx}}{\sqrt{1 + u_x^2}} + c \frac{u_{xx} u_x}{\sqrt{1 + u_x^2}}$$

which implies that at the origin we have

$$D_t \tilde{\theta}(0, 0, 0) = u_{xt}(0, 0) = \tau \cdot \partial_y c + \kappa \partial_\theta c = D_\tau c$$

Derivating now (3.21) with respect to x, we get at the origin

$$u_{xxt}(0, 0) = (0, u_{xx}) \cdot \partial_y c + (1, 0) \otimes (1, 0) : \partial_{yy}^2 c + 2(1, 0) \cdot \partial_{y\theta}^2 c \, u_{xx} + \partial_{\theta\theta}^2 c \, u_{xx}^2 + \partial_\theta c \, u_{xxx} + c u_{xx}^2$$

$$= \kappa n \cdot \partial_y c + \tau \otimes \tau : \partial_{yy}^2 c + 2\kappa \tau \cdot \partial_{y\theta}^2 c + \kappa^2 \partial_{\theta\theta}^2 c + c\kappa^2 + \partial_\theta c \, u_{xxx}$$

$$= a_\kappa + \partial_\theta c \, \frac{\partial}{\partial x} \left(\frac{u_{xx}}{(1 + u_x^2)^{\frac{3}{2}}} \right)$$

with a_κ given in (2.7). This shows that

$$D_t \tilde{\kappa}(0, 0, 0) = u_{xxt}(0, 0) = a_\kappa + (\partial_\theta c)(\tau \cdot \partial_y \tilde{\kappa})$$

This ends the proof of the Lemma.

Lemma 3.4 *With the notation of Lemma 3.2, we have*

$$-\int_{\Gamma_t} (\partial_\theta c)(\tau \cdot \partial_y \tilde{\kappa}) \, \partial_\kappa \varphi = \int_{\Gamma_t} D_\tau(\varphi \, \partial_\theta c) \quad with \quad D_\tau = \tau \cdot \partial_y + \kappa \partial_\theta$$

where the quantities in the integrals are evaluated at (t, y), $\theta(t, y)$, $\kappa(t, y)$.

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Proof of Lemma 3.4

Let us consider a parametrization γ by the curvilinear abscissa s of the connected connected curve Γ_t, and set $\gamma(s)$ the point associated to $s \in \mathbb{R}/(L\mathbb{Z})$, with L the length of Γ_t and $\dfrac{d\gamma}{ds} = \tau$. Then we have $\tau \cdot \partial_y \tilde{\kappa} = \dfrac{d\kappa}{ds}(s)$. With an obvious abuse of notation, we denote by $\kappa(s)$, $\theta(s)$, respectively the curvature and the angle associated to s, i.e. $\kappa(s) = \kappa(t, \gamma(s))$, $\theta(s) = \theta(t, \gamma(s))$. For a general C^1 function $G(y, \theta, \kappa)$, we have

(3.22)
$$\frac{d}{ds}\left(G(\gamma(s), \theta(s), \kappa(s))\right) = \left\{\tau \cdot \partial_y + \kappa\partial_\theta + \frac{d\kappa}{ds}\partial_\kappa\right\}G$$

where we have in particular used the fact that $\dfrac{d\theta}{ds} = \kappa$ (i.e. $\dfrac{d\tau}{ds} = \kappa n$). We deduce that

$$
\int_{\Gamma_t} (\partial_\theta c)\,\frac{d\kappa}{ds}\,\partial_\kappa\varphi \;=\; \int_{\mathbb{R}/(L\mathbb{Z})} ds\,(\partial_\theta c)\left\{\frac{d\kappa}{ds}\,\partial_\kappa\varphi + \kappa\partial_\theta\varphi + \tau \cdot \partial_y\varphi - D_\tau\varphi\right\}
$$

$$
= \int_{\mathbb{R}/(L\mathbb{Z})} ds\,(\partial_\theta c)\,\frac{d\varphi}{ds} - \int_{\mathbb{R}/(L\mathbb{Z})} ds\,(\partial_\theta c)\,D_\tau\varphi
$$

$$
= -\int_{\mathbb{R}/(L\mathbb{Z})} ds\left\{\left(\frac{d}{ds}(\partial_\theta c)\right)\varphi + (\partial_\theta c)\,D_\tau\varphi\right\}
$$

$$
= -\int_{\mathbb{R}/(L\mathbb{Z})} ds\left\{(D_\tau(\partial_\theta c))\,\varphi + (\partial_\theta c)\,D_\tau\varphi\right\}
$$

$$
= -\int_{\Gamma_t} D_\tau\,(\varphi\partial_\theta c)
$$

where we have used (3.22) with $G = \varphi$ at the second line, we have made an integration by parts at the third line, and used (3.22) with $G = \partial_\theta c$ at the fourth line. This ends the proof of the Lemma.

Lemma 3.5 *For a general g solution of (2.5), we have*

(3.23)
$$(D_\tau g)_t + \partial_y \cdot (a_y D_\tau g) + \partial_\theta(a_\theta D_\tau g) + \partial_\kappa(D_\tau(a_\kappa g)) = 0$$

Proof of Lemma 3.5

For two vector fields V_1, V_2 and g a given distribution, we recall the definition of brackets $[V_1, V_2]$:
$$[V_1, V_2]g = V_1(V_2 g) - V_2(V_1 g)$$

We first compute the following brackets of vector fields for general g:

$$
\begin{cases}
[D_\tau, \partial_y \cdot (a_y\cdot)]\,g = (D_\tau(\partial_y \cdot a_y))g + (D_\tau a_y) \cdot \partial_y g + a_y \cdot [D_\tau, \partial_y]\,g \\[2mm]
[D_\tau, \partial_\theta\,(a_\theta\cdot)]\,g = (D_\tau(\partial_\theta a_\theta))g + (D_\tau a_\theta)\partial_\theta g + a_\theta\,[D_\tau, \partial_\theta]\,g
\end{cases}
$$

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and

$$\begin{cases} [D_\tau, \partial_y] = 0 \\[2mm] [D_\tau, \partial_\theta] = -n \cdot \partial_y \\[2mm] [D_\tau, \partial_\kappa] = -\partial_\theta \end{cases}$$

Applying the vector field D_τ to the equation

$$-g_t = a_0 g + \partial_y \cdot (a_y g) + \partial_\theta (a_\theta g) + \partial_\kappa (a_\kappa g)$$

we get

$$\begin{aligned} -(D_\tau g)_t &= D_\tau (a_0 g) + \partial_y \cdot (a_y D_\tau g) + \partial_\theta (a_\theta D_\tau g) + \partial_\kappa (D_\tau (a_\kappa g)) \\ &\quad + [D_\tau, \partial_y \cdot (a_y \cdot)] \, g + [D_\tau, \partial_\theta (a_\theta \cdot)] \, g + [D_\tau, \partial_\kappa] \, (a_\kappa g) \end{aligned}$$

i.e.

$$\begin{aligned} &-(D_\tau g)_t - \partial_y \cdot (a_y D_\tau g) - \partial_\theta (a_\theta D_\tau g) - \partial_\kappa (D_\tau (a_\kappa g)) \\[2mm] &= D_\tau (a_0 g) + [D_\tau, \partial_y \cdot (a_y \cdot)] \, g + [D_\tau, \partial_\theta (a_\theta \cdot)] \, g + [D_\tau, \partial_\kappa] \, (a_\kappa g) \\[4mm] &= (D_\tau a_0) g + a_0 D_\tau g \\ &\quad + (D_\tau (\partial_y \cdot a_y)) g + (D_\tau a_y) \cdot \partial_y g \\ &\quad + (D_\tau (\partial_\theta a_\theta)) g + (D_\tau a_\theta) \partial_\theta g - a_\theta \, n \cdot \partial_y g \\ &\quad - \partial_\theta (a_\kappa g) \\[4mm] &= (D_\tau a_0 + D_\tau (\partial_y \cdot a_y) + D_\tau (\partial_\theta a_\theta) - \partial_\theta a_\kappa) g \\ &\quad + (a_0 \tau + D_\tau a_y - a_\theta \, n) \cdot \partial_y g \\ &\quad + (a_0 \kappa + D_\tau a_\theta - a_\kappa) \, \partial_\theta g \\[4mm] &= 0 \end{aligned}$$

because, on the other hand, we compute

$$\begin{cases} D_\tau a_\theta = a_\kappa - a_0 \kappa \\ D_\tau a_y = a_\theta \, n - a_0 \tau \\ D_\tau a_0 + D_\tau (\partial_y \cdot a_y) + D_\tau (\partial_\theta a_\theta) - \partial_\theta a_\kappa = 0 \end{cases}$$

where the last line is a consequence of the following computations

$$\begin{cases} \partial_y \cdot a_y + \partial_\theta a_\theta = 2n \cdot \partial_y c \\ D_\tau (\partial_y \cdot a_y + \partial_\theta a_\theta) = -2 \left(\kappa \tau \cdot \partial_y c - \kappa n \cdot \partial^2_{y\theta} c - \partial^2_{yy} c \cdot (\tau, n) \right) \\ D_\tau a_0 - \partial_\theta a_\kappa = \tau \cdot \partial_y a_0 - \partial_\theta (D_\tau a_\theta) = 2 \left(\kappa \tau \cdot \partial_y c - \kappa n \cdot \partial^2_{y\theta} c - \partial^2_{yy} c \cdot (\tau, n) \right) \end{cases}$$

This ends the proof of the Lemma.

Proof of Theorem 2.1

Putting together the results of Lemmata 3.2, 3.3 and 3.4, we get for $g = g_\Gamma$

$$\begin{aligned} < g_t, \varphi > &= < g, -c\kappa\varphi + +cn \cdot \partial_y \varphi + (D_\tau c) \, \partial_\theta \varphi + a_\kappa \, \partial_\kappa \varphi + (\partial_\theta c)(\tau \cdot \partial_y \tilde{\kappa}) \, \partial_\kappa \varphi > \\[2mm] &= < g, -c\kappa\varphi + +cn \cdot \partial_y \varphi + (D_\tau c) \, \partial_\theta \varphi + a_\kappa \, \partial_\kappa \varphi - \partial_\theta c \, D_\tau \varphi - \varphi D_\tau (\partial_\theta c) > \\[2mm] &= < g, -a_0 \varphi + a_y \partial_y \varphi + a_\theta \partial_\theta \varphi + a_\kappa \partial_\kappa \varphi > \end{aligned}$$

which leads to equation (2.5).

Integrating equation (3.23) with respect to κ, leads to (2.8).

Finally, let us compute for g_Γ defined in (2.3) and $\overline{\psi}(t, y, \theta, \kappa) = \psi(t, y, \theta)$

$$
\begin{aligned}
< \hat{g}_\Gamma, \psi > &= < D_\tau g, \overline{\psi} > \\[2mm]
&= - < g_\Gamma, D_\tau \overline{\psi} > \\[2mm]
&= - < g_\Gamma, \tau \cdot \partial_y \overline{\psi} + \kappa \partial_\theta \overline{\psi} + \frac{d\kappa}{ds} \partial_\kappa \overline{\psi} > \\[2mm]
&= - \int_0^T dt \int_{\Gamma_t} \tau \cdot \partial_y \tilde{\psi}(t, y) \\[2mm]
&= 0
\end{aligned}
$$

where at the thord line, we have used the fact that $\overline{\psi}$ is independent on κ, and we have used (3.22) with $G = \overline{\psi}$ at the fourth line. Finally we have set $\tilde{\psi}(t, y) = \overline{\psi}(t, y, \theta(t, y), \kappa(t, y))$. This proves that $\hat{g}_\Gamma = 0$. This ends the proof of the Theorem.

Proof of Propositon 2.2

Let us start to multiply by τ the equation (2.5) satisfied by g. We get

$$
0 = (\tau g)_t + a_0 \tau g + \partial_y \cdot (a_y \otimes \tau g) + \partial_\theta (a_\theta \tau g) - a_\theta n g + \partial_\kappa (a_\kappa \tau g)
$$

Let us compute

$$
\begin{aligned}
&a_0 \tau g + \partial_y \cdot (a_y \otimes \tau g) - a_\theta n g + \partial_\theta (c\kappa n g) \\[2mm]
=\; &\partial_y \cdot (a_y \otimes \tau g) - a_\theta n g + n \partial_\theta (c\kappa g) + \kappa (\partial_{\theta\theta}^2 c) \tau g + (\tau \cdot \partial_{y\theta}^2 c) \tau g \\[2mm]
=\; &(n \cdot \partial_y)(c \tau g) - (\tau \cdot \partial_y)((\partial_\theta c) \tau g) - (\tau \cdot \partial_y c) n g \\
&+ n c \kappa \partial_\theta g + n \kappa (\partial_\theta c) g + \{\partial_\theta (\kappa(\partial_\theta c) \tau g) - \kappa(\partial_\theta c) n g - \kappa(\partial_\theta c) \tau \partial_\theta g\} + (\tau \cdot \partial_{y\theta}^2 c) \tau g \\[2mm]
=\; &n c D_\tau g - n c \tau \cdot \partial_y g - (\tau \cdot \partial_y c) n g + (n \cdot \partial_y)(c \tau g) - (\tau \cdot \partial_y)((\partial_\theta c) \tau g) \\
&+ \partial_\theta (\kappa(\partial_\theta c) \tau g) - (\partial_\theta c) \tau D_\tau g + (\partial_\theta c) \tau (\tau . \partial_y g) + (\tau \cdot \partial_{y\theta}^2 c) \tau g \\[2mm]
=\; &\partial_\theta (\kappa(\partial_\theta c) \tau g) + (D_\tau g)(c n - \tau \partial_\theta c) - n(\tau \cdot \partial_y)(c g) + \tau(n \cdot \partial_y)(c g) \\[2mm]
=\; &\partial_\theta (\kappa(\partial_\theta c) \tau g) + (D_\tau g)(c n - \tau \partial_\theta c) - \partial_y^\perp(c g)
\end{aligned}
$$

where we have used the explicit expression of a_0 in the second line, the explicit expressions of a_y and a_θ in the third line, and we have introduced the expression of $D_\tau g = \tau \cdot \partial_y g + \kappa \partial_\theta g$ in the fourth line. Therefore we get

$$
0 = (\tau g)_t - \partial_y^\perp(c g) + (D_\tau g) a_y + \partial_\theta ((a_\theta + \kappa \partial_\theta c) \tau g - c \kappa n g) + \partial_\kappa (a_\kappa \tau g)
$$

This ends the proof of the Proposition.

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4   Result in dimension $N \geq 2$

Let us now trurn back to our main result in dimension $N \geq 2$. We use the notation of subsection 1.3. We consider a closed connected and oriented hypersurface Γ_t for $t \in [0, T)$ for some fixed $T > 0$, with the normal pointing out the bounded set whose boundary is the hypersurface. At a point y of Γ_t, we recall that we call $n(t, y)$ the unit normal and $K(t, y) \in \mathbb{R}^{N \times N}_{sym}$ the curvature matrix (negative for a ball).
We define for $(t, y, n, K) \in [0, T) \times \mathbb{R}^N \times \mathbb{S}^{N-1} \times \mathbb{R}^{N \times N}_{sym}$, the measure $g_\Gamma(t, y, n, K)$ by

$$(4.24) \qquad < g_\Gamma, \varphi > = \int_0^T dt \int_{\Gamma_t} \varphi(t, y, n(t, y), K(t, y))$$

for any test function $\varphi \in C_c^\infty \left([0, T) \times \mathbb{R}^N \times \mathbb{S}^{N-1} \times \mathbb{R}^{N \times N}_{sym} \right)$. Given any distribution g (with compact support in the variable $K \in \mathbb{R}^{N \times N}_{sym}$), we also define formally the distribution $\hat{g}(t, y, n)$ by

$$" \ \hat{g} := \int_{\mathbb{R}^{N \times N}_{sym}} dK \ \{D_\tau g + n(I : K)g\} \ " \quad \text{with} \quad D_\tau = - (I - n \otimes n) \cdot \partial_y + K \cdot \partial_n$$

i.e. rigorously, for any $\psi \in C_c^\infty \left([0, T) \times \mathbb{R}^N \times \mathbb{S}^{N-1} \right)$

$$(4.25) \qquad < \hat{g}, \psi > = < D_\tau g + n(I : K)g, \overline{\psi} > \quad \text{with} \quad \overline{\psi}(t, y, n, K) = \psi(t, y, n)$$

Then we have

Theorem 4.1 (Equivalence geometric motion/ linear transport, $N \geq 2$)
Under the regularity assumption (1.2), if $(\Gamma_t)_t$ solves equation (1.1) on the time interval $[0, T)$, then the distribution $g_\Gamma(t, y, n, K)$ defined by (4.24) solves the following equation

$$(4.26) \qquad g_t + div \ (ag) + a_0 g = 0 \quad in \quad \mathcal{D}' \left((0, T) \times \mathbb{R}^N \times \mathbb{S}^{N-1} \times \mathbb{R}^{N \times N}_{sym} \right)$$

with

$$(4.27) \qquad div \ (ag) = \partial_y \cdot (a_y g) + \partial_n \cdot (a_n g) + \partial_K : (a_K g) \qquad for \qquad a = (a_y, a_n, a_K)$$

and
$$(4.28)$$
$$\begin{cases} a_0 = c(I : K) + K : \partial^2_{nn} c - I : \partial^2_{ny} c, \quad a_y = cn + \partial_n c, \quad a_n = -(I - n \otimes n) \cdot \partial_y c, \\[2mm] a_K \ = \ cK^2 + K \cdot \partial^2_{nn} c \cdot K + (I - n \otimes n) \cdot \partial^2_{yy} c \cdot (I - n \otimes n) \\ \qquad \quad + (n \cdot \partial_y c) \ K - K \cdot \partial^2_{ny} c \cdot (I - n \otimes n) - (I - n \otimes n) \cdot \partial^2_{yn} c \cdot K \\ \qquad \quad + K \cdot \partial_y c \otimes n + n \otimes K \cdot \partial_y c \end{cases}$$

Moreover we have for $\alpha := K \cdot n \ g$

$$(4.29) \qquad \alpha_t + div \ (a \otimes \alpha) + A_0 \cdot \alpha = 0 \quad in \quad \left(\mathcal{D}' \left((0, T) \times \mathbb{R}^N \times \mathbb{S}^{N-1} \times \mathbb{R}^{N \times N}_{sym} \right) \right)^N$$

with

$$(4.30) \qquad -A_0 = cK + K \cdot \partial_{nn} c - (I - n \otimes n) \cdot \partial^2_{yn} c + n \otimes \partial_y c + (2 (n \cdot \partial_y c) - a_0) I$$

Finally, for g_Γ defined by (4.24) we have $K \cdot n \ g_\Gamma = 0$ and $\hat{g}_\Gamma$ defined in (4.25) satisfies $\hat{g}_\Gamma = 0$.

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Let now make a few comments on this Theorem.

The invariant manifold.
We first remark that in the expression (4.28) of a_K, the two last terms are new, in comparison to (2.7). Moreover these terms are the only terms not perpandicular to n. Their existence is due to our choice of writting the equations on $\mathbb{R}^N \times \mathbb{S}^{N-1} \times \mathbb{R}_{sym}^{N \times N}$, in order to preserve $K \cdot n = 0$ on the support of g for all time, if it is true at the initial time (see the justification of equation (4.29)). This really means that we are interested in measures with support on the natural manifold

$$\mathcal{M} = \left\{(y, n, K) \in \mathbb{R}^N \times \mathbb{S}^{N-1} \times \mathbb{R}_{sym}^{N \times N}, \quad K \cdot n = 0\right\}$$

whose the dimension is the same as $\mathbb{R}^N \times \mathbb{S}^{N-1} \times \mathbb{R}_{sym}^{(N-1) \times (N-1)}$. This is obviously related to the fact (easy to check) that for any $t \in [0, T)$, the vector field

$$a^t(X) = a(t, X) \quad \text{with} \quad X = (y, n, K)$$

is tangent to the manifold $\mathcal{M}$. This means that it should be possible to represent (but probably less simple to write) the kinetic equation as some transport equation on the manifold $\mathcal{M}$, similar to equation (4.26). This also means that, while we keep our description on the space $\mathbb{R}^N \times \mathbb{S}^{N-1} \times \mathbb{R}_{sym}^{N \times N}$, there are several equivalent kinetic formulations, because for what we have in mind, in (4.28), K can be replaced by $K \cdot (I - n \otimes n)$, or $(I - n \otimes n) \cdot K$, or even $(I - n \otimes n) \cdot K \cdot (I - n \otimes n)$.

Explicit solution based on the characteristics
It is known (see for instance Leveque [20]) that that the solution g of (4.26) is given (at least formally with the measure $g(0, \cdot)$ at the initial time) for $X = (y, n, K)$ by

$$g(t, Y) = g(0, X(0; t, Y))e^{-\int_0^t b_0(s;t,Y) \ ds} \quad \text{with} \quad b_0(s; t, Y) = (a_0 + \text{div } a)(s, X(s; t, Y))$$

and

$$\frac{d}{ds}X(s; t, Y) = a(s, X(s; t, Y)), \quad X(t; t, Y) = Y$$

In particular, defining for $t \geq 0$

$$\hat{\Gamma}_t = \left\{ \begin{array}{l} (y, n(t, y), K(t, y)) \in \mathbb{R}^N \times \mathbb{S}^{N-1} \times \mathbb{R}_{sym}^{N \times N}, \\[2mm] \text{for } y \in \Gamma_t \text{ with normal } n(t, y) \text{ and curvature } K(t, y) \end{array} \right\}$$

we have

$$\text{supp } g_\Gamma(t, \cdot) = \hat{\Gamma}_t$$

Therefore, if $Y \in \hat{\Gamma}_0$, then $X(t; 0, Y) \in \hat{\Gamma}_t$, which shows (at least formally) that the solutions $X(t; 0, Y)$ are the characteristics of the evolution. In particular, we get, as a result, the value of the *evolution of the curvature*, which has to be put in relation with curvature estimates in Cannarsa, Frankowska [5], Cannarsa, Cardaliaguet [4] or Alvarez, Cardaliaguet, Monneau [1]. Moreover, at least the first two components $(cn + \partial_n c, -(I - n \otimes n) \cdot \partial_y c)$ of this vector field are similar to the characteristics of classical Hamilton-Jacobi equations with Hamiltonian $c(t, y, n)$, see Evans [9]).

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The mathematical difficulty for long time existence.
The principal mathematical difficulty to solve equation (4.26), is the quadratic growth of the vector field a as a function of the curvature K. This means that, even for initial data with compact support, the support of the solution can go to infinity in K in finite time (which corresponds to the apparition of geometric singularities of the fronts, revealed by infinite curvature). On the contrary, in the particular case where the hypersurface is transported by a vector field $V(t, y)$, which means that the normal velocity is given by $c(t, y, n) = n \cdot V(t, y)$, then we can check that the vector field a is at most linear in the curvature K, and $a_y = V$ is exactly the original vector field. This is natural, because its is well-known that linear transport equations do not create singularities in finite time. See also the Appendix (Subsection 6.3 and Remark 6.3) for indications to overcome these difficulties using a different kinetic formulation.

We do not know if there are conservative quantities (other than $K \cdot ng$, like maybe $\hat{g}$) that can be derived from this kinetic formulation of moving fronts. We do not know neither what is the best regularity that we can assume on g, in order to satisfy the natural constraints like the conserved quantities.

5   Proofs in dimension $N \geq 2$

Let us start with the following result which, for sake of completeness, is proved in the Appendix (Section 6).

Lemma 5.1 *Let $\psi \in C_c^1\left((0, T) \times \mathbb{R}^N\right)$ and $(\Gamma_t)_t$ be a smooth evolution with normal velocity $c(t, y)$. Then we have*

$$(5.31) \qquad \frac{d}{dt}\left(\int_{\Gamma_t} \psi\right) = \int_{\Gamma_t} D_t\psi - c(I : K)\psi \qquad for \qquad D_t\psi := \psi_t + cn \cdot \partial_y\psi$$

where ψ, n, K and c are evaluated at the current point (t, y), and $I : K$ denotes the trace of the matrix K.

Then, we have the following result

Lemma 5.2 *For any $\varphi \in C_c^\infty\left((0, T) \times \mathbb{R}^N \times \mathbb{S}^{N-1} \times \mathbb{R}_{sym}^{N \times N}\right)$, we have for $g = g_\Gamma$*

$$(5.32) \qquad < g_t, \varphi > = < g, -c(I : K)\varphi + + cn \cdot \partial_y\varphi + (D_t\tilde{n}) \cdot \partial_n\varphi + (D_t\tilde{K}) : \partial_K\varphi >$$

for any C^1 extension $\tilde{n}$ (resp. $\tilde{K}$), which restricted to Γ is equal to n (resp. K).

Proof of Lemma 5.2
The proof is similar to the proof of Lemma 3.2, with (3.18) replaced by (5.31). This ends the proof of the Lemma.

Lemma 5.3 *Let us consider a local parametrization of the hypersurface Γ as*

$$z = u(t, x) \quad for \quad y = (x, z) \in U \subset \mathbb{R}^{N-1} \times \mathbb{R}$$

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where U is an open set. Then we have with $u_x = \dfrac{\partial u}{\partial x} \in \mathbb{R}^{N-1}$ and $u_{xx} = \dfrac{\partial^2 u}{\partial x^2} \in \mathbb{R}^{(N-1)\times(N-1)}_{sym}$

$$n(t,x,u) = \frac{1}{\sqrt{1+u_x^2}}(-u_x, 1) \in \mathbb{R}^{N-1} \times \mathbb{R} \quad and \quad K(t,x,u) = F(u_{xx}, u_x) \in \mathbb{R}^{N\times N}_{sym}$$

with

$$F(M,p) = \frac{1}{\sqrt{1+p^2}} \left\{ \begin{array}{l} \mathcal{I}(M) + \dfrac{p\cdot M \cdot p}{(1+p^2)^2}\,(-p,1)\otimes(-p,1) \\[2mm] + \left(\dfrac{M\cdot p}{1+p^2}, 0\right)\otimes(-p,1) + (-p,1)\otimes\left(\dfrac{M\cdot p}{1+p^2}, 0\right) \end{array} \right\}$$

where

$$(\mathcal{I}(M))_{ij} = \left\{ \begin{array}{ll} M_{ij} & if \quad i,j \in \{1,...,n-1\} \\ 0 & otherwise \end{array} \right.$$

Proof of Lemma 5.3

The only thing to prove is the expression of the curvature.

In the case where $n = (0,1) \in \mathbb{R}^{N-1}\times\mathbb{R}$, i.e. $u_x = 0$, it is clear that we have $F(M,0) = \mathcal{I}(M)$. In the general case where $n \neq (0,1)$, i.e. $u_x \neq 0$, we need to make a rotation of the coordinates in the plane generated by n and $(0,1)$, such that in the new coordinates the surface Γ is represented by $Z = v(t,X)$ with $v_X = 0$. Then a full computation (based on the inverse function theorem) is possible.

Let us now make the details of the computation.

Let us drop the time coordinate which does not play any role in the present computation. Up to a translation we can assume that we work close to the origin where $u_x(0) \neq 0$. Let us choose an orthonormal basis $(e_1, ..., e_{N-1})$ such that

$$e_1 = \frac{u_x}{|u_x|}(0)$$

Let us set

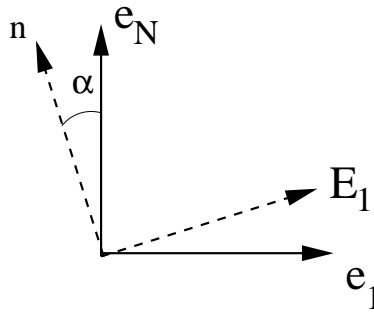$$x' = (x_2, ..., x_{N-1}) = X'$$

Figure 1: New coordinates by rotation

We will consider the unit vector E_1 such that orthonormal basis $(E_1, n(0))$ of the plane is obtained by a rotation of the basis (e_1, e_N) (see Fig. 1), and define the new coordinates X_1, X_N such that

$$X_1 E_1 + X_N n(0) = x_1 e_1 + x_N e_N \quad and \quad X = (X_1, X', X_N)$$

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i.e.
$$X_1 = x_1 \, \cos \alpha + x_N \, \sin \alpha, \quad X_N = -x_1 \, \sin \alpha + x_N \, \cos \alpha$$

for
$$\alpha = \text{angle} \ (e_N, n(0)) \in (0, \pi/2), \quad \text{i.e.} \quad \cos \alpha = \frac{1}{\sqrt{1 + u_1^2}}, \quad \sin \alpha = \frac{u_1}{\sqrt{1 + u_1^2}}$$

with $u_1 = \dfrac{\partial u}{\partial x_1}$. We see that

$$x_N = u(x_1, x') \iff X_N = v(X_1, X')$$

for some new function v which satisfies

$$v \, (x_1 \, \cos \alpha + u(x_1, x') \, \sin \alpha, x') = -x_1 \, \sin \alpha + u(x_1, x') \, \cos \alpha$$

We compute by derivation

$$\begin{cases} v_1(\cos \alpha + u_1 \sin \alpha) = -\sin \alpha + u_1 \cos \alpha \\[2mm] v_1 u_i \sin \alpha + v_i = u_i \cos \alpha, \quad i = 2, ..., N-1 \end{cases}$$

In particular we have $v_X(0) = 0$. Derivating once again, we get at the origin

$$\begin{cases} v_{11} = \dfrac{u_{11}}{(1 + u_1^2)^{\frac{3}{2}}} \\[2mm] v_{1i} = \dfrac{u_{1i}}{1 + u_1^2}, \quad i = 2, ..., N-1 \\[2mm] v_{ij} = \dfrac{u_{ij}}{(1 + u_1^2)^{\frac{1}{2}}}, \quad i, j = 2, ..., N-1 \end{cases}$$

Then (with $E_1 = \cos \alpha \, e_1 + \sin \alpha \, e_N$, $E_1' = \cos \alpha E_1$)

$$K(1 + u_1^2)^{\frac{1}{2}} = v_{11} E_1 \otimes E_1 + \sum_{i=2,...,N-1} v_{1i} \left(E_1 \otimes e_i + e_i \otimes E_1\right) + \sum_{i,j=2,...,N-1} v_{ij} e_i \otimes e_j$$

$$= \sum_{i,j=1,...,N-1} u_{ij} e_i \otimes e_j + \sum_{i=1,...,N-1} u_{1i} \left((E_1' - e_1) \otimes e_i + e_i \otimes (E_1' - e_1)\right)$$
$$+ u_{11} (E_1' - e_1) \otimes (E_1' - e_1)$$

Using the fact that
$$E_1' - e_1 = \frac{u_1}{1 + u_1^2} \left(-u_1 e_1 + e_N\right))$$

we see that we exactly get
$$K = F(u_{xx}, u_1 e_1)$$

which ends the proof of the Lemma.

Lemma 5.4 *With the notation of Lemma 5.2, we have on* Γ

$$\begin{cases} D_t \tilde{n} = -(I - n \otimes n) \cdot \partial_y c + K \cdot \partial_n c =: D_\tau c \\[3mm] D_t \tilde{K} = a_K - \partial_n c \cdot \partial_y \tilde{K} \end{cases}$$

with a_K *given in (4.28), and on* Γ

(5.33)
$$(I - n \otimes n) \cdot \partial_y \tilde{n} = -K$$

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Proof of Lemma 5.4

We procceed exactly as in the proof of Lemma 3.3.

Let us consider $(t_0, y_0) \in \Gamma$. Up to a translation and rotations of the coordinates, we can assume that $t_0 = 0 = y_0$ and consider a local representation of Γ as

$$z = u(t, x) \quad \text{for} \quad y = (x, z) \quad \text{with} \quad u_x(0, 0) = 0$$

where u_x denotes $\dfrac{\partial u}{\partial x}$ and the normal at $(0, 0)$ is $(0, 1)$ in the (x, y) coordinates. We set $u_{xx} = \dfrac{\partial^2 u}{\partial x^2}$, $u_{xt} = \dfrac{\partial^2 u}{\partial x \partial t}$, and $u_{xxt} = \dfrac{\partial^3 u}{\partial^2 x \partial t}$. In these coordinates, the angle n and the curvature K are given by Lemma 5.3.

Let us recall that locally in a neighborhood of $(0, 0)$, the function u satisfies

(5.34) $$u_t = c(t, x, u, n) \sqrt{1 + u_x^2}$$

For t in a neighborhood of zero, let us define the curve γ contained in Γ by:

$$\gamma(t) = (t, 0, u(t, 0))$$

Then

$$
\begin{aligned}
D_t \tilde{n}(0, 0, 0) &= \frac{d}{dt}(\tilde{n} \circ \gamma)(0) = \frac{d}{dt}(n(t, 0, u(t, 0)))_{|t=0} \\
&= \frac{d}{dt}\left(\frac{1}{\sqrt{1 + u_x^2(t, 0)}}(-u_x(t, 0), 1)\right)_{|t=0} = (-u_{xt}(0, 0), 0)
\end{aligned}
$$

Similarly

$$
\begin{aligned}
D_t \tilde{K}(0, 0, 0) &= \frac{d}{dt}(\tilde{K} \circ \gamma)(0) = \frac{d}{dt}(F(u_{xx}(t, 0), u_x(t, 0))_{|t=0} \\
&= F'_M(\cdot, u_x(0, 0))u_{xxt}(0, 0) + F'_p(u_{xx}(0, 0), u_x(0, 0))u_{xt}(0, 0) \\
&= \mathcal{I}(u_{xxt}) + (u_{xx} \cdot u_{xt}, 0) \otimes (0, 1) + (0, 1) \otimes (u_{xx} \cdot u_{xt}, 0)
\end{aligned}
$$

where all the quantities are evaluated at the origin $(0, 0)$.

Derivating (3.20) with respect to x, we get

(5.35) $$u_{xt} = (\partial_x c + u_x \partial_{y_N} c + \partial_n c \cdot n_x) \sqrt{1 + u_x^2} + c\frac{u_{xx} \cdot u_x}{\sqrt{1 + u_x^2}}$$

where

$$n_x = -\frac{1}{(1 + u_x^2)^{\frac{3}{2}}}(-u_x, 1) \otimes u_{xx} \cdot u_x + \frac{1}{\sqrt{1 + u_x^2}}(-u_{xx}, 0)$$

which implies that at the origin we have

$$D_t \tilde{n}(0, 0, 0) = (-u_{xt}(0, 0), 0) = -(I - n \otimes n) \cdot \partial_y c + K \cdot \partial_n c = D_\tau c$$

and

$$(I - n \otimes n) \cdot \partial_y \tilde{n} = -K$$

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Derivating now (5.35) with respect to x, we get at the origin (with contraction in n in expressions like $n_x \cdot \partial^2_{nx} c$ or $n_x \cdot \partial^2_{nn} c \cdot n_x$)

$$u_{xxt}(0,0) = \partial^2_{xx} c + \partial^2_{xn} c \cdot n_x + u_{xx} \partial_{yN} c + n_x \cdot \partial^2_{nx} c + n_x \cdot \partial^2_{nn} c \cdot n_x + \partial_n c \cdot n_{xx} + c u_{xx} \cdot u_{xx}$$

and then at the origin

$$\mathcal{I}(u_{xxt}) = A + \mathcal{I}(\partial_n c \cdot n_{xx})$$

with

$$A = (I - n \otimes n) \cdot \{\partial^2_{yy} c - \partial^2_{yn} c \cdot K + K(n \cdot \partial_y c) - K \cdot \partial^2_{ny} c + K \cdot \partial^2_{nn} c \cdot K + cK \cdot K\} \cdot (I - n \otimes n)$$

Hence

$$D_t \tilde{K}(0,0,0) = B + C$$

with

$$B = A + K \cdot D_\tau c \otimes n + n \otimes K \cdot D_\tau c + K^2 \cdot \partial_n c \otimes n + n \otimes K^2 \cdot \partial_n c = a_K$$

where a_K given in (4.28). and

$$C = \mathcal{I}(\partial_n c \cdot n_{xx}) - K^2 \cdot \partial_n c \otimes n - n \otimes K^2 \cdot \partial_n c$$

On the one hand, let us compute at the origin

$$n_{xx} = -\{(0,1) \otimes u_{xx} \cdot u_{xx} + (u_{xxx}, 0)\}$$

and then at the origin (because $\partial_n c$ is orthogonal to n)

$$\partial_n c \cdot n_{xx} = -u_{xxx} \cdot \partial_n c$$

On the other hand, let us compute at the origin

$$(F(u_{xx}, u_x))_x = F'_M(\cdot, u_x) u_{xxx} + F'_p(u_{xx}, u_x) u_{xx}$$
$$= \mathcal{I}(u_{xxx}) + (u_{xx} \cdot u_{xx}, 0) \otimes (0,1) + (0,1) \otimes (u_{xx} \cdot u_{xx}, 0)$$

and then

$$\partial_n c \cdot (F(u_{xx}, u_x))_x = \mathcal{I}(u_{xxx} \cdot \partial_n c) + K^2 \partial_n c \otimes n + n \otimes K^2 \cdot \partial_n c$$

$$= \mathcal{I}(-\partial_n c \cdot n_{xx}) + K^2 \partial_n c \otimes n + n \otimes K^2 \cdot \partial_n c$$

$$= -C$$

This shows that

$$D_t \tilde{K}(0,0,0) = a_K - \partial_n c \cdot \partial_y \tilde{K}$$

This ends the proof of the Lemma.

Lemma 5.5 *Let $\psi \in C^1(\mathbb{R}^N)$, $V \in (C^1(\mathbb{R}^N))^N$. Then we have*

$$-\int_{\Gamma_t} V \cdot (I - n \otimes n) \cdot \partial_y \psi = \int_{\Gamma_t} \psi \{(I - n \otimes n) \cdot \partial_y\} \cdot V + \int_{\Gamma_t} \psi (n \cdot V)(I : K)$$

Proof of Lemma 5.5

Let us introduce for $\varepsilon > 0$

$$\Omega_\varepsilon = \left\{ x \in \mathbb{R}^N, \quad \exists y \in \Gamma_t, \quad x = y + rn(t,y) \quad \text{for some} \quad r \in (-\varepsilon, \varepsilon) \right\}$$

For $\varepsilon > 0$ small enough, and $x \in \Omega_\varepsilon$, there exists a unique $y = y(x)$ such that $y \in \Gamma_t$ and $x = y + rn(t,y)$. Then we can then extend the field n on Ω_ε by

$$\tilde{n}(x) = n(t, y(x))$$

Then we have

$$\int_{\Gamma_t} V \cdot (I - n \otimes n) \cdot \partial_y \psi = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{\Omega_\varepsilon} V \cdot (I - \tilde{n} \otimes \tilde{n}) \cdot \partial_y \psi$$

and

$$-\int_{\Omega_\varepsilon} V \cdot (I - \tilde{n} \otimes \tilde{n}) \cdot \partial_y \psi = \int_{\Omega_\varepsilon} \psi \left\{ (I - \tilde{n} \otimes \tilde{n}) \cdot \partial_y \right\} \cdot V - \int_{\Omega_\varepsilon} \psi \, V \cdot \left\{ (\partial_y \cdot \tilde{n}) \, \tilde{n} + (\tilde{n} \cdot \partial_y) \, \tilde{n} \right\}$$

Hence

$$-\int_{\Gamma_t} V \cdot (I - n \otimes n) \cdot \partial_y \psi = \int_{\Gamma_t} \psi \left\{ (I - n \otimes n) \cdot \partial_y \right\} \cdot V - \int_{\Gamma_t} \psi \, V \cdot \left\{ (\partial_y \cdot \tilde{n}) \, \tilde{n} + (\tilde{n} \cdot \partial_y) \, \tilde{n} \right\}$$

For our choice of the extension $\tilde{n}$, we have $(\tilde{n} \cdot \partial_y) \, \tilde{n} = 0$ and in the coordinates $(\tilde{n}, \tilde{n}^\perp)$, we see from (5.33) that $\partial_y \cdot \tilde{n} = -I : K$, which implies the result. This ends the proof of the Lemma.

Lemma 5.6 *With the notation of Lemma 5.2, we have*

$$\int_{\Gamma_t} \partial_n c \cdot \partial_y \tilde{K} : \partial_K \varphi = \int_{\Gamma_t} D_\tau \cdot (\varphi \, \partial_n c) \quad \text{with} \quad D_\tau = -(I - n \otimes n) \cdot \partial_y + K \cdot \partial_n$$

where the quantities in the integrals are evaluated at (t,y), $n(t,y)$, $K(t,y)$.

Proof of Lemma 5.6

For a general smooth function G, we define $\tilde{G}(t,y) := G(t, y, \tilde{n}(t,y), \tilde{K}(t,y))$, and we have on Γ

$$(5.36) \quad (I - n \otimes n) \cdot \partial_y \tilde{G}(t,y) = \left\{ (I - n \otimes n) \cdot \partial_y - K \cdot \partial_n + (I - n \otimes n) \cdot \partial_y \tilde{K} : \partial_K \right\} G$$

where we have in particular used (5.33). We deduce that

$$\int_{\Gamma_t} \partial_n c \cdot \partial_y \tilde{K} : \partial_K \varphi = \int_{\Gamma_t} \partial_n c \cdot \left\{ (I - n \otimes n) \cdot \partial_y \tilde{K} : \partial_K \varphi - K \cdot \partial_n \varphi + (I - n \otimes n) \cdot \partial_y \varphi + D_\tau \varphi \right\}$$

$$= \int_{\Gamma_t} \partial_n c \cdot (I - n \otimes n) \cdot \partial_y \tilde{\varphi} + \int_{\Gamma_t} \partial_n c \cdot D_\tau \varphi$$

$$= \int_{\Gamma_t} \left\{ -\varphi \, ((I - n \otimes n) \cdot \partial_y) \cdot \widetilde{\partial_n c} + \partial_n c \cdot D_\tau \varphi \right\}$$

$$= \int_{\Gamma_t} \left\{ \varphi \, D_\tau \cdot \partial_n c + \partial_n^2 c \cdot D_\tau \varphi \right\}$$

$$= \int_{\Gamma_t} D_\tau \cdot (\varphi \partial_n c)$$

where we have used (5.36) with $G = \varphi$ in the second line, Lemma 5.5 at the third line, and (5.36) with $G = \partial_n c$ in the fourth line. This ends the proof of the Lemma.

Lemma 5.7 *For a general g solution of (4.26), we have for $\alpha := K \cdot n \ g$*

(5.37) $$\alpha_t + \partial_y \cdot (a_y \otimes \alpha) + \partial_n \cdot (a_n \otimes \alpha) + \partial_K : (a_K \otimes \alpha) + A_0 \cdot \alpha = 0$$

with A_0 given in (4.30).

Proof of Lemma 5.7
Let us compute

$$
\begin{aligned}
-(K \cdot n \ g)_t \ &= \ -K \cdot n \ g_t \\[2mm]
&= \ K \cdot n \ \{a_0 g + \partial_y \cdot (a_y g) + \partial_n \cdot (a_n g) + \partial_K : (a_K g)\} \\[2mm]
&= \ a_0 \ K \cdot n \ g + \partial_y \cdot (a_y \otimes K \cdot n \ g) + \partial_n \cdot (a_n \otimes K \cdot n \ g) + \partial_K : (a_K \otimes K \cdot n \ g) \\
&\quad -(a_n g \cdot \partial_n)(K \cdot n) - (a_K g : \partial_K)(K \cdot n)
\end{aligned}
$$

i.e. $\alpha := K \cdot n \ g$ satisfies

$$
\begin{aligned}
&\alpha_t + a_0 \alpha + \partial_y \cdot (a_y \otimes \alpha) + \partial_n \cdot (a_n \otimes \alpha) + \partial_K : (a_K \otimes \alpha) \\[2mm]
&= (a_n g \cdot \partial_n)(K \cdot n) + (a_K g : \partial_K)(K \cdot n) \\[2mm]
&= K \cdot a_n \ g + a_K \cdot n \ g \\[2mm]
&= -(A_0 - a_0 I) \cdot \alpha
\end{aligned}
$$

with

$$
-(A_0 - a_0 I) \ = \ cK + K \cdot \partial_{nn}^2 c - (I - n \otimes n) \cdot \partial_{yn}^2 c + n \otimes \partial_y c + 2 \ (n \cdot \partial_y c) \ I
$$

This ends the proof of the Lemma.

Proof of Theorem 4.1
The proof is completely similar to the one of Theorem 2.1.
Putting together the results of Lemmata 5.2, 5.4 and 5.6, we get for $g = g_\Gamma$

$$
\begin{aligned}
< g_t, \varphi > \ &= \ < g, -c(I : K)\varphi + cn \cdot \partial_y \varphi + D_\tau c \cdot \partial_n \varphi + a_K : \partial_K \varphi - \partial_n c \cdot \partial_y \tilde{K} : \partial_K \varphi > \\[2mm]
&= \ < g, -c(I : K)\varphi + cn \cdot \partial_y \varphi + D_\tau c \cdot \partial_n \varphi + a_K : \partial_K \varphi - \partial_n c \cdot D_\tau \varphi - \varphi D_\tau \cdot \partial_n c > \\[2mm]
&= \ < g, -a_0 \varphi + a_y \partial_y \varphi + a_n \partial_n \varphi + a_K \partial_K \varphi >
\end{aligned}
$$

which leads to equation (4.26).
Equation (5.37) is exactly (4.29).

Finally, let us compute still for g_Γ defined in (4.24) and $\overline{\psi}(t, y, n, K) = \psi(t, y, n)$

$$
\begin{aligned}
< \hat{g}_\Gamma, \psi > \; &= \; < D_\tau g + n(I:K)g, \overline{\psi} > \\
&= \; < g, n(I:K)\overline{\psi} - D_\tau \overline{\psi} > \\
&= \; < g, n(I:K)\overline{\psi} + (I - n \otimes n) \cdot \partial_y \overline{\psi} - K \cdot \partial_n \overline{\psi} + (I - n \otimes n) \cdot \partial_y \tilde{K} : \partial_K \overline{\psi} > \\
&= \; \int_0^T dt \int_{\Gamma_t} \left\{ n(I:K)\tilde{\psi}(t, y) + (I - n \otimes n) \cdot \partial_y \tilde{\psi}(t, y) \right\} \\
&= \; 0
\end{aligned}
$$

where in the third line, we have used the fact that $\overline{\psi}$ is indenpendent on the curvature K, we have used in the fourth line equation (5.36) with $G = \overline{\psi}$ and $\tilde{\psi}(t, y) = \overline{\psi}(t, y, n(t, y), K(t, y))$. For the last line, we have used Lemma 5.5 with $V = e_i$, $i = 1, ..., N$, succesively. This proves that $\hat{g}_\Gamma = 0$.
This ends the proof of the Theorem.

Remark 5.8 *Let us remark that if $K \cdot ng = 0$, then $(I - n \otimes n) \cdot K \cdot \partial_n g = K \cdot \partial_n g + (I:K) \, ng$. This last relation explains why the contribution of the term in $n(I:K)$ to the definition of $\hat{g}_\Gamma$ does not vanish in general, even if $\hat{g}_\Gamma = 0$.*

6 Appendix

6.1 Proof of Lemma 5.1

Proof of Lemma 5.1
Let us consider a point $(t_0, y_0) \in \Gamma$, and a local parametrization $\gamma : B_{r_0} \longrightarrow \mathbb{R}^N$ of Γ_{t_0} in a neighborhood of y_0, with $B_{r_0} = B_{r_0}(0) \subset \mathbb{R}^{N-1}$ and $r_0 > 0$. Then, in a neighborhood U_{t_0, y_0}, we can parametrize Γ_{t+h} for h small, by γ^{t_0+h} defined by

$$
\gamma^{t_0+h}(x) = \gamma(x) + r(h, x)n(x)
$$

where $n(x)$ is the normal n to γ at the point $\gamma(x)$.
We will prove that formula (5.31) holds, assuming moreover that

$$
(6.38) \hspace{4cm} \text{supp } \psi \subset U_{t_0, y_0}
$$

Finally, using a partition of unity, we recover the full formula (5.31) as in the Lemma.
Let us now prove (5.31) assuming (6.38).
We have $r(0, x) = 0$ and $r_h(0, x) = c(t, \gamma(x))$ (where r_h stands for $\dfrac{\partial r}{\partial h}$, r_i stands for $\dfrac{\partial r}{\partial x_i}$, and r_{ih} stands for $\dfrac{\partial^2 r}{\partial x_i \partial h}$). We compute the jacobian

$$
J_h = |a_h| \quad \text{with} \quad < a_h, \cdot > = \det \left(\gamma_1^{t_0+h}, ..., \gamma_{N-1}^{t_0+h}, \cdot \right)
$$

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where we denote

$$(6.39) \qquad \gamma_i^{t_0+h} = \frac{\partial \gamma}{\partial x_i}^{t_0+h} = \gamma_i + r_i n + r \frac{\partial n}{\partial x_i}$$

and similarly $\gamma_i = \dfrac{\partial \gamma}{\partial x_i}$ for $i = 1, ..., N-1$.

$$\int_{\Gamma_{t_0+h}} \psi(t_0 + h, \cdot) = \int_{B_{r_0}} dx \ \psi(t_0 + h, \gamma^{t_0+h}(x)) J_h$$

$$= \int_{B_{r_0}} dx \ \psi(t + h, \gamma(x) + r(h, x)n) J_h$$

Therefore

$$(6.40) \qquad \frac{d}{dh} \left(\int_{\Gamma_{t_0+h}} \psi(t + h, \cdot) \right)_{|h=0} = \int_{B_{r_0}} dx \ \{ (\psi_t + (r_h(0, x)) \ n \cdot \partial_y \psi) J_0 + \psi \ J_0' \}$$

with

$$J_0' = \left(\frac{dJ_h}{dh} \right)_{|h=0} = \frac{a_0 \cdot a_0'}{|a_0|}$$

with $a_0' = \left(\dfrac{da_h}{dh} \right)_{|h=0}$. On the other hand, we have

$$|a_h|^2 = \det \left(\gamma_1^{t_0+h}, ..., \gamma_{N-1}^{t_0+h}, a_h \right)$$

which by derivation gives

$$2a_0 \cdot a_0' = \det \ (\gamma_1, ..., \gamma_{N-1}, a_0') + \sum_{i=1}^{N-1} \det \ \left(\gamma_1, ..., \gamma_i^{t_0'}, ..., \gamma_{N-1}, a_0 \right)$$

with (from (6.38))

$$\gamma_i^{t_0'} = \left(\frac{\partial \gamma_i}{\partial h}^{t_0+h} \right)_{|h=0} = r_{ih}(0, x)n + r_h(0, x)\frac{\partial n}{\partial x_i}$$

Therefore

$$a_0 \cdot a_0' = \sum_{i=1}^{N-1} \det \ \left(\gamma_1, ..., r_{ih}n + r_h \frac{\partial n}{\partial x_i}, ..., \gamma_{N-1}, a_0 \right)$$

Using in particular the fact that a_0 is parallel to n, we deduce that

$$a_0 \cdot a_0' = r_h(0, x)|a_0|^2 k \quad \text{with} \quad k := \sum_{i=1}^{N-1} \frac{< \gamma_i, \frac{\partial n}{\partial x_i} >}{|\gamma_i|^2}$$

A direct computation in local coordinates allows to see that $k = -I : K$, which gives

$$J_0' = -c(I : K)J_0$$

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Finally (6.40) implies

$$\frac{d}{dh}\left(\int_{\Gamma_{t_0+h}} \psi(t+h,\cdot)\right)_{|h=0} = \int_{B_{r_0}} dx\ J_0\{\psi_t + c\ n\cdot\partial_y\psi - c(I:K)\psi\}$$

$$= \int_{\Gamma_{t_0}} \{\psi_t + c\ n\cdot\partial_y\psi - c(I:K)\psi\}$$

This ends the proof of the Lemma.

6.2 Case of a curve in dimension $\overline{N}$

Let us consider a curve $\overline{\Gamma}_t$ evolving in $\mathbb{R}^{\overline{N}}$ along a vector field $\overline{c}(t,\overline{y}) \in \mathbb{R}^{\overline{N}}$. Here we do not consider the case of "normal velocity", possibly depending on the unit tangent vector to the curve $\overline{\Gamma}_t$, that we call $\overline{\tau} \in \mathbb{S}^{\overline{N}-1}$.

We call $\overline{\kappa} \in \mathbb{R}^{\overline{N}}$ the curvature vector $\overline{\kappa} = \dfrac{d\overline{\tau}}{ds}$ of the curve and set

$$g_{\overline{\Gamma}}(t,\overline{y},\overline{\tau},\overline{\kappa}) = \delta_{\overline{\Gamma}_t}(\overline{y})\delta_0(\overline{\tau} - \overline{\tau}(t,\overline{y}))\delta_0(\overline{\kappa} - \overline{\kappa}(t,\overline{y}))$$

Indeed, in the case where the velocity depends on $\overline{\tau}$ (see Remark 6.1), we get in the expression of $< g_t, \varphi >$ (with $g = g_{\overline{\Gamma}}$) a term

$$(I - \overline{\tau}\otimes\overline{\tau})\cdot\left(\frac{d\overline{\kappa}}{ds}\cdot\partial_{\overline{\tau}}\overline{c}\right)\cdot\partial_{\overline{\kappa}}\varphi$$

where $\partial_{\overline{\tau}}c$ is a matrix. Then we can not use the trick applied in the case of hypersurfaces, where we rewrote this term, essentially as $\partial_{\overline{\tau}}c$ times the term $\dfrac{d\overline{\kappa}}{ds}\cdot\partial_{\overline{\kappa}}\varphi$, and then conclude by an integration by parts.

Nevertheless, in the case where $\overline{c}$ is independent on $\overline{\tau}$, we can check (see Remark 6.1) that $g_{\overline{\Gamma}}$ solves the following equation

(6.41)
$$g_t + \text{div}\ (\overline{a}g) + \overline{a}_0 g = 0$$

with

$$\begin{cases} \overline{a}_0 = \overline{c}\cdot\overline{\kappa}, \quad \overline{a}_{\overline{y}} = (I - \overline{\tau}\otimes\overline{\tau})\cdot\overline{c}, \quad \overline{a}_{\overline{\tau}} = (I - \overline{\tau}\otimes\overline{\tau})\cdot(\overline{\tau}\cdot\partial_{\overline{y}}\overline{c}) \\[2mm] \overline{a}_{\overline{\kappa}} = (I - \overline{\tau}\otimes\overline{\tau})\cdot\{\overline{\kappa}\cdot\partial_{\overline{y}}\overline{c} + \overline{\tau}\otimes\overline{\tau}:\partial_{\overline{yy}}^2\overline{c}\} - (\overline{\kappa}\otimes\overline{\tau} + \overline{\tau}\otimes\overline{\kappa})\cdot(\overline{\tau}\cdot\partial_{\overline{y}}\overline{c}) \end{cases}$$

Remark 6.1 *To get equation (6.41), we simply follows the lines of our approach for hyper-surfaces, and use the following formulas for $g = g_{\overline{\Gamma}}$ and a general velocity $\overline{c}(t,\overline{y},\overline{\tau})$ (which is not assumed perpendicular to $\overline{\tau}$)*

$$\frac{d}{dt}\left(\int_{\overline{\Gamma}_t}\psi\right) = \int_{\overline{\Gamma}_t} D_t\psi - \overline{c}\cdot\overline{\kappa}\psi \quad \text{with} \quad D_t\psi = \psi_t + \overline{c}\cdot(I - \overline{\tau}\otimes\overline{\tau})\cdot\partial_{\overline{y}}\psi$$

$$< g_t, \varphi > = < g, -\overline{c} \cdot \overline{\kappa} \varphi + \overline{c} \cdot (I - \overline{\tau} \otimes \overline{\tau}) \cdot \partial_{\overline{y}} \varphi + D_t \tilde{\overline{\tau}} \cdot \partial_{\overline{\tau}} \varphi + D_t \tilde{\overline{\kappa}} \cdot \partial_{\overline{\kappa}} \varphi >$$

$$\begin{cases} D_t \tilde{\overline{\tau}} = D_\tau c \quad \text{with} \quad D_\tau = (I - \overline{\tau} \otimes \overline{\tau}) \cdot (\overline{\tau} \cdot \partial_{\overline{y}} + \overline{\kappa} \cdot \partial_{\overline{\tau}}) \\[2mm] D_t \tilde{\overline{\kappa}} = a_{\overline{\kappa}} + (I - \overline{\tau} \otimes \overline{\tau}) \cdot \left(\dfrac{d\overline{\kappa}}{ds} \cdot \partial_{\overline{\tau}} \overline{c} \right) \\[2mm] \begin{aligned} a_{\overline{\kappa}} = \ & (I - \overline{\tau} \otimes \overline{\tau}) \cdot \left\{ \overline{\kappa} \cdot \partial_{\overline{y}} \overline{c} + \overline{\tau} \otimes \overline{\tau} : \partial_{\overline{yy}}^2 \overline{c} + 2\overline{\tau} \cdot \partial_{\overline{y\tau}}^2 \overline{c} \cdot \overline{\kappa} + \overline{\kappa} \otimes \overline{\kappa} : \partial_{\overline{\tau\tau}}^2 \overline{c} \right\} \\ & - (\overline{\kappa} \otimes \overline{\tau} + \overline{\tau} \otimes \overline{\kappa}) \cdot (\overline{\tau} \cdot \partial_{\overline{y}} \overline{c} + \overline{\kappa} \cdot \partial_{\overline{\tau}} \overline{c}) \end{aligned} \end{cases}$$

6.3 An application to another kinetic formulation of a curves moving in the plane

Let us remark that the formalism of subsection 6.2 can be applied to the particular case of physical curves Γ_t moving in the plane with a normal velocity $c(t, y, \theta)$, to which we associate the corresponding curve in the space/angle coordinates

$$\overline{\Gamma}_t = \begin{cases} \overline{y} = (y, \theta(t, y)) \in \mathbb{R}^2 \times (\mathbb{R}/(2\pi\mathbb{Z})) \\[2mm] \text{for} \quad y \in \Gamma_t \quad \text{with unit tangent vector} \quad \tau(t, y) = (\sin \theta(t, y), -\cos \theta(t, y)) \end{cases}$$

The curve $\overline{\Gamma}_t$ moves with velocity

$$\overline{c}(t, \overline{y}) = a'(t, y, \theta) \quad \text{with} \quad \overline{y} = (y, \theta) \in \mathbb{R}^2 \times (\mathbb{R}/(2\pi\mathbb{Z})$$

where $a' = (a_y, a_\theta)$ is defined in Theorem 2.1 $(N = 2)$. Here this velocity does not depend on $\overline{\tau}$, the unit tangent vector to $\overline{\Gamma}_t$, which is given by

$$\overline{\tau}(t, \overline{y}) = \frac{1}{\sqrt{1 + \kappa^2}} (\tau, \kappa)$$

where $\kappa(t, y)$ is the (scalar) curvature of the physical curve Γ_t.
The advantage to consider $\overline{\Gamma}_t$ in place of Γ_t, is that the curve $\overline{\Gamma}_t$ always stays regular for all time (this is the wavefront, see for instance Osher et al. [22] for the wavefront associated to curves and Leung, Qian, Osher [19] for the wavefront associated to surfaces), while Γ_t can become singular in finite time. At the level of the kinetic equation (6.41) satisfied by $g_{\overline{\Gamma}}$ (with $\overline{N} = 3$), the nice property is that the vector field $\overline{a}$ has at most a linear growth in $\overline{\kappa}$. The consequence is the existence of solutions for all time to this kinetic equation.

Remark 6.2 *From the previous point of view, it is natural to ask if there are some relations between first order evolution of curves in $\mathbb{R}^3$, and second order evolution of curves in the plane.*

Remark 6.3 *Similarly, if could be interesting to consider*

$$\overline{\Gamma}_t = \begin{cases} \overline{y} = (y, n(t, y)) \in \mathbb{R}^N \times \mathbb{S}^{N-1} \\[2mm] \text{for} \quad y \in \Gamma_t \quad \text{with unit normal vector} \quad n(t, y) \end{cases}$$

and to try to write a kinetic equation for a measure

$$g_{\overline{\Gamma}}(t, \overline{y}, \overline{n}, \overline{K}) = \delta_{\overline{\Gamma}_t}(\overline{y})\delta_0(\overline{n} - \overline{n}(t, \overline{y}))\delta_0(\overline{K} - \overline{K}(t, \overline{y}))$$

where $\overline{n}(t, \overline{y})$ defines the tangent space (of dimension $2N - 1$) to $\overline{\Gamma}_t$ at $\overline{y}$, and $\overline{K}(t, \overline{y})$ defines its curvature.
It would be also interesting to see how to extend this method for the evolution of general submanifolds of arbitrary codimensions.

Aknowledgements

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