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CHANGING THE BRANCHING MECHANISM OF A CONTINUOUS STATE BRANCHING PROCESS USING IMMIGRATION

ROMAIN ABRAHAM AND JEAN-FRANÇOIS DELMAS

Abstract. We construct a continuous state branching process with immigration (CBI) whose immigration depends on the CBI itself and we recover a continuous state branching process (CB). This provides a dual construction of the pruning at nodes of CB introduced by the authors in a previous paper [1]. This construction is a natural way to model neutral mutation. Using exponential formula, we compute the probability of extinction of the original type population in a critical or sub-critical quadratic branching, conditionally on the non extinction of the total population.

1. Introduction

We consider an initial Eve-population whose size evolves as a continuous state branching process (CB), $Y^0 = (Y_t^0, t \geq 0)$, with branching mechanism ψ defined by formula (1) (see [6] for a definition of CB as limit of Galton-Watson processes). We assume this population gives birth to a population of (irreversible) mutants. The new mutants population can be seen as an immigration process with rate proportional to the size of the Eve-population. This second population also evolves according to the same branching mechanism as the Eve-population (i.e. the mutations are neutral). This population of mutants gives birth to a population of other (irreversible) mutants, with rate proportional to its size. And so on. We are interested in the law of the total population size $X = (X_t, t \geq 0)$, which is a CB with immigration (CBI) proportional to its own size. If the mutations are neutral, we expect X to be a CB. We give in Theorem 3.2 the law of the CBI X. Then we check, Corollary 4.1, that under some natural condition on the immigration process, the CBI X is indeed a CB whose branching mechanism is given by a shift of the branching mechanism of the Eve-population: more precisely, the branching mechanism of X is

$$\psi_{-\theta}(\lambda) = \psi(\lambda - \theta) - \psi(-\theta), \lambda > 0,$$

for some $\theta > 0$. This corresponds in some sense, see Corollary 4.2, to the dual of the pruning at nodes introduced in [1] for critical or sub-critical CB.

Then we use this model to compute the joint law of the Eve-population and the whole population at given time: (Y_t^0, X_t). In particular, we compute $\mathbb{P}(Y_t^0 = 0 | X_t > 0)$, the probability for the Eve-type to have disappeared at time t, conditionally on the survival of the total population at time t, see Remark 5.3, as well as $\lim_{s\to\infty} \mathbb{P}(Y_t^0 = 0 | X_s > 0)$, the probability for the Eve-type to have disappeared, conditionally on the population to never be extinct, see Proposition 5.5.

In the particular case of CB with quadratic branching mechanism ($\psi(u) = \beta u^2$, $\beta > 0$), similar results are given in [10] (using genealogical structure for CB) and in [11] (using a decomposition of Feller diffusion and [9]). In the critical ($\psi'(0^+) = 0$) or sub-critical ($\psi'(0^+) > 0$) case one could have used the genealogical process associated to CB introduced

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by [7] and to CBI developed by [5] to prove the present result. This presentation would have been more natural in view of the pruning method used in [1]. Our choice not to rely on this presentation was motivated by the possibility to consider super-critical cases ($\psi'(0^+) < 0$).

The paper is organized as follows: In Section 2, we recall some well known facts on continuous state branching processes (CB) and continuous state branching processes with immigration (CBI). In Section 3, we built a CBI X whose branching mechanism is ψ and immigration rate at time t proportional to X_t. We prove in Section 4 this process is, under some condition on the immigration process, again a CB and we give some link with the pruning at nodes of CB introduced in [1]. Eventually, we compute the joint law of the Eve-population and the whole population in Section 5, as well as some related quantities.

2. CB and CB with immigration

The results from this section can be found in [4] (see also [8] for a survey on CB and CBI, and the references therein). Let ψ be a branching mechanism of a CB: for $\lambda \geq 0$,

$$\psi(\lambda) = \alpha_0 \lambda + \beta\lambda^2 + \int_{(0,\infty)} \pi(d\ell) \left[\mathrm{e}^{-\lambda\ell} - 1 + \lambda\ell \mathbf{1}_{\{\ell \leq 1\}} \right], \tag{1}$$

where $\alpha_0 \in \mathbb{R}$, $\beta \geq 0$ and π is a Radon measure on $(0, \infty)$ such that $\int_{(0,\infty)} (1 \wedge \ell^2)\, \pi(d\ell) < \infty$. Notice ψ is smooth on $(0, \infty)$ and convex. We have $\psi'(0^+) \in [-\infty, +\infty)$, and $\psi'(0^+) = -\infty$ if and only if $\int_{(1,\infty)} \ell\, \pi(d\ell) = \infty$. In order to consider only conservative CB, we shall also assume that for all $\varepsilon > 0$ small enough

$$\int_0^\varepsilon \frac{1}{|\psi(u)|}\, du = \infty. \tag{2}$$

2.1. **CB.** Let $\mathbb{P}_x$ be the law of a CB $Z = (Z_t, t \geq 0)$ started at $x \geq 0$ and with branching mechanism ψ. The process Z is a Feller process and thus càdlàg. Thanks to (2), the process is conservative, that is a.s. for all $t \geq 0$, $Z_t < +\infty$. For every $\lambda > 0$, for every $t \geq 0$, we have

$$\mathbb{E}_x \left[\mathrm{e}^{-\lambda Z_t} \right] = \mathrm{e}^{-x u(t,\lambda)} \tag{3}$$

where the function u is the unique non-negative solution of

$$u(t, \lambda) + \int_0^t \psi\big(u(s, \lambda)\big) ds = \lambda, \quad \lambda \geq 0, \quad t \geq 0. \tag{4}$$

This equation is equivalent to

$$\int_{u(t,\lambda)}^{\lambda} \frac{dr}{\psi(r)} = t \quad \lambda \geq 0, \quad t \geq 0. \tag{5}$$

The process Z is infinitely divisible. Let Q be its canonical measure. In particular, under $\mathbb{P}_x$, Z is distributed as $\sum_{i\in I} Z^i$, where $\sum_{i \in I} \delta_{Z^i}$ is a Poisson measure with intensity $xQ(dZ)$.

The CB is called critical (resp. super-critical, resp. sub-critical) if $\psi'(0^+) = 0$ (resp. $\psi'(0^+) < 0$, resp. $\psi'(0^+) > 0$).

We shall need inhomogeneous notation. For $t < 0$, we set $Z_t = 0$. Let $\mathbb{P}_{x,t}$ denote the law of $(Z_{s-t}, s \in \mathbb{R})$ under $\mathbb{P}_x$, and let Q_t be the distribution of $(Z_{s-t}, s \in \mathbb{R})$ under Q.

For μ a positive measure on $\mathbb{R}$, we set $H^\mu = \sup\{r \in \mathbb{R}; \mu([r, \infty)) > 0\}$ the maximal element of its support.

Proposition 2.1. *Let μ be a finite positive measure on $\mathbb{R}$ with support bounded from above (i.e. H^μ is finite). Then we have for all $s \in \mathbb{R}$, $x \geq 0$,*

$$\mathbb{E}_x \left[\mathrm{e}^{-\int Z_{r-s}\, \mu(dr)} \right] = \mathrm{e}^{-xw(s)}, \tag{6}$$

where the function w is a measurable locally bounded non-negative solution of the equation

$$w(s) + \int_s^\infty \psi(w(r))dr = \int_{[s,\infty)} \mu(dr), \quad s \leq H^\mu \quad \text{and} \quad w(s) = 0, \quad s > H^\mu. \tag{7}$$

If $\psi'(0^+) > -\infty$ or if $\mu(\{H^\mu\}) > 0$, then (7) has a unique measurable locally bounded non-negative solution.

Proof. Let $n \geq 1$. We set $Z_t^{(n),s} = Z_{\frac{i+1}{2^n}-s}$ for $t \in [i/2^n, (i+1)/2^n)$. Using that Z is càdlàg, we get a.s. $\lim_{n\to\infty} Z_t^{(n),s} = Z_{t-s}$ for all $t, s \in \mathbb{R}$. Since the process Z is finite, we get by dominated convergence theorem a.s. for all $s \in \mathbb{R}$

$$\int_{[-s,H^\mu]} Z_{r-s}\, \mu(dr) = \lim_{n\to\infty} \int_{[-s,H^\mu]} Z_r^{(n),s}\, \mu(dr).$$

Using the Markov property of Z, we get that

$$\mathbb{E}_x \left[\mathrm{e}^{-\int Z_r^{(n),s}\, \mu(dr)} \right] = \mathrm{e}^{-xw^{(n)}(s)},$$

where $w^{(n)}$ is the unique non-negative solution of

$$w^{(n)}(s) + \int_s^{([H^\mu 2^n]+1)/2^n} \psi(w^{(n)}(r))\, dr = \int_{[k/2^n,\infty)} \mu(dr),$$

with k s.t. $k/2^n < s \leq (k+1)/2^n$.

Let $T > H^\mu + 1$. Notice that for all $s \in [-T, T]$, we have $\int Z_r^{(n),s} \mu(dr) \leq \sup\{Z_t, t \in [0, 2T]\}\mu([-T, H^\mu]) < \infty$ a.s. Let C be defined by $\mathrm{e}^{-C} = \mathbb{E}_x[\mathrm{e}^{-\sup\{Z_t, t\in[0,2T]\}\mu([-T,H^\mu])}]$. Notice $C < \infty$. This implies that for all $n \geq 1$, $s \in [-T, T]$,

$$0 \leq w^{(n)}(s) \leq C < \infty.$$

By dominated convergence theorem, $w^{(n)}(s)$ converges to $w(s) = -\log \left(\mathbb{E}_1[\mathrm{e}^{-\int Z_{r-s}\mu(dr)}] \right)$, which lies in $[0, C]$, for all $s \in [-T, T]$. By dominated convergence theorem, we deduce that w solves (7). Since T is arbitrary, the Proposition is proved but for the uniqueness of solutions of (7).

If $\psi'(0^+) > -\infty$, then ψ is locally Lipschitz. This implies there exists a unique locally bounded non-negative solution of (7).

If $\psi'(0^+) = -\infty$, and $\mu(\{H^\mu\}) > 0$, we get that $\int Z_{r-s}\, \mu(dr) \geq aZ_{H^\mu - s}$, where $a = \mu(\{H^\mu\}) > 0$. This implies that $w(s) \geq u(H^\mu - s, a) > 0$ for $s \in \mathbb{R}$. The function $u(\cdot, a)$ is strictly positive on $\mathbb{R}_+$ because of condition (2) and equation (5). Since ψ is locally Lipschitz on $(0, \infty)$, we deduce there exists a unique locally bounded non-negative solution of (7). □

2.2. **CBI.** Let $x > 0$, $\alpha \geq 0$, ν be a Radon measure on $(0, \infty)$ such that $\int_{(0,\infty)} (1 \wedge x)\, \nu(dx) < \infty$. Let $\mathcal{B}_+$ denote the set of non-negative measurable functions defined on $\mathbb{R}$. Let $h \in \mathcal{B}_+$ be locally bounded. We consider the following independent processes.

- $\sum_{i\in I} \delta_{t^i,x^i,Z^i}$, a Poisson measure with intensity $h(t)\mathbf{1}_{\{t\geq 0\}} dt\, \nu(dx)\, \mathbb{P}_{x,t}(dZ)$.
- $\tilde{Z}$, distributed according to $\mathbb{P}_x$.
- $\sum_{j\in J} \delta_{t^j,\hat{Z}^j}$, a Poisson measure with intensity $\alpha h(t)\mathbf{1}_{\{t\geq 0\}} dt\, Q_t(dZ)$.

For $t \in \mathbb{R}$, let $Y_t = \tilde{Z}_t + \sum_{i\in I} Z_t^i + \sum_{j\in J} \hat{Z}_t^j \in [0,\infty]$. We say $Y = (Y_t, t \geq 0)$ is a continuous state branching process with immigration (CBI) started at x, whose branching mechanism is ψ and immigration is characterized with (h, ϕ) where the immigration mechanism, ϕ, is defined by

$$\phi(\lambda) = \alpha\lambda + \int_{(0,\infty)} \nu(dx)(1 - \mathrm{e}^{-\lambda x}), \quad \lambda \geq 0.$$

One gets Y is a conservative Hunt process when h is constant, see [4]. Notice that Y is a non-homogeneous Markov processes. We also have $Y_0 = x$, and $Y_t = 0$ for $t < 0$.

Using Poisson measure property, one can construct on the same probability space two CBI, Y^1 and Y^2, with same branching process ψ, same starting point and immigration characterized by (h^1, ϕ) and (h^2, ϕ) such that $Y_t^1 \leq Y_t^2$ for all $t \leq T$ as soon as $h^1(t) \leq h^2(t)$ for all $t \leq T$. We can apply this with $h^1 = h$ and $h^2(t) = \sup\{h(s); s \in [0,T]\}$ for $t \in \mathbb{R}$ and some $T > 0$, and use that Y^2 is conservative (see [4]) to get that Y^1 has a locally bounded version over $[0,T]$. Since T is arbitrary, this implies that any CBI has a locally bounded version. We shall work with this version.

The following Lemma is a direct consequence of the exponential formula for Poisson measure.

Lemma 2.2. *Let μ be a finite positive measure on $\mathbb{R}$ with support bounded from above (i.e. H^μ is finite). We have for $s \in \mathbb{R}$:*

$$\mathbb{E}\left[\mathrm{e}^{-\int Y_{r-s}\,\mu(dr)}\right] = \mathrm{e}^{-xw(s) - \int_0^\infty h(t)\phi(w(s+t))dt}, \tag{8}$$

where the function w is defined by (6).

3. State dependent immigration

3.1. **Induction formula.** Let $(x_k)_{k\in\mathbb{N}}$ a sequence of non-negative real numbers. Let Y^0 be a CB with branching mechanism ψ starting at x_0. We construct by induction Y^n, $n \geq 1$, as the CBI started at x_n, with branching mechanism ψ and immigration characterized by (Y^{n-1}, ϕ).

Lemma 3.1. *Let $(\mu_k, k \in \mathbb{N})$ be a family of finite measures on $\mathbb{R}$ with support bounded from above. We have for all $n \in \mathbb{N}$, $s \in \mathbb{R}$,*

$$\mathbb{E}\left[\mathrm{e}^{-\sum_{k=0}^n \int Y_{r-s}^k\,\mu_k(dr)}\right] = \mathrm{e}^{-\sum_{k=0}^n x_{n-k} w_k^{(n)}(s)},$$

where $w_0^{(n)}$ is defined by (6) with μ replaced by μ_n, and for $k \geq 1$, $w_k^{(n)}$ is defined by (6) with μ replaced by $\mu_{n-k}(dr) + \phi(w_{k-1}(r))\,dr$. In particular, w_k is a locally bounded non-negative solution of the equation

$$w(s) + \int_s^\infty \psi(w(r))dr = \int_{[s,\infty)} \mu_{n-k}(dr) + \int_s^\infty \phi(w_{k-1}^{(n)}(r))\,dr, \quad s \in \mathbb{R}. \tag{9}$$

(Notice $w_k(s) = 0$ for $s > \max\{H^{\mu_{k'}}, k' \in \{0,\ldots,k\}\}$.)

Proof. This is a consequence of the computation of $\mathbb{E}\left[\mathrm{e}^{-\sum_{k=0}^n \int Y_{r-s}^k\,\mu_k(dr)} \Big| Y^0,\ldots,Y^{n-1}\right]$, using Proposition 2.1. This also implies that (9) holds. Then, by induction, one deduces from (9) that w_k is locally bounded. $\square$

3.2. **Convergence towards a total mass process.** We consider the sequence $(Y^n, n \geq 0)$ defined in the previous section with $x_0 = x \geq 0$ and $x_n = 0$ for $n \geq 1$. We set $X_t^n = \sum_{k=0}^n Y_t^k$ for $t \in \mathbb{R}$. Let X_t be the increasing limit of $X^n = (X_t^n, n \geq 0)$ for all $t \in \mathbb{R}$. We have $X_t \in [0, +\infty]$. We call $X = (X_t, t \in \mathbb{R})$ a CBI with branching mechanism ψ and immigration process (X, ϕ).

The process Y^0 describes the size process of the Eve-population, Y^1 the size process of the population of mutants born from the Eve-population Y^0, Y^2 the size process of the population of mutants born from mutant population Y^1, and so on. The size process of the total population is given by $X = \sum_{k \geq 0} Y^k$. In neutral mutation case, it is natural to assume that all the processes Y^k have the same branching mechanism. Since we assume $x_k = 0$ for all $k \geq 1$, this means only the Eve-population is present at time 0.

Theorem 3.2. *The process X, which is a CBI with branching mechanism ψ and immigration process (X, ϕ), is a CB with branching mechanism $\psi - \phi$.*

Remark 3.3. For $\lambda \geq 0$, we have

$$\psi(\lambda) - \phi(\lambda) = \Big(\alpha_0 - \alpha - \int_{(0,1]} \ell \, \nu(d\ell)\Big)\lambda + \beta\lambda^2 + \int_{(0,\infty)} (\pi(d\ell) + \nu(d\ell)) \left[\mathrm{e}^{-\lambda\ell} - 1 + \lambda\ell \mathbf{1}_{\{\ell \leq 1\}}\right],$$

which shows that $\psi - \phi$ is a branching mechanism.

Remark 3.4. As a consequence of Theorem 3.2, X is a Markov processes. Notice that $(Y^0, \ldots, Y^n)$ is also Markov but not X_t^n for $n \geq 1$.

Proof. Let μ be a finite measure on $\mathbb{R}$ with support bounded from above (i.e. $H^\mu < \infty$). We shall assume that $\mu(\{H^\mu\}) = a > 0$.

We keep the notations of Lemma 3.1, with $\mu_k = \mu$. In particular we see from (9) that $w_k^{(n)}$ does not depend on n. We shall denote it by w_k. By monotone convergence, we have

$$\mathbb{E}\left[\mathrm{e}^{-\int X_{r-s} \, \mu(dr)}\right] = \lim_{n \to \infty} \mathbb{E}\left[\mathrm{e}^{-\sum_{k=0}^n \int Y_{r-s}^k \, \mu(dr)}\right] = \lim_{n \to \infty} \mathrm{e}^{-x w_n(s)},$$

where the limits are non-increasing. This implies that $(w_n, n \geq 0)$ increases to a non-negative function w_∞. By monotone convergence theorem (for $\int_s^{H^\mu} \psi(w(r)) \mathbf{1}_{\{w_n(r) > 0\}} \, dr$ and the integral with ϕ) and dominated convergence theorem (for $\int_s^{H^\mu} \psi(w(r)) \mathbf{1}_{\{w_n(r) \leq 0\}} \, dr$), we deduce from (9), that w_∞ solves $w(s) = 0$ for $s > H^\mu$ and

$$w(s) + \int_s^{H^\mu} \psi(w(r)) dr = \int_{[s,\infty)} \mu(dr) + \int_s^{H^\mu} \phi(w(r)) \, dr, \quad s \leq H^\mu. \tag{10}$$

Notice that $w_\infty(s) \in [0, \infty]$ and the two sides of the previous equality may be infinite.

Thanks to Proposition 2.1, and since $\psi - \phi$ is a branching mechanism (see Remark 3.3), there exists a unique locally bounded non-negative solution of (10), which we shall call $\bar{w}$. Therefore to prove that $w_\infty = \bar{w}$, it is enough to check that w_∞ is locally bounded. This will be the case if we check that $w_\infty \leq \bar{w}$. In particular, we get $w_\infty = \bar{w}$, if we can prove that $w_n \leq \bar{w}$ for all $n \in \mathbb{N}$. We shall prove this by induction.

We consider the measure $\mu^0(dr) = \mu(dr) + \phi(\bar{w}(r)) \mathbf{1}_{\{r \leq H^\mu\}} \, dr$. Notice $H^{\mu_0} = H^\mu$ and $\mu^0(\{H^{\mu^0}\}) = \mu(\{H^\mu\}) = a > 0$. We define $\bar{w}_0$ by

$$\mathrm{e}^{-x\bar{w}_0(s)} = \mathbb{E}\left[\mathrm{e}^{-\int Y_{r-s}^0 \, \mu^0(dr)}\right].$$

The function $\bar{w}_0$ is a locally bounded non-negative function which solves

$$w(s) + \int_s^{H^\mu} \psi(w(r))dr = \int_{[s,\infty)} \mu(dr) + \int_s^{H^\mu} \phi(\bar{w}(r))\, dr, \quad s \leq H^\mu.$$

Thanks to Proposition 2.1, $\bar{w}_0$ is unique. Since $\bar{w}$ solves the same equation, we deduce that $\bar{w}_0 = \bar{w}$. We also have

$$\mathrm{e}^{-xw_0(s)} = \mathbb{E}\left[\mathrm{e}^{-\int Y^0_{r-s}\, \mu(dr)}\right] \geq \mathbb{E}\left[\mathrm{e}^{-\int Y^0_{r-s}\, \mu^0(dr)}\right].$$

This implies that $w_0 \leq \bar{w}_0 = \bar{w}$.

Assume we proved that $w_{n-1} \leq \bar{w}$ for some $n \geq 1$. Then we can consider the measure $\mu^n(dr) = \mu(dr) + [\phi(\bar{w}(r)) - \phi(w_{n-1}(r))]\mathbf{1}_{\{r \leq H^\mu\}}\, dr$. Notice $H^{\mu^n} = H^\mu$ and $\mu^n(\{H^{\mu^n}\}) = a > 0$. Recall $x = x_0 \geq 0$ and $x_k = 0$ for $k \geq 1$. We define $\bar{w}_n$ by

$$\mathrm{e}^{-x\bar{w}_n(s)} = \mathbb{E}\left[\mathrm{e}^{-\int Y^n_{r-s}\, \mu^n(dr)}\right].$$

The function $\bar{w}_n$ is a locally bounded non-negative function which solves for $s \leq H^\mu$

$$\begin{aligned} w(s) + \int_s^{H^\mu} \psi(w(r))dr &= \int_{[s,\infty)} \mu^n(dr) + \int_s^{H^\mu} \phi(w_{n-1}(r))\, dr \\ &= \int_{[s,\infty)} \mu(dr) + \int_s^{H^\mu} \phi(\bar{w}(r))\, dr. \end{aligned}$$

Thanks to Proposition 2.1, $\bar{w}_n$ is unique. Since $\bar{w}$ solves the same equation, we deduce that $\bar{w}_n = \bar{w}$. We also have

$$\mathrm{e}^{-xw_n(s)} = \mathbb{E}\left[\mathrm{e}^{-\int Y^n_{r-s}\, \mu(dr)}\right] \geq \mathbb{E}\left[\mathrm{e}^{-\int Y^n_{r-s}\, \mu^n(dr)}\right].$$

This implies that $w_n \leq \bar{w}$. Therefore, this holds for all $n \geq 0$, which according to our previous remark entails that $w_\infty = \bar{w}$.

By taking $\mu(dr) = \sum_{k=1}^K \lambda_k \delta_{t_k}(dr)$ for $K \in \mathbb{N}^*$, $\lambda_1, \ldots, \lambda_K \in [0,\infty)$ and $0 \leq t_1 \leq \ldots \leq t_K$, we deduce that X has the same finite marginals distribution as a CB with branching mechanism $\psi - \phi$. Hence X is a CB with branching mechanism $\psi - \phi$.

□

4. The dual to the pruning at node

For $\theta \in \mathbb{R}$, we consider the group of operators $(T_\theta, \theta \in \mathbb{R})$ on the set of real measurable functions defined by

$$T_\theta(f)(\cdot) = f(\theta + \cdot) - f(\theta).$$

In [1], see Theorem 6.1, a pruning with intensity $\theta > 0$, for the continuous random tree (CRT), with branching mechanism ψ is introduced and gives a CRT with branching mechanism $\psi_\theta = T_\theta(\psi)$.

Recall the local time of the height process of the CRT is a (sub-critical or critical) CB process with branching mechanism ψ (see [3]). The above pruning procedure gives a natural construction of a CB process of branching mechanism ψ_θ, which we shall called a pruned CB with intensity $\theta > 0$, from a CB process of branching mechanism ψ. Notice this construction was done under the assumption that $\beta = 0$ (see also [2] when $\beta > 0$ and $\pi = 0$). The case $\beta > 0$ and $\pi \neq 0$ could certainly be handled in a similar way. Intuitively, the pruning correspond to removing all the descendants of an individual with a probability $\mathrm{e}^{-\theta\Delta}$, where Δ correspond to the "number" of its children.

Using the previous Section, we can give a probabilistic interpretation to $T_\theta(\psi)$ as a branching mechanism for some negative values of θ.

Let $\theta_0 = \sup\{\theta \geq 0; \int_{(1,\infty)} \mathrm{e}^{\theta\ell}\ \pi(d\ell) < \infty\}$. Notice that $\theta_0 = 0$ if $\psi'(0^+) = -\infty$, as $\psi'(0^+) = -\infty$ is equivalent to $\int_{(1,\infty)} \ell\ \pi(d\ell) = +\infty$. We assume $\theta_0 > 0$. and we set $\Theta = (0, \theta_0]$ if $\int_{(1,\infty)} \mathrm{e}^{\theta_0\ell}\ \pi(d\ell) < \infty$ and $\Theta = (0, \theta_0)$ otherwise. Let $\theta \in \Theta$. Recall the Lévy measure π in the definition (1) of ψ. We define

$$\phi_\theta(\lambda) = 2\beta\theta\lambda + \int_{(0,\infty)} (\mathrm{e}^{\theta x} - 1)(1 - \mathrm{e}^{-\lambda x})\ \pi(dx)$$

and $\psi_{-\theta} = \psi - \phi_\theta$. It is straightforward to check that ϕ_θ can be seen as an immigration mechanism and that $\psi_{-\theta} = T_{-\theta}(\psi)$. The next Corollary is a direct consequence of the previous Section.

Corollary 4.1. *Let $\theta \in \Theta$. A CBI process Y with branching mechanism ψ and immigration (Y, ϕ_θ) is a CB with branching mechanism $T_{-\theta}(\psi)$.*

In a certain sense the immigration is the dual to the pruning at node. In particular, to build a CB process of branching mechanism ψ from a CB process of branching mechanism $\psi_\theta = T_\theta(\psi)$, with $\theta > 0$, one can add an immigration at time t which rate is proportional to the size of the population at time t and immigration mechanism $\tilde{\phi}_\theta$ defined by:

$$\tilde{\phi}_\theta(\lambda) = \psi_\theta(\lambda) - \psi(\lambda) = 2\beta\theta\lambda + \int_{(0,\infty)} (1 - \mathrm{e}^{-\theta x})(1 - \mathrm{e}^{-\lambda x})\ \pi(dx), \quad \text{for } \lambda \geq 0.$$

can be seen as an immigration mechanism.

In conclusion, we get the following result, whose first part comes from Theorem 6.1 in [1]. As in [1], we assume only for the next Corollary that $\beta = 0$ and $\int_{(0,1)} \ell\ \pi(d\ell) > 0$.

Corollary 4.2. *Let X be a critical or sub-critical CB process with branching mechanism ψ. Let $X^{(\theta)}$ be the pruned CB of X with intensity $\theta > 0$: $X^{(\theta)}$ is a CB process with branching mechanism ψ_θ. The CBI process, Y, with branching mechanism ψ_θ and immigration $(Y, \psi_\theta - \psi)$ is distributed as X.*

5. Application : law of the initial process

We consider a population whose size evolves as $X = (X_t, t \geq 0)$, a CB with branching mechanism ψ. We assume ψ satisfies the hypothesis of Section 2. This population undergoes some irreversible mutations with constant rate. Each mutation produce a new type of individual. The mutation is described by the pruning at node, with rate $\theta > 0$, given in [1] if $\beta = 0$ or in [2] if $\pi = 0$. In the quadratic case ($\pi = 0$) this corresponds to the limit of the Wright-Fisher model, but for the fact that the "size" of the population is not constant but is distributed as a CB. In the case $\beta = 0$, intuitively a mutation occurs to an individual with probability $1 - \mathrm{e}^{-\theta\Delta}$, where Δ is its "number" of children.

We assume the population at time 0 has the same original Eve-type. We are interested in the law of $Y^0 = (Y_t^0, t \geq 0)$, the "size" of the sub-population with the original type knowing the size of the whole population. In particular, we shall compute $\mathbb{P}(Y_t^0 = 0 | X_t > 0)$, the probability for the Eve-type to have disappeared, conditionally on the survival of the total population at time t.

We keep the notation of the previous Sections. Following the idea of Corollary 4.2, we shall assume Y^0 is a CB with branching mechanism $\psi_\theta = T_\theta(\psi)$, where $\theta > 0$, and X the CBI with immigration $(X, \psi_\theta - \psi)$ considered in Section 3.2. Thus, we model the mutations by an immigration process with rate proportional to the size of the population. This interpretation may be more natural than the a posteriori pruning.

The joint law of (X_t, Y_t^0) can be easily characterized by the following Lemma.

Lemma 5.1. *Let $t \geq 0$, $\lambda_1, \lambda_2 \in \mathbb{R}_+$. We assume $X_0 = Y_0^0 = x \geq 0$. We have*

$$\mathbb{E}\left[\mathrm{e}^{-\lambda_1 X_t - \lambda_2 Y_t^0}\right] = \mathrm{e}^{-xw(0)},$$

where (w, w^*) is the unique measurable non-negative solution on $(-\infty, t]$ of

$$w(s) + \int_s^t \psi\big(w(r)\big) dr = \lambda_1 + \lambda_2 + \int_s^t \phi\big(w^*(r)\big) dr, \tag{11}$$

$$w^*(s) + \int_s^t \psi\big(w^*(r)\big) dr = \lambda_1 + \int_s^t \phi\big(w^*(r)\big) dr. \tag{12}$$

Proof. Recall notation of Section 3.2. In particular $x_0 = x$ and $x_n = 0$ for all $n \geq 1$. Let us apply Lemma 3.1 with

$$\mu_0(dr) = (\lambda_1 + \lambda_2)\delta_t(dr),$$
$$\mu_k(dr) = \lambda_1 \delta_t(dr) \quad \text{for } k \geq 1.$$

We get

$$\mathbb{E}\left[\mathrm{e}^{-(\lambda_1 X_t^n + \lambda_2 Y_t^0)}\right] = \mathrm{e}^{-xw_n^{(n)}(0)},$$

where for $s \leq t$,

$$w_0^{(n)}(s) + \int_s^t \psi\big(w_0^{(n)}(r)\big) dr = \lambda_1,$$
$$w_k^{(n)}(s) + \int_s^t \psi\big(w_k^{(n)}(r)\big) dr = \lambda_1 + \int_s^t \phi\big(w_{k-1}^{(n)}(r)\big) dr \quad \text{for } 1 \leq k \leq n-1,$$
$$w_n^{(n)}(s) + \int_s^t \psi\big(w_n^{(n)}(r)\big) dr = \lambda_1 + \lambda_2 + \int_s^t \phi\big(w_{n-1}^{(n)}(r)\big) dr.$$

We let n goes to infinity and use similar arguments as in the proof of Theorem 3.2 to get the result. □

Some more explicit computations can be made in the case of quadratic branching mechanism (see also [11] when $\alpha_0 = 0$). Let $\alpha_0 \geq 0$ and $\theta > 0$ and set

$$\psi(u) = \alpha_0 u + u^2, \quad \psi_\theta(u) = (\alpha_0 + 2\theta)u + u^2.$$

The CB which models the total population is critical ($\alpha_0 = 0$) or sub-critical ($\alpha_0 > 0$). The immigration mechanism is $\psi_\theta(u) - \psi(u) = 2\theta u$.

We set $b = (\alpha_0 + 2\theta)$ and for $t \geq 0$,

$$h(t) = \begin{cases} 1 + \lambda_1 \frac{1 - \mathrm{e}^{-\alpha_0 t}}{\alpha_0} & \text{if } \alpha_0 > 0, \\ 1 + \lambda_1 t & \text{if } \alpha_0 = 0. \end{cases} \tag{13}$$

Proposition 5.2. *Let $t \geq 0$, $\lambda_1, \lambda_2 \in \mathbb{R}_+$. We have*

$$\mathbb{E}\left[\mathrm{e}^{-\lambda_1 X_t - \lambda_2 Y_t^0}\right] = \mathrm{e}^{-xv_0(t)},$$

where

$$v_0(t) = \mathrm{e}^{-bt}\, h(t)^{-2} \left(\frac{1}{\lambda_2} + \int_0^t \mathrm{e}^{-br}\, h(r)^{-2} dr\right)^{-1} + \lambda_1\, \mathrm{e}^{-\alpha_0 t}\, h(t)^{-1}.$$

Proof. By the previous lemma, we have

$$\mathbb{E}\left[\mathrm{e}^{-\lambda_1 X_t - \lambda_2 Y_t^0}\right] = \mathrm{e}^{-xw(0)} \tag{14}$$

where for $s \leq t$,

$$\begin{aligned} w(s) + \int_s^t w(r)\big(w(r)+b\big)dr &= \lambda_1 + \lambda_2 + 2\theta \int_s^t w^*(r)dr, \\ w^*(s) + \int_s^t w^*(r)\big(w^*(r)+\alpha_0\big)dr &= \lambda_1. \end{aligned} \tag{15}$$

The last equation is equivalent to

$$(w^*)' - w^*(w^* + \alpha_0) = 0 \quad \text{on } (-\infty, t], \quad w^*(t) = \lambda_1. \tag{16}$$

The function $z^* := \dfrac{1}{w^*}$ is thus the unique solution of

$$(z^*)' + \alpha_0 z^* + 1 = 0 \quad \text{on } (-\infty, t], \quad z^*(t) = \frac{1}{\lambda_1}.$$

If $\alpha_0 > 0$, this leads to

$$z^*(s) = \frac{1}{\alpha_0}\left(\mathrm{e}^{\alpha_0(t-s)} - 1\right) + \frac{1}{\lambda_1}\,\mathrm{e}^{\alpha_0(t-s)}.$$

If $\alpha_0 = 0$, we have $z^*(s) = t - s + \dfrac{1}{\lambda_1}$. We get

$$w^*(s) = h'(t-s)h(t-s)^{-1} = \lambda_1\,\mathrm{e}^{-\alpha_0(t-s)}\,h(t-s)^{-1}. \tag{17}$$

Equation (15) is equivalent to

$$w' - w(w+b) = -2\theta w^* \quad \text{on } (-\infty, t], \quad w(t) = \lambda_1 + \lambda_2.$$

Set $y = w - w^*$ and use the differential equation (16), to get that y solves

$$y' - y^2 - y(2w^* + b) = 0 \quad \text{on } (-\infty, t], \quad y(t) = \lambda_2.$$

Then the function $z := 1/y$ is the unique solution of

$$z' + (2w^* + b)z + 1 = 0 \quad \text{on } (-\infty, t], \quad z(t) = \frac{1}{\lambda_2}.$$

One solution of the homogeneous differential equation $z_0' = -(2w^*+b)z_0$ is $z_0(s) = \mathrm{e}^{b(t-s)}\,h(t-s)^2$. Looking for solutions of the form $z(s) = C(s)z_0(s)$ gives

$$z(s) = z_0(s)\left(\frac{1}{\lambda_2} + \int_s^t z_0(u)^{-1}du\right).$$

We conclude using (14) and $w = w^* + z^{-1}$. □

Remark 5.3. We can compute the conditional probability of the non extinction of the Eve-population: $\mathbb{P}(Y_t^0 > 0 | X_t > 0)$. However, this computation can be done without the joint law of (X_t, Y_t^0) as

$$\mathbb{P}(Y_t^0 > 0 | X_t > 0) = \frac{\mathbb{P}(Y_t^0 > 0, X_t > 0)}{\mathbb{P}(X_t > 0)} = \frac{\mathbb{P}(Y_t^0 > 0)}{\mathbb{P}(X_t > 0)} = \frac{1 - \mathbb{P}(Y_t^0 = 0)}{1 - \mathbb{P}(X_t = 0)},$$

with $\mathbb{P}(X_t = 0) = \lim_{\lambda_1\to\infty} \mathbb{E}[\mathrm{e}^{-\lambda_1 X_t}] = \mathrm{e}^{-xg(\alpha_0,t)^{-1}}$ and $\mathbb{P}(Y_t^0 = 0) = \lim_{\lambda_2\to\infty} \mathbb{E}[\mathrm{e}^{-\lambda_2 Y_t^0}] = \mathrm{e}^{-xg(b,t)^{-1}}$, where

$$g(a,t) = \begin{cases} \frac{\mathrm{e}^{at}-1}{a} & \text{if } a > 0, \\ t & \text{if } a = 0. \end{cases} \tag{18}$$

The same kind of computation allows also to compute the joint law at different times.

Proposition 5.4. *Let $0 \leq u < t$, $\lambda_1, \lambda_2 \in \mathbb{R}_+$. We have*

$$\mathbb{E}\left[\mathrm{e}^{-\lambda_1 X_t - \lambda_2 Y_u^0}\right] = \mathrm{e}^{-xv_1(u,t)},$$

where

$$v_1(u,t) = \mathrm{e}^{-bt}\, h(t)^{-2} \left(\frac{\mathrm{e}^{-b(t-u)}\, h(t-u)^{-2}}{\lambda_2} + \int_{t-u}^t \mathrm{e}^{-br}\, h(r)^{-2} dr \right)^{-1} + \lambda_1\, \mathrm{e}^{-\alpha_0 t}\, h(t)^{-1}.$$

Proof. Recall notation of Section 3.2. In particular $x_0 = x$ and $x_n = 0$ for all $n \geq 1$. Let us apply Lemma 3.1 with

$$\begin{aligned} \mu_0 &= \lambda_1 \delta_t + \lambda_2 \delta_u, \\ \mu_k &= \lambda_1 \delta_t \qquad \text{for } k \geq 1. \end{aligned}$$

Let n goes to infinity as in the proof of Lemma 5.1 to get that

$$\mathbb{E}\left[\mathrm{e}^{-\lambda_1 X_t - \lambda_2 Y_u^0}\right] = \mathrm{e}^{-xw(0)}, \tag{19}$$

where (w, w^*) is the unique non-negative solution on $(-\infty, t]$ of

$$\begin{aligned} w(s) + \int_s^t \psi\big(w(r)\big) dr &= \lambda_1 + \lambda_2 \mathbf{1}_{\{s\leq u\}} + \int_s^t \phi\big(w^*(r)\big) dr, \\ w^*(s) + \int_s^t \psi\big(w^*(r)\big) dr &= \lambda_1 + \int_s^t \phi\big(w^*(r)\big) dr. \end{aligned} \tag{20}$$

Notice w^* is still given by (17). For $s > u$, we have $w(s) = w^*(s)$ and, for $s \leq u$, Equation (20) is equivalent to

$$w' - w(w+b) = -2\theta w^* \quad \text{on } (-\infty, u], \quad w(t) = w^*(u) + \lambda_2.$$

From the proof of Proposition 5.2, we get

$$\frac{1}{w(s) - w^*(s)} = \mathrm{e}^{b(t-s)}\, h(t-s)^2 \left(\frac{\mathrm{e}^{-b(t-u)}\, h(t-u)^{-2}}{\lambda_2} + \int_s^u \mathrm{e}^{-b(t-r)}\, h(t-r)^{-2} dr \right).$$

We conclude using (19). □

At this stage, we can give the joint distribution of the extinction time of X, $\tau = \inf\{t > 0; X_t = 0\}$, and of Y^0, $\sigma = \inf\{t > 0; Y_t^0 = 0\}$. For $u \leq t$, we have $\mathbb{P}(\tau \leq t, \sigma \leq u) = \lim_{\lambda_1\to\infty,\, \lambda_2\to\infty} v_1(u,t)$ that is

$$\mathbb{P}(\tau \leq t, \sigma \leq u) = \exp -x \left(\mathrm{e}^{-bt} (\int_{t-u}^t \mathrm{e}^{-br}\, g(\alpha_0, t)^2 g(\alpha_0, r)^{-2}\, dr)^{-1} + g(\alpha_0, t)^{-1} \right).$$

We can compute the probability of the Eve-population to live up to time τ conditionally on the value of τ:

$$\mathbb{P}(\sigma = \tau | \tau = t) = 1 - \frac{\lim_{u\uparrow t} \partial_t \mathbb{P}(\sigma \leq u, \tau \leq t)}{\partial_t \mathbb{P}(\tau \leq t)} = \mathrm{e}^{-2(\alpha_0+\theta)t}.$$

See also proposition 5 in [11], where $\alpha_0 = 0$.

We can deduce from the latter Proposition the law of Y_u^0 conditionally on the non-extinction of the whole population. We set

$$A(b,u) = \frac{1}{\lambda_2}\,\mathrm{e}^{bu} + g(b,u).$$

Proposition 5.5. *Let $u \geq 0$, $\lambda_2 \in \mathbb{R}_+$. We have*

$$\lim_{t\to+\infty} \mathbb{E}\left[\mathrm{e}^{-\lambda_2 Y_u^0} \,\middle|\, X_t > 0\right] = \mathrm{e}^{-xA(b,u)^{-1}} \left(1 - A(b,u)^{-2} G(\alpha_0,u)\right),$$

where

$$G(a,u) = \frac{2}{\lambda_2}\,\mathrm{e}^{bu}\, g(\alpha_0,u) + \begin{cases} 2\frac{g(b+\alpha_0,u)-g(b,u)}{\alpha_0} & \text{if } a>0, \\ 2\partial_1 g(b,u) & \text{if } a=0. \end{cases}$$

Proof. We have

$$\mathbb{E}\left[\mathrm{e}^{-\lambda_2 Y_u^0} \,\middle|\, X_t > 0\right] = \frac{\mathbb{E}\left[\mathrm{e}^{-\lambda_2 Y_u^0}\right] - \mathbb{E}\left[\mathrm{e}^{-\lambda_2 Y_u^0}\,\mathbf{1}_{\{X_t=0\}}\right]}{\mathbb{P}(X_t>0)}.$$

Using Proposition 5.4

$$\mathbb{E}\left[\mathrm{e}^{-\lambda_2 Y_u^0}\,\mathbf{1}_{\{X_t=0\}}\right] = \lim_{\lambda_1\to+\infty} \mathbb{E}\left[\mathrm{e}^{-\lambda_2 Y_u^0 - \lambda_1 X_t}\right] = \mathrm{e}^{-x\bar{v}_1(u,t)},$$

with $\bar{v}_1(u,t) = \lim_{\lambda_1\to+\infty} v_1(u,t)$.

Definition (18) implies

$$\bar{v}_1(u,t) = \left(\frac{\mathrm{e}^{bu}\, g(\alpha_0,t)^2}{\lambda_2 g(\alpha_0,t-u)^2} + \mathrm{e}^{bt}\int_{t-u}^{t} \mathrm{e}^{-br}\,\frac{g(\alpha_0,t)^2}{g(\alpha_0,r)^2}dr\right)^{-1} + \mathrm{e}^{-\alpha_0 t}\, g(\alpha_0,t)^{-1}.$$

Performing an asymptotic expansion of $\bar{v}_1$ as t goes to ∞ leads to the result. □

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