Discontinuous Galerkin methods for Friedrichs' systems. Part III. Multi-field theories with partial coercivity

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DISCONTINUOUS GALERKIN METHODS FOR FRIEDRICHS' SYSTEMS. PART III. MULTI-FIELD THEORIES WITH PARTIAL COERCIVITY

A. ERN* AND J.-L. GUERMOND†

Abstract. This paper is the third and last part of a work attempting to give a unified analysis of Discontinuous Galerkin methods. The purpose of this paper is to extend the framework that has been developed in part II for two-field Friedrichs' systems associated with second-order PDE's. We now consider two-field Friedrichs' systems with partial L^2-coercivity and three-field Friedrichs' systems with an even weaker L^2-coercivity hypothesis. In particular, this work generalizes the Discontinuous Galerkin methods of part II to compressible and incompressible linear continuum mechanics. We also show how the stabilizing parameters of the method must be set when the two-field Friedrichs' system is composed of terms that may be of different magnitude, thus accounting for instance for advection–diffusion equations at high Péclet numbers.

Key words. Friedrichs' systems, Finite Elements, Partial Differential Equations, Discontinuous Galerkin Method

AMS subject classifications. 65N30, 65M60, 35F15

1. Introduction.

The framework of Friedrichs' systems [13] is well adapted to the approximation of first-order PDE's by means of Discontinuous Galerkin (DG) methods, since for such systems boundary conditions can be enforced weakly through boundary integrals. The analysis of the approximation of Friedrichs' systems by DG methods has been initiated by Lesaint and Raviart [15, 16] and Johnson et al. [14]. A thorough systematic analysis generalizing [14, 15, 16] has been undertaken in part I [10] and part II [11] of this work. Part I deals with the DG approximation of Friedrichs' systems in general form. Part II specializes the setting to two-field Friedrichs' systems associated with elliptic-like PDE's in mixed form; that is, Friedrichs' systems having a particular two-field structure in which the unknown z can be decomposed into $z = (z^\sigma, z^u)$ and where the σ-component can be eliminated to yield a system of second-order PDE's for the u-component. The two-field DG methods studied in part II are such that z^σ can be locally eliminated on each mesh cell.

The goal of the present work is to extend the analysis of part II in three directions by weakening the L^2-coercivity on which the theory of the two-field Friedrichs' systems is based. First, the L^2-coercivity is assumed to hold only on the σ-component of the field $z = (z^\sigma, z^u)$. Examples include advection–diffusion equations and compressible linear continuum mechanics problems. Second, further weakening of the partial coercivity framework is done by introducing a three-field theory of Friedrichs' systems. This framework encompasses incompressible linear continuum mechanics, e.g., Stokes and Oseen flows. Third, the two-field DG method is revisited by performing a singular perturbation analysis. The goal of this third extension is to determine how the stabilizing parameters of the method must be set when the elliptic-like PDE associated with the two-field Friedrichs' system under scrutiny is composed of a second-order term and a first-order term that may be of different magnitude. The situation covered by this theory is that of advection-diffusion equations at high Péclet numbers.

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This paper is organized as follows. §2 sets the notation and briefly reviews the main results obtained in parts I and II. This section can be skipped by readers who are familiar with the material introduced in parts I and II. §3 treats two-field Friedrichs' systems for which L^2-coercivity holds only on the σ-component. The key difference with part II is that a Poincaré-like inequality must be invoked to transfer the L^2-stability from z^σ to z^u. §4 deals with three-field Friedrichs' systems where the partial coercivity framework is further weakened. In both cases, the well–posedness of the Friedrichs' systems is established and the convergence of their DG approximation is analyzed under general design conditions. Finally, §5 presents a singular perturbation analysis relevant to second-order PDE's where first- and second-order terms are not of the same magnitude. Sections §3, §4, and §5 are independent and can be read separately.

2. DG approximation of Friedrichs' systems. The objective of this section is to set the notation and briefly restate the main results of parts I and II. The reader familiar with this material can jump to §3.

2.1. One-field Friedrichs' systems. Let Ω be a bounded, open, and connected Lipschitz domain in $\mathbb{R}^d$. Let m be a positive integer and set $L = [L^2(\Omega)]^m$ with inner product $(\cdot,\cdot)_L$. The two ingredients to build a Friedrichs' system are an operator $K \in \mathcal{L}(L;L)$ and a family $\{\mathcal{A}^k\}_{1\le k\le d}$ of d functions on Ω with values in $\mathbb{R}^{m,m}$ s.t.

$$\forall k \in \{1,\dots,d\},\ \mathcal{A}^k \in [L^\infty(\Omega)]^{m,m} \quad \text{and} \quad \textstyle\sum_{k=1}^d \partial_k \mathcal{A}^k \in [L^\infty(\Omega)]^{m,m}, \tag{A1}$$

$$\forall k \in \{1,\dots,d\},\ \mathcal{A}^k = (\mathcal{A}^k)^t \ \text{ a.e. in } \Omega, \tag{A2}$$

$$\exists \mu_0 > 0, \quad \forall z \in L, \quad ((K + K^* - \nabla\cdot A)z, z)_L \ge 2\mu_0 \|z\|_L^2, \tag{A3}$$

where K^* is the adjoint of K in $\mathcal{L}(L;L)$ and $\nabla\cdot A \in \mathcal{L}(L;L)$ is defined such that $\nabla\cdot A(z) = (\sum_{k=1}^d \partial_k \mathcal{A}^k) z$ for all $z \in L$.

Let $\mathfrak{D}(\Omega)$ denote the space of $\mathfrak{C}^\infty$ functions that are compactly supported in Ω. A function z in L is said to have an A-weak derivative in L if the linear form $[\mathfrak{D}(\Omega)]^m \ni \phi \longmapsto -\int_\Omega \sum_{k=1}^d z^t \partial_k(\mathcal{A}^k \phi) \in \mathbb{R}$ is bounded on L. In this case, the function in L that can be associated with the above linear form by means of the Riesz representation theorem is denoted by Az. The so-called graph space $W = \{z \in L;\, Az \in L\}$ is endowed with a Hilbert structure when equipped with the scalar product $(z,y)_L + (Az,Ay)_L$. Define the operators $A \in \mathcal{L}(W;L)$ and $\tilde{A} \in \mathcal{L}(W;L)$ by

$$Az = \sum_{k=1}^d \mathcal{A}^k \partial_k z, \qquad \tilde{A} z = -\sum_{k=1}^d \partial_k(\mathcal{A}^k z), \tag{2.1}$$

and set $T = K + A$, $\tilde{T} = K^* + \tilde{A}$. $\tilde{A}$ and $\tilde{T}$ are the formal adjoints of A and T, respectively. Assumption (A3), which implies that $T + \tilde{T}$ is L-coercive on L, is the full L^2-coercivity property alluded to in §1.

Let $f \in L$ and consider the PDE system $Tz = f$. An important question we are facing now is to equip this problem with proper boundary conditions. The key idea underlying the theory of Friedrichs' systems is that boundary conditions can be enforced by making use of a boundary operator $M \in \mathcal{L}(W;W')$ such that

$$\forall z \in W,\ \langle Mz, z\rangle_{W',W} \ge 0, \tag{M1}$$

$$W = \mathrm{Ker}(D - M) + \mathrm{Ker}(D + M), \tag{M2}$$

where $D \in \mathcal{L}(W; W')$ is defined by

$$\forall (z, y) \in W \times W, \quad \langle Dz, y \rangle_{W',W} = (Az, y)_L - (z, \tilde{A}y)_L. \tag{2.2}$$

Observe that (2.2) is just an integration by parts formula and that D is self-adjoint by construction. It is shown in [12] that by setting $V = \mathrm{Ker}(D - M)$ and $V^* = \mathrm{Ker}(D + M^*)$ where M^* is the adjoint operator of M, the following problems are well-posed:

$$\text{Seek } z \in V \text{ such that } Tz = f, \qquad \text{Seek } z^* \in V^* \text{ such that } \tilde{T}z^* = f. \tag{2.3}$$

The key idea sustaining the entire DG theory developed in parts I, II, and hereafter, is that it is possible to enforce boundary conditions weakly by introducing the following bilinear forms on $W \times W$,

$$a(z, y) = (Tz, y)_L + \tfrac{1}{2}\langle (M - D)z, y \rangle_{W',W}, \tag{2.4}$$

$$a^*(z, y) = (\tilde{T}z, y)_L + \tfrac{1}{2}\langle (M^* + D)z, y \rangle_{W',W}, \tag{2.5}$$

and by reformulating (2.3) as follows:

$$\text{Seek } z \in W \text{ such that } a(z, y) = (f, y)_L, \ \forall y \in W, \tag{2.6}$$

$$\text{Seek } z^* \in W \text{ such that } a^*(z^*, y) = (f, y)_L, \ \forall y \in W. \tag{2.7}$$

The key well–posedness result established in part I is the following

THEOREM 2.1. *Assume* (A1)–(A3) *and* (M1)–(M2). *Then, there are unique solutions to* (2.6) *and* (2.7) *and these solutions solve* (2.3).

We finish this section by giving local representations of the operators D and M. Let $n = (n_1, \ldots, n_d)^t$ be the unit outward normal to $\partial\Omega$. Whenever the fields $\{\mathcal{A}^k\}_{1 \le k \le d}$ are sufficiently smooth for the field $\mathcal{D} = \sum_{k=1}^{d} n_k \mathcal{A}^k : \partial\Omega \longrightarrow \mathbb{R}^{m,m}$ to be meaningful at the boundary, the following representation of D holds:

$$\langle Dz, y \rangle_{W',W} = \int_{\partial\Omega} y^t \mathcal{D} z, \tag{2.8}$$

for every smooth functions z and y. Likewise, we henceforth assume that there is a field $\mathcal{M} : \partial\Omega \longrightarrow \mathbb{R}^{m,m}$ such that following representation of M holds for every smooth functions z and y:

$$\langle Mz, y \rangle_{W',W} = \int_{\partial\Omega} y^t \mathcal{M} z. \tag{2.9}$$

2.2. Two-field Friedrichs' systems. We now briefly recall the two-field theory developed in part II. Elliptic-like PDE's in mixed form lead to Friedrichs' systems with the following 2×2 structure: There are two positive integers m_σ and m_u such that $m = m_\sigma + m_u$ and $L = L_\sigma \times L_u$, where $L_\sigma = [L^2(\Omega)]^{m_\sigma}$ and $L_u = [L^2(\Omega)]^{m_u}$, yielding the decomposition $v = (v^\sigma, v^u)$ for all $v \in L$. With obvious notation this yields the following block decompositions

$$K = \begin{bmatrix} K^{\sigma\sigma} & K^{\sigma u} \\ K^{u\sigma} & K^{uu} \end{bmatrix}, \qquad \mathcal{A}^k = \begin{bmatrix} \mathcal{A}^{\sigma\sigma,k} & \mathcal{B}^k \\ (\mathcal{B}^k)^t & \mathcal{C}^k \end{bmatrix}, \tag{2.10}$$

where for all $k \in \{1, \ldots, d\}$, $\mathcal{B}^k$ is an $m_\sigma \times m_u$ matrix field and $\mathcal{C}^k$ an $m_u \times m_u$ matrix field. Assume now that the block $K^{\sigma\sigma}$ has a local representation, i.e., there is $\mathcal{K}^{\sigma\sigma} \in$

$[L^\infty(\Omega)]^{m_\sigma,m_\sigma}$ such that $K^{\sigma\sigma}y^\sigma = \mathcal{K}^{\sigma\sigma}y^\sigma$ for all $y^\sigma \in L_\sigma$ (this localization hypothesis is needed to locally eliminate the σ-component in the two-field DG method described below). The two key-hypotheses on which the two-field theory is based are

$$\mathcal{A}^{\sigma\sigma,k} = 0, \tag{A4}$$

$$\exists k_0 > 0, \quad \mathcal{K}^{\sigma\sigma} \geq k_0 \mathcal{I}_{m_\sigma}, \tag{A5}$$

where $\mathcal{I}_{m_\sigma}$ is the identity matrix in $\mathbb{R}^{m_\sigma,m_\sigma}$. Assumptions (A4)–(A5) allow to eliminate the σ-component of z in the PDE system $Tz = f$ leading to an elliptic-like PDE for the u-component. Furthermore, assumption (A4) yields

$$\mathcal{D} = \left[\begin{array}{c|c} 0 & \mathcal{D}^{\sigma u} \\ \hline \mathcal{D}^{u\sigma} & \mathcal{D}^{uu} \end{array}\right], \tag{2.11}$$

with $\mathcal{D}^{\sigma u} = \sum_{k=1}^d n_k \mathcal{B}^k$, $\mathcal{D}^{u\sigma} = (\mathcal{D}^{\sigma u})^t$, and $\mathcal{D}^{uu} = \sum_{k=1}^d n_k \mathcal{C}^k$.

Henceforth, boundary conditions are enforced by taking

$$\mathcal{M} = \left[\begin{array}{c|c} 0 & -\alpha\mathcal{D}^{\sigma u} \\ \hline \alpha\mathcal{D}^{u\sigma} & \mathcal{M}^{uu} \end{array}\right], \tag{2.12}$$

where $\mathcal{M}^{uu} \in \mathbb{R}^{m_u,m_u}$ is positive and $\alpha \in \{-1,+1\}$. The choice $\alpha = +1$ leads to the Dirichlet boundary condition $z^u \in \mathrm{Ker}(\mathcal{D}^{\sigma u}) \cap \mathrm{Ker}(\mathcal{M}^{uu} - \mathcal{D}^{uu})$. The choice $\alpha = -1$ yields the Robin-type boundary condition $2\mathcal{D}^{u\sigma}z^\sigma + (\mathcal{D}^{uu} - \mathcal{M}^{uu})z^u = 0$; the boundary condition is of Neumann-type if $\mathcal{M}^{uu} = \mathcal{D}^{uu}$ provided $\mathcal{D}^{uu}$ is positive. In practice (see the examples in §3.3 and §3.4), $\mathrm{Ker}(\mathcal{D}^{\sigma u}) = \{0\}$, so that the Dirichlet boundary condition amounts to $z^u = 0$, while the Robin-type boundary condition is enforced by taking $\mathcal{M}^{uu} = |\mathcal{D}^{uu}|$.

It will prove convenient in the sequel to define the operators $B = \sum_{k=1}^d \mathcal{B}^k \partial_k$, $B^\dagger = \sum_{k=1}^d [\mathcal{B}^k]^t \partial_k$, and $C = \sum_{k=1}^d \mathcal{C}^k \partial_k$.

2.3. The discrete setting. Let $\{\mathcal{T}_h\}_{h>0}$ be a family of meshes of Ω. To simplify, we assume that the meshes are affine and that Ω is a polyhedron. For all $K \in \mathcal{T}_h$, $n_K = (n_{K,1}, \ldots, n_{K,d})^t$ denotes the unit outward normal to K and h_K is the diameter of K. We set $h = \max_{K\in\mathcal{T}_h} h_K$ and we denote by $\mathfrak{h}$ the piecewise constant function such that for all $K \in \mathcal{T}_h$, $\mathfrak{h}|_K = h_K$. Henceforth, the notation $\xi \lesssim \zeta$ means that there is a positive c, *independent of* h, such that $\xi \leq c\zeta$.

We denote by $\mathcal{F}_h^{\mathrm{i}}$ the set of mesh interfaces, i.e., $F \in \mathcal{F}_h^{\mathrm{i}}$ if F is a $(d-1)$-manifold and there are $K_1(F)$ and $K_2(F) \in \mathcal{T}_h$ such that $F = K_1(F) \cap K_2(F)$. For $F \in \mathcal{F}_h^{\mathrm{i}}$, we set $\mathcal{T}(F) = K_1(F) \cup K_2(F)$. We denote by $\mathcal{F}_h^\partial$ the set of the faces that separate the mesh from the exterior of Ω, i.e., $F \in \mathcal{F}_h^\partial$ if F is a $(d-1)$-manifold and there is $K(F) \in \mathcal{T}_h$ such that $F = K(F) \cap \partial\Omega$. For $F \in \mathcal{F}_h^\partial$, we set $\mathcal{T}(F) = K(F)$. For all $F \in \mathcal{F}_h^{\mathrm{i}}$, n_F is the unit normal vector on F pointing from $K_1(F)$ to $K_2(F)$, and for all $F \in \mathcal{F}_h^\partial$, n_F is the unit normal vector on F pointing outside Ω. Finally, we set $\mathcal{F}_h = \mathcal{F}_h^{\mathrm{i}} \cup \mathcal{F}_h^\partial$ and for all $F \in \mathcal{F}_h$, h_F denotes the diameter of F. The sole assumption we make on the matching of element faces is that for all $F \in \mathcal{F}_h$, $\max_{K\in\mathcal{T}(F)} h_K \lesssim h_F$. This assumption implies, in particular, that the mesh family $\{\mathcal{T}_h\}_{h>0}$ is shape-regular.

For any measurable subset E of Ω, $(\cdot,\cdot)_{L,E}$ denotes the usual scalar product in $[L^2(E)]^m$. For ease of notation, we define the operators B_h, $B_h^\dagger$, and C_h as the elementwise versions of B, $B^\dagger$, and C, respectively; for instance, for v smooth enough, $(B_h v)|_K = \sum_{k=1}^d \mathcal{B}^k \partial_k(v|_K)$ for all $K \in \mathcal{T}_h$.

Let p be a *non-negative* integer and consider the DG finite element space

$$W_h = [P_{h,p}]^m, \qquad P_{h,p} = \{v_h \in L^2(\Omega); \forall K \in \mathcal{T}_h,\ v_h|_K \in \mathbb{P}_p\}, \tag{2.13}$$

$\mathbb{P}_p$ denoting the vector space of polynomials with real coefficients and with total degree less than or equal to p. Observe that the usual inverse and trace inverse inequalities hold in W_h. For a function v that admits a possibly two-valued trace on mesh interfaces, we define the jump and mean-value of v on $F \in \mathcal{F}_h^{\mathrm{i}}$ as

$$[\![v]\!] = v^1 - v^2, \qquad \{v\} = \tfrac{1}{2}(v^1 + v^2), \tag{2.14}$$

where $v^\gamma(x) = \lim_{y\to x} v(y)|_{K_\gamma(F)}$, $\gamma \in \{1,2\}$. The field $\mathcal{D}$ is extended to $\mathcal{F}_h$ by setting for all $K \in \mathcal{T}_h$, $\mathcal{D} = \sum_{k=1}^d n_{K,k}\mathcal{A}^k$ a.e. on ∂K. Observe that $\mathcal{D}$ is two-valued on $\mathcal{F}_h^{\mathrm{i}}$ with $\{\mathcal{D}\} = 0$ on $\mathcal{F}_h^{\mathrm{i}}$ and that $|\mathcal{D}|$ is well-defined and single-valued since $\mathcal{D}$ is symmetric. We also define $\mathcal{D}_F = \sum_{k=1}^d n_{F,k}\mathcal{A}^k$.

To write a DG method starting from (2.6), we introduce three families of boundary and interface operators $\{M_F\}_{\mathcal{F}_h^\partial}$, $\{S_F\}_{\mathcal{F}_h^{\mathrm{i}}}$, and $\{R_F\}_{\mathcal{F}_h}$. For all $F \in \mathcal{F}_h^\partial$, the role of the operator $M_F \in \mathcal{L}([L^2(F)]^m; [L^2(F)]^m)$ is to weakly enforce the boundary conditions on F. For all $F \in \mathcal{F}_h^{\mathrm{i}}$, the role of the operator $S_F \in \mathcal{L}([L^2(F)]^m; [L^2(F)]^m)$ is to penalize the jump of the discrete unknowns across F. For all $F \in \mathcal{F}_h$, the operator $R_F \in \mathcal{L}([L^2(\mathcal{F}_h)]^m; [L^2(F)]^m)$ is user-defined so as to facilitate the implementation of the method. The design of these operators depends on whether the one-field, the two-field, or the three-field approach is used. Let $W(h) = W_h + [H^1(\Omega)]^m$ and define on $W(h) \times W(h)$ the DG bilinear form

$$\begin{aligned} a_h(z,y) = \sum_{K\in\mathcal{T}_h} (Tz,y)_{L,K} + \sum_{F\in\mathcal{F}_h^\partial} \tfrac{1}{2}(M_F(z) - \mathcal{D}z, y)_{L,F} - \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} 2(\{\mathcal{D}z\}, \{y\})_{L,F} \\ + \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} (S_F([\![z]\!]), [\![y]\!])_{L,F} + \sum_{F\in\mathcal{F}_h} (R_F([\![z]\!]), [\![y]\!])_{L,F}. \end{aligned} \tag{2.15}$$

The first and second terms in the right-hand side are the discrete counterparts of (2.4). The third term is a consistency term; it is zero whenever z is smooth and it is meant to guarantee the L-coercivity of a_h (recall that (A3) and (M1) imply that a is L-coercive). The fourth term is used to control the jump of the discrete solution across interfaces. The last term is a user-defined perturbation whose role may be to alleviate the implementation of the method. The default option is to take $R_F = 0$ in general for the one-field approach, but a nonzero choice must be made if, when using the multi-field approach, one insists on obtaining an Interior Penalty-like method [3], cf. Part II. The reader can take $R_F = 0$ in a first-reading.

The discrete counterpart of (2.6) is formulated as follows:

$$\text{Seek } z_h \in W_h \text{ such that } a_h(z_h, y_h) = (f, y_h)_L,\ \forall y_h \in W_h. \tag{2.16}$$

Problem (2.16) can be equivalently reformulated in local form by introducing the notion of flux: Seek $z_h \in W_h$ such that for all $K \in \mathcal{T}_h$ and for all $y \in [\mathbb{P}_p(K)]^m$,

$$(Kz_h, y)_{L,K} + (z_h, \tilde{A}y)_{L,K} + (\phi_{\partial K}(z_h), y)_{L,\partial K} = (f, y)_{L,K}, \tag{2.17}$$

where the element fluxes are defined on a face $F \subset \partial K$ by

$$\phi_{\partial K}(z)|_F = \begin{cases} \frac{1}{2}\mathcal{D}_F z + \frac{1}{2}M_F(z) + R_F(z), & \text{if } F \in \mathcal{F}_h^\partial, \\ n_F{\cdot}n_K(\mathcal{D}_F\{z\} + S_F([\![z]\!]) + R_F([\![z]\!])), & \text{if } F \in \mathcal{F}_h^{\mathrm{i}}. \end{cases} \tag{2.18}$$

2.4. One-field DG approximation. For the one-field DG method, the operators $\{R_F\}_{\mathcal{F}_h}$ are generally set to zero and the operators $\{M_F\}_{\mathcal{F}_h^\partial}$ and $\{S_F\}_{\mathcal{F}_h^{\mathrm{i}}}$ are designed as follows: For all $v, w \in [L^2(F)]^m$,

$$\mathrm{Ker}(\mathcal{M}-\mathcal{D}) \subset \mathrm{Ker}(M_F - \mathcal{D}), \tag{DG1a}$$

$$(M_F(v), v)_{L,F} \geq 0, \tag{DG1b}$$

$$|(M_F(v) - \mathcal{D}v, w)_{L,F}| \lesssim |v|_{M,F} \|w\|_{L,F}, \tag{DG1c}$$

$$|(M_F(v) + \mathcal{D}v, w)_{L,F}| \lesssim \|v\|_{L,F} |w|_{M,F}, \tag{DG1d}$$

$$S_F = (S_F)^* \text{ and } |\mathcal{D}| \lesssim S_F \lesssim \mathcal{I}_m, \tag{DG1e}$$

where $|v|^2_{M,F} = (M_F(v), v)_{L,F}$, $\mathcal{I}_m$ is the identity matrix in $\mathbb{R}^{m,m}$, and $(S_F)^*$ is the adjoint operator of S_F. Assumption (DG1a) is a consistency assumption meaning that for all $F \in \mathcal{F}_h^\partial$ and for all $v \in [L^2(F)]^m$, $\mathcal{M}v = \mathcal{D}v$ implies $M_F(v) = \mathcal{D}v$. Design conditions slightly more general than (DG1b)–(DG1e) are stated in part I.

To formulate the convergence result, we equip $W(h)$ with the following norms:

$$\|y\|^2_{h,A} = \|y\|^2_L + |y|^2_J + |y|^2_M + \sum_{K\in\mathcal{T}_h} h_K \|Ay\|^2_{L,K}, \tag{2.19}$$

$$\|y\|^2_{h,\frac{1}{2}} = \|y\|^2_{h,A} + \sum_{K\in\mathcal{T}_h} [h_K^{-1}\|y\|^2_{L,K} + \|y\|^2_{L,\partial K}], \tag{2.20}$$

with $|y|^2_J = \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} |[\![y]\!]|^2_{J,F}$, $|y|^2_{J,F} = (S_F([\![y]\!]), [\![y]\!])_{L,F}$, and $|y|^2_M = \sum_{F\in\mathcal{F}_h^\partial} |y|^2_{M,F}$.

The main convergence result derived in part I is the following

THEOREM 2.2. *Assume* (DG1a)–(DG1e), $\mathcal{A}^k \in [\mathfrak{C}^{0,\frac{1}{2}}(K)]^{m,m}$ *for all* $K \in \mathcal{T}_h$ *and all* $1 \leq k \leq d$. *Let* $z \in [H^1(\Omega)]^m \cap V$ *solve* (2.6) *and let* z_h *solve* (2.16). *Then,*

$$\|z - z_h\|_{h,A} \lesssim \inf_{y_h \in W_h} \|z - y_h\|_{h,\frac{1}{2}}. \tag{2.21}$$

Theorem 2.2 yields $(p+\frac{1}{2})$-order convergence in the L-norm and p-order convergence in the broken graph norm if the mesh family is quasi-uniform and z is in $[H^{p+1}(\Omega)]^m \cap V$.

2.5. Two-field DG approximation. Let $p_u > 0$ be a *positive* integer and take $p_\sigma \in \mathbb{N}$ such that $p_u - 1 \leq p_\sigma$. Define the finite element spaces

$$\Sigma_h = [P_{h,p_\sigma}]^{m_\sigma}, \qquad U_h = [P_{h,p_u}]^{m_u}, \qquad W_h = \Sigma_h \times U_h. \tag{2.22}$$

The bilinear form a_h is still defined by (2.15) and the discrete problem is still (2.16).

The design of the operators $\{M_F\}_{F\in\mathcal{F}_h^\partial}$, $\{S_F\}_{F\in\mathcal{F}_h^{\mathrm{i}}}$, and $\{R_F\}_{F\in\mathcal{F}_h^\partial}$ for the two-field DG approximation hinges on that we insist on being able to locally eliminate the discrete component z_h^σ. To this purpose, these operators are designed such that

$$M_F = \left[\begin{array}{c:c} 0 & -\alpha\mathcal{D}^{\sigma u} \\ \hdashline \alpha\mathcal{D}^{u\sigma} & M_F^{uu} \end{array}\right],\ \alpha \in \{-1,+1\}, \quad S_F = \left[\begin{array}{c:c} 0 & 0 \\ \hdashline 0 & S_F^{uu} \end{array}\right], \quad R_F = \left[\begin{array}{c:c} 0 & 0 \\ \hdashline 0 & R_F^{uu} \end{array}\right], \tag{DG2a}$$

$$\text{If } \alpha = +1, \begin{cases} M_F^{uu} = (M_F^{uu})^* \quad \text{and} \quad \mathrm{Ker}(\mathcal{D}^{\sigma u}) \subset \mathrm{Ker}(M_F^{uu} - \mathcal{D}^{uu}), \\ h_F^{-1}(\mathcal{D}^{u\sigma}\mathcal{D}^{\sigma u})^{\frac{1}{2}} + h_F|\mathcal{D}^{uu}| \lesssim M_F^{uu} \lesssim h_F^{-1}\mathcal{I}_{m_u}, \end{cases} \tag{DG2b}$$

$$\text{If } \alpha = -1,\ M_F^{uu}(v) = \mathcal{M}^{uu}v \text{ and } |\mathcal{D}^{uu}| \lesssim \mathcal{M}^{uu} \lesssim \mathcal{I}_{m_u}, \tag{DG2c}$$

$$S_F^{uu} = (S_F^{uu})^* \text{ and } h_F^{-1}(\mathcal{D}^{u\sigma}\mathcal{D}^{\sigma u})^{\frac{1}{2}} + h_F|\mathcal{D}^{uu}| \lesssim S_F^{uu} \lesssim h_F^{-1}\mathcal{I}_{m_u}.. \tag{DG2d}$$

Owing to (DG2A), the σ-component of the element fluxes defined by (2.18) does not depend on z_h^σ, thus allowing for the local elimination of z_h^σ. Design conditions slightly more general than (DG2A)–(DG2D) are stated in part II. Observe that assumptions (DG2A)–(DG2C) imply the consistency condition $\mathrm{Ker}(\mathcal{M}-\mathcal{D}) \subset \mathrm{Ker}(M_F-\mathcal{D})$ and the adjoint-consistency condition $\mathrm{Ker}(\mathcal{M}^t+\mathcal{D}) \subset \mathrm{Ker}(M_F^*+\mathcal{D})$. Indeed, both conditions are evident if $\alpha = -1$ since (DG2A) and (DG2C) imply $M_F = \mathcal{M}$. If $\alpha = +1$, the consistency condition directly results from (DG2B) while the adjoint-consistency condition results from the fact if $z \in \mathrm{Ker}(\mathcal{M}^t + \mathcal{D})$, $\mathcal{D}^{\sigma u} z^u = 0$ and using (DG2B) yields $(\mathcal{M}^{uu})^t z^u = -\mathcal{D}^{uu} z^u = -M_F^{uu}(z^u) = -(M_F^{uu})^*(z^u)$. Finally, we observe that the second part of assumption (DG2C) imposes a condition on the way the Robin–Neumann boundary condition is enforced rather than on the DG setting; in practice, $\mathcal{M}^{uu} = |\mathcal{D}^{uu}|$ (see §3.3.2) so that (DG2C) holds.

To formulate the convergence result, we equip $W(h)$ with the following norms:

$$\|y\|_{h,B}^2 = \|y\|_L^2 + |y^u|_J^2 + |y^u|_M^2 + \|B_h y^u\|_{L_\sigma}^2, \tag{2.23}$$

$$\|y\|_{h,1}^2 = \|y\|_{h,B}^2 + \sum_{K\in\mathcal{T}_h} [h_K^{-2}\|z^u\|_{L_u,K}^2 + h_K^{-1}\|z^u\|_{L_u,\partial K}^2 + h_K\|z^\sigma\|_{L_\sigma,\partial K}^2], \tag{2.24}$$

$$\|y\|_{h,1^+}^2 = \|y\|_{h,1}^2 + \sum_{K\in\mathcal{T}_h} [h_K^2\|y^\sigma\|_{[H^1(K)]^{m_\sigma}}^2 + h_K\|y^\sigma\|_{L_\sigma,\partial K}^2], \tag{2.25}$$

with $|y^u|_J^2 = \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} |y|_{J,F}^2$, $|y^u|_M^2 = \sum_{F\in\mathcal{F}_h^\partial} |y^u|_{M,F}^2$, where

$$|y^u|_{J,F}^2 = (S_F^{uu}([\![y^u]\!]), [\![y^u]\!])_{L_u,F}, \qquad |y^u|_{M,F}^2 = (M_F^{uu}(y^u), y^u)_{L_u,F}. \tag{2.26}$$

The user-dependent operator R_F^{uu} must be designed so that for all $z_h^u \in U_h$ and all $(z^u, y_h^u) \in U(h)\times U_h$

$$\sum_{F\in\mathcal{F}_h} (R_F^{uu}([\![z_h^u]\!]), [\![z_h^u]\!])_{L_u,F} \geq -\frac{1}{4}(|z_h^u|_J^2 + |z_h^u|_M^2) \tag{DG2E}$$

$$\sum_{F\in\mathcal{F}_h} (R_F^{uu}([\![z^u]\!]), [\![y_h^u]\!])_{L_u,F} \leq (|z^u|_J + |z^u|_M)(|y_h^u|_J + |y_h^u|_M). \tag{DG2F}$$

The main convergence result proved in Part II is the following

THEOREM 2.3. *Assume* (DG2A)–(DG2F) *and* $\mathcal{B}^k \in [\mathfrak{C}^{0,1}(K)]^{m_\sigma,m_u}$ *for all* $K \in \mathcal{T}_h$ *and* $1 \leq k \leq d$. *Let* $z \in [H^1(\Omega)]^m \cap V$ *solve* (2.6) *and let* z_h *solve* (2.16). *Then,*

$$\|z - z_h\|_{h,B} \lesssim \inf_{y_h\in W_h} \|z - y_h\|_{h,1}. \tag{2.27}$$

Moreover, if for every $y^u \in L_u$, *the solution* $\psi \in V^*$ *to the dual problem* $\tilde{T}\psi = (0, y^u)$ *is such that* $\|\psi^u\|_{[H^2(\Omega)]^{m_u}} + \|\psi^\sigma\|_{[H^1(\Omega)]^{m_\sigma}} \lesssim \|y^u\|_{L_u}$, *then*

$$\|z^u - z_h^u\|_{L_u} \lesssim h \inf_{y_h\in W_h} \|z - y_h\|_{h,1^+}. \tag{2.28}$$

If the exact solution is in $[H^{p_u}(\Omega)]^{m_\sigma} \times [H^{p_u+1}(\Omega)]^{m_u}$, Theorem 2.3 yields p_u-order convergence in the L_σ-norm for the σ-component and (p_u+1)-order convergence in the L_u-norm and p_u-order convergence in the broken graph norm for the u-component.

3. Two-field theory with L_σ-coercivity only. The goal of this section is to weaken assumption (A3), so as to be able to account for two-field Friedrichs' systems with no L_u-coercivity on the u-component. The model problems we have in mind are advection-diffusion equations with no zero-order term, i.e., no reaction (see §3.3), and compressible linear continuum mechanics (see §3.4).

3.1. Two-field Friedrichs' systems with L_σ-coercivity only. We assume

$$\exists \mu_0 > 0, \quad \forall z \in L, \quad ((K + K^* - \nabla{\cdot}A)z, z)_L \geq 2\mu_0 \|z^\sigma\|^2_{L_\sigma}, \tag{A3A}$$

$$\exists \gamma_1 > 0, \quad \begin{cases} \forall z \in V,\ \gamma_1 \|z^u\|_{L_u} \leq \|z^\sigma\|_{L_\sigma} + \|Tz\|_L, \\ \forall z \in V^*,\ \gamma_1 \|z^u\|_{L_u} \leq \|z^\sigma\|_{L_\sigma} + \|\tilde{T}z\|_L. \end{cases} \tag{A3B}$$

Assumption (A3A) implies that T is L_σ-coercive on $V = \mathrm{Ker}(D - M)$ since the definition of D and assumption (M1) yield for all $z \in V$,

$$\begin{aligned} (Tz, z)_L &= \tfrac{1}{2}[(Tz, z)_L + (z, \tilde{T}z)_L] + \tfrac{1}{2}\langle Dz, z\rangle_{W',W} \\ &= \tfrac{1}{2}[(Tz, z)_L + (z, \tilde{T}z)_L] + \tfrac{1}{2}\langle Mz, z\rangle_{W',W} \geq \mu_0 \|z^\sigma\|^2_{L_\sigma}. \end{aligned} \tag{3.1}$$

Similarly, one proves that $\tilde{T}$ is L_σ-coercive on V^* and that the bilinear forms a and a^* defined by (2.4)–(2.5) are L_σ-coercive on W. One way to establish (A3B), for instance, is to use the Petree–Tartar Lemma by proving that the canonical injection from $W^u = \{z^u \in L_u;\ Bz^u \in L_\sigma\}$ into L_u is compact and that the operator T (resp., $\tilde{T}$) is injective on V (resp., V^*); see Lemma 3.6.

THEOREM 3.1. *The conclusions of Theorem* 2.1 *still hold if assumption* (A3) *is replaced by assumptions* (A3A)–(A3B).

Proof. (1) Let us first prove that $T : V \to L$ is an isomorphism by using the so-called Banach–Nečas–Babuška (BNB) Theorem which states that the bijectivity of $T \in \mathcal{L}(V; L)$ is equivalent to the following conditions [9, p. 85]:

$$\exists \gamma > 0, \quad \forall z \in V, \quad \sup_{y \in L\setminus\{0\}} \frac{(Tz, y)_L}{\|y\|_L} = \|Tz\|_L \geq \gamma \|z\|_W, \tag{3.2}$$

$$\forall y \in L, \quad ((Tz, y)_L = 0,\ \forall z \in V) \implies (y = 0). \tag{3.3}$$

Recall that the graph norm is $\|z\|_W = \|z\|_L + \|Az\|_L$ with $\|z\|_L = \|z^\sigma\|_{L_\sigma} + \|z^u\|_{L_u}$ and $\|Az\|_L = \|Bz^u\|_{L_\sigma} + \|B^\dagger z^\sigma + Cz^u\|_{L_u}$.
(1a) Proof of (3.2). Let $z \in V$. Combining (3.1) together with (A3B) yields

$$\mu_0 \|z^\sigma\|^2_{L_\sigma} \leq \frac{(Tz, z)_L}{\|z\|_L}(\|z^\sigma\|_{L_\sigma} + \|z^u\|_{L_u}) \lesssim \|Tz\|_L(\|z^\sigma\|_{L_\sigma} + \|Tz\|_L),$$

whence it follows that $\|z^\sigma\|_{L_\sigma} \lesssim \|Tz\|_L$. Using again (A3B) leads to $\|z\|_L \lesssim \|Tz\|_L$ and hence, $\|z\|_W = \|z\|_L + \|Az\|_L \lesssim \|z\|_L + \|Tz\|_L \lesssim \|Tz\|_L$.
(1b) Proof of (3.3). Assume that $y \in L$ is such that $(Tz, y)_L = 0$ for all $z \in V$. Following the same arguments as in the proof of Theorem 2.5 in part I or Corollary 5.8 in [9], we infer that $y \in V^*$ and $\tilde{T}y = 0$. The L_σ-coercivity of $\tilde{T}$ on V^* yields $y^\sigma = 0$. That $y^u = 0$ is then a direct consequence of (A3B).
(2) Since $T : V \to L$ is an isomorphism and $V = \mathrm{Ker}(D - M)$, a solution to (2.6) is readily constructed by setting $z = T^{-1}f$. To prove uniqueness, let us prove that the only solution to (2.6) with $f = 0$ is $z = 0$. Since a is L_σ-coercive on W, $z^\sigma = 0$. In addition, taking $y \in [\mathfrak{D}(\Omega)]^m$ in (2.6) yields $Tz = 0$ in L. Using (A3B) yields $z^u = 0$.
(3) Proceed similarly to prove that problem (2.7) is well–posed. ◻

A somewhat simpler framework relevant to elliptic-like PDE's in mixed form consists of replacing assumption (A3B) by

$$K^{\sigma u} = (K^{u\sigma})^* = 0 \text{ and the fields } \mathcal{B}^k \text{ are constant over } \Omega, \tag{A3B'}$$

$$\exists \gamma_1 > 0, \quad \begin{cases} \forall z \in V,\ \gamma_1 \|z^u\|_{L_u} \leq (Tz, z)_L^{\frac{1}{2}} + \|Bz^u\|_{L_\sigma}, \\ \forall z \in V^*,\ \gamma_1 \|z^u\|_{L_u} \leq (\tilde{T}z, z)_L^{\frac{1}{2}} + \|Bz^u\|_{L_\sigma}. \end{cases} \tag{A3B''}$$

Observe that (A3B") is meaningful since T (resp., $\tilde{T}$) is L_σ-coercive on V (resp., V^*).

PROPOSITION 3.2. *Assumptions* (A3B')–(A3B") *imply* (A3B).

Proof. Let $z \in V$. Since $K^{\sigma u} = 0$ owing to (A3B'), one infers that $Bz^u = (Tz)^\sigma - \mathcal{K}^{\sigma\sigma} z^\sigma$. Hence, $\|Bz^u\|_{L_\sigma} \le c(\|z^\sigma\|_{L_\sigma} + \|Tz\|_L)$. Then, (A3B") implies

$$\gamma_1 \|z^u\|_{L_u} \le (Tz, z)_L^{\frac{1}{2}} + c(\|z^\sigma\|_{L_\sigma} + \|Tz\|_L) \le \tfrac{\gamma_1}{2}\|z^u\|_{L_u} + c_{\gamma_1}(\|z^\sigma\|_{L_\sigma} + \|Tz\|_L),$$

whence (A3B) immediately follows. The proof is similar for $z \in V^*$ since (A3B') implies that $Bz^u = (\tilde{T}z)^\sigma - (\mathcal{K}^{\sigma\sigma})^t z^\sigma$. ∎

3.2. Two-field DG approximation with L_σ-coercivity only. Consider the two-field DG method introduced in §2.5 and assume that conditions (DG2A)–(DG2F) are fulfilled. The objective of this section is to analyze the convergence of the two-field DG approximation in the framework of the partial coercivity assumptions (A3A)–(A3B')–(A3B"). The discrete counterpart of assumption (A3B") is

$$\forall z_h \in W_h, \quad \|z_h^u\|_{L_u}^2 \lesssim a_h(z_h, z_h) + \|B_h z_h^u\|_{L_\sigma}^2. \tag{3.4}$$

Recall that the norm $\|\cdot\|_{h,B}$ is defined by (2.23).

LEMMA 3.3. *Assume* (A3A)–(A3B')–(A3B"), (DG2A)–(DG2F), *and* (3.4). *Then,*

$$\forall z_h \in W_h, \qquad \|z_h\|_{h,B} \lesssim \sup_{y_h \in W_h \setminus \{0\}} \frac{a_h(z_h, y_h)}{\|y_h\|_{h,B}}. \tag{3.5}$$

Proof. Let $z_h \in W_h$. Owing to the definition of a_h, (DG2A), (DG2E), and (A3A),

$$\|z_h^\sigma\|_{L_\sigma}^2 + |z_h^u|_J^2 + |z_h^u|_M^2 \lesssim a_h(z_h, z_h). \tag{3.6}$$

Set $\varpi_h = (B_h z_h^u, 0)$ and observe that $\varpi_h \in W_h$ since the fields $\mathcal{B}^k$ are constant over Ω and $p_u - 1 \le p_\sigma$. Moreover,

$$\|B_h z_h^u\|_{L_\sigma}^2 = a_h(z_h, \varpi_h) - (\mathcal{K}^{\sigma\sigma} z_h^\sigma, B_h z_h^u)_{L_\sigma} - \sum_{F \in \mathcal{F}_h^\partial} \tfrac{\alpha+1}{2} (\mathcal{D}^{\sigma u} z_h^u, B_h z_h^u)_{L_\sigma, F}$$
$$+ \sum_{F \in \mathcal{F}_h^{\mathrm{i}}} 2(\{\mathcal{D}^{\sigma u} z_h^u\}, \{B_h z_h^u\})_{L_\sigma, F} := a_h(z_h, \varpi_h) + R_1 + R_2 + R_3.$$

Clearly, $|R_1| \lesssim \|z_h^\sigma\|_{L_\sigma}^2 + \gamma\|B_h z_h^u\|_{L_\sigma}^2$, where $\gamma > 0$ can be chosen as small as needed. If $\alpha = +1$, use (DG2B) and a trace inverse inequality to infer

$$|R_2| \lesssim \sum_{F \in \mathcal{F}_h^\partial} h_F^{\frac{1}{2}} |z_h^u|_{M,F} h_F^{-\frac{1}{2}} \|B_h z_h^u\|_{L_\sigma, \mathcal{T}(F)} \lesssim |z_h^u|_M^2 + \gamma \|B_h z_h^u\|_{L_\sigma}^2,$$

while if $\alpha = -1$, $R_2 = 0$. Finally, using $\{\mathcal{D}^{\sigma u}\} = 0$ and (DG2D) leads to $|R_3| \lesssim |z_h^u|_J^2 + \gamma\|B_h z_h^u\|_{L_\sigma}^2$. Collecting the above bounds yields

$$\|B_h z_h^u\|_{L_\sigma}^2 \lesssim a_h(z_h, \varpi_h) + a_h(z_h, z_h),$$

and owing to (3.4) and (3.6), it is inferred that $\|z_h\|_{h,B}^2 \lesssim a_h(z_h, \varpi_h) + a_h(z_h, z_h)$. Conclude using the fact that $\|\varpi_h\|_{h,B} = \|B_h z_h^u\|_{L_\sigma} \lesssim \|z_h\|_{h,B}$. ∎

It is now straightforward to verify the following convergence result.

THEOREM 3.4. *The statement of Theorem* 2.3 *remains valid under the assumptions of Lemma* 3.3.

3.3. Example 1: Advection–diffusion. Let $f \in L^2(\Omega)$ and let $\beta \in [L^\infty(\Omega)]^d$ with $\nabla{\cdot}\beta \in L^\infty(\Omega)$. Let $\mu \in L^\infty(\Omega)$ and let $\kappa = (\kappa_{kl})_{1\le k,l\le d}$ be a symmetric positive definite tensor-valued field defined on Ω whose lowest eigenvalue is uniformly bounded away from zero. Consider the PDE $-\nabla{\cdot}(\kappa\nabla u) + \beta{\cdot}\nabla u + \mu u = f$ in mixed form

$$\begin{cases} \kappa^{-1}\sigma + \nabla u = 0, \\ \mu u + \nabla{\cdot}\sigma + \beta{\cdot}\nabla u = f. \end{cases} \tag{3.7}$$

Letting $m = d+1$, $m_\sigma = d$, and $m_u = 1$, the mixed formulation (3.7) fits the two-field framework by setting for all $z \in L$ and for all $k \in \{1, \dots, d\}$,

$$K(z) = \left[\begin{array}{c:c} \kappa^{-1} & 0 \\ \hdashline 0 & \mu \end{array}\right] z, \qquad \mathcal{A}^k = \left[\begin{array}{c:c} 0 & e^k \\ \hdashline (e^k)^t & \beta^k \end{array}\right], \tag{3.8}$$

where e^k is the k-th vector in the canonical basis of $\mathbb{R}^d$, and β^k is the k-th component of β. Clearly, hypotheses (A1)–(A2)–(A4)–(A5) hold. We further assume that

$$\inf \text{ess}_\Omega(\mu - \tfrac{1}{2}\nabla{\cdot}\beta) \ge 0 \tag{3.9}$$

so that (A3) does not hold, but (A3A) holds instead with μ_0 equal to the reciprocal of the largest eigenvalue of κ. This situation covers, in particular, the Laplace/Poisson equation where $\mu = 0$ and $\beta = 0$.

The graph space is $W = H(\text{div};\Omega) \times H^1(\Omega)$ and the boundary operator D is such that for all $z, y \in W$,

$$\langle Dz, y\rangle_{W',W} = \langle z^\sigma{\cdot}n, y^u\rangle_{-\frac{1}{2},\frac{1}{2}} + \langle y^\sigma{\cdot}n, z^u\rangle_{-\frac{1}{2},\frac{1}{2}} + \int_{\partial\Omega} (\beta{\cdot}n) z^u y^u, \tag{3.10}$$

where $\langle , \rangle_{-\frac{1}{2},\frac{1}{2}}$ denotes the duality pairing between $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$. Dirichlet boundary conditions can be enforced by setting

$$\langle Mz, y\rangle_{W',W} = \langle z^\sigma{\cdot}n, y^u\rangle_{-\frac{1}{2},\frac{1}{2}} - \langle y^\sigma{\cdot}n, z^u\rangle_{-\frac{1}{2},\frac{1}{2}}, \tag{3.11}$$

yielding $V = H(\text{div};\Omega) \times H^1_0(\Omega)$. Furthermore, mixed Robin–Neumann boundary conditions can be enforced by setting

$$\langle Mz, y\rangle_{W',W} = -\langle z^\sigma{\cdot}n, y^u\rangle_{-\frac{1}{2},\frac{1}{2}} + \langle y^\sigma{\cdot}n, z^u\rangle_{-\frac{1}{2},\frac{1}{2}} + \int_{\partial\Omega} (2\varrho + \beta{\cdot}n) z^u y^u, \tag{3.12}$$

where $\varrho \in L^\infty(\partial\Omega)$ is such that $2\varrho + \beta{\cdot}n \ge 0$ a.e. on $\partial\Omega$. Then $V = \{z \in W;\, z^\sigma{\cdot}n - \varrho z^u|_{\partial\Omega} = 0\}$ and $V^* = \{z \in W;\, z^\sigma{\cdot}n + (\varrho + \beta{\cdot}n)z^u|_{\partial\Omega} = 0\}$. In terms of boundary fields, (3.10), (3.11), (3.12) respectively yield

$$\mathcal{D} = \left[\begin{array}{c:c} 0 & n \\ \hdashline n^t & \beta{\cdot}n \end{array}\right], \qquad \mathcal{M} = \left[\begin{array}{c:c} 0 & -n \\ \hdashline n^t & 0 \end{array}\right], \qquad \mathcal{M} = \left[\begin{array}{c:c} 0 & n \\ \hdashline -n^t & 2\varrho + \beta{\cdot}n \end{array}\right]. \tag{3.13}$$

Observe that $\text{Ker}(\mathcal{D}^{\sigma u}) = \{0\}$. Furthermore, a possible choice for the mixed Robin–Neumann boundary condition is $\varrho = -\min(\beta{\cdot}n, 0)$ (or, equivalently, $\mathcal{M}^{uu} = |\beta{\cdot}n| = |\mathcal{D}^{uu}|$) yielding the usual Robin (inflow) condition $(z^\sigma + \beta z^u){\cdot}n = 0$ on $\partial\Omega^- = \{x \in \partial\Omega;\, \beta(x){\cdot}n(x) < 0\}$ and the usual Neumann (outflow) condition $z^\sigma{\cdot}n = 0$ on $\partial\Omega\backslash\partial\Omega^-$.

3.3.1. Well–posedness. Let us verify that the advection–diffusion equation equipped with the above boundary conditions fits the theoretical framework analyzed in §3.1.

PROPOSITION 3.5. *Assumptions* (A3A)–(A3B')–(A3B") *hold for Dirichlet boundary conditions and for mixed Robin–Neumann boundary conditions provided either* $\mu - \frac{1}{2}\nabla\cdot\beta \neq 0$ *or* $\varrho + \frac{1}{2}\beta\cdot n \neq 0$ *(this means that either* $\mu - \frac{1}{2}\nabla\cdot\beta$ *or* $\varrho + \frac{1}{2}\beta\cdot n$ *is uniformly bounded away from zero on a measurable subset of* Ω *of nonzero measure).*

Proof. Assumptions (A3A)–(A3B') are evident. Observe also that $Bz^u = \nabla z^u$.
(1) For Dirichlet boundary conditions, (A3B") directly results from the Poincaré inequality since $z \in V = V^*$ implies $z^u \in H^1_0(\Omega)$.
(2) For mixed Robin–Neumann boundary conditions, observe that for all $z \in V$,

$$(Tz, z)_L \geq \int_\Omega (\mu - \tfrac{1}{2}\nabla\cdot\beta)(z^u)^2 + \int_{\partial\Omega} (\varrho + \tfrac{1}{2}\beta\cdot n)(z^u)^2.$$

We apply Lemma 3.6 below; it is a simple variant of the Petree–Tartar Lemma (the proof is omitted for brevity). Take $X = H^1(\Omega)$, $Y = [L^2(\Omega)]^d$, $Z = L^2(\Omega)$, $Fx = \nabla x$, $Gx = x$, and $\Phi(x) = \|(\mu - \frac{1}{2}\nabla\cdot\beta)^{\frac{1}{2}} x\|_{L^2(\Omega)} + \|(\varrho + \frac{1}{2}\beta\cdot n)^{\frac{1}{2}} x\|_{L^2(\partial\Omega)}$. Properties (i) and (iii) are evident, property (iv) holds for $\delta = \frac{1}{2}$, while property (ii) results from the fact that if $\|Fx\|_Y + \Phi(x) = 0$, then x is constant (since Ω is connected) and $\Phi(x) = 0$ implies that $x = 0$ since $\mu - \frac{1}{2}\nabla\cdot\beta \neq 0$ or $\varrho + \frac{1}{2}\beta\cdot n \neq 0$. Hence, Lemma 3.6 yields $\gamma_2\|z^u\|_{L_u} \leq (Tz, z)_L^{\frac{1}{2}} + \|\nabla z^u\|_{L_\sigma}$. Proceed similarly for $z \in V^*$. ∎

LEMMA 3.6. *Let* X, Y, Z *be Banach spaces, let* $F \in \mathcal{L}(X; Y)$, *and let* $G \in \mathcal{L}(X; Z)$. *Let* $\Phi : X \to \mathbb{R}_+$ *be a semi-norm. Assume that:*
(i) *G is compact.*
(ii) *For all* $x \in X$, $(\|Fx\|_Y + \Phi(x) = 0) \Rightarrow (x = 0)$.
(iii) *There is* $\gamma_1 > 0$ *such that for all* $x \in X$, $\gamma_1\|x\|_X \leq \|Fx\|_Y + \|Gx\|_Z$.
(iv) *There is* $\delta \in (0, 1)$ *such that for all* $x \in X$, $\Phi(x) \leq c\|Gx\|_Z^\delta \|x\|_X^{1-\delta}$.
Then, there is $\gamma_2 > 0$ *such that for all* $x \in X$, $\gamma_2\|x\|_X \lesssim \|Fx\|_Y + \Phi(x)$.

Remark 3.1. When mixed Robin–Neumann boundary conditions are enforced and $\mu - \frac{1}{2}\nabla\cdot\beta = 0$ and $2\varrho + \beta\cdot n = 0$, the analysis proceeds as follows. If $\mu \neq 0$ or $\varrho \neq 0$, it is easily verified that T is injective on V and that $\tilde{T}$ is injective on V^*; then, using Lemma 3.6 yields (A3B) and thus well–posedness. If $\mu = \nabla\cdot\beta = 0$ and $\varrho = \beta\cdot n = 0$, then T is no longer injective on V, the compatibility condition $\langle f\rangle_\Omega = 0$ must be imposed on the data, and the solution u is subjected to the constraint $\langle u\rangle_\Omega = 0$ (here, for a function $\phi \in L^2(\Omega)$, $\langle\phi\rangle_\Omega := \frac{1}{|\Omega|}\int_\Omega \phi$ where $|\Omega|$ denotes the measure of Ω). Hence, we modify (3.7) as follows:

$$\begin{cases} \kappa^{-1}\sigma + \nabla u = 0, \\ \nabla\cdot\sigma + \beta\cdot\nabla u + \langle u\rangle_\Omega = f, \end{cases} \tag{3.14}$$

equipped with the boundary condition $\sigma\cdot n|_{\partial\Omega} = 0$. Since $\langle\nabla\cdot\sigma + \beta\cdot\nabla u\rangle_\Omega = \langle f\rangle_\Omega = 0$, the second PDE implies $\langle u\rangle_\Omega = 0$, i.e., (3.14) is equivalent to (3.7). Moreover,

$$(Tz, z)_L \geq |\Omega|\langle z^u\rangle_\Omega^2,$$

for all $z \in W$, so that (A3B") results from the fact that for all $\phi \in H^1(\Omega)$, $\|\phi\|_{L_u} \lesssim \langle\phi\rangle_\Omega + \|\nabla\phi\|_{L_\sigma}$, yielding again well–posedness.

3.3.2. Two-field DG approximation. When Dirichlet boundary conditions are enforced, let $\eta_1 > 0$, $\eta_2 > 0$ (these parameters can vary from face to face), and set

$$M_F^{uu}(v) = \eta_1 h_F^{-1} v, \qquad S_F^{uu}(v) = \eta_2 h_F^{-1} v, \qquad R_F^{uu} \equiv 0. \tag{3.15}$$

Since $\mathcal{D}^{u\sigma}\mathcal{D}^{\sigma u} = 1$ and $\mathcal{D}^{uu} = \beta{\cdot}n$, properties (DG2A)–(DG2F) hold. Many other choices can be considered for M_F^{uu}, S_F^{uu}, and R_F^{uu}; see part II and [2] for details.

PROPOSITION 3.7. *Property* (3.4) *holds.*

Proof. Direct consequence of the fact that for all $v_h \in U_h$,

$$\|v_h\|_{L_u}^2 \lesssim \sum_{K\in\mathcal{T}_h} \|\nabla v_h\|_{L_\sigma,K}^2 + \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} h_F^{-1}\|[\![v_h]\!]\|_{L_u,F}^2 + \sum_{F\in\mathcal{F}_h^{\partial}} h_F^{-1}\|v_h\|_{L_u,F}^2.$$

See [1, 4] or [9, p. 134] for the proof. ∎

When mixed Robin–Neumann boundary conditions are enforced, (DG2C) holds for $M_F^{uu}(v) = \mathcal{M}^{uu}v = (2\varrho + \beta{\cdot}n)v$ provided $\varrho \geq -\min(\beta{\cdot}n, 0)$, while S_F^{uu} can be chosen as in (3.15).

PROPOSITION 3.8. *Assume* (3.9) *and either* $\mu - \frac{1}{2}\nabla{\cdot}\beta \neq 0$ *or* $\varrho + \frac{1}{2}\beta{\cdot}n \neq 0$ *and that* Ω *is such that* $H^{\frac{3}{2}+\epsilon}$*-elliptic regularity holds,* $\epsilon > 0$*. Then, property* (3.4) *holds.*

Proof. Let v_h be an arbitrary function in U_h. Let $\psi \in H^1(\Omega)$ solve

$$(\mu - \tfrac{1}{2}\nabla{\cdot}\beta)\psi - \Delta\psi = v_h, \qquad \partial_n\psi|_{\partial\Omega} = -(\rho + \tfrac{1}{2}\beta{\cdot}n)\psi.$$

This problem is well–posed (proceed as in the proof of Proposition 3.5), and the $H^{\frac{3}{2}+\epsilon}$-elliptic regularity hypothesis means that $\|\psi\|_{H^{\frac{3}{2}+\epsilon}} \lesssim \|v_h\|_{L_u}$. Testing with v_h yields

$$\begin{aligned}
\|v_h\|_{L_u}^2 &= ((\mu - \tfrac{1}{2}\nabla{\cdot}\beta)\psi, v_h)_{L_u} + \sum_{K\in\mathcal{T}_h} (\nabla\psi, \nabla v_h)_{L_\sigma,K} \\
&\quad - \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} \int_F 2\nabla\psi{\cdot}\,\{nv_h\} + \sum_{F\in\mathcal{F}_h^{\partial}} \int_F (\varrho + \tfrac{1}{2}\beta{\cdot}n)\psi v_h \\
&\lesssim \|\psi\|_{H^{\frac{3}{2}+\epsilon}} \Bigg(\|(\mu - \tfrac{1}{2}\nabla{\cdot}\beta)^{\frac{1}{2}} v_h\|_{L_u} + \Big(\sum_{K\in\mathcal{T}_h} \|\nabla v_h\|_{L_\sigma,K}^2\Big)^{\frac{1}{2}} + \|v_h\|_J \\
&\qquad + \|(\varrho + \tfrac{1}{2}\beta{\cdot}n)^{\frac{1}{2}} v_h\|_{L_u,\partial\Omega} \Bigg).
\end{aligned}$$

The conclusion follows readily. ∎

3.4. Example 2: Compressible linear continuum mechanics. Let $\beta \in [L^\infty(\Omega)]^d$ with $\nabla{\cdot}\beta \in L^\infty(\Omega)$, let λ, $\gamma \in L^\infty(\Omega)$, and let $f \in [L^2(\Omega)]^d$. Consider the following set of PDE's

$$\begin{cases} \sigma + p\mathcal{I}_d - \frac{1}{2}(\nabla u + (\nabla u)^t) = 0, \\ \mathrm{tr}(\sigma) + (d+\gamma)p = 0, \\ -\frac{1}{2}\nabla{\cdot}(\sigma + \sigma^t) + \beta{\cdot}\nabla u + \lambda u = f, \end{cases} \tag{3.16}$$

where σ is $\mathbb{R}^{d,d}$-valued, p is scalar-valued, and u is $\mathbb{R}^d$-valued. Assuming $\gamma \neq 0$, the first and second equations in (3.16) imply $p = -\gamma^{-1}\nabla{\cdot}u$ and $\sigma = \frac{1}{2}(\nabla u + (\nabla u)^t) + \gamma^{-1}(\nabla{\cdot}u)\mathcal{I}_d$. Equations (3.16) are encountered in flows governed by the linearized

Euler equations and in models of linear continuum mechanics (when $\lambda = 0$ and $\beta = 0$). Henceforth, the tensor σ in $\mathbb{R}^{d,d}$ is identified with the vector $\overline{\sigma} \in \mathbb{R}^{d^2}$ by setting $\overline{\sigma}_{[ij]} = \sigma_{ij}$ with $1 \le i, j \le d$ and $[ij] = d(j-1) + i$.

Set $m = d^2 + 1 + d$, $m_\sigma = d^2 + 1$, and $m_u = d$. Thus, the σ-component in the two-field Friedrichs' system is the pair $(\overline{\sigma}, p)$, and the u-component corresponds to u. The mixed formulation (3.16) fits in the framework of two-field Friedrichs' systems by setting for all $z \in L$ and for all $k \in \{1, \dots, d\}$,

$$K(z) = \left[\begin{array}{cc|c} \mathcal{I}_{d^2} & \mathcal{Z} & 0 \\ (\mathcal{Z})^t & (d+\gamma) & 0 \\ \hline 0 & 0 & \lambda\mathcal{I}_d \end{array}\right] z, \qquad \mathcal{A}^k = \left[\begin{array}{cc|c} 0 & 0 & \mathcal{E}^k \\ 0 & 0 & 0 \\ \hline (\mathcal{E}^k)^t & 0 & \mathcal{C}^k \end{array}\right], \tag{3.17}$$

where $\mathcal{Z} \in \mathbb{R}^{d^2}$ is such that $\mathcal{Z}_{[ij]} = \delta_{ij}$ with $1 \le i, j \le d$, and for all $k \in \{1, \dots, d\}$, $\mathcal{C}^k = \beta^k \mathcal{I}_d \in \mathbb{R}^{d,d}$ and $\mathcal{E}^k \in \mathbb{R}^{d^2,d}$ is such that $\mathcal{E}^k_{[ij],l} = -\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ with $1 \le i, j, l \le d$; here, the δ's are Kronecker symbols. Clearly, hypotheses (A1)–(A2)–(A4)–(A5) hold. We further assume that

$$\lambda_0 := \inf \operatorname{ess}_\Omega(\lambda - \tfrac{1}{2}\nabla{\cdot}\beta) \ge 0, \qquad \gamma_0 := \inf \operatorname{ess}_\Omega \gamma > 0. \tag{3.18}$$

so that (A3) does not hold (note that (A3) would hold if $\inf \operatorname{ess}_\Omega(\lambda - \frac{1}{2}\nabla{\cdot}\beta) > 0$). The case $\lambda_0 \ge 0$ covers, in particular, the usual compressible solid mechanics problems for which $\lambda = 0$ and $\beta = 0$. The incompressible limit $\gamma_0 = 0$ is treated in §4.3.

The graph space is $W = H_{\overline{\sigma}} \times L^2(\Omega) \times [H^1(\Omega)]^d$ with $H_{\overline{\sigma}} = \{\overline{\sigma} \in [L^2(\Omega)]^{d^2};\ \nabla{\cdot}(\sigma + \sigma^t) \in [L^2(\Omega)]^d\}$ and the boundary operator D is such that $\forall z, y \in W$ with $z = (\overline{\sigma}, p, u)$ and $y = (\overline{\tau}, q, v)$,

$$\langle Dz, y\rangle_{W',W} = -\langle \tfrac{1}{2}(\tau{+}\tau^t){\cdot}n, u\rangle_{-\frac{1}{2},\frac{1}{2}} - \langle \tfrac{1}{2}(\sigma{+}\sigma^t){\cdot}n, v\rangle_{-\frac{1}{2},\frac{1}{2}} + \int_{\partial\Omega} (\beta{\cdot}n)uv, \tag{3.19}$$

where $\langle , \rangle_{-\frac{1}{2},\frac{1}{2}}$ denotes the duality pairing between $[H^{-\frac{1}{2}}(\partial\Omega)]^d$ and $[H^{\frac{1}{2}}(\partial\Omega)]^d$. An homogeneous Dirichlet boundary conditions on the u-component is obtained by setting

$$\langle Mz, y\rangle_{W',W} = \langle \tfrac{1}{2}(\tau{+}\tau^t){\cdot}n, u\rangle_{-\frac{1}{2},\frac{1}{2}} - \langle \tfrac{1}{2}(\sigma{+}\sigma^t){\cdot}n, v\rangle_{-\frac{1}{2},\frac{1}{2}} \tag{3.20}$$

yielding $V = V^* = H_{\overline{\sigma}} \times L^2(\Omega) \times [H^1_0(\Omega)]^d$. Similarly, a mixed Robin–Neumann boundary condition is obtained by setting

$$\langle Mz, y\rangle_{W',W} = -\langle \tfrac{1}{2}(\tau{+}\tau^t){\cdot}n, u\rangle_{-\frac{1}{2},\frac{1}{2}} + \langle \tfrac{1}{2}(\sigma{+}\sigma^t){\cdot}n, v\rangle_{-\frac{1}{2},\frac{1}{2}} + \int_{\partial\Omega} (2\varrho{+}\beta{\cdot}n)uv, \tag{3.21}$$

where $\varrho \in L^\infty(\partial\Omega)$ is such that $2\varrho + \beta{\cdot}n \ge 0$ a.e. on $\partial\Omega$. Then $V = \{(\overline{\sigma}, p, u) \in W;\ \frac{1}{2}(\sigma{+}\sigma^t){\cdot}n - \varrho u|_{\partial\Omega} = 0\}$ and $V^* = \{(\overline{\sigma}, p, u) \in W;\ \frac{1}{2}(\sigma{+}\sigma^t){\cdot}n + (\varrho{+}\beta{\cdot}n)u|_{\partial\Omega} = 0\}$. A standard choice for ϱ is $\varrho = -\min(\beta{\cdot}n, 0)$.

In terms of boundary fields, letting $\mathcal{N} = \sum_{k=1}^d n_k \mathcal{E}^k \in \mathbb{R}^{d^2,d}$, (3.19), (3.20), and (3.21) respectively yield

$$\mathcal{D} = \left[\begin{array}{cc|c} 0 & 0 & \mathcal{N} \\ 0 & 0 & 0 \\ \hline \mathcal{N}^t & 0 & (\beta{\cdot}n)\mathcal{I}_d \end{array}\right], \quad \mathcal{M} = \left[\begin{array}{cc|c} 0 & 0 & -\mathcal{N} \\ 0 & 0 & 0 \\ \hline \mathcal{N}^t & 0 & 0 \end{array}\right], \quad \mathcal{M} = \left[\begin{array}{cc|c} 0 & 0 & \mathcal{N} \\ 0 & 0 & 0 \\ \hline -\mathcal{N}^t & 0 & (2\varrho{+}\beta{\cdot}n)\mathcal{I}_d \end{array}\right]. \tag{3.22}$$

Observe that $\operatorname{Ker}(\mathcal{D}^{\sigma u}) = \{0\}$ and that $\mathcal{M}^{uu} = |\mathcal{D}^{uu}|$ in the Robin–Neumann case provided $\varrho = -\min(\beta{\cdot}n, 0)$.

3.4.1. Well–posedness. For the sake of brevity, we restrict ourselves to homogeneous Dirichlet boundary conditions. The case of mixed Robin–Neumann boundary conditions can be treated by proceeding as in §3.3.1.

PROPOSITION 3.9. *Assume $\lambda_0 = 0$ and $\gamma_0 > 0$ and that Dirichlet boundary conditions are enforced. Then, assumptions* (A3A)–(A3B')–(A3B") *hold.*

Proof. Assumptions (A3A)–(A3B') are evident. Moreover, since Dirichlet boundary conditions are enforced on the u-component, $z \in V = V^*$ implies $z^u \in [H^1_0(\Omega)]^d$; hence, (A3B") results from Korn's First Inequality. □

3.4.2. Two-field DG approximation. We assume again for simplicity that homogeneous Dirichlet boundary conditions are enforced. Let $\eta_1 > 0$, $\eta_2 > 0$ (these parameters can vary from face to face), and

$$M^{uu}_F(v) = \eta_1 h_F^{-1} v, \qquad S^{uu}_F(v) = \eta_2 h_F^{-1} v. \tag{3.23}$$

Since $\mathcal{D}^{u\sigma}\mathcal{D}^{\sigma u} = \frac{1}{2}(\mathcal{I}_d + n\otimes n)$ and $\mathcal{D}^{uu} = (\beta{\cdot}n)\mathcal{I}_d$, properties (DG2A)–(DG2D) hold. Many other choices can be considered for M^{uu}_F and S^{uu}_F.

PROPOSITION 3.10. *Property* (3.4) *holds.*

Proof. Direct consequence of the fact that for all $v_h \in U_h$,

$$\|v_h\|^2_{L_u} \lesssim \sum_{K\in\mathcal{T}_h} \|\nabla v_h + (\nabla v_h)^t\|^2_{L_\sigma,K} + \sum_{F\in\mathcal{F}^{\mathrm{i}}_h} h_F^{-1}\|[\![v_h]\!]\|^2_{L_u,F} + \sum_{F\in\mathcal{F}^{\partial}_h} h_F^{-1}\|v_h\|^2_{L_u,F}.$$

See [5, 8] for a proof. □

4. Three-field theory. The goal of this section is to weaken even further the set of hypotheses (A3A)–(A3B')–(A3B") to account for situations that are similar to incompressible linear continuum mechanics. To this purpose, we introduce a three-field theory of Friedrichs' systems and we adapt the DG approximation to this setting. Thus, we assume that z can be decomposed into three fields z^σ, z^p, and z^u, and we have in mind to locally eliminate z^σ.

4.1. Three-field Friedrichs' systems. We now assume that $L = L_\sigma \times L_p \times L_u$, where $L_\sigma = [L^2(\Omega)]^{m_\sigma}$, $L_p = [L^2(\Omega)]^{m_p}$, and $L_u = [L^2(\Omega)]^{m_u}$, with $m = m_\sigma + m_p + m_u$. We also assume that the operator K and the matrices $\mathcal{A}^k$, $1 \le k \le d$, admit the following 3×3 structure:

$$K = \left[\begin{array}{c:c:c} K^{\sigma\sigma} & K^{\sigma p} & 0 \\ \hdashline K^{p\sigma} & K^{pp} & 0 \\ \hdashline 0 & 0 & K^{uu} \end{array}\right] \qquad \text{and} \qquad \mathcal{A}^k = \left[\begin{array}{c:c:c} 0 & 0 & \mathcal{B}^k \\ \hdashline 0 & 0 & 0 \\ \hdashline [\mathcal{B}^k]^t & 0 & \mathcal{C}^k \end{array}\right]. \tag{4.1}$$

We assume that the block $K^{\sigma\sigma}$ has a local representation, i.e., there is $\mathcal{K}^{\sigma\sigma} \in [L^\infty(\Omega)]^{m_\sigma,m_\sigma}$ such that $K^{\sigma\sigma}(y^\sigma) = \mathcal{K}^{\sigma\sigma}y^\sigma$ for all $y^\sigma \in L_\sigma$ (this local representation is needed in the three-field DG method described below to locally eliminate the discrete σ-component). We denote by K^σ (resp., K^p) the canonical projection of K onto L_σ (resp., L_p). We use similar notation for the adjoint of K, say $K^{*\sigma}$ and K^{*p}.

Finally, we assume that there exists an operator $\pi \in \mathcal{L}(L_\sigma; L_\sigma)$ such that

$$\exists \mu_0 > 0, \quad \forall z \in L, \quad ((K + K^* - \nabla\cdot A)z, z)_L \geq 2\mu_0 \|\pi z^\sigma\|^2_{L_\sigma}, \tag{A3c}$$

$$\exists \gamma_1 > 0, \quad \begin{cases} \forall z \in V,\ (Tz, z)_L^{\frac{1}{2}} + \|Bz^u\|_{L_\sigma} \ \geq \gamma_1 \|z^u\|_{L_u}, \\ \forall z \in V^*,\ (\tilde{T}z, z)_L^{\frac{1}{2}} + \|Bz^u\|_{L_\sigma} \geq \gamma_1 \|z^u\|_{L_u}. \end{cases} \tag{A3d}$$

$$\exists \gamma_2 > 0, \quad \begin{cases} \forall z \in V,\ (Tz, z)_L^{\frac{1}{2}} + \|B^\dagger z^\sigma\|_{L_u} \ \geq \gamma_2 \|z^\sigma\|_{L_\sigma}, \\ \forall z \in V^*,\ (\tilde{T}z, z)_L^{\frac{1}{2}} + \|B^\dagger z^\sigma\|_{L_u} \geq \gamma_2 \|z^\sigma\|_{L_\sigma}. \end{cases} \tag{A3e}$$

$$\exists \gamma_3 > 0, \quad \forall z \in W, \quad \|z^u\|_{L_u} + \|Bz^u\|_{L_\sigma} \geq \gamma_3 \|Cz^u\|_{L_u}, \tag{A3f}$$

$$\exists \gamma_4 > 0, \quad \forall z \in L, \quad \begin{cases} \|K^p z\|_{L_p} + \|\pi z^\sigma\|_{L_\sigma} \geq \gamma_4 \|K^\sigma z\|_{L_\sigma}, \\ \|K^{*p} z\|_{L_p} + \|\pi z^\sigma\|_{L_\sigma} \geq \gamma_4 \|K^{*\sigma} z\|_{L_\sigma}, \end{cases} \tag{A3g}$$

$$\exists \gamma_5 > 0, \quad \forall z \in L, \quad \min(\|K^{pp} z^p\|_{L_p}, \|K^{*pp} z^p\|_{L_p}) \geq \gamma_5 \|z^p\|_{L_p}, \tag{A3h}$$

$$\forall k \in \{1, \dots, k\}, \quad \mathcal{B}^k \text{ is constant over } \Omega. \tag{A3i}$$

Observe that the assumptions (M1) and (A3C) yield

$$\forall z \in V, \qquad (Tz, z)_L \geq \mu_0 \|\pi z^\sigma\|^2_{L_\sigma}, \tag{4.2}$$

$$\forall z \in V^*, \qquad (\tilde{T}z, z)_L \geq \mu_0 \|\pi z^\sigma\|^2_{L_\sigma}. \tag{4.3}$$

Theorem 4.1. *The conclusions of Theorem* 2.1 *still hold if assumption* (A3) *is replaced by assumptions* (A3C)–(A3I).

Proof. The proof, which is similar to that of Theorem 3.1, is only sketched. We only prove that $T : V \longrightarrow L$ is an isomorphism, since the rest of the proof is unchanged or goes along the same lines.

(1) Proof of (3.2). Let $z \in V$. Property (4.2) immediately implies $\|\pi z^\sigma\|_{L_\sigma} \lesssim (Tz, z)_L^{\frac{1}{2}}$. Since $Bz^u + K^\sigma z = (Tz)^\sigma$ and $K^p z = (Tz)^p$, we derive using (A3G) and the above bound that

$$\begin{aligned} \|Bz^u\|_{L_\sigma} &\leq \|Bz^u + K^\sigma z\|_{L_\sigma} + \|K^\sigma z\|_{L_\sigma} \leq \|Tz\|_L + \tfrac{1}{\gamma_4}(\|K^p z\|_{L_p} + \|\pi z^\sigma\|_{L_\sigma}) \\ &\lesssim \|Tz\|_L + (Tz, z)_L^{\frac{1}{2}}. \end{aligned}$$

Owing to (A3D) and (A3F), it is inferred that

$$\|z^u\|_{L_u} + \|Cz^u\|_{L_u} \lesssim \|Tz\|_L + (Tz, z)_L^{\frac{1}{2}}.$$

Now we use (A3E), the above bounds, and the fact that $B^\dagger z^\sigma = (Tz)^u - Cz^u - K^{uu} z^u$ to deduce

$$\gamma_2 \|z^\sigma\|_{L_\sigma} \leq \|B^\dagger z^\sigma\|_{L_u} + (Tz, z)_L^{\frac{1}{2}} \lesssim \|Tz\|_L + (Tz, z)_L^{\frac{1}{2}}.$$

To derive a bound on $\|z^p\|_{L_p}$ we use (A3H), the above bounds, and the fact that $K^p z = (Tz)^p$ to infer

$$\gamma_5 \|z^p\|_{L_p} \leq \|K^{pp} z^p\|_{L_p} \leq \|K^p z\|_{L_p} + \|K^{p\sigma} z^\sigma\|_{L_p} \lesssim \|Tz\|_L + (Tz, z)_L^{\frac{1}{2}}.$$

Combining the above bounds yields $\|z\|_L \lesssim \|Tz\|_L + (Tz, z)_L^{\frac{1}{2}}$ and we conclude as usual.

(2) Proof of (3.3). Assume that $y \in L$ is such that $(Tz, y)_L = 0$ for all $z \in V$. Then, $y \in V^*$ and $\tilde{T}y = 0$. Hence (4.3) implies $\pi y^\sigma = 0$. Since $K^{*p}y = 0$ and $\pi y^\sigma = 0$, (A3G) implies $K^{*\sigma}y = 0$. Then, observing that $0 = (\tilde{T}y)^\sigma = K^{*\sigma}y - By^u$ since the fields $\mathcal{B}^k$ are constant, we infer $By^u = 0$. Then using (A3D), this yields $y^u = 0$. Using $0 = (\tilde{T}y)^u$ and the fact that the fields $\mathcal{B}^k$ are constant, we then infer $B^\dagger y^\sigma = 0$ so that (A3E) implies $y^\sigma = 0$. Finally, since $0 = (\tilde{T}y)^p = K^{*pp}y^p$, using (A3H) we infer $y^p = 0$, thus completing the proof. □

4.2. Three-field DG approximation. We analyze in this section a DG method to approximate the three-field Friedrichs' systems introduced in §4.1. We assume that hypotheses (A3C)–(A3I) hold so that the continuous problem is well–posed. The key property of the three-field DG approximation developed hereafter is that the discrete σ-component can be locally eliminated. This strategy differs from the two-field DG approximation of compressible problems analyzed in §3.2, where the pair (σ, p) can be locally eliminated.

Let $p_u > 0$ be a *positive* integer and let p_σ and p_p be such that

$$p_u - 1 \le p_\sigma \le p_u + 1, \qquad p_\sigma \le p_p. \tag{4.4}$$

Consider the finite elements spaces

$$\Sigma_h = [P_{h,p_\sigma}]^{m_\sigma}, \quad P_h = [P_{h,p_p}]^{m_p}, \quad U_h = [P_{h,p_u}]^{m_u}, \quad W_h = \Sigma_h \times P_h \times U_h. \tag{4.5}$$

Consider the discrete problem (2.16) with the bilinear form still defined by (2.15).

We now design the operators M_F and S_F so that the discrete σ-component can be locally eliminated. We consider either Dirichlet boundary conditions or mixed Robin–Neumann boundary conditions enforced by setting $\mathcal{M}^{uu} = |\mathcal{D}^{uu}|$, see §3.3 and §3.4. The design conditions of the three-field DG method are the following:

$$M_F = \begin{bmatrix} 0 & 0 & -\alpha\mathcal{D}^{\sigma u} \\ 0 & 0 & 0 \\ \alpha\mathcal{D}^{u\sigma} & 0 & M_F^{uu} \end{bmatrix}, \quad S_F = \begin{bmatrix} 0 & 0 & 0 \\ 0 & S_F^{pp} & 0 \\ 0 & 0 & S_F^{uu} \end{bmatrix}, \quad R_F = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & R_F^{uu} \end{bmatrix}. \tag{DG3A}$$

$$\text{If } \alpha = +1, \begin{cases} M_F^{uu} = (M_F^{uu})^* \quad \text{and} \quad \mathrm{Ker}(\mathcal{D}^{\sigma u}) \subset \mathrm{Ker}(M_F^{uu} - \mathcal{D}^{uu}), \\ h_F^{-1}(\mathcal{D}^{u\sigma}\mathcal{D}^{\sigma u})^{\frac{1}{2}} + h_F^{-1}|\mathcal{D}^{uu}| \lesssim M_F^{uu} \lesssim h_F^{-1}\mathcal{I}_{m_u}, \end{cases} \tag{DG3B}$$

$$\text{If } \alpha = -1,\ M_F^{uu}(v) = \mathcal{M}^{uu}v, \tag{DG3C}$$

$$S_F^{uu} = (S_F^{uu})^*, \text{ and } h_F^{-1}(\mathcal{D}^{u\sigma}\mathcal{D}^{\sigma u})^{\frac{1}{2}} + h_F^{-1}|\mathcal{D}^{uu}| \lesssim S_F^{uu} \lesssim h_F^{-1}\mathcal{I}_{m_u}, \tag{DG3D}$$

$$S_F^{pp} = (S_F^{pp})^*, \quad \text{and} \quad h_F\mathcal{I}_{m_p} \lesssim S_F^{pp} \lesssim h_F\mathcal{I}_{m_p}. \tag{DG3E}$$

where $\alpha \in \{-1, +1\}$ in the definition of M_F in (DG3A).

Our aim is to control the approximation error in the norm $\|\cdot\|_{h,B}$ defined by

$$\|y\|_{h,B}^2 = \|y\|_L^2 + |y^u|_M^2 + |y^u|_{J^u}^2 + |y^p|_{J^p}^2 + \|B_h y^u\|_{L_\sigma}^2, \tag{4.6}$$

with $|y^u|_{J^u}^2 = \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} |y^u|_{J^u,F}^2$, $|y^p|_{J^p}^2 = \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} |y^p|_{J^p,F}^2$, $|y^u|_M^2 = \sum_{F\in\mathcal{F}_h^\partial} |y^u|_{M,F}^2$,

$$|y^u|_{J^u,F}^2 = (S_F^{uu}([\![y^u]\!], [\![y^u]\!])_{L_u,F}, \quad |y^p|_{J^p,F}^2 = (S_F^{pp}([\![y^p]\!], [\![y^p]\!])_{L_p,F}, \tag{4.7}$$

$$|y^u|_{M,F}^2 = (M_F^{uu}(y^u), y^u)_{L_u,F}. \tag{4.8}$$

The user-dependent operator R_F^{uu} must be designed so that for all $z_h^u \in U_h$ and all $(z^u, y_h^u) \in U(h)\times U_h$

$$\sum_{F\in\mathcal{F}_h} (R_F^{uu}([\![z_h^u]\!]), [\![z_h^u]\!])_{L_u,F} \geq -\frac{1}{4}(|z_h^u|^2_{J^u} + |z_h^u|^2_M) \tag{DG3F}$$

$$\sum_{F\in\mathcal{F}_h} (R_F^{uu}([\![z^u]\!]), [\![y_h^u]\!])_{L_u,F} \leq (|z^u|_{J^u} + |z^u|_M)(|y_h^u|_{J^u} + |y_h^u|_M). \tag{DG3G}$$

The discrete counterpart of assumption (A3D) is still (3.4), while the discrete counterpart of assumption (A3E) is the following: For all $z_h \in W_h$,

$$\|z_h^\sigma\|^2_{L_\sigma} \lesssim a_h(z_h, z_h) + \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} h_F \|[\![z_h^\sigma - \pi z_h^\sigma]\!]\|^2_{L_\sigma,F} + \left(\sup_{0\neq v_h^u\in U_h} \frac{\mathbb{B}_h(z_h^\sigma, v_h^u)}{\|(0,0,v_h^u)\|_{h,B}}\right)^2, \tag{4.9}$$

where $\mathbb{B}_h(z_h^\sigma, v_h^u) := a_h((0,0,z_h^u),(0,0,v_h^u)) - a_h(z_h,(0,0,v_h^u))$, i.e.,

$$\mathbb{B}_h(z_h^\sigma, v_h^u) = (z_h^\sigma, B_h v_h^u)_{L_u} - \tfrac{\alpha+1}{2} \sum_{F\in\mathcal{F}_h^\partial} (\mathcal{D}^{u\sigma} z_h^\sigma, v_h^u)_{L_u,F}.$$

Finally, the discrete counterpart of assumption (A3F) is

$$\forall z_h \in W_h, \qquad \|C_h z_h^u\|_{L_u} \lesssim \|B_h z_h^u\|_{L_\sigma} + \|z_h^u\|_{L_u}. \tag{4.10}$$

Since the jumps of the σ-component are not controlled in the three-field DG method (so as to eliminate this component locally), stability must come from the control on the jumps of the p-component. The link between the jumps of the σ- and p-components is provided by the equation for the p-component. This motivates the following additional localization assumptions:

$$K^p(z) = \mathcal{K}^p z \text{ where } \mathcal{K}^p = [\mathcal{K}^{p\sigma}, \mathcal{K}^{pp}, 0] \in \mathbb{R}^{m_p,m} \text{ is constant over } \Omega, \tag{4.11}$$

$$\exists \mathcal{R} \in \mathbb{R}^{m_\sigma,m_p},\ \forall z_h \in W_h,\ [\![z_h^\sigma - \pi z_h^\sigma]\!] = \mathcal{R}[\![\mathcal{K}^{p\sigma} z_h^\sigma]\!]. \tag{4.12}$$

LEMMA 4.2. *Assume that* (A3C)–(A3I) *and the discrete assumptions* (DG3A)–(DG3G)*,* (3.4)*,* (4.4)*,* (4.9)*,* (4.10)*,* (4.11)*, and* (4.12) *hold. Then, the following holds:*

$$\forall z_h \in W_h, \qquad \|z_h\|_{h,B} \lesssim \sup_{y_h\in W_h\setminus\{0\}} \frac{a_h(z_h, y_h)}{\|y_h\|_{h,B}}. \tag{4.13}$$

Proof. Let $z_h \in W_h$ and set $\mathbb{S} = \sup_{y_h\in W_h\setminus\{0\}} \frac{a_h(z_h,y_h)}{\|y_h\|_{h,B}}$.
(1) Owing to the definition of a_h, (DG3A), (DG3F), and (A3C),

$$\|\pi z_h^\sigma\|^2_{L_\sigma} + |z_h^u|^2_M + |z_h^u|^2_{J^u} + |z_h^p|^2_{J^p} \lesssim a_h(z_h, z_h) \leq \mathbb{S}\,\|z_h\|_{h,B}. \tag{4.14}$$

(2) Control on $\mathcal{K}^p z_h$. Set $y_h = (0, \mathcal{K}^p z_h, 0)$ and observe that $y_h \in W_h$ owing to (4.11) and the fact that $p_\sigma \leq p_p$. Moreover, using a trace inverse inequality and the fact that $S_F^{pp} \lesssim h_F \mathcal{I}_{m_p}$ leads to $\|y_h\|_{h,B} \lesssim \|z_h\|_L \leq \|z_h\|_{h,B}$. Furthermore,

$$a_h(z_h, y_h) = \|\mathcal{K}^p z_h\|^2_{L_p} + \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} (S_F^{pp}([\![z_h^p]\!]), [\![\mathcal{K}^p z_h]\!])_{L_p,F},$$

whence it follows that

$$\|\mathcal{K}^p z_h\|_{L_p}^2 \lesssim \mathbb{S}\,\|z_h\|_{h,B}. \tag{4.15}$$

(3) Control on z_h^u, $C_h z_h^u$, and $B_h z_h^u$. Set $y_h = (B_h z_h^u, 0, 0)$ and observe that $y_h \in W_h$ owing to (A3I) and $p_u - 1 \le p_\sigma$. Moreover, $\|y_h\|_{h,B} \lesssim \|z_h\|_{h,B}$. Furthermore,

$$\|B_h z_h^u\|_{L_\sigma}^2 = a_h(z_h, y_h) - (K^\sigma z_h, B_h z_h^u)_{L_\sigma} - \tfrac{\alpha+1}{2} \sum_{F\in\mathcal{F}_h^\partial} (\mathcal{D}^{\sigma u} z_h^u, B_h z_h^u)_{L_\sigma,F}$$
$$+ \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} 2(\{\mathcal{D}^{\sigma u} z_h^u\}, \{B_h z_h^u\})_{L_\sigma,F} := T_1 + T_2 + T_3 + T_4.$$

Clearly, $|T_1| \lesssim \mathbb{S}\,\|z_h\|_{h,B}$. Moreover, using (A3G), (4.14), and (4.15) yields

$$|T_2| \lesssim \|K^\sigma z_h\|_{L_\sigma} \|B_h z_h^u\|_{L_\sigma} \lesssim \mathbb{S}\,\|z_h\|_{h,B} + \gamma \|B_h z_h^u\|_{L_\sigma}^2,$$

where $\gamma > 0$ can be chosen as small as needed. Similarly, using (DG3B), $\{\mathcal{D}^{\sigma u}\} = 0$, (DG3D), and a trace inverse inequality leads to

$$|T_3| + |T_4| \lesssim (|z_h^u|_M + |z_h^u|_{J^u}) \|B_h z_h^u\|_{L_\sigma} \lesssim \mathbb{S}\,\|z_h\|_{h,B} + \gamma \|B_h z_h^u\|_{L_\sigma}^2.$$

Collecting the above bounds and choosing the γ's small enough, it is inferred that $\|B_h z_h^u\|_{L_\sigma}^2 \lesssim \mathbb{S}\,\|z_h\|_{h,B}$. Then, owing to (3.4), (4.10), and (4.14), this in turn implies

$$\|z_h^u\|_{L_u}^2 + \|C_h z_h^u\|_{L_u}^2 + \|B_h z_h^u\|_{L_\sigma}^2 \lesssim \mathbb{S}\,\|z_h\|_{h,B}. \tag{4.16}$$

(4) Control on z_h^σ. The idea is to use (4.9) by controlling the three terms in the right-hand side of (4.9) , say R_1–R_3.
(4.a) Clearly, $R_1 \lesssim \mathbb{S}\,\|z_h\|_{h,B}$.
(4.b) To control R_2, use (4.11), (4.12), a trace inverse inequality, and (DG3E) to infer

$$\|[\![z_h^\sigma - \pi z_h^\sigma]\!]\|_{L_\sigma,F}^2 \lesssim \|[\![\mathcal{K}^p z_h]\!]\|_{L_p,F}^2 + \|[\![\mathcal{K}^{pp} z_h^p]\!]\|_{L_p,F}^2 \lesssim h_F^{-1} \|\mathcal{K}^p z_h\|_{L_p,\mathcal{T}(F)}^2 + h_F^{-1} |z_h^p|_{J^p,F}^2.$$

Hence, owing to (4.15), $R_2 \lesssim \mathbb{S}\,\|z_h\|_{h,B}$.
(4.c) To control R_3, we first prove that for all $(0,0,v_h^u) \in W_h$,

$$a_h((0,0,z_h^u),(0,0,v_h^u)) \lesssim \mathbb{A}_h := (\mathbb{S}\,\|z_h\|_{h,B})^{\frac{1}{2}} \|(0,0,v_h^u)\|_{h,B}. \tag{4.17}$$

Indeed,

$$a_h((0,0,z_h^u),(0,0,v_h^u)) = (K^{uu} z_h^u, v_h^u)_{L_u} + (C_h z_h^u, v_h^u)_{L_u}$$
$$+ \sum_{F\in\mathcal{F}_h^\partial} \tfrac{1}{2}(M_F^{uu}(z_h^u) - \mathcal{D}^{uu} z_h^u, v_h^u)_{L_u,F} - \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} 2(\{\mathcal{D}^{uu} z_h^u\}, \{v_h^u\})_{L_u,F}$$
$$+ \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} ((S_F^{uu} + R_F^{uu})([\![z_h^u]\!]), [\![v_h^u]\!])_{L_u,F} := T_1 + T_2 + T_3 + T_4 + T_5.$$

Using (4.16) yields $|T_1| + |T_2| \lesssim \mathbb{A}_h$, while (4.14) together with (DG3G) readily yields $|T_5| \lesssim \mathbb{A}_h$. Since $\{\mathcal{D}^{uu}\} = 0$, using (DG3D) and a trace inverse inequality leads to

$$|T_4| \lesssim \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} h_F^{\frac{1}{2}} |z_h^u|_{J^u,F} h_F^{-\frac{1}{2}} \|v_h^u\|_{L_u,\mathcal{T}(F)} \lesssim \mathbb{A}_h.$$

Finally, to control T_3, we proceed similarly using (DG3B) if $\alpha = +1$ to infer $|T_3| \lesssim |z_h^u|_M(|v_h^u|_M + \|v_h^u\|_{L_u}) \lesssim \mathbb{A}_h$, while if $\alpha = -1$, we use (DG3C) and the assumption $\mathcal{M}^{uu} = |\mathcal{D}^{uu}|$ to infer $|T_3| \lesssim |z_h^u|_M |v_h^u|_M \lesssim \mathbb{A}_h$. Collecting the bounds for T_1–T_5 yields (4.17) whence the bound $R_3 \lesssim \mathbb{S}^2 + \mathbb{S}\,\|z_h\|_{h,B}$ is readily inferred.
(4.d) Collecting the bounds for R_1–R_3 yields

$$\|z_h^\sigma\|_{L_\sigma}^2 \leq R_1 + R_2 + R_3 \lesssim \mathbb{S}^2 + \mathbb{S}\,\|z_h\|_{h,B}. \tag{4.18}$$

(5) Control on z_h^p. Using (A3H), (4.15), and (4.18) leads to

$$\|z_h^p\|_{L_p}^2 \lesssim \|K^{pp} z_h^p + K^{p\sigma} z_h^\sigma\|_{L_p}^2 + \|K^{p\sigma} z_h^\sigma\|_{L_p}^2 \lesssim \mathbb{S}^2 + \mathbb{S}\,\|z_h\|_{h,B}. \tag{4.19}$$

(6) Conclusion. Collecting the above bounds yields $\|z_h\|_{h,B}^2 \lesssim \mathbb{S}^2 + \mathbb{S}\,\|z_h\|_{h,B}$, whence (4.13) readily follows. □

Remark 4.1. Assumption (DG3D) (resp., (DG3B)) requires a stronger control on $|\mathcal{D}^{uu}|$ with respect to (DG2D) (resp., (DG2B)). This stronger control is needed to prove (4.17) in step (4.c) of the above proof. This is not really a restriction, since in practice $|\mathcal{D}^{uu}| \lesssim (\mathcal{D}^{u\sigma}\mathcal{D}^{\sigma u})^{\frac{1}{2}}$ so that $h_F^{-1}(\mathcal{D}^{u\sigma}\mathcal{D}^{\sigma u})^{\frac{1}{2}} \lesssim S_F^{uu}$ already yields $h_F^{-1}|\mathcal{D}^{uu}| \lesssim S_F^{uu}$ (see the examples in §3.3 and §3.4).

It is now straightforward to verify the following convergence result.

THEOREM 4.3. *The statement of Theorem* 2.3 *remains valid provided assumption* (A3) *is replaced by* (A3C)–(A3I) *and the discrete assumptions* (DG3A)–(DG3G)*,* (3.4)*,* (4.4)*,* (4.9)*,* (4.10)*,* (4.11)*, and* (4.12) *hold.*

4.3. Example: incompressible linear continuum mechanics. Let us consider problem (3.16) in §3.4; but instead of (3.18), we now assume that

$$\lambda_0 \geq 0, \qquad \gamma_0 = 0. \tag{4.20}$$

This setting covers, in particular, problems in solid mechanics with incompressible materials (i.e., $\lambda = 0$, $\beta = 0$, and $\gamma = 0$) and incompressible Stokes or Oseen flows. For the sake of simplicity, we henceforth restrict ourselves to homogeneous Dirichlet boundary conditions.

4.3.1. Well–posedness. Let us first observe that the pressure is defined up to a constant, since Dirichlet boundary conditions are enforced on the u-component. To avoid this arbitrariness, we choose the representative of the pressure which is of zero mean, i.e., $\langle p\rangle_\Omega = 0$. Accordingly, we modify slightly the equations as follows:

$$\begin{cases} \sigma + \frac{1}{d}\langle \mathrm{tr}(\sigma)\rangle_\Omega \mathcal{I}_d + p\mathcal{I}_d - \frac{1}{2}(\nabla u + (\nabla u)^t) = 0, \\ \mathrm{tr}(\sigma) + dp = 0, \\ -\frac{1}{2}\nabla\cdot(\sigma + \sigma^t) + \beta\cdot\nabla u + \lambda u = f, \end{cases} \tag{4.21}$$

Note that by taking the trace of the first equation, accounting for the second one, and integrating over Ω, we obtain $\langle \mathrm{tr}(\sigma)\rangle_\Omega = \langle \nabla\cdot u\rangle_\Omega$. For homogeneous Dirichlet boundary conditions on u, this gives $\langle \mathrm{tr}(\sigma)\rangle_\Omega = 0$. Hence, we are really solving (3.16) with $\gamma = 0$ and $\langle p\rangle_\Omega = 0$. To alleviate the notation, we henceforth use the same symbol for a field with values in $\mathbb{R}^{d,d}$ and the associated field with values in $\mathbb{R}^{d^2}$.

Let us define the operator $\pi \in \mathcal{L}(L_\sigma; L_\sigma)$ such that for all $\sigma \in L_\sigma$

$$\pi\sigma = \sigma - \tfrac{1}{d}(\mathrm{tr}(\sigma) - \langle \mathrm{tr}(\sigma)\rangle_\Omega)\mathcal{I}_d. \tag{4.22}$$

Observe that π is a projection, $\pi^2\sigma = \pi\sigma$, and this projection is orthogonal in L_σ, $(\pi\sigma, \sigma - \pi\sigma)_{L_\sigma} = 0$.

PROPOSITION 4.4. *Assumptions* (A3C)–(A3I) *hold.*

Proof. Assumptions (A3H) and (A3I) are evident, while assumption (A3F) results from Korn's Second Inequality. Assumption (A3D) is a simple consequence of Korn's First Inequality since $V = V^* = H_{\overline{\sigma}} \times L^2(\Omega) \times [H_0^1(\Omega)]^d$. Let us now prove (A3C). Since $(\sigma,\sigma)_{L_\sigma} = (\pi\sigma,\pi\sigma)_{L_\sigma} + (\sigma-\pi\sigma,\sigma-\pi\sigma)_{L_\sigma}$ and $(\sigma-\pi\sigma,\sigma-\pi\sigma)_{L_\sigma} = \frac{1}{d}|\Omega|(\langle \mathrm{tr}(\sigma)^2\rangle_\Omega - \langle \mathrm{tr}(\sigma)\rangle_\Omega^2)$, it is clear that for all $z = (\sigma, p, 0) \in W$,

$$
\begin{aligned}
(Kz, z)_L &\geq \int_\Omega [\sigma^2 + \tfrac{1}{d}\langle \mathrm{tr}(\sigma)\rangle_\Omega \mathrm{tr}(\sigma) + 2p\,\mathrm{tr}(\sigma) + dp^2] \\
&= \int_\Omega [(\pi\sigma)^2 + (\sigma - \pi\sigma)^2 + \tfrac{1}{d}\langle \mathrm{tr}(\sigma)\rangle_\Omega^2 - \tfrac{1}{d}\mathrm{tr}(\sigma)^2 + \tfrac{1}{d}(\mathrm{tr}(\sigma) + dp)^2] \\
&= \int_\Omega [(\pi\sigma)^2 + \tfrac{1}{d}(\mathrm{tr}(\sigma) + dp)^2] \geq \|\pi\sigma\|_{L_\sigma}^2.
\end{aligned}
$$

Since K is self-adjoint and the fields $\mathcal{B}^k$ are constant over Ω, this yields for all $z = (\sigma, p, u) \in W$,

$$
\tfrac{1}{2}((K + K^* - \nabla\cdot A)z, z)_L \geq \|\pi\sigma\|_{L_\sigma}^2 + \lambda_0 \|u\|_{L_u}^2 \geq \|\pi\sigma\|_{L_\sigma}^2,
$$

since $\lambda_0 \geq 0$. This proves (A3C). To prove (A3G), observe that $K^\sigma z = \pi\sigma + \frac{1}{d}(K^p z)\mathcal{I}_d$ and that K is self-adjoint. Finally, let us prove (A3E). Let $z \in V$ be such that $\sigma \neq \pi\sigma$; then, $\mathrm{tr}(\sigma - \pi\sigma) \neq 0$. Since $\mathrm{tr}(\sigma - \pi\sigma) = \mathrm{tr}(\sigma) - \langle \mathrm{tr}(\sigma)\rangle_\Omega \in L_0^2(\Omega)$, there is $0 \neq v \in [H_0^1(\Omega)]^d$ such that $\nabla\cdot v = \mathrm{tr}(\sigma - \pi\sigma)$ and $\|v\|_{[H^1(\Omega)]^d} \lesssim \|\sigma - \pi\sigma\|_{L_\sigma}$. Since $\langle \nabla\cdot v\rangle_\Omega = 0$, we have $Bv - \pi Bv = \sigma - \pi\sigma$. Hence,

$$
\begin{aligned}
\|\sigma - \pi\sigma\|_{L_\sigma} &\lesssim \tfrac{1}{\|v\|_{[H^1(\Omega)]^d}}(Bv - \pi Bv, \sigma - \pi\sigma)_{L_\sigma} \lesssim \tfrac{1}{\|v\|_{[H^1(\Omega)]^d}}(Bv, \sigma - \pi\sigma)_{L_\sigma} \\
&\lesssim \|\pi\sigma\|_{L_\sigma} + \tfrac{1}{\|v\|_{[H^1(\Omega)]^d}}(Bv, \sigma)_{L_\sigma} \lesssim \|\pi\sigma\|_{L_\sigma} + \|B^\dagger\sigma\|_{L_u},
\end{aligned}
$$

whence (A3E) follows using (4.2) and the triangle inequality. □

4.3.2. Three-field DG approximation. Let $\eta_1 > 0$, $\eta_2 > 0$, $\eta_3 > 0$ (these parameters can vary from face to face) and

$$
M_F^{uu}(v) = \eta_1 h_F^{-1} v, \qquad S_F^{uu}(v) = \eta_2 h_F^{-1} v, \qquad S_F^{pp}(q) = \eta_3 h_F q, \qquad R_F^{uu} = 0. \quad (4.23)
$$

Clearly, assumptions (DG3B)–(DG3G) hold. Other choices can be considered for M_F^{uu}, S_F^{uu}, and S_F^{pp}; see, e.g., [7, 6] for a similar DG method to approximate the Stokes and the Oseen equations.

PROPOSITION 4.5. *The discrete assumptions* (3.4)*,* (4.9)*,* (4.10)*,* (4.11)*, and* (4.12) *hold.*

Proof. The discrete Poincaré inequality (3.4) has already been shown to hold in §3.4. Furthermore, assumptions (4.10), (4.11), and (4.12) are evident. It remains to prove (4.9). Let $z_h \in W_h$ such that $z_h^\sigma \neq \pi z_h^\sigma$. Proceeding as in the proof of (A3E) in Proposition 4.4, there is $0 \neq v \in [H_0^1(\Omega)]^d$ such that $\nabla\cdot v = \mathrm{tr}(z_h^\sigma - \pi z_h^\sigma)$, $\|v\|_{[H^1(\Omega)]^d} \lesssim \|z_h^\sigma - \pi z_h^\sigma\|_{L_\sigma}$, and

$$
\|z_h^\sigma\|_{L_\sigma} \leq \|\pi z_h^\sigma\|_{L_\sigma} + \|z_h^\sigma - \pi z_h^\sigma\|_{L_\sigma} \lesssim \|\pi z_h^\sigma\|_{L_\sigma} + \tfrac{1}{\|v\|_{[H^1(\Omega)]^d}}(Bv, z_h^\sigma)_{L_\sigma}.
$$

Since $v \in [H^1_0(\Omega)]^d$, integration by parts yields

$$(Bv, z_h^\sigma)_{L_\sigma} = -(v, B^\dagger z_h^\sigma)_{L_u} + 2 \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} (\{\mathcal{D}^{u\sigma} z_h^\sigma\}, v)_{L_u,F}.$$

Let v_h be the L_u-orthogonal projection of v onto U_h. Then using $p_\sigma \le p_u + 1$ and $\|v - v_h\|_{L_u,\partial K} \lesssim h_K^{\frac{1}{2}} \|v\|_{[H^1(K)]^d}$ for all $K \in \mathcal{T}_h$, we infer

$$\begin{aligned}(Bv, z_h^\sigma)_{L_\sigma} =&\ -(v - v_h, B_h^\dagger z_h^\sigma)_{L_u} + 2 \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} (\{\mathcal{D}^{u\sigma} z_h^\sigma\}, \{v - v_h\})_{L_u,F} \\ & (B_h v_h, z_h^\sigma)_{L_u} - \sum_{F\in\mathcal{F}_h^{\partial}} (\{\mathcal{D}^{u\sigma} z_h^\sigma\}, \{v_h\})_{L_u,F} \\ \lesssim & \left(\|\pi z_h^\sigma\|_{L_\sigma}^2 + \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} h_F \|[\![z_h^\sigma - \pi z_h^\sigma]\!]\|_{L_\sigma,F}^2 \right)^{\frac{1}{2}} \|v\|_{[H^1(\Omega)]^d} + \mathbb{B}_h(z_h^\sigma, v_h),\end{aligned}$$

whence (4.9) readily follows since $\|(0, 0, v_h)\|_{h,B} \lesssim \|v\|_{[H^1(\Omega)]^d}$. □

5. Singular perturbation analysis. The purpose of this section is to show how the stabilizing parameters of the DG method must be set when the Friedrichs' system is composed of terms that are of different magnitude. The situation we want to analyze is that of two-field Friedrichs' systems where the off-diagonal term $\mathcal{B}^k$ coupling the σ- and u-components takes arbitrarily small values. To avoid irrelevant technicalities, we henceforth assume that (A1)–(A5) hold, i.e., full L-coercivity holds. Hypothesis (A3) can be replaced by the weaker hypotheses introduced in §3, but these developments are omitted for brevity. The singular perturbation analysis for the three-field DG approximation will be reported elsewhere.

5.1. The setting. Let $1 \ge \epsilon > 0$ be a positive real number. The setting of §2.2 is modified by considering the following two-field structure:

$$K = \left[\begin{array}{c:c} K^{\sigma\sigma} & K^{\sigma u} \\ \hdashline K^{u\sigma} & K^{uu} \end{array}\right], \qquad \mathcal{A}^k = \left[\begin{array}{c:c} 0 & \epsilon^{\frac{1}{2}}\mathcal{B}^k \\ \hdashline \epsilon^{\frac{1}{2}}[\mathcal{B}^k]^t & \mathcal{C}^k \end{array}\right], \tag{5.1}$$

where it is assumed that all the blocks of the operator K as well as the fields $\mathcal{B}^k$ and $\mathcal{C}^k$ are independent of the parameter ϵ.

Owing to (5.1), the definitions (2.11) and (2.12) are now replaced by

$$\mathcal{D} = \left[\begin{array}{c:c} 0 & \epsilon^{\frac{1}{2}}\mathcal{D}^{\sigma u} \\ \hdashline \epsilon^{\frac{1}{2}}\mathcal{D}^{u\sigma} & \mathcal{D}^{uu} \end{array}\right], \qquad \mathcal{M} = \left[\begin{array}{c:c} 0 & -\alpha\epsilon^{\frac{1}{2}}\mathcal{D}^{\sigma u} \\ \hdashline \alpha\epsilon^{\frac{1}{2}}\mathcal{D}^{u\sigma} & \mathcal{M}^{uu} \end{array}\right]. \tag{5.2}$$

The discrete problem we consider is (2.16) with the bilinear form a_h still defined by (2.15). As in §2.5, we assume that p_u is a *positive* integer and that $p_u - 1 \le p_\sigma \le p_u$.

Henceforth the notation $\xi \lesssim \zeta$ now means that there is a positive c, *independent of h and ϵ*, such that $\xi \le c\zeta$.

5.2. Design of the boundary and jump operators. To avoid unnecessary technicalities we assume that the user-dependent operator R_F^{uu} is zero. Everything that is said hereafter extends to IP-like methods provided the assumptions (DG2E)-(DG2F) are localized. The details are left to the reader. To account for the presence of ϵ, we modify the design conditions (DG2A)–(DG2D) for the operators M_F and S_F as follows:

$$M_F = \left[\begin{array}{c|c} 0 & -\alpha\epsilon^{\frac{1}{2}}\mathcal{D}^{\sigma u} \\ \hline \alpha\epsilon^{\frac{1}{2}}\mathcal{D}^{u\sigma} & M_F^{uu} \end{array}\right],\ \alpha \in \{-1,+1\}, \qquad S_F = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & S_F^{uu} \end{array}\right], \tag{DG2$_\epsilon$A}$$

$$\text{If } \alpha = +1, \begin{cases} M_F^{uu} = (M_F^{uu})^* \quad \text{and} \quad \mathrm{Ker}(\mathcal{D}^{\sigma u}) \subset \mathrm{Ker}(M_F^{uu} - \mathcal{D}^{uu}), \\ \epsilon(\mathcal{D}^{u\sigma}\mathcal{D}^{\sigma u})^{\frac{1}{2}} + h_F|\mathcal{D}^{uu}| \lesssim h_F M_F^{uu} \lesssim \theta_F \mathcal{I}_{m_u}, \end{cases} \tag{DG2$_\epsilon$B}$$

$$\text{If } \alpha = -1,\ M_F^{uu}(v) = \mathcal{M}^{uu}v \text{ and } |\mathcal{D}^{uu}| \lesssim \mathcal{M}^{uu} \lesssim \mathcal{I}_{m_u}, \tag{DG2$_\epsilon$C}$$

$$S_F^{uu} = (S_F^{uu})^* \text{ and } \epsilon(\mathcal{D}^{u\sigma}\mathcal{D}^{\sigma u})^{\frac{1}{2}} + h_F|\mathcal{D}^{uu}| \lesssim h_F S_F^{uu} \lesssim \theta_F \mathcal{I}_{m_u}, \tag{DG2$_\epsilon$D}$$

where we have set

$$\theta_S = \max(\epsilon, h_S), \qquad \forall S \in \mathcal{F}_h \cup \mathcal{T}_h. \tag{5.3}$$

If $\epsilon \ll h$, (DG2$_\epsilon$D) amounts to $|\mathcal{D}^{uu}| \lesssim S_F^{uu} \lesssim \mathcal{I}_{m_u}$, that is, assumption (DG1E) for one-field Friedrichs' systems is recovered for the (uu)-blocks. If $\epsilon \sim 1$, (DG2$_\epsilon$D) leads to assumption (DG2D) for two-field Friedrichs' systems concerning the control on $(\mathcal{D}^{u\sigma}\mathcal{D}^{\sigma u})^{\frac{1}{2}}$ and leads to a slightly stronger control on $\mathcal{D}^{uu}$, namely $|\mathcal{D}^{uu}| \lesssim M_F^{uu}$. The reason for this difference is that in the present analysis, we aim at obtaining a sharper convergence result for the u-component.

5.3. Convergence analysis. For all $z \in W(h)$, we introduce the following norms:

$$\|z\|_{h\epsilon,A}^2 = \|z\|_L^2 + |z^u|_J^2 + |z^u|_M^2 + \|\epsilon^{\frac{1}{2}} B_h z^u\|_{L_\sigma}^2 + \|\mathfrak{h}^{\frac{1}{2}} C_h z^u\|_{L_u}^2, \tag{5.4}$$

$$\|z\|_{h\epsilon,1}^2 = \|z\|_{h\epsilon,A}^2 + \sum_{K\in\mathcal{T}_h} [\theta_K h_K^{-1}(h_K^{-1}\|z^u\|_{L_u,K}^2 + \|z^u\|_{L_u,\partial K}^2) + h_K\|z^\sigma\|_{L_\sigma,\partial K}^2]. \tag{5.5}$$

We denote by $\mathcal{T}_h^+$ the set of mesh cells K such that $h_K \geq \epsilon$. We also denote by $\mathcal{F}_h^{i+}$ the set of faces F such that $\max_{K\in\mathcal{T}(F)} h_K \geq \epsilon$; observe that $h_F \gtrsim \epsilon$ whenever $F \in \mathcal{F}_h^{i+}$. The same definition applies for $\mathcal{F}_h^{\partial+}$.

LEMMA 5.1. *Assume $\mathcal{B}^k \in [\mathfrak{C}^{0,1}(K)]^{m_\sigma,m_u}$ and $\mathcal{C}^k \in [\mathfrak{C}^{0,\frac{1}{2}}(K)]^{m_u,m_u}$ for all $K \in \mathcal{T}_h$ and all $1 \leq k \leq d$, and that*

$$\forall z_h \in W_h, \quad \|C_h z_h^u\|_{L_u} \lesssim \|B_h z_h^u\|_{L_\sigma} + \|z_h^u\|_{L_u}. \tag{5.6}$$

Then,

$$\forall z_h \in W_h, \qquad \|z_h\|_{h\epsilon,A} \lesssim \sup_{y_h \in W_h\setminus\{0\}} \frac{a_h(z_h, y_h)}{\|y_h\|_{h\epsilon,A}}. \tag{5.7}$$

Proof. Let $z_h \in W_h$ and set $\mathbb{S} = \sup_{y_h\in W_h\setminus\{0\}} \frac{a_h(z_h,y_h)}{\|y_h\|_{h\epsilon,A}}$.
(1) Using the definition of a_h, (DG2$_\epsilon$A) and (A3) yields

$$\|z_h\|_L^2 + |z_h^u|_J^2 + |z_h^u|_M^2 \lesssim a_h(z_h, z_h) \leq \mathbb{S}\,\|z_h\|_{h\epsilon,A}. \tag{5.8}$$

(2) Define the field π_h^σ such that for all $K \in \mathcal{T}_h$, $\pi_h^\sigma|_K = \sum_{k=1}^d \epsilon^{\frac{1}{2}} \overline{\mathcal{B}_K^k}\partial_k z_h^u$, where $\overline{\mathcal{B}_K^k}$ denotes the mean-value of $\mathcal{B}^k$ over K. Owing to the regularity of the $\mathcal{B}^k$'s, a standard inverse inequality, and the fact that $\epsilon \leq 1$,

$$\|(\pi_h^\sigma, 0)\|_{h\epsilon,A} = \|\pi_h^\sigma\|_{L_\sigma} \lesssim \|\epsilon^{\frac{1}{2}} B_h z_h^u\|_{L_\sigma} + \epsilon^{\frac{1}{2}}\|z_h^u\|_{L_u} \lesssim \|z_h\|_{h\epsilon,A}. \tag{5.9}$$

From the definition of a_h and ($\text{DG}2_\epsilon\text{A}$), it follows that

$$\begin{aligned}\|\epsilon^{\frac12}B_h z_h^u\|_{L_\sigma}^2 &= a_h(z_h,(\pi_h^\sigma,0)) + (\epsilon^{\frac12}B_h z_h^u, \epsilon^{\frac12}B_h z_h^u - \pi_h^\sigma)_{L_\sigma}\\ &\quad - (K^{\sigma\sigma}z_h^\sigma + K^{\sigma u}z_h^u, \pi_h^\sigma)_{L_\sigma} + \sum_{F\in\mathcal{F}_h^\partial} \tfrac{\alpha+1}{2}\epsilon^{\frac12}(\mathcal{D}^{\sigma u}z_h^u,\pi_h^\sigma)_{L_\sigma,F}\\ &\quad + \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} 2\epsilon^{\frac12}(\{\mathcal{D}^{\sigma u}z_h^u\},\{\pi_h^\sigma\})_{L_\sigma,F}.\end{aligned}$$

Then, by proceeding as in the proof of Lemma 5.5 in part II yields

$$\|\epsilon^{\frac12}B_h z_h^u\|_{L_\sigma}^2 \lesssim \mathbb{S}\,\|z_h\|_{h\epsilon,A}. \tag{5.10}$$

(3) Let $\overline{\mathcal{C}_K^k}$ denote the mean-value of $\mathcal{C}^k$ over K and define the field π_h^u such that $\pi_h^u|_K = \sum_{k=1}^d h_K\overline{\mathcal{C}_K^k}\partial_k z_h^u$ if $h_K \ge \epsilon$ and $\pi_h^u|_K = 0$ otherwise. Owing to the regularity of the $\mathcal{C}^k$'s and a standard inverse inequality,

$$h_K^{-\frac12}\|\pi_h^u\|_{L_u,K} \lesssim h_K^{\frac12}\|C_h z_h^u\|_{L_u,K} + \|z_h^u\|_{L_u,K}, \tag{5.11}$$

whence it is inferred using inverse inequalities, the fact that $h_K \ge \epsilon$ in the support of π_h^u, and the upper bounds in ($\text{DG}2_\epsilon\text{B}$)–($\text{DG}2_\epsilon\text{D}$) that $\|(0,\pi_h^u)\|_{h\epsilon,A} \lesssim \|z_h\|_{h\epsilon,A}$. Set $\mathtt{C} = \sum_{K\in\mathcal{T}_h^+} h_K\|C_h z_h^u\|_{L_u,K}^2$. From the definition of a_h, it follows that

$$\begin{aligned}\mathtt{C} &= a_h(z_h,(0,\pi_h^u)) - (K^{u\sigma}z_h^\sigma + K^{uu}z_h^u,\pi_h^u)_{L_u}\\ &\quad - \sum_{K\in\mathcal{T}_h^+}(\epsilon^{\frac12}B_h^\dagger z_h^\sigma,\pi_h^u)_{L_u,K} + \sum_{K\in\mathcal{T}_h^+}(C_h z_h^u, h_K C_h z_h^u - \pi_h^u)_{L_u,K}\\ &\quad - \sum_{F\in\mathcal{F}_h^{\partial+}}\tfrac{\alpha-1}{2}\epsilon^{\frac12}(\mathcal{D}^{u\sigma}z_h^\sigma,\pi_h^u)_{L_u,F} - \sum_{F\in\mathcal{F}_h^{\partial+}}\tfrac12(M_F^{uu}(z_h^u) - \mathcal{D}^{uu}z_h^u,\pi_h^u)_{L_u,F}\\ &\quad + \sum_{F\in\mathcal{F}_h^{i+}}2\epsilon^{\frac12}(\{\mathcal{D}^{u\sigma}z_h^\sigma\},\{\pi_h^u\})_{L_u,F} + \sum_{F\in\mathcal{F}_h^{i+}}2(\{\mathcal{D}^{uu}z_h^u\},\{\pi_h^u\})_{L_u,F}\\ &\quad - \sum_{F\in\mathcal{F}_h^{i+}}(S_F^{uu}([\![z_h^u]\!]),[\![\pi_h^u]\!])_{L_u,F} = a_h(z_h,(0,\pi_h^u)) + R_1 + \ldots + R_8,\end{aligned}$$

Let us estimate the remainder terms R_i, $1 \le i \le 8$, in the right-hand side. Clearly, $|R_1| \lesssim \|z_h\|_L\|\pi_h^u\|_{L_u}$. Furthermore, using an inverse inequality and the fact that $\epsilon \le h_K$ for $K \in \mathcal{T}_h^+$,

$$|R_2| \lesssim \sum_{K\in\mathcal{T}_h^+}\epsilon^{\frac12}h_K^{-1}\|z_h^\sigma\|_{L_\sigma,K}\|\pi_h^u\|_{L_u,K} \lesssim \sum_{K\in\mathcal{T}_h^+}\|z_h^\sigma\|_{L_\sigma,K}h_K^{-\frac12}\|\pi_h^u\|_{L_u,K},$$

and $|R_3| \lesssim \sum_{K\in\mathcal{T}_h^+} h_K^{\frac12}\|C_h z_h^u\|_{L_u,K}\|z_h^u\|_{L_u,K}$. If $\alpha = +1$, $R_4 = 0$ while if $\alpha = -1$,

$$\begin{aligned}|R_4| &\lesssim \sum_{F\in\mathcal{F}_h^{\partial+}}\epsilon^{\frac12}h_F^{-\frac12}\|z_h^\sigma\|_{L_\sigma,\mathcal{T}(F)}h_F^{-\frac12}\|\pi_h^u\|_{L_u,\mathcal{T}(F)}\\ &\lesssim \sum_{F\in\mathcal{F}_h^{\partial+}}\|z_h^\sigma\|_{L_\sigma,\mathcal{T}(F)}h_F^{-\frac12}\|\pi_h^u\|_{L_u,\mathcal{T}(F)}.\end{aligned}$$

Moreover, since $|\mathcal{D}^{uu}| \lesssim M_F^{uu} \lesssim \mathcal{I}_{m_u}$ for all $F \in \mathcal{F}_h^{\partial+}$ in both cases for α,

$$|R_5| \lesssim \sum_{F\in\mathcal{F}_h^{\partial+}} |z_h^u|_{M,F}(|\pi_h^u|_{M,F} + \|\pi_h^u\|_{L_u,F}) \lesssim \sum_{F\in\mathcal{F}_h^{\partial+}} |z_h^u|_{M,F} h_F^{-\frac{1}{2}} \|\pi_h^u\|_{L_u,\mathcal{T}(F)}.$$

Similarly,

$$|R_6| + |R_7| \lesssim \sum_{F\in\mathcal{F}_h^{i+}} (\|z_h^\sigma\|_{L_\sigma,\mathcal{T}(F)} + |z_h^u|_{J,F}) h_F^{-\frac{1}{2}} \|\pi_h^u\|_{L_u,\mathcal{T}(F)},$$

and $|R_8| \le \sum_{F\in\mathcal{F}_h^{i+}} |z_h^u|_{J,F}|\pi_h^u|_{J,F} \lesssim \sum_{F\in\mathcal{F}_h^{i+}} |z_h^u|_{J,F} h_F^{-\frac{1}{2}} \|\pi_h^u\|_{L_u,\mathcal{T}(F)}$. Collecting the above bounds and using (5.8) and (5.11), we deduce $|R_1|+\ldots+|R_8| \lesssim \gamma\mathtt{C} + a_h(z_h, z_h)$ where γ can be chosen as small as needed. Hence, $\mathtt{C} \lesssim \mathbb{S}\,\|z_h\|_{h\epsilon,A}$, and using (5.6) and (5.10) leads to the same bound for $\|\mathfrak{h}^{\frac{1}{2}} C_h z^u\|_{L_u}^2$.
(4) Collecting the above bounds yields $\|z_h\|_{h\epsilon,A}^2 \lesssim \mathbb{S}\,\|z_h\|_{h\epsilon,A}$ and hence (5.7). ∎

Lemma 5.2. *The following holds:*

$$\forall (z, y_h) \in W(h) \times W_h, \quad a_h(z, y_h) \lesssim \|z\|_{h\epsilon,1} \|y_h\|_{h\epsilon,A}. \tag{5.12}$$

Proof. Use integration by parts to infer

$$\begin{aligned} a_h(z, y_h) = &\sum_{K\in\mathcal{T}_h} (z, \tilde{T} y_h)_{L,K} + \sum_{F\in\mathcal{F}_h^\partial} \tfrac{1}{2}(M_F(z) + \mathcal{D}z, y_h)_{L,F} \\ &+ \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} \tfrac{1}{2}([\![\mathcal{D}z]\!], [\![y_h]\!])_{L,F} + \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} (S_F^{uu}([\![z^u]\!]), [\![y_h^u]\!])_{L_u,F}. \end{aligned}$$

Let R_1 to R_4 be the four terms in the right-hand side. Observe that

$$\begin{aligned} (z, \tilde{T} y_h)_{L,K} \lesssim &\|z\|_{L,K} \|y_h\|_{L,K} + \|z^\sigma\|_{L_\sigma,K} \epsilon^{\frac{1}{2}} \|B_h y_h^u\|_{L_\sigma,K} \\ &+ \|z^u\|_{L_u,K} \epsilon^{\frac{1}{2}} \|B_h^\dagger y_h^\sigma\|_{L_u,K} + \|z^u\|_{L_u,K} \|C_h y_h^u\|_{L_u,K} \\ \lesssim &\|z\|_{L,K} \|y_h\|_{L,K} + \|z^\sigma\|_{L_\sigma,K} \epsilon^{\frac{1}{2}} \|B_h y_h^u\|_{L_\sigma,K} \\ &+ \epsilon^{\frac{1}{2}} h_K^{-1} \|z^u\|_{L_u,K} \|y_h^\sigma\|_{L_\sigma,K} + h_K^{-\frac{1}{2}} \|z^u\|_{L_u,K} h_K^{\frac{1}{2}} \|C_h y_h^u\|_{L_u,K}. \end{aligned}$$

Hence, $|R_1| \lesssim \|z\|_{h\epsilon,1} \|y_h\|_{h\epsilon,A}$. Furthermore, if $\alpha = +1$,

$$\begin{aligned} |R_2| &\le \sum_{F\in\mathcal{F}_h^\partial} [|\epsilon^{\frac{1}{2}} (\mathcal{D}^{u\sigma} z^\sigma, y_h^u)_{L_u,F}| + \tfrac{1}{2}|(M_F^{uu}(z^u) + \mathcal{D}^{uu} z^u, y_h^u)_{L_u,F}|] \\ &\lesssim \sum_{F\in\mathcal{F}_h^\partial} [\|z^\sigma\|_{L_\sigma,F} h_F^{\frac{1}{2}} |y_h^u|_{M,F} + \theta_F^{\frac{1}{2}} h_F^{-\frac{1}{2}} \|z^u\|_{L_u,F} |y_h^u|_{M,F}], \end{aligned}$$

while if $\alpha = -1$, $|R_2| \lesssim \sum_{F\in\mathcal{F}_h^\partial} [\epsilon^{\frac{1}{2}} \|z^u\|_{L_u,F} h_F^{-\frac{1}{2}} \|y_h^\sigma\|_{L_\sigma,\mathcal{T}(F)} + \|z^u\|_{L_u,F} |y_h^u|_{M,F}]$. Hence, in both cases, $|R_2| \lesssim \|z\|_{h\epsilon,1} \|y_h\|_{h\epsilon,A}$. Similarly,

$$\begin{aligned} \epsilon^{\frac{1}{2}} ([\![\mathcal{D}^{\sigma u} z^u]\!], [\![y_h^\sigma]\!])_{L_\sigma,F} &\lesssim \epsilon^{\frac{1}{2}} \|\,\{z^u\}\,\|_{L_u,F} h_F^{-\frac{1}{2}} \|y_h^\sigma\|_{L_\sigma,\mathcal{T}(F)}, \\ \epsilon^{\frac{1}{2}} ([\![\mathcal{D}^{u\sigma} z^\sigma]\!], [\![y_h^u]\!])_{L_u,F} &\lesssim \|\,\{z^\sigma\}\,\|_{L_\sigma,F} h_F^{\frac{1}{2}} |y_h^u|_{J,F}, \\ ([\![\mathcal{D}^{uu} z^u]\!], [\![y_h^u]\!])_{L_u,F} &\lesssim \|\,\{z^u\}\,\|_{L_u,F} |y_h^u|_{J,F}. \end{aligned}$$

Hence, $|R_3| \lesssim \|z\|_{h\epsilon,1}\|y_h\|_{h\epsilon,A}$. Finally, it is clear that $\sum_{F\in\mathcal{F}_h^{\mathrm{i}}}(S_F^{uu}([\![z^u]\!]), [\![y_h^u]\!])_{L_u,F} \leq \|z\|_{h\epsilon,1}\|y_h\|_{h\epsilon,A}$, thereby completing the proof. □

It is now straightforward to derive the following result.

THEOREM 5.3. *Keep the hypotheses of Lemma 5.1. Assume that $z \in [H^1(\Omega)]^m$. Then,*

$$\|z - z_h\|_{h\epsilon,A} \lesssim \inf_{y_h\in W_h} \|z - y_h\|_{h\epsilon,1}. \tag{5.13}$$

In particular, if $z \in [H^{p_u+1}(\Omega)]^m$,

$$\|z - z_h\|_{h\epsilon,A} \lesssim \theta^{\frac{1}{2}} h^{p_u} \|z\|_{[H^{p_u+1}(\Omega)]^m}. \tag{5.14}$$

The convergence estimate in Theorem 5.3 is consistent with that from the two-field DG theory when $1 \sim \epsilon \geq h$ and degenerate into that from the one-field theory for the u-component when $h \geq \epsilon$. Indeed, if $1 \sim \epsilon \geq h$,

$$\|z^u - z_h^u\|_{L_u} + \|B_h(z^u - z_h^u)\|_{L_\sigma} \lesssim h^{p_u}\|z\|_{[H^{p_u+1}(\Omega)]^m}, \tag{5.15}$$

and the L_u-norm error estimate can be improved if elliptic regularity holds, while if $h \geq \epsilon$,

$$\|z^u - z_h^u\|_{L_u} + \|\mathfrak{h}^{\frac{1}{2}} C_h(z^u - z_h^u)\|_{L_u} \lesssim h^{p_u+\frac{1}{2}}\|z\|_{[H^{p_u+1}(\Omega)]^m}. \tag{5.16}$$

5.4. Example: Advection dominated advection–diffusion. Consider the advection–diffusion equation introduced in §3.3 with a diffusion coefficient $\epsilon > 0$. The PDE $-\epsilon\Delta u + \beta\cdot\nabla u + \mu u = f$ in mixed form becomes

$$\begin{cases} \sigma + \epsilon^{\frac{1}{2}}\nabla u = 0, \\ \mu u + \epsilon^{\frac{1}{2}}\nabla\cdot\sigma + \beta\cdot\nabla u = f, \end{cases} \tag{5.17}$$

so that the off-diagonal blocks of $\mathcal{A}^k$ in (3.8) are rescaled by $\epsilon^{\frac{1}{2}}$ while the operator K is unchanged.

In the case of a Dirichlet boundary condition, the boundary and interface operators can be redesigned to fit the above analysis by modifying (3.15) as follows:

$$M_F^{uu}(v) = \eta_1(|\beta\cdot n_F| + \epsilon h_F^{-1})v, \qquad S_F^{uu}(v) = \eta_2(|\beta\cdot n_F| + \epsilon h_F^{-1})v, \tag{5.18}$$

where n_F is a unit normal vector to F and $\eta_1 > 0$, $\eta_2 > 0$ (these two parameters can vary from face to face). It is easily verified that properties (DG2$_\epsilon$A)–(DG2$_\epsilon$D) hold.

Assuming $\varrho \geq -\min(\beta\cdot n, 0)$, mixed Robin–Neumann boundary conditions can be enforced by redesigning the boundary and interface operators as follows:

$$M_F^{uu}(v) = (2\varrho + \beta\cdot n)v, \qquad S_F^{uu}(v) = \eta_2(|\beta\cdot n_F| + \epsilon h_F^{-1})v. \tag{5.19}$$

to satisfy (DG2$_\epsilon$C)–(DG2$_\epsilon$D).

6. Conclusion. We have analyzed various DG methods in parts I, II, and III. We have attempted to give a unified analysis for all these methods. Following the seminal ideas of Lesaint and Raviart [15, 16], we have shown that the framework of symmetric positive Friedrichs' systems is the natural setting for this theory insofar boundary conditions can be enforced weakly for all these systems. The first building block of the theory is the bilinear form (2.4) along with the weak formulation (2.6). All the DG methods that we have analyzed can be put into the unified bilinear form (2.15) and the unified formulation (2.16). The differences between all these methods reside solely in the design of the boundary and interface operators.

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