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Interchange of minimization and integration with measurability constraints

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INTERCHANGE OF MINIMIZATION AND INTEGRATION WITH MEASURABILITY CONSTRAINTS

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Abstract. In [9], Theorem 14.60 shows that without measurability constraints, to minimize an integral functional with respect to all measurable mappings is equivalent to integrate the minimum over the constant mappings of the underlying cost function. We prove here the analogous result under measurability and almost sure constraints. In our case, the interchange introduces a conditional expectation, and we need another tool on conditional expectations of integrands to conclude. We propose moreover some other interchange results on conjugate and on convex normal integrands.

1. Definitions and Propositions

Let T be a polish space, and $\mathcal{F}$ its completed σ-algebra of Borel sets. $(T, \mathcal{F})$ is therefore a measurable space, and we endow it with some probability measure μ. In the following, $\mathcal{A}$ is some complete sub-σ-field of $\mathcal{F}$, i.e. a sub-σ-field of $\mathcal{F}$ such that for all $A \in \mathcal{A}$ with $\mu(A) = 0$, $\mathcal{A}$ contains also all the subsets of A. Let E be a separable Banach space with norm denoted by $\|\cdot\|$. We will denote by $\mathcal{B}$ the Borel sigma-field of E, and by $\mathbb{B} = \{x \in E \ : \ \|x\| \leq 1\}$. We will denote by $\mathcal{X}$ the set of $\mathcal{F}$-measurable mappings $x : T \to E$.

1.1. **Normal integrands and measurable selections.** The presentation reproduced here comes from [9], Chapter 14.

Definition 1.1 (Normal integrand)**.** *The mapping $f : T \times E \to \overline{\mathbb{R}}$ is said to be a $\mathcal{A}$-normal integrand if its epigraphical mapping $S_f : T \rightrightarrows E \times \mathbb{R}$ defined by $S_f(t) = \{(x, y) \in E \times \mathbb{R} \ : f(t, x) \geq y \ \}$ is closed-valued and $\mathcal{A}$-measurable, i.e. if for every open $O \subset E \times \mathbb{R}$, $S_f^{-1}(O) \in \mathcal{A}$. We will denote by $I_f : \mathcal{X} \to \overline{\mathbb{R}}$ the integral mapping $I_f(x) = \int_T f(t, x(t))\mu(dt)$.*

Definition 1.2. *Denote by E' the dual topological space of E, and by $\langle \cdot, \cdot \rangle$ the pairing product. Let $f : T \times E \to \overline{\mathbb{R}}$ be a $\mathcal{A}$-normal integrand. The conjugate of f, denoted by f^*, is the mapping $f^* : T \times E' \to \overline{\mathbb{R}}$ defined by*

$$\forall (t, x') \in T \times E', \ f^*(t, x') = \sup_{x \in E} \left(\langle x, x' \rangle - f(t, x) \right).$$

Proposition 1.3. *Let $f : T \times E \to \overline{\mathbb{R}}$ be a $\mathcal{A}$-normal integrand. Then, its conjugate $f^* : T \times E' \to \overline{\mathbb{R}}$ is a $\mathcal{A}$-normal integrand.*

Proof. This result can be found in [9, Theorem 14.50] when E is finite dimensional, and in [7, 2] or [5, 6] (whithout completeness hypothesis on $\mathcal{A}$ for μ)when E is a separable Banach space. ☐

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The following propositions are given with $E = \mathbb{R}^n$, and $\|\cdot\|$ the euclidean norm.

Proposition 1.4. *The mapping $f : T \times E \to \overline{\mathbb{R}}$ is a $\mathcal{A}$-normal integrand if and only if f is $\mathcal{A} \otimes \mathcal{B}$-measurable and $f(t, \cdot)$ is lower semicontinuous for μ-almost all $t \in T$.*

Proof. cf. [9], Corollary 14.34. □

Proposition 1.5 (Level-set)**.** *For a $\mathcal{A}$-normal integrand $f : T \times \mathbb{R}^n \to \overline{\mathbb{R}}$ and any $\mathcal{A}$-measurable mapping $\alpha : T \to \mathbb{R}$, the mapping $t \mapsto \{x \ : f(t, x) \leq \alpha(t)\ \}$ is closed-valued and $\mathcal{A}$-measurable. In particular, the level-set mappings of f are closed-valued and $\mathcal{A}$-measurable.*

Proof. cf. [9], Proposition 14.33. □

Proposition 1.6 (Measurable selection)**.** *A closed-valued, $\mathcal{A}$-measurable mapping $S : T \rightrightarrows \mathbb{R}^p$ always admits a $\mathcal{A}$-measurable selection: there exists a measurable function $x : \mathrm{dom} S \to \mathbb{R}^p$ such that $x(t) \in S(t)$ for all $t \in T$.*

Proof. cf. [9], Corollary 14.6. □

Proposition 1.7 (Marginal mapping)**.** *For any $\mathcal{A}$-normal integrand $f : T \times \mathbb{R}^n \to \overline{\mathbb{R}}$, the function $p_{\mathcal{A}} : T \to \overline{\mathbb{R}}$ defined as $p_{\mathcal{A}}(t) = \inf_{x \in \mathbb{R}^n} f(t, x)$ is $\mathcal{A}$-measurable.*

Proof. cf. [9], Theorem 14.37. □

Remark 1.8. *The following remark reproduces the Example 14.32 of* [9]*. Let $X : T \rightrightarrows E$ be a closed-valued and $\mathcal{A}$-measurable correspondence. Then, $\chi_X : T \times E \to \overline{\mathbb{R}}$ defined by $\chi_X(t, x) = +\infty$ if $x \notin X(t)$ and $\chi_X(t, x) = 0$ else, is a $\mathcal{A}$-normal integrand. Let f be a $\mathcal{A}$-normal integrand. Hence, the mapping $(t, x) \mapsto f(t, x) + \chi_X(t, x)$ is a normal integrand. Hence, the constrained marginal mapping $p_{\mathcal{A}} : T \to \overline{\mathbb{R}}$ defined as $p_{\mathcal{A}}(t) = \inf_{x \in X(t)} f(t, x)$ is $\mathcal{A}$-measurable.*

1.2. **Conditional expectations.** For all complete sub-σ-field $\mathcal{A}$ of $\mathcal{F}$, and all $p \in \mathbb{N}$, we define the set $\mathcal{L}^p(\mathcal{A}) = \{x \in \mathcal{X} \ : \ \int_T \|x(t)\|^p \mu(dt) < +\infty\}$. Let us recall the definition of a conditional expectation:

Definition 1.9. *Let $x : T \to \mathbb{R}^n$ be $\mathcal{F}$-measurable and $\mathcal{A}$ a complete sub-σ-field of $\mathcal{F}$. A conditional expectation of x with respect to $\mathcal{A}$ is a $\mathcal{A}$-measurable mapping $z : T \to \mathbb{R}^n$ such that for all $A \in \mathcal{A}$, $\int_A x(t)\mu(dt) = \int_A z(t)\mu(dt)$.*

The following result can be found in any probability textbook:

Proposition 1.10. *Let $x \in \mathcal{L}^p(\mathcal{F})$. Then, the conditional expectation of x with respect to $\mathcal{A}$ exists and is unique up to indistinguishability. It is denoted by $x_{\mathcal{A}} : T \to \mathbb{R}^n$, and belongs to $\mathcal{L}^p(\mathcal{A})$.*

The most important tool for our main result is the theorem 1.12, which is a generalization to integrands of the concept of conditional expectation. A natural way to generalize the definition of conditional expectations to integrands is the following.

Definition 1.11. *E denotes a Banach space, and X is a subset of E. Let $f : T \times E \to \overline{\mathbb{R}}$ be a $\mathcal{F}$-normal integrand. Let $\mathcal{A}$ be a complete sub-σ-field of $\mathcal{F}$. A conditional expectation of f with respect to $\mathcal{A}$ is a $\mathcal{A}$-normal integrand $g : T \times E \to \overline{\mathbb{R}}$ such that*

$$\forall A \in \mathcal{A},\ \forall x \in \mathcal{X},\ \int_A f(t, x(t))\mu(dt) = \int_A g(t, x(t))\mu(dt), \tag{1}$$

The question is now to prove the existence and uniqueness of such conditional expectations for various classes of integrands. In [3], such a result is given under the assumption (CHS).

$$\text{(CHS)}\qquad \begin{array}{l}\exists \text{ a sequence } B_n \in \mathcal{B},\ \cup_{n\in\mathbb{N}} B_n = E,\ \forall n\in\mathbb{N},\ \exists m_n : T \to \mathbb{R},\\ \forall (t,x)\in T\times B_n,\ f(t,x) \geq m_n(t) \text{ and } \displaystyle\int_T |m_n(t)|\mu(dt) < +\infty.\end{array}$$

In [4], assumption (DE) is introduced to prove the same result:

$$\text{(DE)}\qquad \exists h : T\to\mathbb{R},\ \forall (t,x)\in T\times E,\ |f(t,x)| \leq h(t) \text{ and } \int_T h(t)\mu(dt) < +\infty.$$

In [10], the assumption (T) appears as a sufficient assumption to prove the existence and uniqueness result, and is referred to as $\mathcal{F}$-quasi integrability of the integrand.

(T)

$$\forall m\in\mathbb{N},\ \exists \alpha_m : T\to\mathbb{R},\ \forall t\in T,\ \inf_{x\in m\mathbb{B}} f(t,x) \geq \alpha_m(t), \text{ and } \int_T |\alpha_m(t)|\mu(dt) < +\infty.$$

Clearly, (DE) $\Rightarrow$ (T) $\Rightarrow$ (CHS). We provide here an existence and uniqueness result for conditional expectation of normal integrands satisfying assumption (CHS), since it is the weakest available assumption.

Theorem 1.12. *E is here a separable Banach space. Let $f : T\times E \to \overline{\mathbb{R}}$ be a $\mathcal{F}$-normal integrand satisfying* (CHS)*. Then, for all complete sub-σ-field $\mathcal{A}$ of $\mathcal{F}$, there exists a $\mathcal{A}\otimes\mathcal{B}$-measurable mapping $f_{\mathcal{A}} : T\times E\to\overline{\mathbb{R}}$ such that*

(i) for all $A\in\mathcal{A}$, and all $\mathcal{A}$-measurable $x : T\to E$,

$$\int_A f(t,x(t))\mu(dt) = \int_A f_{\mathcal{A}}(t,x(t))\mu(dt). \tag{2}$$

(ii) For all $n\in\mathbb{N}$, for all $(t,x)\in T\times B_n$, $f_{\mathcal{A}}(t,x)\geq m_n(t)$.

Moreover, $f_{\mathcal{A}}$ is unique up to indistinguishability and is a $\mathcal{A}$-normal integrand: it is the $\mathcal{A}$-conditional expectation of f.

Proof. cf. [3], Theorem 2.1. and Corollary 2.2. □

Remark 1.13 (Conditional probability)**.** *If there exists a conditional probability measure $\mu_{\mathcal{A}} : T\times\mathcal{F}\to[0,1]$ associated to the complete sub-σ field $\mathcal{A}$, $f_{\mathcal{A}}$ is defined as:*

$$\forall (t,x)\in T\times E,\ f_{\mathcal{A}}(t,x) = \int_T f(t',x)\mu_{\mathcal{A}}(t,dt'),$$

and the preceding result is very analogous to the Theorem 1.2 of [4].

2. Interchange results

2.1. **Interchange of minimization and integration.** We are now able to state and prove our main result:

Theorem 2.1. *Let $\mathcal{A}$ be a complete sub-σ-field of $\mathcal{F}$. Let $X : T \rightrightarrows \mathbb{R}^n$ be a correspondence with closed values, which is $\mathcal{A}$-measurable. Let $f : T\times\mathbb{R}^n\to\mathbb{R}$ be a $\mathcal{F}$-normal integrand which satisfies* (CHS)*. Then, it holds that*

$$\inf_{\substack{x\in\mathcal{X}\\ x\ \mathcal{A}-\text{measurable}\\ \forall t\in T,\ x(t)\in X(t)}} \int_T f(t,x(t))\mu(dt) = \int_T \left(\inf_{x\in X(t)} f_{\mathcal{A}}(t,x)\right)\mu(dt), \tag{3}$$

with $f_{\mathcal{A}} : T\times\mathbb{R}^n\to\mathbb{R}$ defined by Theorem 1.12.

Proof. We prove the theorem by double inequality.
By Theorem 1.12, there exists a unique (up to indistinguishability) $f_{\mathcal{A}} : T \times \mathbb{R}^n \to \overline{\mathbb{R}}$ which is a $\mathcal{A}$-normal integrand and satisfies (2).

- Let $p_{\mathcal{A}} : T \to \overline{\mathbb{R}}$ be its marginal mapping, i.e. $p_{\mathcal{A}}(t) = \inf_{x \in X(t)} f_{\mathcal{A}}(t,x)$. By proposition 1.7, $p_{\mathcal{A}}$ is hence $\mathcal{A}$-measurable. Hence, for all $x : T \to \mathbb{R}^n$ $\mathcal{A}$-measurable, such that $x(t) \in X(t)$ for all $t \in T$,
$$f_{\mathcal{A}}(t, x(t)) \geq p_{\mathcal{A}}(t), \text{ therefore, by integration,}$$
$$\int_T f_{\mathcal{A}}(t, x(t))\mu(dt) \geq \int_T p_{\mathcal{A}}(t)\mu(dt).$$
By definition of $f_{\mathcal{A}}$, the last equation reads
$$\int_T f(t, x(t))\mu(dt) \geq \int_T p_{\mathcal{A}}(t)\mu(dt).$$
Passing to the infimum in feasible x's yields the first inequality.
- For all $k \in \mathbb{N}^*$, define $S_k : T \rightrightarrows \mathbb{R}^n$ as $S_k(t) = \{x \in \mathbb{R}^n \; : \; f_{\mathcal{A}}(t,x) \leq p_{\mathcal{A}}(t) + \frac{1}{k}\}$. By proposition 1.5, S_k is $\mathcal{A}$-measurable and closed valued for all $k \in \mathbb{N}^*$. Let us now consider $Z_k : T \rightrightarrows \mathbb{R}^n$ as $Z_k(t) = S_k(t) \cap X(t)$. As an intersection of $\mathcal{A}$-measurable and closed valued mappings, Z_k is itself $\mathcal{A}$-measurable and closed valued (see e.g. [9], Proposition 14.11). Hence, by proposition 1.6, there are $\mathcal{A}$-measurable mappings $x_k : T \to \mathbb{R}^n$ such that $x_k(t) \in Z_k(t)$ for all $t \in T$. Thus, for all $k \in \mathbb{N}^*$,
$$p_{\mathcal{A}}(t) \geq f_{\mathcal{A}}(t, x_k(t)) - \frac{1}{k}, \text{ and by integration,}$$
$$\int_T p_{\mathcal{A}}(t)\mu(dt) \geq \int_T f_{\mathcal{A}}(t, x_k(t))\mu(dt) - \frac{1}{k} = \int_T f(t, x_k(t))\mu(dt) - \frac{1}{k},$$
using once again Theorem 1.12. By the very definition of the infimum, it yields
$$\int_T p_{\mathcal{A}}(t)\mu(dt) \geq \inf_{\substack{x \in \mathcal{X} \\ x\ \mathcal{A}-\text{measurable} \\ \forall t \in T,\ x(t) \in X(t)}} \int_T f(t, x(t))\mu(dt) - \frac{1}{k}.$$
Making $k \to +\infty$ completes the proof.

□

Remark 2.2 (Particular cases)**.** *We can distinguish one particular case of measurability constraints: the case where $\mathcal{A} = \mathcal{F}$. The equation* (3) *of theorem 2.1 simply reads*
$$\inf_{\substack{x \in \mathcal{X} \\ \forall t \in T,\ x(t) \in X(t)}} \int_T f(t, x(t))\mu(dt) = \int_T \left(\inf_{x \in X(t)} f(t,x) \right) \mu(dt),$$
and is very similar to the result stated in [9]*, Theorem 14.60, which is available without assumption* (CHS)*. The main strength of Theorem 14.60 of* [9] *is that the space $\mathcal{X}$ is not restricted to the space of measurable functions, but may be any so called decomposable space with respect to μ.*

2.2. **Interchange of conjugacy and integration.** First of all, we recall an interchange result between conjugacy and integration:

Theorem 2.3. *Let $f : T \times \mathbb{R}^n \to \overline{\mathbb{R}}$ be a $\mathcal{F}$-normal integrand. Assume that there exists $u \in \mathcal{L}^p(\mathcal{F})$, with $1 < p < \infty$, such that $t \mapsto f(t, u(t))$ is summable, and that there exists $u^* \in \mathcal{L}^q(\mathcal{F})$ with $1/p + 1/q = 1$, such that $t \mapsto f^*(t, u^*(t))$ is summable. Then, I_f and I_{f^*} are well defined, and satisfy the conjugacy relation $(I_f)^* = I_{f^*}$ on $\mathcal{L}^q(\mathcal{F})$.*

Proof. As $\mathcal{L}^p$ spaces are decomposable in the sense of Rockafellar, the result follows in the convex case from [8], Theorem 2. In the general case, since

$$\begin{aligned}
\forall u^* \in \mathcal{L}^q(\mathcal{F}),\ (I_f)^*(u^*) &= \sup_{u \in \mathcal{L}^p(\mathcal{F})} \left(\int_T \langle u(t), u^*(t) \rangle \mu(dt) - \int_T f(t, u(t)) \mu(dt) \right), \\
&= \sup_{u \in \mathcal{L}^p(\mathcal{F})} \int_T \left(\langle u(t), u^*(t) \rangle - f(t, u(t)) \right) \mu(dt),
\end{aligned}$$

the result is a consequence of the general interchange result of [9], Theorem 14.60. □

We can also prove a partial interchange result between the conjugacy and the conditional expectation:

Proposition 2.4. *Let $\mathcal{A}$ be a complete sub σ field of $\mathcal{F}$. Let $f : T \times \mathbb{R}^n \to \overline{\mathbb{R}}$ be a $\mathcal{F}$-normal integrand satisfying* (T)*, and such that there exists $x_0 \in \mathbb{R}^n$ such that $t \mapsto f(t, x_0)$ is integrable. Then, the conjugate $f^* : T \times \mathbb{R}^n \to \overline{\mathbb{R}}$ is a $\mathcal{F}$-normal integrand which admits a unique (up to indistinguishability) conditional expectation. Moreover, there exists a negligeable set $T_{\mathcal{A}} \in \mathcal{A}$ such that*

$$\forall t \notin T_{\mathcal{A}},\ \forall x \in \mathbb{R}^n,\ (f^*)_{\mathcal{A}}(t, x) \geq (f_{\mathcal{A}})^*(t, x).$$

Proof. Proposition 1.3 proves that the conjugate of a normal integrand is a normal integrand. We now prove that f^* satisfies assumption (T) and thus (CHS). It will prove that $(f^*)_{\mathcal{A}}$ exists and is unique up to indistinguishability.

$$\begin{aligned}
&\forall (t, z) \in T \times \mathbb{R}^n,\ f^*(t, z) \geq \langle x_0, z \rangle - f(t, x_0),\ \text{by definition}, \\
&\forall (t, z) \in T \times m\mathbb{B},\ f^*(t, z) \geq \underbrace{-m\|x_0\| - f(t, x_0)}_{\alpha_m(t)}, \\
&\quad \forall t \in T,\ \inf_{z \in m\mathbb{B}} f^*(t, z) \geq \alpha_m(t),
\end{aligned}$$

and hence, f^* satisfies (T) and (CHS): there exists a unique conditional expectation to f^* denoted by $(f^*)_{\mathcal{A}} : T \times \mathbb{R}^n \to \overline{\mathbb{R}}$. We now prove that $(f^*)_{\mathcal{A}}(t, x) \geq (f_{\mathcal{A}})^*(t, x)$ for all $x \in \mathbb{R}^n$ and all $t \notin T_{\mathcal{A}}$, with $T_{\mathcal{A}}$ a negligeable subset of T. Using Lemma 6 of [10], it suffices to prove that for all bounded $y : T \to \mathbb{R}^n$, for all $A \in \mathcal{A}$,

$$\int_A (f^*)_{\mathcal{A}}(t, y_{\mathcal{A}}(t)) \mu(dt) \geq \int_A (f_{\mathcal{A}})^*(t, y_{\mathcal{A}}(t)) \mu(dt).$$

Let $y \in \mathcal{X}$ and y bounded. For all $A \in \mathcal{A}$,

$$\begin{aligned}
\int_A (f_{\mathcal{A}})^*(t, y_{\mathcal{A}}(t)) \mu(dt) &= \int_A \sup_{x \in \mathbb{R}^n} \left(\langle x, y_{\mathcal{A}}(t) \rangle - f_{\mathcal{A}}(t, x) \right) \mu(dt), \\
&= \int_T \sup_{x \in \mathbb{R}^n} \left(1_A(t) \left(\langle x, y_{\mathcal{A}}(t) \rangle - f_{\mathcal{A}}(t, x) \right) \right) \mu(dt), \\
&= -\int_T \inf_{x \in \mathbb{R}^n} \left(h_{\mathcal{A}}(t, x) \right) \mu(dt),
\end{aligned}$$

with $h(t,x) = 1_A(t)\left(f(t,x) - \langle x, y_{\mathcal{A}}(t)\rangle\right)$, which is a $\mathcal{F}$-normal integrand satisfying (T) by boundedness of y. Hence, its conditional expectation $h_{\mathcal{A}}$ exists and is well-defined. Moreover, using Theorem 2.1, we obtain

$$\begin{aligned}
\int_A (f_{\mathcal{A}})^*(t, y_{\mathcal{A}}(t))\mu(dt) &= -\inf_{x\mathcal{A}-\text{measurable}} \int_T h(t,x(t))\mu(dt),\\
&\geq -\inf_{x\in\mathcal{X}} \int_T h(t,x(t))\mu(dt),\\
&= -\int_T \inf_{x\in\mathbb{R}^n} h(t,x)\mu(dt),\ \text{ again by Theorem 2.1},\\
&= \int_A \sup_{x\in\mathbb{R}^n} \left(\langle x, y_{\mathcal{A}}(t)\rangle - f(t,x)\right)\mu(dt),\ \text{ by definition of } h,\\
&= \int_A f^*(t, y_{\mathcal{A}}(t))\mu(dt) = \int_A (f^*)_{\mathcal{A}}(t, y_{\mathcal{A}}(t))\mu(dt).
\end{aligned}$$

Hence, using Lemma 6 of [10], it completes the proof. □

3. Convex case

Following [1] or [4], other interchange results exist in the convex case.

Definition 3.1. *A normal integrand $f : T \times E \to \overline{\mathbb{R}}$ is called convex if and only if for μ-almost all $t \in T$, $f(t,\cdot)$ is convex.*

Proposition 3.2. *Let $f : T \times E \to \overline{\mathbb{R}}$ be a convex $\mathcal{F}$-normal integrand satisfying* (T)*. Then, for any complete sub σ-field $\mathcal{A}$ of $\mathcal{F}$, $f_{\mathcal{A}} : T \times E \to \overline{\mathbb{R}}$ defined by* (2) *is convex.*

Proof. cf. [10] Proposition 15. □

Proposition 3.3 (Artstein)**.** *Let $\mathcal{A}$ be a complete sub σ-field of $\mathcal{F}$. Let $f : T\times E \to \overline{\mathbb{R}}$ be a convex $\mathcal{A}$-normal integrand. It holds that*

$$\forall x \in \mathcal{L}^p(\mathcal{F}),\ \int_T f(t,x(t))\mu(dt) \geq \int_T f(t, x_{\mathcal{A}}(t))\mu(dt).$$

Proof. Since the p-integrable mappings for $p \geq 1$ are also 1-integrable, the result follows from [1], Proposition 3.2. □

Theorem 3.4. *Let $\mathcal{A}$ be a complete sub σ field of $\mathcal{F}$. Let $X : T \rightrightarrows \mathbb{R}^n$ be a correspondence with closed values, which is $\mathcal{A}$-measurable. Let $f : T \times \mathbb{R}^n \to \mathbb{R}$ be a convex $\mathcal{F}$-normal integrand satisfying* (T)*. Then, it holds that*

(4)

$$\inf_{\substack{x \in \mathcal{L}^p(\mathcal{A}),\\ \forall t\in T,\ x(t)\in X(t)}} \int_T f(t,x(t))\mu(dt) = \inf_{\substack{x \in \mathcal{L}^p(\mathcal{F}),\\ \forall t\in T,\ x(t)\in X(t)}} \int_T f_{\mathcal{A}}(t,x(t))\mu(dt),$$

with $f_{\mathcal{A}} : T \times \mathbb{R}^n \to \mathbb{R}$ defined by Theorem 1.12. Moreover, for all x solution to the right hand-side problem of (4)*, $x_{\mathcal{A}}$ is a solution to the two problems.*

Proof. Without loss of generality, we can omit the constraints $x(t) \in X(t)$ by the Remark 1.8. We use the same notations as in the proof of Theorem 2.1. Again, we proceed by double inequality.

- From Theorem 1.12 and Proposition 3.2, $f_{\mathcal{A}}$ is a convex $\mathcal{A}$-normal integrand. Hence From Proposition 3.3, we know that
$$\begin{aligned}\forall x \in \mathcal{L}^p(\mathcal{F}),\ \int_T f_{\mathcal{A}}(t, x(t))\mu(dt) &\geq \int_T f_{\mathcal{A}}(t, x_{\mathcal{A}}(t))\mu(dt),\\ &= \int_T f(t, x_{\mathcal{A}}(t))\mu(dt), \text{ by (2).}\end{aligned}$$
Passing to the infimum over $\mathcal{L}^p(\mathcal{F})$ yields the first inequality.
- On the other hand, since $\mathcal{A}$ is a sub-σ field of $\mathcal{F}$, it holds that
$$\begin{aligned}\inf_{x \in \mathcal{L}^p(\mathcal{F})} \int_T f_{\mathcal{A}}(t, x(t))\mu(dt) &\leq \inf_{x \in \mathcal{L}^p(\mathcal{A})} \int_T f_{\mathcal{A}}(t, x(t))\mu(dt),\\ &= \inf_{x \in \mathcal{L}^p(\mathcal{A})} \int_T f(t, x(t))\mu(dt), \text{ by (2),}\end{aligned}$$
which proves (4).

Assume now that x is solution to the right hand-side problem of (4). Since $f_{\mathcal{A}}$ is a convex $\mathcal{A}$-normal integrand, we obtain with Proposition 3.3 that $x_{\mathcal{A}}$ is also solution. $\square$

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