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Existence and uniqueness of a solution of a non-conservative Burgers type system describing the dynamics of dislocations densities

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Abstract

In this work we study a system of non-conservative Burgers type equations modeling the dynamics of dislocations densities in a crystal. More precisely, we prove that this system admits a global in time solution, unique in the space $H^1_{loc}(\mathbb{R} \times [0, +\infty))$.

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1 Introduction

1.1 Physical motivations and presentation of the model

Real crystals comprise certain defects in the organization of their crystalline structure called dislocations. In a particular case where these defects are parallel straight lines in the three-dimensional space, they can be viewed as points in a plane. Under the effect of exterior constraints, dislocations can move in a certain crystallographic direction called the slip direction. This slip direction is given by a vector called the "Burger's vector". The norm of this vector represents the amplitude of the generated deformation. (We refer to [12] for further physical explanation).

In this work, we are interested in the study of a the model defined by El Hajj, Forcadel [8, Lemme 3.1]. In fact, this is a 1-D sub-model of a problem introduced by Groma and Balogh [11], initially proposed in the two-dimensional case.

This two-dimensional model is characterized by the fact that the dislocations propagate in the plane (x_1, x_2) just following two Burger's vectors $\pm\vec{b}$ with $\vec{b} = (1, 0)$. In this 1-D sub-model we suppose also that dislocations densities depend only on the variable $x = x_1 + x_2$, that transform the 2-D into a 1-D model (see El Hajj, Forcadel [8] for more modeling details).
More precisely this 1-D model is given by the following coupled equations of non-conservative Burgers type :

$$\begin{cases} \dfrac{\partial \rho^+}{\partial t}(x,t) & = -\left(a(t) + (\rho^+ - \rho^-)(x,t) + \alpha \displaystyle\int_0^1 (\rho^+ - \rho^-)(y,t)dy\right) \dfrac{\partial \rho^+}{\partial x}(x,t) \quad \text{in} \quad \mathcal{D}'(\mathbb{R} \times (0,T)), \\ \dfrac{\partial \rho^-}{\partial t}(x,t) & = \quad \left(a(t) + (\rho^+ - \rho^-)(x,t) + \alpha \displaystyle\int_0^1 (\rho^+ - \rho^-)(y,t)dy\right) \dfrac{\partial \rho^-}{\partial x}(x,t) \quad \text{in} \quad \mathcal{D}'(\mathbb{R} \times (0,T)). \end{cases} \tag{1.1}$$

The unknowns ρ^+ and ρ^- that appear in the system are scalar valued functions, that we denote for simplicity by $\rho^\pm$. Their spatial derivatives, $\dfrac{\partial \rho^\pm}{\partial x}$ are the dislocations densities of Burger's vector $\pm\vec{b} = \pm(1, 0)$. The function $a = a(t)$ representing the field of the imposed exterior constraint, is supposed to be independent of x, and the constant α depends on the elastic coefficients and the material size.
We consider the following initial conditions for (1.1):

$$\rho^{\pm}(x,t=0)=\rho_0^{\pm}(x)=\rho_0^{\pm,per}(x)+L_0 x \quad \text{in} \quad \mathbb{R}, \tag{1.2}$$

where $\rho_0^{\pm,per}$ are 1-periodic functions. We model thus a periodic distributions of the $\pm$ dislocations, with a spatial period of length 1, each type of $\pm$ dislocations having a mean density equal to L_0. In fact, the use of the periodic boundary conditions is a way of regarding what is going on in the interior of the material away from its boundary.

1.2 A brief review of some related literature

From a mathematical point of view, system (1.1) is related to other similar models, such as transport equations based on vector fields with low regularity. Such equations were for instance studied by Diperna, Lions in [7]. They proved the existence and uniqueness of a solution (in the renormalized sense), for vector fields in $L^1((0,+\infty);W^{1,1}_{loc}(\mathbb{R}^N))$ whose divergence are in $L^1((0,+\infty);L^\infty(\mathbb{R}^N))$. This study was generalized by Ambrosio [3], who considered vector fields in $L^1((0,+\infty);BV_{loc}(\mathbb{R}^N))$ with bounded divergence. In the present paper, we work in dimension $N=1$ and prove the existence and uniqueness of solutions of the system (1.1)-(1.2) with a vector field (i.e. the velocity) only in $L^\infty((0,+\infty),H^1_{loc}(\mathbb{R}))$.
We also refer to the works of LeFloch and Liu [13, 14] in which they considered the study in the framework of functions of bounded variation for a system of the form:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t)+A(u)\dfrac{\partial u}{\partial x}(x,t)=0 & u(x,t)\in U,\ x\in\mathbb{R},\ t\in(0,T),\\[2ex] u(x,0)=u_0(x) & x\in\mathbb{R}, \end{cases} \tag{1.3}$$

where the space of states U is an open subset of $\mathbb{R}^p$, and A is $(p\times p)$ matrix which of class C^1 on U. Moreover $A(u)$ have p scalar distinct eigenvalues that we denote by: $\lambda_1(u)<\lambda_2(u)<\ldots<\lambda_p(u)$. We remark that this condition on the eigenvalues does not enter in our framework even in the case where $\alpha=a=0$, because we have not sign property on $\rho^+-\rho^-$. LeFloch and Liu proved that if the initial condition u_0 is sufficiently close to a constant state, and if the total variation $TV(u_0)$ is assumed to be small enough, then system (1.3) admits a unique solution in $L^\infty(\mathbb{R}\times(0,+\infty))\cap BV(\mathbb{R}\times(0,+\infty))$, in the sense of weak entropy solutions with respect to admissible function (see LeFloch [13, Definition 3.2]).
When the system is hyperbolic and symmetric, this corresponds to the case $\alpha=a=0$ in our system (1.1), it is proved in Serre [17, Vol I, Th 3.6.1] a result of local existence and uniqueness in $C([0,T);H^s(\mathbb{R}^N))\cap C^1([0,T);H^{s-1}(\mathbb{R}^N))$, with $s>\frac{N}{2}+1$. This result is only local in time, even in dimension $N=1$.
The assumptions of increasing initial conditions was also considered in the study of the Euler equation for compressible fluids in dimension one. With regard to these studies, we refer to Chen and Wang [6, Th 3.1] for an existence and uniqueness result in $C^1(\mathbb{R}\times[0,+\infty))$ based on the method of characteristic. This result of Chen and Wang shows that the Euler equation of compressible fluids does not create shocks, for suitable increasing and $C^1(\mathbb{R})$ initial conditions. In our case, we already knew that solutions of (1.1), are Lipschitz continuous, see El Hajj and Forcadel [8]. Even if this regularity question is not concerned in the present paper, we may expect the some $C^1(\mathbb{R}\times[0,+\infty))$ regularity of the solution for $C^1(\mathbb{R})$ initial data.

1.3 Main result

The main result of this paper is the existence and uniqueness of global in time solutions for the system (1.1)-(1.2), modeling the dynamics of dislocations densities. This result ensures the mathematical well-posedness of the Groma-Balogh model [11] in the particular case we are interested in.

Theorem 1.1 (Existence and uniqueness)
For all $T, L_0 \geq 0$, $\alpha \in \mathbb{R}$ and $\rho_0^\pm \in H^1_{loc}(\mathbb{R})$ and under the following assumptions:

(H1) $\rho_0^\pm(x+1) = \rho_0^\pm(x) + L_0$, *(1-periodic function + linear function)*

(H2) $\dfrac{\partial \rho_0^\pm}{\partial x} \geq 0$, *($\rho_0^\pm$ non-decreasing)*

(H3) $a \in L^\infty(0,T)$

the system (1.1)-(1.2) admits a unique solution $\rho^\pm \in H^1_{loc}(\mathbb{R} \times [0,T))$ such that $\forall t \in [0,T)$, the function $\rho^\pm(.,t) : x \longmapsto \rho^\pm(x,t)$ verifies $(H1)$ and $(H2)$.

The preceding theorem gives a global existence and uniqueness result of the system (1.1). Its proof is based on the following steps. First of all, we regularize the system (1.1). Then, we show the uniform *a priori* estimates in $L^\infty((0,T); H^1_{loc}(\mathbb{R}))$ for this regularized system. These estimates in one hand, lead to a result of existence for long time solution and on the other hand, they assure the passage to the limit by compactness. Finally, the demonstration of uniqueness is done in a direct way.

Theorem 1.2 (Comparison principle for (1.1) with $\alpha = 0$)
Let $a(\cdot)$ satisfy $(H3)$ and $\rho_1^\pm$, $\rho_2^\pm \in H^1_{loc}(\mathbb{R} \times [0,T))$ be two solutions of the system (1.1) with $\alpha = 0$. Moreover, let $\rho_1^\pm(.,t)$, $\rho_2^\pm(.,t)$ verify $(H1)$ and $(H2)$ for all $t \in [0,T)$. Then, if $\rho_1^\pm(\cdot,0) \leq \rho_2^\pm(\cdot,0)$ in $\mathbb{R}$, we have $\rho_1^\pm \leq \rho_2^\pm$ in $\mathbb{R} \times [0,T)$.

This comparison result was crucial in a previous work [8], for the demonstration of existence and uniqueness of Lipschitz solution to problem (1.1), in the sense of viscosity solution, for Lipschitz initial conditions. Here, the interest of this result is a little bit secondary. Indeed, thanks to this comparison principle, we have been able to obtain indirectly $H^1_{loc}(\mathbb{R} \times [0,T))$ estimates. These estimates in their turn lead to a result of existence in $H^1_{loc}(\mathbb{R} \times [0,T))$.
Our work focuses on the study of the dynamics of dislocations densities. In a different direction, let us quote some recent results on the dynamics of dislocations lines, taken individually, that are represented by non-local Hamilton-Jacobi equations (see [2, 9] and [1, 4] for local and global in time results respectively).

Remark 1.3 (Existence and uniqueness for Burgers equation)
We remark that these technics can be applied to the case of classical Burgers equations in $W^{1,p}_{loc}(\mathbb{R} \times [0,T))$ for all $1 \leq p < +\infty$.
Indeed, if we consider, for a given function f, and initial data u_0, the following equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}(f(u)) = 0 & in \quad \mathcal{D}'(\mathbb{R} \times (0,T)) \\[2ex] u(x,0) = u_0(x) & x \in \mathbb{R}, \end{cases} \tag{1.4}$$

then we have the following theorem:

Theorem 1.4 *Let $p \in [1, +\infty)$ and f locally lipschitz and convex, then, for all $T, L_0 \geq 0$, and $u_0 \in W^{1,p}_{loc}(\mathbb{R})$, verifies $(H1)$ and $(H2)$. The equation (1.4) admits a solution $u \in W^{1,p}_{loc}(\mathbb{R} \times [0,T))$, unique in the class of solutions satisfying $(H1)$ and $(H2)$, for all $t \in [0,T)$.*

1.4 Organization of the paper

In section 2, we regularize the function $a(\cdot)$ and the initial conditions. After that, we prove that the system (1.1)-(1.2), modified by the term $(\varepsilon \dfrac{\partial^2 \rho^{\pm}}{\partial x^2})$ admits local in time solutions (in the "Mild" sense), by using an application of a fixed point theorem in the space of functions in $C([0,T); H^1_{loc}(\mathbb{R}))$ and verifying $(H1)$ for all $t \in (0,T)$. Then in section 3, we prove that the obtained solutions are regular and verify $(H2)$ for all $t \in (0,T)$, with initial conditions verifying $(H2)$. In section 4, we prove some uniform *a priori* estimates of the regularized solution obtained in section 3. Then, thanks to these estimates, we also prove in this section the existence of global in time solutions. In section 5, we give the demonstration of Theorem 1.1. And in section 6 we prove a comparison principle result of the system (1.1) in the case $\alpha = 0$. Finally, in section 7 we give an application of the previous results in the case of the classical Burgers equation.

2 Existence of solutions for an approximated system

In this paragraph, we prove a theorem of existence of solutions, local in time, for the system (1.1) modified by the term $\varepsilon \dfrac{\partial^2 \rho^{\pm}}{\partial x^2}$ after the regularization of the function $a(\cdot)$ and the initial conditions. This approximation brings us back to the study, for every $0 < \varepsilon < 1$, of the following system:

$$\begin{cases} \dfrac{\partial \rho^{+,\varepsilon}}{\partial t} - \varepsilon \dfrac{\partial^2 \rho^{+,\varepsilon}}{\partial x^2} = -\left(a^{\varepsilon}(t) + (\rho^{+,\varepsilon} - \rho^{-,\varepsilon}) + \alpha \displaystyle\int_0^1 (\rho^{+,\varepsilon} - \rho^{-,\varepsilon})(y,t)dy \right) \dfrac{\partial \rho^{+,\varepsilon}}{\partial x} & \text{in } \mathcal{D}'(\mathbb{R} \times (0,T)), \\ \dfrac{\partial \rho^{-,\varepsilon}}{\partial t} - \varepsilon \dfrac{\partial^2 \rho^{-,\varepsilon}}{\partial x^2} = \left(a^{\varepsilon}(t) + (\rho^{+,\varepsilon} - \rho^{-,\varepsilon}) + \alpha \displaystyle\int_0^1 (\rho^{+,\varepsilon} - \rho^{-,\varepsilon})(y,t)dy \right) \dfrac{\partial \rho^{-,\varepsilon}}{\partial x} & \text{in } \mathcal{D}'(\mathbb{R} \times (0,T)), \end{cases} \tag{2.5}$$

where $a^{\varepsilon} = \tilde{a} * \eta_{\varepsilon}$, with $\eta_{\varepsilon}(\cdot) = \frac{1}{\varepsilon}\eta(\frac{\cdot}{\varepsilon})$, such that $\eta \in C^{\infty}_c(\mathbb{R})$, positive, and $\int_{\mathbb{R}} \eta = 1$. The function $\tilde{a}(\cdot)$ is an extension in $\mathbb{R}$ of the function $a(\cdot)$ by 0.

We also consider the regularized initial conditions of the system (2.5):

$$\rho^{\pm,\varepsilon}(x,0) = \rho_0^{\pm,\varepsilon}(x) = \rho_0^{\pm,\varepsilon,per}(x) + L_0 x = \rho_0^{\pm,per} * \eta_{\varepsilon}(x) + L_0 x. \tag{2.6}$$

We have the following existence local in time result, for the approximated system:

Theorem 2.1 (Short time existence) *Assume* $(H1)$ *and* $(H3)$. *For all* $\alpha \in \mathbb{R}$ *and* $\rho_0^{\pm} \in H^1_{loc}(\mathbb{R})$ *there exists*

$$T^\star(\|\rho_0^{\pm,per}\|_{H^1(\mathbb{T})}, \|a\|_{L^\infty(0,T)}, L_0, \alpha, \varepsilon) > 0,$$

such that the system (2.5)-(2.6) admits a solution $\rho^{\pm,\varepsilon} \in C([0,T^\star); H^1_{loc}(\mathbb{R}))$ *with* $\rho^{\pm,\varepsilon}(.,t)$ *verifying* $(H1)$.

For the proof of this theorem (see sub-section 2.3), we need some Notation and Preliminary results:

2.1 Notation

In what follows, we are going to use the following notation:

1. $\rho^\varepsilon = \rho^{+,\varepsilon} - \rho^{-,\varepsilon}$,
2. $\rho^{\pm,\varepsilon,per} = \rho^{\pm,\varepsilon} - L_0 x$,
3. $\mathbb{T} = (\mathbb{R}/\mathbb{Z})$ is the $[0,1)$ periodic interval,
4. let $f = (f_1, f_2)$ be a vector such that $f_i \in H^1(\mathbb{T})$ for $i \in \{1,2\}$. The norm of f in $\left(H^1(\mathbb{T})\right)^2$ will be defined by $\|f\|_{H^1(\mathbb{T})} = \max(\|f_1\|_{H^1(\mathbb{T})}, \|f_2\|_{H^1(\mathbb{T})})$.
5. Let f be a function from $\mathbb{R} \times (0,T)$ to $\mathbb{R}$. we note by $f(t) = f(.,t) : x \longmapsto f(x,t)$.

Remark 2.2 (Periodicity) *According to* $(H1)$-$(H2)$, *it is clear that* ρ^ε, $\rho^{\pm,\varepsilon,per}$ *and* $\dfrac{\partial \rho^{\pm,\varepsilon}}{\partial x}$ *are* 1*-periodic in space functions.*

Under the notation of paragraph 2.1, we know that the system (2.5) is equivalent to:

$$\frac{\partial \rho^{\pm,\varepsilon,per}}{\partial t} - \varepsilon \frac{\partial^2 \rho^{\pm,\varepsilon,per}}{\partial x^2} = \overbrace{\mp C_\alpha[\rho^\varepsilon(t)] \frac{\partial \rho^{\pm,\varepsilon,per}}{\partial x}}^{\text{bilinear term}} \overbrace{\mp a^\varepsilon(t) \frac{\partial \rho^{\pm,\varepsilon,per}}{\partial x} \mp L_0 C_\alpha[\rho^\varepsilon(t)]}^{\text{linear term}} \mp L_0 a^\varepsilon(t) \quad \text{in } \mathbb{T}\times(0,T), \tag{2.7}$$

where $C_\alpha[\rho^\varepsilon(t)](x) = \left(\rho^\varepsilon(x,t) + \alpha \displaystyle\int_0^1 \rho^\varepsilon(y,t)dy\right)$,

with the periodic initial conditions

$$\rho^{\pm,\varepsilon,per}(x,0) = \rho_0^{\pm,\varepsilon,per}(x) \quad \text{in} \quad \mathbb{T}. \tag{2.8}$$

2.2 Preliminary results

***Lemma* 2.3 (Properties of the regularized sequence)** *Under the hypothesis* $(H1)$ *and* $(H3)$ *and for every* $\rho_0^{\pm} \in H^1_{loc}(\mathbb{R})$, *we have*

1. *The functions* $\rho_0^{\pm,\varepsilon,per} \in C^\infty(\mathbb{T})$ *and verify the following estimate:*

$$\|\rho_0^{\pm,\varepsilon,per}\|_{H^1(\mathbb{T})} \le \|\rho_0^{\pm,per}\|_{H^1(\mathbb{T})}.$$

2. *The function $a^\varepsilon(\cdot) \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and verifies the following estimate:*

$$\|a^\varepsilon\|_{L^\infty(\mathbb{R})} \leq \|a\|_{L^\infty(0,T)}.$$

3. *The sequence $a^\varepsilon(\cdot)$ strongly converges to $a(\cdot)$ in $L^2(0,T)$. The sequences $\rho_0^{\pm,\varepsilon,per}$ strongly converge to $\rho_0^{\pm,per}$ in $H^1(\mathbb{T})$.*

The proof of this lemma is a classical property of the regularizing sequence $(\eta_\varepsilon)_\varepsilon$.

***Lemma 2.4* (Mild solution)** *Assume (H3). For every $T \geq 0$, if $\rho^{\pm,\varepsilon,per} \in C([0,T); H^1(\mathbb{T}))$ are solutions of the following equation:*

$$\begin{aligned}\rho^{\pm,\varepsilon,per}(x,t) = S_\varepsilon(t)\rho_0^{\pm,\varepsilon,per} \quad &\mp L_0 \int_0^t a^\varepsilon(s)ds \mp \int_0^t S_\varepsilon(t-s)\left(C_\alpha[\rho^\varepsilon(s)]\frac{\partial \rho^{\pm,\varepsilon,per}}{\partial x}(s)\right)ds \\ &\mp \int_0^t S_\varepsilon(t-s)\left(L_0 C_\alpha[\rho^\varepsilon(s)] + a(t)\frac{\partial \rho^{\pm,\varepsilon,per}}{\partial x}(s)\right)ds,\end{aligned} \tag{2.9}$$

where $S_\varepsilon(t) = e^{\varepsilon t\Delta}$ is the heat semi-group, then $\rho^{\pm,\varepsilon,per}$ is a solution of the system (2.7)-(2.8) in the sense of distributions.

For the proof of this lemma, we refer to Pazy [16, Th 5.2. Page 107].

Lemma 2.5* (Fixed point): Let E be a Banach space, B be a continuous bilinear application from $E \times E$ to E and L be a continuous linear application from E to E such that:*

$$\|B(x,y)\|_E \leq \lambda\|x\|_E\|y\|_E \quad \textit{for all} \quad x, y \in E,$$

$$\|L(x)\|_E \leq \mu\|x\|_E \quad \textit{for all} \quad x \in E,$$

where $\lambda > 0$ and $\mu \in (0,1)$ are given constants. Then, for all $x_0 \in E$ such that

$$\|x_0\|_E < \frac{1}{4\lambda}(\mu - 1)^2,$$

the equation $x = x_0 + B(x,x) + L(x)$ admits a solution in E.

For the proof of this lemma we refer to Cannone [5, Lemma 4.2.14].

In order to show the existence of a solution within the framework of Lemma 2.4, we apply Lemma 2.5 in the space $E = \left(L^\infty((0,T); H^1(\mathbb{T}))\right)^2$, where x_0, B and L are defined, for $u = (u_1, u_2),\ v = (v_1, v_2) \in E$, by:

$$x_0 = S_\varepsilon(t)\rho_{0,vec}^\varepsilon + L_0\vec{i}\int_0^t a^\varepsilon(s)ds, \quad \text{where} \quad \rho_{0,vec}^\varepsilon = (\rho_0^{+,\varepsilon,per}, \rho_0^{-,\varepsilon,per}),\ \vec{i} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{2.10}$$

$$B(u,v)(t) = \bar{I}_1 \int_0^t S_\varepsilon(t-s)\left(C_\alpha[u_1(s) - u_2(s)]\frac{\partial v}{\partial x}(s)\right)ds, \quad \text{where} \quad \bar{I}_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.11}$$

$$L(u)(t) = L_0\vec{i}\int_0^t S_\varepsilon(t-s)C_\alpha[u_1(s) - u_2(s)]ds + \bar{I}_1\int_0^t S_\varepsilon(t-s)\left(a^\varepsilon(s)\frac{\partial u}{\partial x}(s)\right)ds. \tag{2.12}$$

***Lemma* 2.6 (Decreasing estimates)** *If $f \in L^q(\mathbb{T})$ with $2 \le q \le +\infty$ and $g \in L^2(\mathbb{T})$, then for all $t > 0$ we have the following estimates:*

(i)

$$\|S_\varepsilon(t)(fg)\|_{L^4(\mathbb{T})} \le Ct^{-\frac{3}{4}}\|f\|_{L^2(\mathbb{T})}\|g\|_{L^2(\mathbb{T})},$$

(ii)

$$\left\|\frac{\partial}{\partial x}(S_\varepsilon(t)f)\right\|_{L^2(\mathbb{T})} \le Ct^{-\frac{1}{2}}\|S_\varepsilon(\frac{t}{2})f\|_{L^2(\mathbb{T})},$$

(iii)

$$\left\|\frac{\partial}{\partial x}\left(S_\varepsilon(t)(fg)\right)\right\|_{L^2(\mathbb{T})} \le Ct^{-(\frac{1}{2}+\frac{1}{q})}\|f\|_{L^q(\mathbb{T})}\|g\|_{L^2(\mathbb{T})},$$

where $C = C(\varepsilon)$ is a positive constant depending on ε.

For the proof of this lemma, see Pazy [16, Lemma 1.1.8, Th 6.4.5].

Proposition 2.7 (Bilinear operator)*: Let $F_T = \left(L^\infty((0,T);H^1(\mathbb{T}))\right)^2$. Then for every $T \ge 0$, $\alpha \in \mathbb{R}$, $u = (u_1, u_2) \in F_T$ and $v = (v_1, v_2) \in F_T$ the bilinear operator B defined in (2.11), is continuous from $F_T \times F_T$ to F_T. Moreover, there exists a positive constant $C = C(\alpha, \varepsilon)$ such that for all $u, v \in F_T$ we have:*

$$\|B(u,v)\|_{F_T} \le CT^{\frac{1}{4}}\|u\|_{F_T}\|v\|_{F_T}.$$

Proof of Proposition 2.7

We have

$$\begin{aligned}
\|B(u,v)(t)\|_{H^1(\mathbb{T})} &= \left\|\bar{I}_1\int_0^t S_\varepsilon(t-s)\left(C_\alpha[u_1(s)-u_2(s)]\frac{\partial v}{\partial x}(s)\right)ds\right\|_{H^1(\mathbb{T})} \\
&\le \int_0^t \left\|S_\varepsilon(t-s)\left(C_\alpha[u_1(s)-u_2(s)]\frac{\partial v}{\partial x}(s)\right)\right\|_{H^1(\mathbb{T})} ds \\
&\le \int_0^t \left\|S_\varepsilon(t-s)\left(C_\alpha[u_1(s)-u_2(s)]\frac{\partial v}{\partial x}(s)\right)\right\|_{L^4(\mathbb{T})} ds \\
&+\int_0^t \left\|\frac{\partial}{\partial x}S_\varepsilon(t-s)\left(C_\alpha[u_1(s)-u_2(s)]\frac{\partial v}{\partial x}(s)\right)\right\|_{L^2(\mathbb{T})} ds.
\end{aligned}$$

Using Lemma 2.6 (i) for the first term and Lemma 2.6 (iii) with $q = 4$ for the second term, we can conclude that:

$$\begin{aligned}
\|B(u,v)(t)\|_{H^1(\mathbb{T})} &\le C\int_0^t \frac{1}{(t-s)^{\frac{3}{4}}}\|C_\alpha[u_1(s)-u_2(s)]\|_{L^4(\mathbb{T})}\left\|\frac{\partial v}{\partial x}(s)\right\|_{L^2(\mathbb{T})} ds \\
&\le C\sup_{0\le t<T}\left(\|u(t)\|_{H^1(\mathbb{T})}\right)\sup_{0\le t<T}\left(\|v(t)\|_{H^1(\mathbb{T})}\right)\int_0^t \frac{1}{(t-s)^{\frac{3}{4}}}ds.
\end{aligned}$$

Then for all $t \in (0,T)$, we have:

$$\begin{aligned} \|B(u,v)(t)\|_{H^1(\mathbb{T})} &\leq Ct^{\frac{1}{4}}\|u\|_{L^\infty((0,T);H^1(\mathbb{T}))^2}\|v\|_{L^\infty((0,T);H^1(\mathbb{T}))^2} \\ &\leq CT^{\frac{1}{4}}\|u\|_{L^\infty((0,T);H^1(\mathbb{T}))^2}\|v\|_{L^\infty((0,T);H^1(\mathbb{T}))^2}. \quad \blacksquare \end{aligned} \tag{2.13}$$

Proposition 2.8 (Linear operator): *Let* $F_T = \left(L^\infty((0,T);H^1(\mathbb{T}))\right)^2$ *and* $a(\cdot)$ *satisfying* $(H3)$. *Then for all* L_0, $T \geq 0$ *and* $u = (u_1,u_2) \in F_T$, *the linear operator* L *defined in (2.12), is continuous from* F_T *to* F_T. *Moreover, there exists a positive constant* $C = C(\alpha, \varepsilon, \|a\|_{L^\infty(0,T)}, L_0)$ *such that:*

$$\|L(u)\|_{F_T} \leq CT^{\frac{1}{4}}\|u\|_{F_T}.$$

The proof of Proposition 2.8 is similar of the one used in Proposition 2.7.

***Lemma* 2.9** *For all* L_0, $T \geq 0$ *and* $a(\cdot)$ *satisfying* $(H3)$, *if*

$$X_{a^\varepsilon}(t) = L_0\vec{i}\int_0^t a^\varepsilon(s)ds,\ t \in (0,T),$$

then

$$\|X_{a^\varepsilon}\|_{(L^\infty(0,T))^2} \leq L_0T\|a\|_{L^\infty(0,T)}.$$

The proof of Lemma 2.9 is trivial (from Lemma 2.3 (2)).

***Lemma* 2.10 (Continuity of the semi-group)** *For all* $f \in W^{2,2}(\mathbb{T})$ *and* $0 \leq \theta < t$, *we have the following estimates:*

(i)

$$\|(S_\varepsilon(t-\theta) - Id)f\|_{L^2(\mathbb{T})} \leq C(t-\theta)\|\frac{\partial^2 f}{\partial x^2}\|_{L^2(\mathbb{T})},$$

(ii)

$$\|(S_\varepsilon(t-\theta) - Id)f\|_{L^2(\mathbb{T})} \leq 2\|f\|_{L^2(\mathbb{T})},$$

where $C = C(\varepsilon)$ *is a positive constant depending on* ε.

We refer to Pazy [16, Lemma 6.2 Page 151] for the proof of this lemma.

***Lemma* 2.11 (Time continuity)** *Assume* $(H3)$. *If* $\rho_{0,vec} = (\rho_0^{+,per}, \rho_0^{-,per}) \in (H^1(\mathbb{T}))^2$, *then for all* $T \geq 0$ *and* $u = (u_1,u_2) \in \left(L^\infty((0,T);H^1(\mathbb{T}))\right)^2$, *the following applications are* $\left(C([0,T);H^1(\mathbb{T}))\right)^2$*:*

(A1): $t \to X_{a^\varepsilon}(t)$,

(A2): $t \to S_\varepsilon(t)\rho^\varepsilon_{0,vec}$, *where* $\rho^\varepsilon_{0,vec} = (\rho_0^{+,\varepsilon,per}, \rho_0^{-,\varepsilon,per})$,

(A3): $t \to B(u,u)(t)$,

(A4): $t \to L(u)(t)$,

where X_{a^ε}, B *and* L *are defined in Lemma 2.9, (2.11) and (2.12) respectively.*

Proof of Lemma 2.11

The continuity of $(A1)$ is trivial because $a \in L^\infty(0,T)$. From the fact that the semi-group $S_\varepsilon(\cdot)$ is continuous from $[0,T)$ to $(H^1(\mathbb{T}))^2$ we deduce the continuity of $(A2)$.
It remains to prove the continuity of $(A3)$ and $(A4)$. Indeed, the continuity of $(A3)$ at 0 is a consequence of inequality (2.13). Now, we are going to prove the continuity of $(A3)$ for all $\theta \in (0,T)$. For all t, such that $\theta < t \le \min(T, \frac{3\theta}{2})$, we write $t = (1+\gamma)\theta$ and denote $\tau = (1-\gamma)\theta$ (where $0 < \gamma \le \frac{1}{2}$) and we write

$$\begin{aligned}
B(u,u)(t) - B(u,u)(\theta) \;&=\; \int_0^\tau (S(t-s) - S(\theta-s))\left(C_\alpha[u_1(s)-u_2(s)]\frac{\partial u}{\partial x}(s)\right)ds \\
&\quad + \int_\tau^\theta (S(t-s) - S(\theta-s))\left(C_\alpha[u_1(s)-u_2(s)]\frac{\partial u}{\partial x}(s)\right)ds \\
&\quad + \int_\theta^t S(t-s)\left(C_\alpha[u_1(s)-u_2(s)]\frac{\partial u}{\partial x}(s)\right)ds
\end{aligned}$$

$$\begin{aligned}
= \;& \overbrace{\int_0^\tau ((S(t-\theta) - Id)S(\theta-s))\left(C_\alpha[u_1(s)-u_2(s)]\frac{\partial u}{\partial x}(s)\right)ds}^{I_1} \\
& + \overbrace{\int_\tau^\theta ((S(t-\theta) - Id)S(\theta-s))\left(C_\alpha[u_1(s)-u_2(s)]\frac{\partial u}{\partial x}(s)\right)ds}^{I_2} \\
& + \int_\theta^t S(t-s)\left(C_\alpha[u_1(s)-u_2(s)]\frac{\partial u}{\partial x}(s)\right)ds.
\end{aligned}$$

We apply Lemma 2.10 (i) and Lemma 2.6 (ii) to find an upper bound to I_1. We then apply Lemma 2.10 (ii) to find an upper bound to I_2. After that, we follow the same steps of the proof of Proposition 2.7 to conclude that:

$$\begin{aligned}
\|B(u,u)(t) - B(u,u)(\theta)\|_{H^1} \le \;& C(t-\theta)\|u\|^2_{(L^\infty((0,T);H^1(\mathbb{T})))^2}\int_0^\tau \frac{1}{(\theta-s)^{\frac{7}{4}}}ds \\
& + C\|u\|^2_{(L^\infty((0,T);H^1(\mathbb{T})))^2}\int_\tau^\theta \frac{1}{(\theta-s)^{\frac{3}{4}}}ds \\
& + C\|u\|^2_{(L^\infty((0,T);H^1(\mathbb{T})))^2}\int_\theta^t \frac{1}{(t-s)^{\frac{3}{4}}}ds.
\end{aligned}$$

After the computation of each integral we deduce that:

$$\|B(u,u)(t)-B(u,u)(\theta)\|_{H^1} \leq \; C(t-\theta)\left(\frac{1}{(\theta-\tau)^{\frac{3}{4}}}-\frac{1}{\theta^{\frac{3}{4}}}\right)\|u\|^2_{(L^\infty((0,T);H^1(\mathbb{T})))^2}$$

$$+C\left((\theta-\tau)^{\frac{1}{4}}+(t-\theta)^{\frac{1}{4}}\right)\|u\|^2_{(L^\infty((0,T);H^1(\mathbb{T})))^2}.$$

Observing that $t-\theta=\theta-\tau=\gamma\theta$ we finally obtain the following inequality:

$$\|B(u,u)(t)-B(u,u)(\theta)\|_{H^1} \leq C(\theta,\gamma)\left((t-\theta)^{\frac{1}{4}}+(t-\theta)\right)\|u\|^2_{(L^\infty((0,T);H^1(\mathbb{T})))^2},$$

hence the continuity of $(A3)$. In the same way we get the continuity in time of $(A4)$. ∎

2.3 Proof of Theorem 2.1

Proof of Theorem 2.1
We rewrite the system (2.9) in the following vectorial form:

$$\rho^\varepsilon_{vec}(\cdot,t) = \; S_\varepsilon(t)\rho^\varepsilon_{0,vec}+L_0\vec{i}\int_0^t a^\varepsilon(s)ds+\bar{I}_1\int_0^t S_\varepsilon(t-s)\left(C_\alpha[\rho^\varepsilon(s)]\frac{\partial\rho^\varepsilon_{vec}}{\partial x}(s)\right)ds$$

$$+L_0\vec{i}\int_0^t S_\varepsilon(t-s)C_\alpha[\rho^\varepsilon(s)]ds+\bar{I}_1\int_0^t S_\varepsilon(t-s)\left(a^\varepsilon(s)\frac{\partial\rho^\varepsilon_{vec}}{\partial x}(s)\right)ds.$$

Such that ρ^ε_{vec} is the vector $(\rho^{+,\varepsilon,per},\rho^{-,\varepsilon,per})$ and $\rho^\varepsilon_{0,vec}$ is the vector $(\rho_0^{+,\varepsilon,per},\rho_0^{-,\varepsilon,per})$. $\vec{i}$ and $\bar{I}_1$ are defined in (2.10) and (2.11) respectively.
This altogether leads to the following equation:

$$\rho^\varepsilon_{vec}(\cdot,t)=S_\varepsilon(t)\rho^\varepsilon_{0,vec}+X_{a^\varepsilon}(t)+B(\rho^\varepsilon_{vec},\rho^\varepsilon_{vec})(t)+L(\rho^\varepsilon_{vec})(t), \tag{2.14}$$

where B is the bilinear application and L is the linear application defined in (2.11) and (2.12) respectively and X_{a^ε} is defined in Lemma 2.9. Moreover, according to Lemmas 2.9 and 2.3 we know that:

$$\|S(t)\rho^\varepsilon_{0,vec}+X_{a^\varepsilon}(t)\|_{(L^\infty((0,T);H^1(\mathbb{T})))^2} \; \leq \|\rho^\varepsilon_{0,vec}\|_{H^1(\mathbb{T})}+L_0T\|a^\varepsilon\|_{L^\infty(\mathbb{R})}$$

$$\leq \|\rho_{0,vec}\|_{H^1(\mathbb{T})}+L_0T\|a\|_{L^\infty(0,T)}.$$

In order to apply Lemma 2.5, we want, for a well chosen time T, that the following inequality holds:

$$\|\rho^0_{vec}\|_{H^1(\mathbb{T})}+L_0T\|a\|_{L^\infty(0,T)}<\frac{1}{4CT^{\frac{1}{4}}}(CT^{\frac{1}{4}}-1)^2, \text{ and } CT^{\frac{1}{4}}<1, \tag{2.15}$$

where C is the largest constant between the two constants computed in Propositions 2.8 and 2.7. For:

$$(T^\star)^{\frac{1}{4}}(\|\rho_{0,vec}\|_{H^1(\mathbb{T})},\|a\|_{L^\infty(0,T)},L_0,\varepsilon)=\min\left(1,\frac{1}{2C},\frac{1}{16C(\|\rho^0_{vec}\|_{H^1(\mathbb{T})}+L_0\|a\|_{L^\infty(0,T)})}\right), \tag{2.16}$$

we can easily verify that $T^\star$ satisfies the inequality (2.15). We apply Lemma 2.5 over the space $F_{T^\star} = \left(L^\infty((0,T^\star);H^1(\mathbb{T}))\right)^2$, to prove the existence of a solution for the system (2.14) in $F_{T^\star}$. Then, according to Lemma 2.11, we deduce that the obtained solution is $\left(C([0,T^\star);H^1(\mathbb{T}))\right)^2$. This proves, by Lemma 2.4, the existence of a solution in the sense of distributions for the system (2.5)-(2.6) in $C([0,T^\star);H^1_{loc}(\mathbb{R}))$ that verifies $(H1)$. ∎

3 Properties of the solution to the approximated system

In this section we show that, the solutions of system (2.5)-(2.6) obtained in the previous section, are regular and verify $(H2)$, provided with initial conditions verify $(H2)$.

***Lemma* 3.1 (Regularity of the solution)** *Assume $(H1)$, $(H3)$ and $\rho_0^\pm \in H^1_{loc}(\mathbb{R})$, if $\rho^{\pm,\varepsilon} \in C([0,T);H^1_{loc}(\mathbb{R}))$ are solutions of the system (2.5)-(2.6), then $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}\times[0,T))$.*

Proof of Lemma 3.1
If we denote the second term of the system (2.7) by

$$f^\pm_{a^\varepsilon,\alpha}[\rho^\varepsilon(t)] = \mp a^\varepsilon(t)\left(L_0 + \frac{\partial \rho^{\pm,\varepsilon,per}}{\partial x}\right) \mp C_\alpha[\rho^\varepsilon(t)]\left(\frac{\partial \rho^{\pm,\varepsilon,per}}{\partial x} + L_0\right),$$

we know that $f^\pm_{a^\varepsilon,\alpha}[\rho^\varepsilon] \in L^2(\mathbb{T}\times(0,T))$. Moreover, we know that the initial conditions $\rho_0^{\pm,\varepsilon,per} \in C^\infty(\mathbb{T})$, which allows us to apply the L^2 regularity of the heat equation over the system (2.7)-(2.8) (see Lions-Magenes [15, Th.8.2]). Then we deduce by induction that the solution is $C^\infty(\mathbb{T}\times[0,T))$. ∎

***Lemma* 3.2 (Monotonicity of the solution in space)** *Assume $(H1)$, $(H2)$, $(H3)$ and $\rho_0^\pm \in H^1_{loc}(\mathbb{R})$, if $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}\times[0,T))$ are solutions of the system (2.5)-(2.6), then $\rho^{\pm,\varepsilon}(.,t)$ verifies $(H2)$ for all $t \in (0,T)$.*

Proof of Lemma 3.2
First, we remark that if $\dfrac{\partial \rho_0^\pm}{\partial x} \geq 0$, then $\dfrac{\partial \rho_0^{\pm,\varepsilon}}{\partial x} \geq 0$. Indeed, we have

$$\begin{aligned}\frac{\partial \rho_0^{\pm,\varepsilon}}{\partial x} = \frac{\partial \rho_0^{\pm,per}}{\partial x} * \eta_\varepsilon + L_0 \quad &= \left(\frac{\partial \rho_0^{\pm,per}}{\partial x} + L_0\right) * \eta_\varepsilon \\ &= \left(\frac{\partial \rho_0^{\pm}}{\partial x}\right) * \eta_\varepsilon \geq 0, \quad \text{because } \eta \text{ is positive.}\end{aligned}$$

We apply the maximum principle over the derived system of (2.5)-(2.6):

$$\begin{cases} \dfrac{\partial \theta^\pm}{\partial t} - \varepsilon \dfrac{\partial^2 \theta^\pm}{\partial x^2} \pm (C_\alpha[\rho^\varepsilon(t)] + a^\varepsilon(t))\dfrac{\partial \theta^\pm}{\partial x} \pm (\theta^+ - \theta^-)\theta^\pm = 0 \quad \text{in} \quad \mathbb{T}\times(0,T), \\ \\ \theta^\pm(x,0) = \dfrac{\partial \rho_0^{\pm,\varepsilon}}{\partial x}, \end{cases},$$

where $\theta^\pm = \dfrac{\partial \rho^{\pm,\varepsilon}}{\partial x}$ (see Gilbarg-Trudinger [10, Th.8.1])). Since $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}\times[0,T))$, we deduce that $\theta^\pm \geq 0$ belongs to $\mathbb{T}\times(0,T)$. ∎

Corollary 3.3 (Short time existence of non-decreasing regular solutions) *For all* $\alpha \in \mathbb{R}$ *and* $\rho_0^\pm \in H^1_{loc}(\mathbb{R})$*, under the assumptions* $(H1)$*,* $(H2)$ *and* $(H3)$*, there exists*

$$T^\star(\|\rho_0^{\pm,per}\|_{H^1(\mathbb{T})}, \|a\|_{L^\infty(0,T)}, L_0, \alpha, \varepsilon) > 0,$$

such that the system (2.5)-(2.6) admits a solution $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}\times[0,T^\star))$ *with* $\rho^{\pm,\varepsilon}(.,t)$ *verifying* $(H1)$ *and* $(H2)$*.*

The proof of Corollary 3.3 is a consequence of Theorem 2.1 and Lemmas 3.1 and 3.2 (with $T = T^\star$).

Remark 3.4 *Here, we remark that the case of non-decreasing solutions corresponds to a non-shock case in Burgers equation. On the other hand, the decreasing solutions represent the shock case.*

4 *A priori* estimates and long time existence for the approximated system

In this paragraph, we are going to show some ε-uniform estimates on the solutions of the system (2.5)-(2.6). These estimates will be used in section 4 for the passage to the limit as ε tends to zero.

***Lemma* 4.1 (L^2 estimates over the space derivatives of the solutions)** *Assume* $(H1)$*,* $(H2)$*,* $(H3)$ *and* $\rho_0^\pm \in H^1_{loc}(\mathbb{R})$*, if* $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}\times[0,T))$ *is a solution of the system (2.5)-(2.6) for all* $T \geq 0$*, then*

$$\left\|\frac{\partial \rho^{+,\varepsilon}}{\partial x}\right\|^2_{L^\infty((0,T);L^2(\mathbb{T}))} + \left\|\frac{\partial \rho^{-,\varepsilon}}{\partial x}\right\|^2_{L^\infty((0,T);L^2(\mathbb{T}))} \leq B_0,$$

with $B_0 = 8\left(\left\|\frac{\partial \rho_0^+}{\partial x}\right\|^2_{L^2(\mathbb{T})} + \left\|\frac{\partial \rho_0^-}{\partial x}\right\|^2_{L^2(\mathbb{T})}\right)$.

Proof of Lemma 4.1
If we denote $\rho^\varepsilon = \rho^{+,\varepsilon} - \rho^{-,\varepsilon}$ and $k^\varepsilon = \rho^{+,\varepsilon} + \rho^{-,\varepsilon}$ then, according to $(H1)$, it is clear that ρ^ε, $\frac{\partial \rho^\varepsilon}{\partial x}$ and $\frac{\partial k^\varepsilon}{\partial x}$ are 1-periodic functions. Moreover, by Lemma 3.2, we know that $\frac{\partial k^\varepsilon}{\partial x} \geq 0$.
If we take into consideration the equations of the system (2.5), we can conclude that ρ^ε and k^ε verify the following system:

$$\left\{\begin{array}{lll} \frac{\partial \rho^\varepsilon}{\partial t} - \varepsilon\frac{\partial^2 \rho^\varepsilon}{\partial x^2} & = -\left(\rho^\varepsilon + \alpha\int_0^1 \rho^\varepsilon dx + a^\varepsilon(t)\right)\frac{\partial k^\varepsilon}{\partial x} & \text{in} \quad \mathcal{D}'(\mathbb{R}\times(0,T)), \\ \frac{\partial k^\varepsilon}{\partial t} - \varepsilon\frac{\partial^2 k^\varepsilon}{\partial x^2} & = -\left(\rho^\varepsilon + \alpha\int_0^1 \rho^\varepsilon dx + a^\varepsilon(t)\right)\frac{\partial \rho^\varepsilon}{\partial x} & \text{in} \quad \mathcal{D}'(\mathbb{R}\times(0,T)). \end{array}\right. \tag{4.17}$$

We derive the first equation of the system (4.17) with respect to x, then we multiply the result by $\dfrac{\partial \rho^\varepsilon}{\partial x}$ and finally we integrate in space. For all $t \in (0,T)$, we then obtain:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial \rho^\varepsilon}{\partial x}(t)\right\|^2_{L^2(\mathbb{T})} + \varepsilon \left\|\frac{\partial^2 \rho^\varepsilon}{\partial x^2}(t)\right\|^2_{L^2(\mathbb{T})} &= -\int_0^1 (\frac{\partial \rho^\varepsilon}{\partial x})^2 \frac{\partial k^\varepsilon}{\partial x} - \int_0^1 \rho^\varepsilon \frac{\partial \rho^\varepsilon}{\partial x}\frac{\partial^2 k^\varepsilon}{\partial x^2} \\ &\quad - \left(\alpha \int_0^1 \rho^\varepsilon + a^\varepsilon(t)\right)\int_0^1 \frac{\partial^2 k^\varepsilon}{\partial x^2}\frac{\partial \rho^\varepsilon}{\partial x}.\end{aligned}$$

Now, we proceed in the same way of the previous equation, but we multiply the second equation of the system (4.17) by $\dfrac{\partial k^\varepsilon}{\partial x}$. For every $t \in (0,T)$, we obtain:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial k^\varepsilon}{\partial x}(t)\right\|^2_{L^2(\mathbb{T})} + \varepsilon \left\|\frac{\partial^2 k^\varepsilon}{\partial x^2}(t)\right\|^2_{L^2(\mathbb{T})} &= -\int_0^1 (\frac{\partial \rho^\varepsilon}{\partial x})^2 \frac{\partial k^\varepsilon}{\partial x} - \int_0^1 \rho^\varepsilon \frac{\partial k^\varepsilon}{\partial x}\frac{\partial^2 \rho^\varepsilon}{\partial x^2} \\ &\quad - \left(\alpha \int_0^1 \rho^\varepsilon + a^\varepsilon(t)\right)\int_0^1 \frac{\partial^2 \rho^\varepsilon}{\partial x^2}\frac{\partial k^\varepsilon}{\partial x}.\end{aligned}$$

Now, we add the two previous equations. Thanks to the periodicity of ρ^ε and $\dfrac{\partial k^\varepsilon}{\partial x}$, we deduce that:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial \rho^\varepsilon}{\partial x}(t)\right\|^2_{L^2(\mathbb{T})} + \frac{1}{2}\frac{d}{dt}\left\|\frac{\partial k^\varepsilon}{\partial x}(t)\right\|^2_{L^2(\mathbb{T})} &\le -\int_0^1 (\frac{\partial \rho^\varepsilon}{\partial x})^2 \frac{\partial k^\varepsilon}{\partial x} - \int_0^1 \frac{\partial}{\partial x}\left(\rho^\varepsilon \frac{\partial \rho^\varepsilon}{\partial x}\frac{\partial k^\varepsilon}{\partial x}\right) \\ &\quad - \left(\alpha \int_0^1 \rho^\varepsilon + a^\varepsilon(t)\right)\int_0^1 \frac{\partial}{\partial x}\left(\frac{\partial \rho^\varepsilon}{\partial x}\frac{\partial k^\varepsilon}{\partial x}\right) \\ &\le -\int_0^1 (\frac{\partial \rho^\varepsilon}{\partial x})^2 \frac{\partial k^\varepsilon}{\partial x} \le 0.\end{aligned}$$

We integrate in time and we use the fact that $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}\times[0,T))$ and Lemma 2.3. We obtain in particular

$$\sup_{t\in(0,T)}\left\|\frac{\partial \rho^\varepsilon}{\partial x}(t)\right\|^2_{L^2(\mathbb{T})} + \sup_{t\in(0,T)}\left\|\frac{\partial k^\varepsilon}{\partial x}(t)\right\|^2_{L^2(\mathbb{T})} \le 2\left(\left\|\frac{\partial(\rho_0^+ - \rho_0^-)}{\partial x}\right\|^2_{L^2(\mathbb{T})} + \left\|\frac{\partial(\rho_0^+ + \rho_0^-)}{\partial x}\right\|^2_{L^2(\mathbb{T})}\right).$$

That leads to the desired result. ∎

***Lemma* 4.2** (**L^2 estimates of the solutions**) *Assume $(H1)$, $(H2)$, $(H3)$ and $\rho_0^\pm \in H^1_{loc}(\mathbb{R})$, if $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}\times[0,T))$ are solutions of the system (2.5)-(2.6) for every $T \ge 0$, then*

$$\|\rho^{+,\varepsilon}\|^2_{L^\infty((0,T);L^2(0,1))} + \|\rho^{-,\varepsilon}\|^2_{L^\infty((0,T);L^2(0,1))} \le 8\left(M_0 + (B_0 + \|a\|^2_{L^\infty(0,T)})\right)e^{4L_0(1+\alpha^2)T},$$

where B_0 is defined in Lemma 4.1, and $M_0 = \left(\|\rho_0^+\|^2_{L^2(0,1)} + \|\rho_0^-\|^2_{L^2(0,1)}\right)$.

Proof of Lemma 4.2

We will use the same procedure of the proof of Lemma 4.1. We multiply the first equation of the system (4.17) by ρ^ε then we integrate in space. For every $t \in (0,T)$, we obtain:

$$\frac{1}{2}\frac{d}{dt}\|\rho^\varepsilon(t)\|^2_{L^2(\mathbb{T})} + \varepsilon\left\|\frac{\partial \rho^\varepsilon}{\partial x}(t)\right\|^2_{L^2(\mathbb{T})} = -\int_0^1 (\rho^\varepsilon)^2\frac{\partial k^\varepsilon}{\partial x} - \left(\alpha\int_0^1 \rho^\varepsilon + a^\varepsilon(t)\right)\int_0^1 \rho^\varepsilon\frac{\partial k^\varepsilon}{\partial x}.$$

Similarly, we multiply the second equation of the system (4.17) by k^ε and we integrate in space. For every $t \in (0,T)$, we obtain:

$$\frac{1}{2}\frac{d}{dt}\|k^\varepsilon(t)\|^2_{L^2(0,1)} + \varepsilon\left\|\frac{\partial k^\varepsilon}{\partial x}(t)\right\|^2_{L^2(\mathbb{T})} = -\int_0^1 \rho^\varepsilon\frac{\partial \rho^\varepsilon}{\partial x}k^\varepsilon - \left(\alpha\int_0^1 \rho^\varepsilon + a^\varepsilon(t)\right)\int_0^1 k^\varepsilon\frac{\partial \rho^\varepsilon}{\partial x}.$$

Now, we add the two previous equations and get:

$$\begin{aligned}
\frac{1}{2}\left(\frac{d}{dt}\|\rho^\varepsilon(t)\|^2_{L^2(\mathbb{T})} + \frac{d}{dt}\|k^\varepsilon(t)\|^2_{L^2(0,1)}\right) &\leq -\int_0^1\left((\rho^\varepsilon)^2\frac{\partial k^\varepsilon}{\partial x} + \frac{1}{2}k^\varepsilon\frac{\partial(\rho^\varepsilon)^2}{\partial x}\right)\\
&\quad - \left(\alpha\int_0^1\rho^\varepsilon + a^\varepsilon(t)\right)\left(\int_0^1 k^\varepsilon\frac{\partial\rho^\varepsilon}{\partial x} + \int_0^1\rho^\varepsilon\frac{\partial k^\varepsilon}{\partial x}\right)\\
&\leq -\frac{1}{2}\int_0^1(\rho^\varepsilon)^2\frac{\partial k^\varepsilon}{\partial x} - \frac{1}{2}\int_0^1\frac{\partial((\rho^\varepsilon)^2k^\varepsilon)}{\partial x}\\
&\quad - \left(\alpha\int_0^1\rho^\varepsilon + a^\varepsilon(t)\right)\int_0^1\frac{\partial(k^\varepsilon\rho^\varepsilon)}{\partial x}\\
&\leq -\left(\alpha\int_0^1\rho^\varepsilon + a^\varepsilon(t)\right)\int_0^1\frac{\partial(k^\varepsilon\rho^\varepsilon)}{\partial x}.
\end{aligned}$$

But we know from $(H1)$ that ρ^ε and $(k^\varepsilon - 2L_0x)$ are 1-periodic functions, which implies that:

$$\int_0^1\frac{\partial(k^\varepsilon\rho^\varepsilon)}{\partial x} = \int_0^1\frac{\partial((k^\varepsilon - 2L_0x)\rho^\varepsilon)}{\partial x} + 2L_0\int_0^1\frac{\partial(x\rho^\varepsilon)}{\partial x} = 2L_0\int_0^1 x\frac{\partial\rho^\varepsilon}{\partial x} + 2L_0\int_0^1\rho^\varepsilon.$$

We use Lemmas 4.1, 2.3, and the fact that ($ab \leq \frac{1}{2}(a^2+b^2)$ and $(a+b)^2 \leq 2(a^2+b^2)$), to deduce that:

$$\begin{aligned}
\frac{d}{dt}\left(\|k^\varepsilon(t)\|^2_{L^2(0,1)} + \|\rho^\varepsilon(t)\|^2_{L^2(\mathbb{T})}\right) &\leq 4L_0\left(|\alpha|\,\|\rho^\varepsilon(t)\|_{L^2(\mathbb{T})} + \|a\|_{L^\infty(0,T)}\right)\left(\left\|\frac{\partial\rho^\varepsilon}{\partial x}(t)\right\|_{L^2(\mathbb{T})} + \|\rho^\varepsilon(t)\|_{L^2(\mathbb{T})}\right)\\
&\leq 4L_0\left(\|a\|^2_{L^\infty(0,T)} + (1+\alpha^2)\|\rho^\varepsilon(t)\|^2_{L^2(\mathbb{T})} + \left\|\frac{\partial\rho^\varepsilon}{\partial x}(t)\right\|^2_{L^2(\mathbb{T})}\right)\\
&\leq 4L_0\left(B_0 + \|a\|^2_{L^\infty(0,T)}\right) + 4L_0(1+\alpha^2)\left(\|\rho^\varepsilon(t)\|^2_{L^2(\mathbb{T})} + \|k^\varepsilon(t)\|^2_{L^2(0,1)}\right),
\end{aligned}$$

Using the previous estimate and the fact that $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}\times[0,T))$, we finally obtain:

$$\|\rho^\varepsilon\|^2_{L^\infty((0,T);L^2(\mathbb{T}))} + \|k^\varepsilon\|^2_{L^\infty((0,T)L^2(0,1))} \leq 8\left(M_0 + B_0 + \|a\|^2_{L^\infty(0,T)}\right)e^{4L_0(1+\alpha^2)T}.$$

This leads to the desired result. $\blacksquare$

***Lemma* 4.3** **(L^2 estimate on the time derivatives of the solutions)** *Assume $(H1)$, $(H2)$, $(H3)$ and $\rho_0^{\pm} \in H^1_{loc}(\mathbb{R})$, if $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}\times[0,T))$ is a solution of the system (2.5)-(2.6) for every $T \geq 0$, then there exists a constant $C\left(T, L_0, \alpha, \|a\|_{L^\infty(0,T)}, M_0, B_0\right)$ independent of ε such that:*

$$\left\| \frac{\partial \rho^{\pm,\varepsilon}}{\partial t} \right\|_{L^2(\mathbb{T}\times(0,T))} \leq C.$$

Proof of Lemma 4.3

For the proof of Lemma 4.3, it is sufficient to show that the second term of the system (2.5)

$$f^{\pm}_{a^\varepsilon,\alpha}[\rho^\varepsilon(t)] = \mp \left(a^\varepsilon(t) + \rho^\varepsilon + \alpha \int_0^1 \rho^\varepsilon dx \right) \frac{\partial \rho^{\pm,\varepsilon}}{\partial x},$$

is bounded in $L^\infty((0,T);L^2(\mathbb{T}))$ uniformly in ε. Indeed,

$$\begin{aligned} \left\| f^{\pm}_{a^\varepsilon,\alpha}[\rho^\varepsilon] \right\|_{L^\infty((0,T^\star);L^2(\mathbb{T}))} &\leq \left\| \left(a^\varepsilon(\cdot) + \rho^\varepsilon + \alpha \int_0^1 \rho^\varepsilon dx \right) \frac{\partial \rho^{\pm,\varepsilon}}{\partial x} \right\|_{L^\infty((0,T);L^2(\mathbb{T}))} \\ &\leq C\left(\|\rho^\varepsilon\|_{L^\infty(\mathbb{T}\times(0,T))} + \|a\|_{L^\infty(0,T)} \right) \left\| \frac{\partial \rho^{\pm,\varepsilon}}{\partial x} \right\|_{L^\infty((0,T);L^2(\mathbb{T}))}. \end{aligned}$$

We use the Lemmas 4.1, 4.2 and the Sobolev injections to deduce that there exists a constant $C\left(T, L_0, \alpha, \|a\|_{L^\infty(0,T)}, M_0, B_0\right)$ such that

$$\left\| f^{\pm}_{a^\varepsilon,\alpha}[\rho^\varepsilon] \right\|_{L^\infty((0,T);L^2(\mathbb{T}))} \leq C.$$

To end up, we multiply the first and the second equations of the system (2.5) by $\dfrac{\partial \rho^{+,\varepsilon}}{\partial t}$, $\dfrac{\partial \rho^{-,\varepsilon}}{\partial t}$ respectively and we integrate in space. We deduce that for every $t \in (0,T)$ we have:

$$\left\| \frac{\partial \rho^{\pm,\varepsilon}}{\partial t}(t) \right\|^2_{L^2(\mathbb{T})} + \frac{\varepsilon}{2}\frac{d}{dt} \left\| \frac{\partial \rho^{\pm,\varepsilon}}{\partial x}(t) \right\|^2_{L^2(\mathbb{T})} = \int_0^1 f^{\pm}_{a^\varepsilon,\alpha}[\rho^\varepsilon(t)] \frac{\partial \rho^{\pm,\varepsilon}}{\partial t}.$$

We integrate in time and we use the fact that $\rho^{\pm,\varepsilon} \in C(\mathbb{R}\times[0,T))$ for all $T \geq 0$, we get:

$$\left\| \frac{\partial \rho^{\pm,\varepsilon}}{\partial t} \right\|^2_{L^2(\mathbb{T}\times(0,T))} + \frac{\varepsilon}{2} \left\| \frac{\partial \rho^{\pm,\varepsilon}}{\partial x}(T) \right\|^2_{L^2(\mathbb{T})} = \int_0^T \int_0^1 f^{\pm}_{a^\varepsilon,\alpha}[\rho^\varepsilon(t)] \frac{\partial \rho^{\pm,\varepsilon}}{\partial t} + \frac{\varepsilon}{2} \left\| \frac{\partial {\rho_0}^{\pm}}{\partial x} \right\|^2_{L^2(\mathbb{T})}.$$

We apply Hölder's inequality and the fact that $\varepsilon < 1$ and $ab \leq \frac{1}{2}(a^2+b^2)$, to obtain that:

$$\begin{aligned} \left\| \frac{\partial \rho^{\pm,\varepsilon}}{\partial t} \right\|^2_{L^2(\mathbb{T}\times(0,T))} &\leq \left\| f^{\pm}_{a^\varepsilon,\alpha}[\rho^\varepsilon] \right\|_{L^2(\mathbb{T}\times(0,T))} \left\| \frac{\partial \rho^{\pm,\varepsilon}}{\partial t} \right\|_{L^2(\mathbb{T}\times(0,T))} + \left\| \frac{\partial {\rho_0}^{\pm}}{\partial x} \right\|^2_{L^2(\mathbb{T})} \\ &\leq \frac{1}{2} \left\| f^{\pm}_{a^\varepsilon,\alpha}[\rho^\varepsilon] \right\|^2_{L^2(\mathbb{T}\times(0,T))} + \frac{1}{2} \left\| \frac{\partial \rho^{\pm,\varepsilon}}{\partial t} \right\|^2_{L^2(\mathbb{T}\times(0,T))} + \left\| \frac{\partial {\rho_0}^{\pm}}{\partial x} \right\|^2_{L^2(\mathbb{T})}. \end{aligned}$$

that leads to

$$\frac{1}{2}\left\|\frac{\partial \rho^{\pm,\varepsilon}}{\partial t}\right\|^2_{L^2(\mathbb{T}\times(0,T))} \le \frac{1}{2}\left\|f^{\pm}_{a^\varepsilon,\alpha}[\rho^\varepsilon]\right\|^2_{L^2(\mathbb{T}\times(0,T))} + \left\|\frac{\partial \rho_0{}^{\pm}}{\partial x}\right\|^2_{L^2(\mathbb{T})}$$

$$\le \frac{T}{2}\left\|f^{\pm}_{a^\varepsilon,\alpha}[\rho^\varepsilon]\right\|^2_{L^\infty((0,T);L^2(\mathbb{T}))} + \left\|\frac{\partial \rho_0{}^{\pm}}{\partial x}\right\|^2_{L^2(\mathbb{T})} \le C,$$

where $C\left(T, L_0, \alpha, \|a\|_{L^\infty(0,T)}, M_0, B_0\right)$. ∎

Remark 4.4 (The sense of the initial conditions) *According to Lemma 4.3, we have $\rho^{\pm,\varepsilon,per} \in C([0,T), L^2(\mathbb{T}))$ uniformly in ε. This will give a sense to the limit of the initial conditions.*

Theorem 4.5 (Long time existence) *Assume $(H1)$, $(H2)$ and $(H3)$, for all L_0, $T \ge 0$, $\alpha \in \mathbb{R}$ and $\rho_0^{\pm} \in H^1_{loc}(\mathbb{R})$, the system (2.5)-(2.6) admits the solutions $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}\times[0,T))$, with $\rho^{\pm,\varepsilon}(.,t)$ verifying $(H1)$ and $(H2)$. Moreover, there exists a constant $C\left(T, L_0, \alpha, \|a\|_{L^\infty(0,T)}, M_0, B_0\right)$ independent of ε, with B_0 and M_0 defined in Lemmas 4.1 and 4.2 respectively, such that:*

$$\left\|\rho^{\pm,\varepsilon,per}\right\|_{L^\infty((0,T);L^2(\mathbb{T}))} + \left\|\frac{\partial \rho^{\pm,\varepsilon}}{\partial x}\right\|_{L^\infty((0,T);L^2(\mathbb{T}))} + \left\|\frac{\partial \rho^{\pm,\varepsilon}}{\partial t}\right\|_{L^2(\mathbb{T}\times(0,T))} \le C, \tag{4.18}$$

where $\rho^{\pm,\varepsilon,per} = \rho^{\pm,\varepsilon} - L_0 x$.

Proof of Theorem 4.5
We are going to prove that the local in time solutions obtained by Corollary 3.3 can be extended to global in time solutions for the same system.
We argue by contradiction: Assume that there exists a maximum time T_{max} such that, we have the existence of solutions of the system (2.5)-(2.6) in the function space $C^\infty(\mathbb{R}\times[0,T_{max}))$.
For every $\delta > 0$, we consider the system (2.5) with the initial conditions

$$\rho^{\pm,\varepsilon}_{\delta,max} = \rho^{\pm,\varepsilon}(x, T_{max}-\delta).$$

We apply for the second time the same technic of Corollary 3.3 to deduce that there exists a time

$$T^\star_{\delta,max}(\|\rho^{\pm,\varepsilon,per}_{\delta,max}\|_{H^1(\mathbb{T})}, \|a\|_{L^\infty(0,T)}, L_0, \alpha, \varepsilon) > 0, \text{ where } \rho^{\pm,\varepsilon,per}_{\delta,max} = \rho^{\pm,\varepsilon}_{\delta,max} - L_0 x,$$

such that the system (2.5)-(2.6) admits a solution defined until the time

$$T_0 = (T_{max}-\delta) + T^\star_{\delta,max}.$$

Moreover, according to Lemmas 4.1 and 4.2, we know that $\rho^{\pm,\varepsilon,per}_{\delta,max}$ are δ-uniformly bounded in $H^1(\mathbb{T})$. We use (2.16) to deduce that there exists a constant $C(\varepsilon, T_{max}, \alpha, \|a\|_{L^\infty(0,T)}, L_0) > 0$ independent of δ such that $T^\star_{\delta,max} \ge C > 0$, then $\lim_{\delta\to 0} T^\star_{\delta,max} \ge C > 0$ which implies that $T_0 > T_{max}$ and so a contradiction.
The estimation (4.18) is a consequence of Lemmas 4.1, 4.2 and 4.3. ∎

5 Existence and uniqueness of the solution of (1.1)-(1.2)

In this paragraph, we are going to prove that the system (1.1)-(1.2) admits a unique solution $\rho^{\pm}$ (in the distribution sense) which is the limit as $\varepsilon \to 0$ of $\rho^{\pm,\varepsilon}$ given by Theorem 4.5. In order to do that, we pass to the limit when ε tends to 0 in the system (2.7)-(2.8), and we use (4.18) in order to assure the compactness. The proof of the uniqueness uses direct arguments.
Proof of Theorem 1.1
We first prove the existence and then establish the uniqueness.
Step 1 (Existence):

Let $\rho^{\pm,\varepsilon}$ be the solution of the system (2.5) given by Theorem 4.5. According to (4.18) we know that $\rho^{\pm,\varepsilon,per}$ are ε-uniformly bounded in $H^1(\mathbb{T}\times(0,T))$, then we can extract a sub-sequence that converges weakly in $H^1(\mathbb{T}\times(0,T))$. Knowing that $H^1(\mathbb{T}\times(0,T))$ is compact in $L^2(\mathbb{T}\times(0,T))$, this sub-sequence strongly converges in $L^2(\mathbb{T}\times(0,T))$. If we denote by $\rho^{\pm,per}$ the limit of this sub-sequence, we have to prove that $\rho^{\pm,per}+L_0x$ is a solution of the system (1.1)-(1.2) in the sense of distribution. Indeed, by Lemma 2.3, the term $\mp(L_0a^\varepsilon)$ of the equation (2.7) converges strongly to $(\mp L_0 a)$ in $L^2(0,T)$.
The linear term

$$\mp\left(L_0C_\alpha[\rho^\varepsilon]+a^\varepsilon(t)\frac{\partial\rho}{\partial x}^{\pm,\varepsilon,per}\right)$$

of the equation (2.7), weakly converges in $L^1(\mathbb{T}\times(0,T))$ and the reason is, in the one hand that $\dfrac{\partial\rho}{\partial x}^{\pm,\varepsilon,per}$ are ε-uniformly bounded in $L^2(\mathbb{T}\times(0,T))$ that gives us the weak convergence in $L^2(\mathbb{T}\times(0,T))$ and on the other hand, that a^ε strongly converges in $L^2(0,T)$. Then, the linear term converges in the sense of distributions (i.e. in $\mathcal{D}'(\mathbb{T}\times(0,T))$). It remains to prove that the bilinear term

$$C_\alpha[\rho^\varepsilon]\frac{\partial\rho}{\partial x}^{\pm,\varepsilon,per}$$

of the equation (2.7), also converges in the sense of distributions. We have:

1. The sequence $C_\alpha[\rho^\varepsilon]$ is compact in $L^2(\mathbb{T}\times(0,T))$.
2. The functions $\dfrac{\partial\rho}{\partial x}^{\pm,\varepsilon,per}$ are ε-uniformly bounded in $L^2(\mathbb{T}\times(0,T))$,

that gives us a strong convergence in $L^2(\mathbb{T}\times(0,T))$ times a weak convergence in $L^2(\mathbb{T}\times(0,T))$ and hence a weak convergence of the product in $L^1(\mathbb{T}\times(0,T))$. This leads, as a consequence, to the convergence in the distribution sense. This, altogether, shows that $\rho^{\pm,per}+L_0x$ is a solution in the sense of distribution of the system (1.1)-(1.2) and $\rho^{\pm,per}$ verifies estimate (4.18).

It remains to prove that the initial condition is satisfied by the limit function $\rho^{\pm,per}$. In fact, according to the estimate (4.18) on $\rho^{\pm,\varepsilon,per}$, $\dfrac{\partial\rho}{\partial t}^{\pm,\varepsilon}$ and $\dfrac{\partial\rho}{\partial x}^{\pm,\varepsilon}$, we see that $\rho^{\pm,\varepsilon,per}$ is ε-uniformly bounded in $H^1(\mathbb{T}\times(0,T))$.
From the fact that the injection of $H^1(\mathbb{T}\times(0,T))$ in $C([0,T);L^2(\mathbb{T}))$ is continuous and compact by classical arguments, we see that, for all $v\in L^2(\mathbb{T})$, the application $\gamma: U\longmapsto \int_{\mathbb{T}} U(0)v$ is a continuous linear form for $U\in C([0,T);L^2(\mathbb{T}))$ and hence $\gamma(\rho^{\pm,\varepsilon,per})\to\gamma(\rho^{\pm,per})$ as $\varepsilon\to 0$,

because up to a subsequence $\rho^{\pm,\varepsilon,per}$ converges strongly in $C([0,T);L^2(\mathbb{T}))$. This altogether proves that the solution verifies the initial conditions (1.2).

Step 2 (Uniqueness):

Let $\rho_1^\pm$ and $\rho_2^\pm$ be two solutions of the system (1.1), such that $\rho_1^\pm(\cdot,0)=\rho_2^\pm(\cdot,0)=\rho_0^\pm$. According to the previous paragraph, we know that $\rho_i^\pm$ verify the estimate (4.18) for $i=1,2$.

If we denote $\rho_i=\rho_i^+-\rho_i^-$, $k_i=\rho_i^++\rho_i^-$ for $i=1,2$, then it is clear that $(\rho_1-\rho_2)$ and (k_1-k_2) are 1-periodic functions in space and ρ_i, k_i verify the following system for $i=1,2$:

$$\left\{\begin{aligned}\frac{\partial\rho_i}{\partial t}&=-\left(\rho_i+\alpha\int_0^1\rho_i dx+a(t)\right)\frac{\partial k_i}{\partial x}\quad\text{in}\quad\mathcal{D}'(\mathbb{R}\times(0,T)),\\ \frac{\partial k_i}{\partial t}&=-\left(\rho_i+\alpha\int_0^1\rho_i dx+a(t)\right)\frac{\partial \rho_i}{\partial x}\quad\text{in}\quad\mathcal{D}'(\mathbb{R}\times(0,T)).\end{aligned}\right.\tag{5.19}$$

We substract the two systems to obtain that:

$$\left\{\begin{aligned}\frac{\partial(\rho_1-\rho_2)}{\partial t}&=-\left(\rho_1+\alpha\int_0^1\rho_1 dx\right)\frac{\partial k_1}{\partial x}+\left(\rho_2+\alpha\int_0^1\rho_2 dx\right)\frac{\partial k_2}{\partial x}-a(t)\frac{\partial(k_1-k_2)}{\partial x},\\ \frac{\partial(k_1-k_2)}{\partial t}&=-\left(\rho_1+\alpha\int_0^1\rho_1 dx\right)\frac{\partial \rho_1}{\partial x}+\left(\rho_2+\alpha\int_0^1\rho_2 dx\right)\frac{\partial \rho_2}{\partial x}-a(t)\frac{\partial(\rho_1-\rho_2)}{\partial x}.\end{aligned}\right.$$

The previous system is equivalent to:

$$\left\{\begin{aligned}\frac{\partial(\rho_1-\rho_2)}{\partial t}&=-\left((\rho_1-\rho_2)+\alpha\int_0^1(\rho_1-\rho_2)dx\right)\frac{\partial k_1}{\partial x}-\left(\rho_2+\alpha\int_0^1\rho_2 dx\right)\frac{\partial(k_1-k_2)}{\partial x}\\ &\quad -a(t)\frac{\partial(k_1-k_2)}{\partial x},\\ \frac{\partial(k_1-k_2)}{\partial t}&=-\left((\rho_1-\rho_2)+\alpha\int_0^1(\rho_1-\rho_2)dx\right)\frac{\partial \rho_1}{\partial x}-\left(\rho_2+\alpha\int_0^1\rho_2 dx\right)\frac{\partial(\rho_1-\rho_2)}{\partial x}\\ &\quad -a(t)\frac{\partial(\rho_1-\rho_2)}{\partial x}.\end{aligned}\right.$$

We multiply the first equation of this system by $(\rho_1-\rho_2)$ and we integrate in space to obtain, for almost every t, that:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|(\rho_1-\rho_2)(t)\|^2_{L^2(\mathbb{T})}=&-\int_0^1\left((\rho_1-\rho_2)^2\frac{\partial k_1}{\partial x}\right)-\alpha\left(\int_0^1(\rho_1-\rho_2)\right)\int_0^1\left((\rho_1-\rho_2)\frac{\partial k_1}{\partial x}\right)\\ &-\int_0^1\left((\rho_1-\rho_2)\left(\rho_2+\alpha\int_0^1\rho_2\right)\frac{\partial(k_1-k_2)}{\partial x}\right)\\ &-a(t)\int_0^1\left((\rho_1-\rho_2)\frac{\partial(k_1-k_2)}{\partial x}\right).\end{aligned}$$

Similarly, we multiply the second equation by $(k_1 - k_2)$ and we integrate in space to get for almost every time t:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|(k_1-k_2)(t)\|^2_{L^2(\mathbb{T})} = & -\int_0^1 \left((\rho_1-\rho_2)(k_1-k_2)\frac{\partial \rho_1}{\partial x}\right) - \alpha\left(\int_0^1 (\rho_1-\rho_2)\right)\int_0^1 (k_1-k_2)\frac{\partial \rho_1}{\partial x} \\
& -\int_0^1 \left((k_1-k_2)\left(\rho_2 + \alpha\int_0^1 \rho_2\right)\frac{\partial(\rho_1-\rho_2)}{\partial x}\right) \\
& -a(t)\int_0^1 \left((k_1-k_2)\frac{\partial(\rho_1-\rho_2)}{\partial x}\right).
\end{aligned}$$

We add the two previous equations to obtain, for almost every time t:

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\left(\|(\rho_1-\rho_2)(t)\|^2_{L^2(\mathbb{T})} + \|(k_1-k_2)(t)\|^2_{L^2(\mathbb{T})}\right) \\
= & -\int_0^1 \left((\rho_1-\rho_2)^2\frac{\partial k_1}{\partial x}\right) - \alpha\left(\int_0^1 (\rho_1-\rho_2)\right)\int_0^1 \left((\rho_1-\rho_2)\frac{\partial k_1}{\partial x}\right) \\
& -\alpha\left(\int_0^1 (\rho_1-\rho_2)\right)\int_0^1 \left((k_1-k_2)\frac{\partial \rho_1}{\partial x}\right) - \int_0^1 \left(\frac{\partial}{\partial x}\left((\rho_1-\rho_2)(k_1-k_2)\left(\rho_2+\alpha\int_0^1\rho_2\right)\right)\right) \\
& -\int_0^1 \left((\rho_1-\rho_2)(k_1-k_2)\frac{\partial(\rho_1-\rho_2)}{\partial x}\right) - a(t)\int_0^1 \left(\frac{\partial}{\partial x}\left((\rho_1-\rho_2)(k_1-k_2)\right)\right).
\end{aligned}$$

From the fact that ρ_i, $i = 1, 2$ and $(k_1 - k_2)$ are 1-periodic functions in space, the previous equation becomes:

$$\begin{aligned}
= & \overbrace{-\int_0^1 \left((\rho_1-\rho_2)^2\frac{\partial k_1}{\partial x}\right)}^{I_1} \; \overbrace{-\alpha\left(\int_0^1 (\rho_1-\rho_2)\right)\int_0^1 \left((\rho_1-\rho_2)\frac{\partial k_1}{\partial x}\right)}^{I_2} \\
& \overbrace{-\alpha\left(\int_0^1 (\rho_1-\rho_2)\right)\int_0^1 \left((k_1-k_2)\frac{\partial \rho_1}{\partial x}\right)}^{I_3} \; \overbrace{-\int_0^1 \left((\rho_1-\rho_2)(k_1-k_2)\frac{\partial(\rho_1-\rho_2)}{\partial x}\right)}^{I_4}.
\end{aligned}$$

And since $\frac{\partial k_i}{\partial x} \geq 0$ for $i = 1, 2$, we know that:

$$\begin{aligned}
I_1 + I_4 & = -\int_0^1 \left((\rho_1-\rho_2)^2\frac{\partial k_1}{\partial x}\right) - \frac{1}{2}\int_0^1 \left((k_1-k_2)\frac{\partial}{\partial x}\left((\rho_1-\rho_2)^2\right)\right) \\
& = -\int_0^1 \left((\rho_1-\rho_2)^2\frac{\partial k_1}{\partial x}\right) + \frac{1}{2}\int_0^1 \left((\rho_1-\rho_2)^2\frac{\partial (k_1-k_2)}{\partial x}\right) \\
& = -\frac{1}{2}\int_0^1 \left((\rho_1-\rho_2)^2\frac{\partial (k_1+k_2)}{\partial x}\right) \leq 0.
\end{aligned}$$

Moreover, from (4.18), we have for almost every t:

$$
\begin{aligned}
I_2 &\leq |\alpha| \|(\rho_1-\rho_2)(t)\|_{L^2(\mathbb{T})} \|(\rho_1-\rho_2)(t)\|_{L^2(\mathbb{T})} \left\| \frac{\partial k_1}{\partial x}(t) \right\|_{L^2(\mathbb{T})} \\
&\leq C \|(\rho_1-\rho_2)(t)\|^2_{L^2(\mathbb{T})}.
\end{aligned}
$$

Similarly, from (4.18), we have, for almost every t, that:

$$
\begin{aligned}
I_3 &\leq |\alpha| \|(\rho_1-\rho_2)(t)\|_{L^2(\mathbb{T})} \|(k_1-k_2)(t)\|_{L^2(\mathbb{T})} \left\| \frac{\partial \rho_1}{\partial x}(t) \right\|_{L^2(\mathbb{T})} \\
&\leq C \left(\|(\rho_1-\rho_2)(t)\|^2_{L^2(\mathbb{T})} + \|(k_1-k_2)(t)\|^2_{L^2(\mathbb{T})} \right)
\end{aligned}
$$

Then

$$
\frac{d}{dt} \left(\|(\rho_1-\rho_2)(t)\|^2_{L^2(\mathbb{T})} + \|(k_1-k_2)(t)\|^2_{L^2(\mathbb{T})} \right) \leq C \left(\|(\rho_1-\rho_2)(t)\|^2_{L^2(\mathbb{T})} + \|(k_1-k_2)(t)\|^2_{L^2(\mathbb{T})} \right).
$$

Now, we integrate in time and we use the fact that $\rho_i,\ k_i \in C([0,T), L^2_{loc}(\mathbb{R}))$, $\rho_1(\cdot,0) = \rho_2(\cdot,0)$ and $k_1(\cdot,0) = k_2(\cdot,0)$ to obtain that:

$$
\sup_{t\in(0,T)} \|(\rho_1-\rho_2)(t)\|^2_{L^2(\mathbb{T})} + \sup_{t\in(0,T)} \|(k_1-k_2)(t)\|^2_{L^2(\mathbb{T})} \leq 0.
$$

This achieves the proof of uniqueness. ∎

Remark 5.1 *In Theorem 1.1, we have proved a result of existence and uniqueness in $H^1_{loc}(\mathbb{R} \times [0,T))$ depending on some uniform estimates in this space. These estimates give a sufficient compactness in order to ensure the passage to the limit as ε tends to 0 in the bilinear term. However, the space $W^{1,1}_{loc}(\mathbb{R} \times [0,T))$ does not give enough compactness. On the other hand, the space of functions $L^2_{loc}(\mathbb{R} \times [0,T))$ having their derivatives in $L^\infty((0,T); \left(L^1 \log L^1\right)_{loc}(\mathbb{R}))$ requires the minimal properties to ensure the passage to the limit in the bilinear term. The result of existence in this space will be the core of a paper in preparation.*

6 Further properties: comparison principle with case $\alpha = 0$

In this section, we are going to prove a comparison principle result of the system (1.1) in the case $\alpha = 0$ (i.e. the Theorem 1.2). In under to do this, first we prove in the following subsection the same result for the approximate system (2.5). After that, we give the proof of Theorem 1.2.

6.1 Comparison principle for the regularized system with case $\alpha = 0$

***Lemma* 6.1 (Comparison principle)** *Let $a(\cdot)$ satisfies $(H3)$ and $\rho_1^{\pm,\varepsilon}$, $\rho_2^{\pm,\varepsilon} \in C^\infty(\mathbb{R}\times[0,T))$ be two solutions of the system (2.5) with $\alpha = 0$. Moreover, let $\rho_1^{\pm,\varepsilon}(.,t)$, $\rho_2^{\pm,\varepsilon}(.,t)$ verify $(H1)$ and $(H2)$ for all $t \in [0,T)$. Then, if $\rho_1^{\pm,\varepsilon}(\cdot,0) \leq \rho_2^{\pm,\varepsilon}(\cdot,0)$ in $\mathbb{R}$, we have $\rho_1^{\pm,\varepsilon} \leq \rho_2^{\pm,\varepsilon}$ in $\mathbb{R} \times [0,T)$.*

Proof of Lemma 6.1
We know that $\rho_1^{\pm,\varepsilon}$ and $\rho_2^{\pm,\varepsilon}$ verify the following systems:

$$\begin{cases} \dfrac{\partial \rho_1^{+,\varepsilon}}{\partial t} - \varepsilon \dfrac{\partial^2 \rho_1^{+,\varepsilon}}{\partial x^2} = & -\left(\rho_1^{+,\varepsilon} - \rho_1^{-,\varepsilon} + a^\varepsilon(t)\right) \dfrac{\partial \rho_1^{+,\varepsilon}}{\partial x} \quad \text{in } \mathcal{D}'(\mathbb{R}\times(0,T)), \\ \dfrac{\partial \rho_1^{-,\varepsilon}}{\partial t} - \varepsilon \dfrac{\partial^2 \rho_1^{-,\varepsilon}}{\partial x^2} = & \left(\rho_1^{+,\varepsilon} - \rho_1^{-,\varepsilon} + a^\varepsilon(t)\right) \dfrac{\partial \rho_1^{-,\varepsilon}}{\partial x} \quad \text{in } \mathcal{D}'(\mathbb{R}\times(0,T)), \end{cases}$$
$$\begin{cases} \dfrac{\partial \rho_2^{+,\varepsilon}}{\partial t} - \varepsilon \dfrac{\partial^2 \rho_2^{+,\varepsilon}}{\partial x^2} = & -\left(\rho_2^{+,\varepsilon} - \rho_2^{-,\varepsilon} + a^\varepsilon(t)\right) \dfrac{\partial \rho_2^{+,\varepsilon}}{\partial x} \quad \text{in } \mathcal{D}'(\mathbb{R}\times(0,T)), \\ \dfrac{\partial \rho_2^{-,\varepsilon}}{\partial t} - \varepsilon \dfrac{\partial^2 \rho_2^{-,\varepsilon}}{\partial x^2} = & \left(\rho_2^{+,\varepsilon} - \rho_2^{-,\varepsilon} + a^\varepsilon(t)\right) \dfrac{\partial \rho_2^{-,\varepsilon}}{\partial x} \quad \text{in } \mathcal{D}'(\mathbb{R}\times(0,T)). \end{cases}$$

respectively.

If we denote $w^{\pm,\varepsilon}$ by $\tilde{\rho}_2^{\pm,\varepsilon} - \tilde{\rho}_1^{\pm,\varepsilon}$, where,

$$\tilde{\rho}_2^{\pm,\varepsilon} = \rho_2^{\pm,\varepsilon} e^{-\gamma t}, \quad \tilde{\rho}_1^{\pm,\varepsilon} = \rho_1^{\pm,\varepsilon} e^{-\gamma t} \quad \text{with} \quad \gamma > 0,$$

we can easily check that $w^{\pm,\varepsilon}$ are solutions of the following system:

$$\begin{cases} \dfrac{\partial w^{+,\varepsilon}}{\partial t} - \varepsilon \dfrac{\partial^2 w^{+,\varepsilon}}{\partial x^2} + \gamma w^{+,\varepsilon} = & -e^{\gamma t}\left(w^{+,\varepsilon} - w^{-,\varepsilon}\right) \dfrac{\partial \tilde{\rho}_2^{+,\varepsilon}}{\partial x} - e^{\gamma t}\left(\tilde{\rho}_1^{+,\varepsilon} - \tilde{\rho}_1^{-,\varepsilon} + e^{-\gamma t} a^\varepsilon(t)\right) \dfrac{\partial w^{+,\varepsilon}}{\partial x}, \\ \dfrac{\partial w^{-,\varepsilon}}{\partial t} - \varepsilon \dfrac{\partial^2 w^{-,\varepsilon}}{\partial x^2} + \gamma w^{-,\varepsilon} = & e^{\gamma t}\left(w^{+,\varepsilon} - w^{-,\varepsilon}\right) \dfrac{\partial \tilde{\rho}_2^{-,\varepsilon}}{\partial x} + e^{\gamma t}\left(\tilde{\rho}_1^{+,\varepsilon} - \tilde{\rho}_1^{-,\varepsilon} + e^{-\gamma t} a^\varepsilon(t)\right) \dfrac{\partial w^{-,\varepsilon}}{\partial x}. \end{cases} \tag{6.20}$$

We are interested in the $\min_{(k,x,t)\in\{+,-\}\times\mathbb{T}\times(0,T)} \left(w^{k,\varepsilon}(x,t)\right)$. Our result follows if we can prove that this minimum is positive. However, this minimum is attained at a point $(k_0, x_0, t_0) \in \{+,-\}\times\mathbb{T}\times[0,T]$ (because $w^{+,\varepsilon}$ et $w^{-,\varepsilon}$ are $C^\infty(\mathbb{T}\times(0,T))$).

Two cases may occur:

1. In the case $t_0 = 0$, we have

   $$\min_{(k,x,t)\in\{+,-\}\times\mathbb{T}\times(0,T)} (w^{k,\varepsilon}(x,t)) = w^{k_0,\varepsilon}(x_0,t_0) = \left(\rho_2^{k_0,\varepsilon}(x_0,0) - \rho_1^{k_0,\varepsilon}(x_0,0)\right) e^{-\gamma t_0} \geq 0;$$

   and we are done.

2. In the case $t_0 \in (0,T]$, we have: (k_0,x_0,t_0) is a minimum point, then:

   $$\frac{\partial^2 w^{k_0,\varepsilon}}{\partial x^2}(x_0,t_0) \geq 0, \tag{6.21}$$

   $$\frac{\partial w^{k_0,\varepsilon}}{\partial t}(x_0,t_0) \leq 0, \tag{6.22}$$

   $$\frac{\partial w^{k_0,\varepsilon}}{\partial x}(x_0,t_0) = 0. \tag{6.23}$$

We combine (6.21), (6.22), (6.23) and we take into consideration that $w^{\pm,\varepsilon}$ verifies the system (6.20), we obtain that:

$$\begin{aligned}\gamma w^{k_0,\varepsilon}(x_0,t_0) &\geq e^{\gamma t_0} sign(w^{+,\varepsilon}(x_0,t_0)-w^{-,\varepsilon}(x_0,t_0))(w^{+,\varepsilon}(x_0,t_0)-w^{-,\varepsilon}(x_0,t_0))\frac{\partial\tilde{\rho}_2^{k_0,\varepsilon}}{\partial x}\\ &\geq e^{\gamma t_0}|w^{+,\varepsilon}(x_0,t_0)-w^{-,\varepsilon}(x_0,t_0)|\frac{\partial\tilde{\rho}_2^{k_0,\varepsilon}}{\partial x}\geq 0.\end{aligned}$$

Then $\tilde{\rho}_1^{\pm,\varepsilon}\leq\tilde{\rho}_2^{\pm,\varepsilon}$ in $\mathbb{R}\times(0,T)$, which gives $\rho_1^{\pm,\varepsilon}\leq\rho_2^{\pm,\varepsilon}$. ∎

We now give the proof of Theorem 1.2:

6.2 Proof of Theorem 1.2

Let

$$\rho_1^{\pm}(x,0)=\rho_{1,0}^{\pm}(x)=\rho_{1,0}^{\pm,per}(x)+L_0x \quad\text{and}\quad \rho_2^{\pm}(x,0)=\rho_{2,0}^{\pm}(x)=\rho_{2,0}^{\pm,per}(x)+L_0x.$$

If we denote

$$\rho_{1,0}^{\pm,\varepsilon}(x)=\rho_{1,0}^{\pm,per}*\eta_\varepsilon(x)+L_0x \quad\text{and}\quad \rho_{2,0}^{\pm,\varepsilon}(x)=\rho_{2,0}^{\pm,per}*\eta_\varepsilon(x)+L_0x,$$

where η_ε is a regularization sequence, we can easily check that $\rho_{1,0}^{\pm,\varepsilon}\leq\rho_{2,0}^{\pm,\varepsilon}$.
Moreover, according to the uniqueness of the solution, we know that there exist $\rho_1^{\pm,\varepsilon}$, $\rho_2^{\pm,\varepsilon}\in C^\infty(\mathbb{R}\times[0,T))$, verifying $(H2)$ for all $t\in(0,T)$, that are solutions of the system (2.5), such that

$$\rho_1^{\pm}=\lim_{\varepsilon\to 0}\rho_1^{\pm,\varepsilon},\quad \rho_2^{\pm}=\lim_{\varepsilon\to 0}\rho_2^{\pm,\varepsilon},$$

$$\rho_1^{\pm,\varepsilon}(x,0)=\rho_{1,0}^{\pm,\varepsilon}(x) \quad\text{and}\quad \rho_2^{\pm,\varepsilon}(x,0)=\rho_{2,0}^{\pm,\varepsilon}(x).$$

We apply Lemma 6.1 to obtain that $\rho_1^{\pm,\varepsilon}\leq\rho_2^{\pm,\varepsilon}$. We pass to the limit as $\varepsilon\to 0$ to deduce that $\rho_1^{\pm}\leq\rho_2^{\pm}$. ∎

Remark 6.2 *Thanks to this comparison result, we proved in a previous paper [8] the existence and the uniqueness of a solution (in the viscosity sense). Here, this comparison result is an indirect explanation of our estimates obtained in Lemmas 4.1, 4.2 and 4.3 that have ensured our principal Theorem 1.1.*

7 Application in the case of classical Burgers equation

In this paragraph we are going to prove that these technics can be also applied to the classical Burgers equation, even in the frame of functions in $W^{1,p}_{loc}(\mathbb{R}\times(0,T))$ for all $1\leq p<+\infty$, constituting the proof of Theorem 1.4:
Proof of Theorem 1.4
For the proof of this theorem, it suffices to show an estimation over the space derivatives of the solution (i.e. a result similar to that of Lemma 4.1).

First of all, we put ourselves in the hypothesis of Lemma 4.1. We derive the equation (1.4) with respect to x, then we multiply it by $\left(\frac{\partial u}{\partial x}\right)^{p-1}$ and finally we integrate over $(0,1)$, since u verifies $(H2)$, we obtain that:

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\|u(t)\|^p_{L^p(\mathbb{T})} &= -\int_0^1 f''(u)\frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial x}\right)^p - \int_0^1 f'(u)\frac{\partial^2 u}{\partial x^2}\left(\frac{\partial u}{\partial x}\right)^{p-1} \\
&= -\int_0^1 \frac{\partial(f'(u))}{\partial x}\left(\frac{\partial u}{\partial x}\right)^p - \frac{1}{p}\int_0^1 f'(u)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)^p \\
&= -\frac{1}{p}\int_0^1 \frac{\partial}{\partial x}\left(f'(u)\left(\frac{\partial u}{\partial x}\right)^p\right) - (1-\frac{1}{p})\int_0^1 f''(u)\frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial x}\right)^p \leq 0,
\end{aligned}$$

because f is convex, u verifies $(H2)$ and $p \geq 1$. To terminate the demonstration, we follow the same steps of the proof of Theorem 1.1. We remark that here we do not need the L^2 bound over the solution and also the compactness in the passage to the limit, because the equation (1.4) is in the conservative form which was not the case of our study.

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