

# Pricing double barrier Parisian Options using Laplace transforms

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# PRICING DOUBLE BARRIER PARISIAN OPTIONS USING LAPLACE TRANSFORMS

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**ABSTRACT.** In this work, we study a double barrier version of the standard Parisian options. We give closed formulae for the Laplace transforms of their prices with respect to the maturity time. We explain how to invert them numerically and prove a result on the accuracy of the numerical inversion.

**Key words:** double barrier, Parisian option, Laplace transform, numerical inversion, Brownian excursions, Euler summation.

## 1. INTRODUCTION

The pricing and hedging of vanilla options is now part of the common knowledge and the general interest has moved on to more complex products. So, practitioners need to be able to price these new products. Among them, there are the so-called path-dependent options. The ones we study in this paper are called double barrier Parisian options. They are a version with two barriers of the standard Parisian options introduced by Marc Chesney, Monique Jeanblanc and Marc Yor in 1997 (see Chesney et al. (1997)). Before introducing double barrier Parisian options, we first recall the definition of Parisian options. Parisian options can be seen as barrier options where the condition involves the time spent in a row above or below a certain level, and not only an exiting time. Double barrier Parisian options are options where the conditions imposed on the asset involve the time spent out of the range defined by the two barriers.

The valuation of single barrier Parisian options can be done by using several different methods: Monte Carlo simulations, lattices, Laplace transforms or partial differential equations. As for standard barrier options, using simulations leads to a biased problem, due to the choice of the discretisation time step in the Monte Carlo algorithm. The problem of improving the performance of Monte Carlo methods in exotic pricing has drawn much attention and has particularly been developed by Andersen and Brotherton-Ratcliffe (1996). Concerning lattices, we refer the reader to the work Avellaneda and Wu (1999). The idea of

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using Laplace transforms to price single barrier Parisian options is owed to Chesney et al. (1997). The Formulae of the Laplace transforms of all the different Parisian option prices can be found in Labart and Lelong (2005). Schröder (2003) and Hartley (2002) have also studied these options using Laplace transforms. An approach based on partial differential equations has been developed by Haber et al. (1999) and Wilmott (1998). Double Parisian options have already been priced by Baldi et al. (2000) using Monte Carlo simulations corrected by the means of sharp large deviation estimates.

In this paper, we compute the prices of double barrier Parisian options by using Laplace transforms. First, we give a detailed computation of the Laplace transforms of the prices with respect to the maturity time. Then, we establish a formula for the inverse of the Laplace transforms using contour integrals. Since it cannot be computed exactly, we give an upper bound of the error between the approximated price and the exact one. We improve the approximation by using the Euler summation to get a fast and accurate numerical inversion. The paper is organised as follows. In section 2, we introduce the general framework and give precise definitions of double barrier Parisian option prices. In section 3, we establish a Call Put parity relationship, which enables us to deduce the price of put options from the prices of call options. In section 4, we carry out the computation of the Laplace transforms of double barrier Parisian option prices. In section 5, we give a formula for the inversion of the Laplace transforms and state some results concerning the accuracy of the method. The technique we use to prove these results is based on the regularity of option price (see Appendix A). In section 6, we draw some graphs and compare the Laplace transform technique with the corrected Monte Carlo method of Baldi et al. (2000). For the comparison, we have used the implementation of the algorithm of Baldi et al. (2000) available in PREMIA<sup>1</sup>.

## 2. DEFINITIONS

**2.1. Some notations.** Let  $S = \{S_t, t \geq 0\}$  denote the price of an underlying asset. We assume that under the risk neutral measure  $\mathbb{Q}$ , the dynamics of S is given by

$$dS_t = S_t((r - \delta)dt + \sigma dW_t), \quad S_0 = x$$

where  $W = \{W_t, t \geq 0\}$  is a  $\mathbb{Q}$ -Brownian motion,  $x > 0$ , the volatility  $\sigma$  is a positive constant,  $r$  denotes the interest rate. The parameter  $\delta$  is the dividend rate if the underlying is a stock or the foreign interest rate in case of a currency. We assume that both  $r$  and  $\delta$  are constant. It follows that

$$S_t = x e^{(r-\delta-\sigma^2/2)t+\sigma W_t}.$$

We introduce

$$(2.1) \quad m = \frac{1}{\sigma} \left( r - \delta - \frac{\sigma^2}{2} \right).$$

Under  $\mathbb{Q}$ , the dynamics of the asset is given by  $S_t = x e^{\sigma(mt+W_t)}$ . From now on, we consider that every option has a finite maturity time  $T$ . Relying on Girsanov's Theorem (see Revuz and Yor (1999)), we can introduce a new probability  $\mathbb{P}$  — defined by  $\frac{d\mathbb{P}}{d\mathbb{Q}|_{\mathcal{F}_T}} = e^{mZ_T - \frac{m^2}{2}T}$  — which makes  $Z = \{Z_t = W_t + mt, 0 \leq t \leq T\}$  a  $\mathbb{P}$ -Brownian motion. Thus, S rewrites  $S_t = x e^{\sigma Z_t}$  under  $\mathbb{P}$ . Without any further indications, all the processes and expectations are considered under  $\mathbb{P}$ .

<sup>1</sup>PREMIA is a pricing software developed the MathFi team of INRIA Rocquencourt, see <http://www.premia.fr>.

**2.2. Double barrier Parisian option.** There are two different ways of measuring the time spent above or below a barrier. Either, one only counts the time spent in a row and resets the counting each time the stock price crosses the barrier(s) — we call it the *continuous* manner — or one adds the time spent in the relevant excursions without resuming the counting from 0 each time the stock price crosses the barrier(s) — we call it the *cumulative* manner. In practice, these two ways of counting time raise different questions about the paths of Brownian motion. In this work, we only focus on continuous style options.

**2.2.1. Knock Out.** A knock out double barrier Parisian call (respectively put) is lost if  $S$  makes an excursion outside the range  $(L_1, L_2)$  older than  $D$  before  $T$  otherwise it pays at maturity time  $T$   $(S_T - K)_+$  (respectively  $(K - S_T)_+$ ) where  $K$  is the strike.

We introduce  $b_1$  and  $b_2$  the barriers corresponding to  $L_1$  and  $L_2$  for the Brownian motion  $Z$

$$b_1 = \frac{1}{\sigma} \log \left( \frac{L_1}{x} \right), \quad b_2 = \frac{1}{\sigma} \log \left( \frac{L_2}{x} \right).$$

For some level  $b$ , let us introduce the following notations

$$\begin{aligned} g_t^b &= g_t^b(Z) = \sup \{u \leq t \mid Z_u = b\}, \\ T_b^- &= T_b^-(Z) = \inf \{t > 0 \mid (t - g_t^b) \mathbf{1}_{\{Z_t < b\}} > D\}, \\ T_b^+ &= T_b^+(Z) = \inf \{t > 0 \mid (t - g_t^b) \mathbf{1}_{\{Z_t > b\}} > D\}. \end{aligned}$$

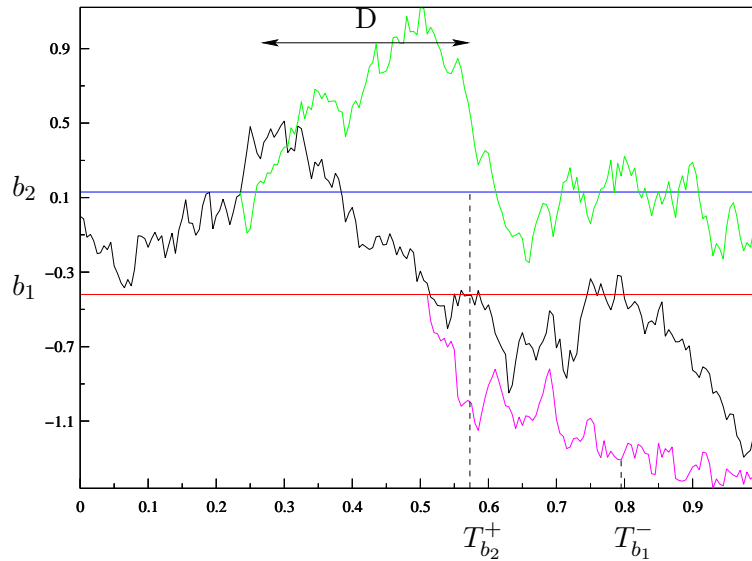


FIGURE 1. Brownian paths

Hence, the price of a knock out double barrier Parisian call (DPOC) is given by

$$(2.2) \quad DPOC(x, T; K, L_1, L_2; r, \delta) = e^{-(\frac{m^2}{2} + r)T} \mathbb{E} \left[ e^{mZ_T} (S_T - K)_+ \mathbf{1}_{\{T_{b_1}^- > T\}} \mathbf{1}_{\{T_{b_2}^+ > T\}} \right].$$

The two indicators can be rewritten

$$\mathbf{1}_{\{T_{b_1}^- > T\}} \mathbf{1}_{\{T_{b_2}^+ > T\}} = 1 - \mathbf{1}_{\{T_{b_1}^- < T\}} - \mathbf{1}_{\{T_{b_2}^+ < T\}} + \mathbf{1}_{\{T_{b_1}^- < T\}} \mathbf{1}_{\{T_{b_2}^+ < T\}}.$$

Since the r.v.  $T_b^+$  and  $T_b^-$  have a density w.r.t the Lebesgue measure (see Appendix B), one can use either strict or non-strict inequalities in the previous formula.

Dealing with inequalities of the type  $\mathbf{1}_{\{T_b^\pm < T\}}$  is much simpler than  $\mathbf{1}_{\{T_b^\pm > T\}}$  since we can condition w.r.t.  $\mathcal{F}_{T_b^\pm}$  and use the Strong Markov property. Consequently, Equation (2.2)

can be split into four terms using the prices of single barrier Parisian options. To describe single barrier Parisian options, we use the following notations : PDOC means Parisian Down and Out Call, whereas PUIP stands for Parisian Up and In Put and so on. BSC simply denotes the price of a standard call option.

$$(2.3) \quad \begin{aligned} DPOC(x, T; K, L_1, L_2; r, \delta) = & BSC(x, T; K; r, \delta) - PDIC(x, T; K, L_1; r, \delta) \\ & - PUIC(x, T; K, L_2; r, \delta) + e^{-(\frac{m^2}{2}+r)T} A, \end{aligned}$$

where

$$(2.4) \quad A = \mathbb{E} \left[ e^{mZ_T} (S_T - K)_+ \mathbf{1}_{\{T_{b_1}^- < T\}} \mathbf{1}_{\{T_{b_2}^+ < T\}} \right],$$

For any function  $f$  of the maturity  $T$ , we introduce the “star” notation

$$*f(T) = e^{(r+\frac{1}{2}m^2)T} f(T).$$

The computation of  $*DPOC$  will be done using numerical inversion of its Laplace transform with respect to  $T$ . Explicit formulae for the Laplace transforms of the first three terms in (2.3) —  $*\widehat{BSC}$ ,  $*\widehat{PDIC}$ ,  $*\widehat{PUIC}$  — can be found in Labart and Lelong (2005) and are recalled in Appendix D for the sake of clearness. We only need to compute  $\widehat{A} = \int_0^\infty \mathbb{E} \left[ e^{mZ_u} (S_u - K)_+ \mathbf{1}_{\{T_{b_1}^- < u\}} \mathbf{1}_{\{T_{b_2}^+ < u\}} \right] e^{-\lambda u} du$ . A detailed computation can be found in Section 4.

**2.2.2. Knock In.** A knock in double barrier Parisian call (respectively put) pays at maturity time  $T$   $(S_T - K)_+$  (respectively  $(K - S_T)_+$ ) if  $S$  makes an excursion outside the range  $(L_1, L_2)$  longer than  $D$  before  $T$  and is lost otherwise.

The price of such an option (DPIC) is given by

$$\begin{aligned} DPIC(x, T; K, L_1, L_2; r, \delta) = & e^{-(\frac{m^2}{2}+r)T} \\ & \mathbb{E} \left[ e^{mZ_T} (S_T - K)_+ \left( \mathbf{1}_{\{T_{b_1}^- < T\}} + \mathbf{1}_{\{T_{b_2}^+ < T\}} - \mathbf{1}_{\{T_{b_1}^- < T\}} \mathbf{1}_{\{T_{b_2}^+ < T\}} \right) \right]. \end{aligned}$$

It is quite obvious that  $DPIC$  can be expressed in terms of single barrier Parisian option prices

$$(2.5) \quad \begin{aligned} DPIC(x, T; K, L_1, L_2; r, \delta) = & PDIC(x, T; K, L_1; r, \delta) \\ & + PUIC(x, T; K, L_2; r, \delta) - e^{-(\frac{m^2}{2}+r)T} A, \end{aligned}$$

where  $A$  is defined by (2.4).

### 3. A CALL PUT PARITY RELATIONSHIP

As for single barrier Parisian options, a parity relationship between calls and puts holds. The basic idea of the relationship is that the processes  $Z$  and  $-Z$  have the same law. Therefore, introducing the new Brownian motion  $\tilde{Z} = -Z$  enables to rewrite the price of double barrier Parisian puts

$$(3.1) \quad \begin{aligned} DPOP(x, T; K, L_1, L_2, D, r, \delta) = & Kx e^{-(r+\frac{m^2}{2})T} \\ & \mathbb{E} \left( e^{-(m+\sigma)\tilde{Z}_T} \left( \frac{1}{x} e^{\sigma\tilde{Z}_T} - \frac{1}{K} \right)^+ \mathbf{1}_{\{T_{-b_2}^+ > T\}} \mathbf{1}_{\{T_{-b_1}^- > T\}} \right). \end{aligned}$$

Let us introduce

$$m' = -(m + \sigma), \quad \delta' = r, \quad r' = \delta, \quad b'_1 = -b_2, \quad b'_2 = -b_1.$$

One can easily check that  $m' = \frac{1}{\sigma} \left( r' - \delta' - \frac{\sigma^2}{2} \right)$  and that  $r' + \frac{m'^2}{2} = r + \frac{m^2}{2}$ . Moreover, by noticing that the barrier  $L'$  corresponding to  $b' = -b$  is  $\frac{1}{L}$ , it becomes clear that the expectation on the right hand side of (3.1) can be interpreted as

$$xK \text{ DPOC} \left( \frac{1}{x}, T; \frac{1}{K}, \frac{1}{L_2}, \frac{1}{L_1}, D, \delta, r \right).$$

The same kind of relation holds for knock in options

$$\text{DPIP}(x, T; K, L_1, L_2, D, r, \delta) = xK \text{ DPIC} \left( \frac{1}{x}, T; \frac{1}{K}, \frac{1}{L_2}, \frac{1}{L_1}, D, \delta, r \right).$$

#### 4. COMPUTATION OF LAPLACE TRANSFORMS

The computation of  $\ast\text{DPOC}$  will be done using numerical inversion of its Laplace transform with respect to maturity time. As explained above, the computation of the Laplace transform of  $\text{DPOC}$  boils down to the one of  $A$ . First, we split  $A$  into two terms depending on the relative position of  $T_{b_1}^-$  and  $T_{b_2}^+$ .

$$\begin{aligned} A = & \mathbb{E} \left[ \mathbf{1}_{\{T_{b_1}^- < T\}} \mathbb{E} \left[ \mathbf{1}_{\{T_{b_1}^- \leq T_{b_2}^+ < T\}} e^{mZ_T} (xe^{\sigma Z_T} - K)_+ \middle| \mathcal{F}_{T_{b_1}^-} \right] \right] \\ & + \mathbb{E} \left[ \mathbf{1}_{\{T_{b_2}^+ < T\}} \mathbb{E} \left[ \mathbf{1}_{\{T_{b_2}^+ \leq T_{b_1}^- < T\}} e^{mZ_T} (xe^{\sigma Z_T} - K)_+ \middle| \mathcal{F}_{T_{b_2}^+} \right] \right] \triangleq A_1 + A_2. \end{aligned}$$

The computation of  $\widehat{A}_2$  being quite similar to the one of  $\widehat{A}_1$ , we only focus on  $\widehat{A}_1$ . See Appendix C for a computation of  $\widehat{A}_2$ .

The computation of  $\widehat{A}_1$  is quite lengthy, so we split it into two separate steps. First, we give a global formula for  $\widehat{A}_1$  (see Theorem 4.1). Then, we carry out a detailed computation of the different terms appearing in the expression of  $\widehat{A}_1$  in the case  $K \leq L_2$ . The reader is referred to Appendix C for the other cases.

Computations are quite long but not difficult, that's why we omit further details.

**4.1. Global formula for  $\widehat{A}_1$ .** Before giving a global formula for  $\widehat{A}_1$ , we state a theorem, which ensues a corollary giving the global formula for  $\widehat{A}_1$ . The rest of the paragraph is devoted to the proof of the theorem.

**Theorem 4.1.** *In the case  $L_1 \leq x \leq L_2$  (i.e.  $b_1 \leq 0 \leq b_2$ ), we have*

$$A_1 = \int_k^{+\infty} e^{my} (xe^{\sigma y} - K) h(T, y) dy,$$

where  $k = \frac{1}{\sigma} \ln\left(\frac{K}{x}\right)$ . The function  $h(t, y)$  is characterised by its Laplace transform

$$(4.1) \quad \widehat{h}(\lambda, y) = \frac{e^{(2b_1 - b_2)\sqrt{2\lambda}}}{\sqrt{2\lambda D} \psi^2(\sqrt{2\lambda D})} \psi(-\sqrt{2\lambda D}) \int_0^{+\infty} x e^{-\frac{x^2}{2D} - \sqrt{2\lambda}|x + b_2 - y|} dx,$$

where

$$(4.2) \quad \psi(z) = \int_0^{+\infty} x e^{-\frac{x^2}{2} + zx} dx = 1 + z\sqrt{2\pi} e^{\frac{z^2}{2}} \mathcal{N}(z).$$

**Corollary 4.1.** *In the case  $L_1 \leq x \leq L_2$  (i.e.  $b_1 \leq 0 \leq b_2$ ), we have*

$$(4.3) \quad \widehat{A}_1 = \frac{e^{(2b_1 - b_2)\sqrt{2\lambda}}}{\sqrt{2\lambda D} \psi^2(\sqrt{2\lambda D})} \psi(-\sqrt{2\lambda D}) \int_k^{+\infty} e^{my} (xe^{\sigma y} - K) H(y - b_2) dy,$$

where  $H(z) = \int_0^{+\infty} x e^{-\frac{x^2}{2D} - \sqrt{2\lambda}|x - z|} dx$ .

*Proof of Theorem 4.1.* In the first part of the proof, we show that  $A_1$  can be written as  $\int_k^{+\infty} e^{my}(xe^{\sigma y} - K)h(T, y)dy$  for a certain function  $h$ . Then, we compute the Laplace transform of  $h$  w.r.t.  $t$  to get (4.1).

Step 1: Computation of  $A_1$ . We can write

$$(4.4) \quad A_1 = \mathbb{E} \left[ \mathbf{1}_{\{T_{b_1}^- < T\}} \mathbb{E} \left[ \mathbf{1}_{\{T_{b_1}^- \leq T_{b_2}^+ < T\}} e^{m(Z_T - Z_{T_{b_1}^-} + Z_{T_{b_1}^-})} (xe^{\sigma(Z_T - Z_{T_{b_1}^-} + Z_{T_{b_1}^-})} - K)_+ \middle| \mathcal{F}_{T_{b_1}^-} \right] \right] \\ \triangleq \mathbb{E}[\mathbf{1}_{\{T_{b_1}^- < T\}} A_{11}].$$

Let us introduce a new Brownian motion  $B = \{B_t = Z_{t+T_{b_1}^-} - Z_{T_{b_1}^-}, t \geq 0\}$  independent of  $\mathcal{F}_{T_{b_1}^-}$  thanks to the Strong Markov property. On the set  $\{T_{b_1}^- < T\}$ , the indicator  $\mathbf{1}_{\{T_{b_1}^- \leq T_{b_2}^+ < T\}}$  can be rewritten using  $B$

$$(4.5) \quad A_{11} = \mathbb{E} \left[ \mathbf{1}_{\{T_{b_2}^+ - Z_{T_{b_1}^-}(B) \leq T - T_{b_1}^-\}} e^{m(B_{T - T_{b_1}^-} + Z_{T_{b_1}^-})} (xe^{\sigma(B_{T - T_{b_1}^-} + Z_{T_{b_1}^-})} - K)_+ \middle| \mathcal{F}_{T_{b_1}^-} \right], \\ = \mathbb{E} \left[ \mathbf{1}_{\{T_{b_2}^+ - z(B) \leq T - \tau\}} e^{m(B_{T - \tau} + z)} (xe^{\sigma(B_{T - \tau} + z)} - K)_+ \right]_{|z=Z_{T_{b_1}^-}, \tau=T_{b_1}^-}, \\ \triangleq E \left( Z_{T_{b_1}^-}, T_{b_1}^- \right).$$

Once again, conditioning w.r.t.  $\mathcal{F}_{T_{b_2}^+}$  and introducing a new Brownian motion  $\tilde{B} = \{\tilde{B}_t = B_{t+T_{b_2}^+} - B_{T_{b_2}^+}, t \geq 0\}$  yields

$$E(z, \tau) = \mathbb{E} \left[ \mathbf{1}_{\{T_{b_2}^+ - z(B) \leq T - \tau\}} \mathbb{E} \left[ e^{m(\tilde{B}_{T - \tau - t} + y + z)} (xe^{\sigma(\tilde{B}_{T - \tau - t} + y + z)} - K)_+ \middle| y=B_{T_{b_2}^+}, t=T_{b_2}^+ \right] \right], \\ = \mathbb{E} \left[ \mathbf{1}_{\{T_{b_2}^+ - z(B) \leq T - \tau\}} \mathcal{P}_{T - \tau - T_{b_2}^+}(f_x)(B_{T_{b_2}^+} + z) \right],$$

where  $f_x(z) = e^{mz}(xe^{\sigma z} - K)_+$ , and  $\mathcal{P}_t(f_x)(z) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} f_x(u) e^{-\frac{(u-z)^2}{2t}} du$ . As recalled by Chesney et al. (1997), the random variables  $B_{T_{b_2}^+}$  and  $T_{b_2}^+$  are independent. Let  $\nu_+(du)$  denote the law of  $B_{T_{b_2}^+}$ , we have

$$E(z, \tau) = \int_{-\infty}^{+\infty} \mathbb{E} \left[ \mathbf{1}_{\{T_{b_2}^+ - z(B) \leq T - \tau\}} \mathcal{P}_{T - \tau - T_{b_2}^+}(f_x)(u + z) \right] \nu_+(du), \\ = \int_{-\infty}^{+\infty} f_x(y) h_{b_2 - z}(T - \tau, y - z) dy,$$



where  $h_b(t, y) = \int_{-\infty}^{+\infty} \mathbb{E} \left[ \mathbf{1}_{\{T_b^+ \leq t\}} \frac{e^{-\frac{(y-u)^2}{2(t-T_b^+)}}}{\sqrt{2\pi(t-T_b^+)}} \right] \nu_+(du)$ . By using (4.4) and (4.5), we get

$$\begin{aligned} A_1 &= \mathbb{E} \left[ \mathbf{1}_{\{T_{b_1}^- < T\}} \int_{-\infty}^{+\infty} f_x(y) h_{b_2-Z_{T_{b_1}^-}}(T - T_{b_1}^-, y - Z_{T_{b_1}^-}) dy \right], \\ &= \int_{-\infty}^{+\infty} f_x(y) h(T, y) dy, \end{aligned}$$

where  $h(t, y) = \int_{-\infty}^{+\infty} \mathbb{E} \left[ \mathbf{1}_{\{T_{b_1}^- < T\}} h_{b_2-z}(t - T_{b_1}^-, y - z) \right] \nu_-(dz)$  and  $\nu_-(dz)$  denotes the density of  $Z_{T_{b_1}^-}$ .

Step 2: Laplace transform of  $h$  w.r.t.  $t$ . Before computing  $\widehat{h}(\lambda, y)$ , we give a more explicit formula for the function  $h$ . Using the law of  $B_{T_{b_2-z}^+}$  (see Chesney et al. (1997)), we have

$$h_{b_2-z}(t, y) = \int_{b_2-z}^{+\infty} du \frac{u - (b_2 - z)}{D} e^{-\frac{(u-(b_2-z))^2}{2D}} \gamma(t, u - y),$$

where  $\gamma(t, x) = \mathbb{E} \left[ \mathbf{1}_{\{T_{b_2-z}^+(B) \leq t\}} \frac{e^{-\frac{x^2}{2(t-T_{b_2-z}^+(B))}}}{\sqrt{2\pi(t-T_{b_2-z}^+(B))}} \right]$ . Using the expression of  $h$  and the explicit formula of  $\nu_-(dz)$  yields

(4.6)

$$h(t, y) = \int_{-\infty}^{b_1} dz \int_{b_2-z}^{+\infty} du \frac{b_1 - z}{D} \frac{u - (b_2 - z)}{D} e^{-\frac{(b_1-z)^2}{2D}} e^{-\frac{(u-(b_2-z))^2}{2D}} \gamma_0(t, u - (y - z)),$$

where  $\gamma_0(t, x) = \mathbb{E} \left[ \mathbf{1}_{\{T_{b_1}^- \leq t\}} \gamma(t - T_{b_1}^-, x) \right]$ .

In view of (4.6), computing  $\widehat{h}(\lambda, y)$  boils down more or less to computing  $\widehat{\gamma}_0(\lambda, u - (y - z))$ . By doing some changes of variables, we get

$$\widehat{\gamma}_0(\lambda, x) = \mathbb{E} \left[ e^{-\lambda T_{b_1}^-} \right] \int_0^{+\infty} e^{-\lambda v} \gamma(v, x) dv = \mathbb{E} \left[ e^{-\lambda T_{b_1}^-} \right] \mathbb{E} \left[ e^{-\lambda T_{b_2-z}^+(B)} \right] \int_0^{+\infty} e^{-\lambda u} \frac{e^{-\frac{x^2}{2u}}}{\sqrt{2\pi u}} du.$$

One can easily prove that  $\int_0^{+\infty} e^{-\lambda u} \frac{e^{-\frac{x^2}{2u}}}{\sqrt{2\pi u}} du = \frac{1}{\sqrt{2\lambda}} e^{-\sqrt{2\lambda}|x|}$ . Furthermore, the values of  $\mathbb{E} \left[ e^{-\lambda T_{b_1}^-} \right]$  and  $\mathbb{E} \left[ e^{-\lambda T_{b_2-z}^+(B)} \right]$  are explicitly known (see Appendix Chesney et al. (1997)

for a proof). Then,  $\widehat{\gamma}_0(\lambda, x) = \frac{e^{(b_1-|x|-b_2+z)\sqrt{2\lambda}}}{\sqrt{2\lambda}\psi^2(\sqrt{2\lambda D})}$ , and Equation (4.1) follows.  $\square$

**4.2. Computation of  $\widehat{A}_1$ .** For the sake of clearness, in the following we write  $\theta = \sqrt{2\lambda}$ . In this part, we state and prove a theorem giving the value of  $\widehat{A}_1$  in the case  $K \leq L_2$ . We refer the reader to Appendix C for the case  $K > L_2$ .

**Theorem 4.2.** *In the case  $K \leq L_2$ , we have the following result*

$$\begin{aligned} \widehat{A}_1 &= \frac{K e^{2(b_1-b_2)\theta}}{\theta \psi^2(\theta\sqrt{D})} \psi^2(-\theta\sqrt{D}) e^{(m+\theta)k} \left[ \frac{1}{m+\theta} - \frac{1}{m+\theta+\sigma} \right] \\ &\quad + \frac{2e^{2b_1-b_2}\theta}{\psi^2(\theta\sqrt{D})} \psi(-\theta\sqrt{D}) e^{mb_2} \left[ \frac{K\psi(m\sqrt{D})}{m^2-\theta^2} - \frac{L_2\psi((m+\sigma)\sqrt{D})}{(m+\sigma)^2-\theta^2} \right]. \end{aligned}$$

*Proof.* We want to compute  $\widehat{A}_1$ , so we need to evaluate

$$(4.7) \quad I \triangleq \int_k^{+\infty} e^{my}(xe^{\sigma y} - K)H(y - b_2)dy.$$

Standard computations lead to the following formula for the function  $H$  (see Corollary 4.1 for the definition of  $H$ ).

$$H(x) = \begin{cases} e^{\theta x} D\psi(-\theta\sqrt{D}) & \text{if } x \leq 0, \\ e^{-\theta x} D\psi(\theta\sqrt{D}) - D\theta\sqrt{2\pi D}e^{\lambda D} \left\{ \mathcal{N}\left(\theta\sqrt{D} - \frac{x}{\sqrt{D}}\right)e^{-\theta x} + \mathcal{N}\left(-\theta\sqrt{D} - \frac{x}{\sqrt{D}}\right)e^{\theta x} \right\} & \text{otherwise.} \end{cases}$$

Using this result, we can compute  $I$ .

$$(4.8) \quad I = \int_k^{b_2} e^{my}(xe^{\sigma y} - K)H(y - b_2)dy + \int_{b_2}^{+\infty} e^{my}(xe^{\sigma y} - K)H(y - b_2)dy \triangleq I_1 + I_2.$$

Computation of  $I_1$ . In view of the definition of  $H$ , this case is the simpler one. Easy computations give

$$I_1 = D\psi(-\theta\sqrt{D})e^{-\theta b_2} \left\{ e^{(m+\theta)b_2} \left[ \frac{L_2}{m+\sigma+\theta} - \frac{K}{m+\theta} \right] + Ke^{(m+\theta)b_2} \left[ \frac{1}{m+\theta} - \frac{1}{m+\sigma+\theta} \right] \right\}.$$

Computation of  $I_2$ . The second integral in (4.8) can be split into three terms

$$\begin{aligned} I_{21} &= D\psi(\theta\sqrt{D}) \int_{b_2}^{\infty} e^{my}(xe^{\sigma y} - K)e^{\theta(b_2-y)}dy, \\ I_{22} &= -D\theta\sqrt{2\pi D}e^{\lambda D} \int_{b_2}^{\infty} e^{my}(xe^{\sigma y} - K)e^{\theta(b_2-y)}\mathcal{N}\left(\theta\sqrt{D} + \frac{b_2-y}{\sqrt{D}}\right)dy, \\ I_{23} &= -D\theta\sqrt{2\pi D}e^{\lambda D} \int_{b_2}^{\infty} e^{my}(xe^{\sigma y} - K)e^{\theta(y-b_2)}\mathcal{N}\left(-\theta\sqrt{D} + \frac{b_2-y}{\sqrt{D}}\right)dy. \end{aligned}$$

For  $I_{21}$ , we simply get  $I_{21} = \psi(\theta\sqrt{D})De^{mb_2} \left[ \frac{K}{m-\theta} - \frac{L_2}{m+\sigma-\theta} \right]$ .

$I_{22}$  and  $I_{23}$  are computed in the following way: we change variables (we introduce  $v = \theta\sqrt{D} + \frac{b_2-y}{\sqrt{D}}$  (for the valuation of  $I_{22}$ )) and we use the following equality  $\int_{-\infty}^a \mathcal{N}(v)e^{bv}dv = \frac{1}{b}(\mathcal{N}(a)e^{ab} - e^{\frac{b^2}{2}}\mathcal{N}(a-b))$ , for  $a, b \in \mathbb{R}$ ,  $b \neq 0$ . We get

$$\begin{aligned} I_{22} &= -D\theta\sqrt{2\pi D}e^{mb_2}e^{\lambda D}\mathcal{N}(\theta\sqrt{D}) \left[ \frac{K}{m-\theta} - \frac{L_2}{m+\sigma-\theta} \right] \\ &\quad + D\theta\sqrt{2\pi D}e^{mb_2} \left[ \frac{K}{m-\theta}\mathcal{N}(m\sqrt{D})e^{\frac{m^2 D}{2}} - \frac{L_2}{m+\sigma-\theta}\mathcal{N}((m+\sigma)\sqrt{D})e^{\frac{(m+\sigma)^2 D}{2}} \right], \\ I_{23} &= -D\theta\sqrt{2\pi D}e^{mb_2}e^{\lambda D}\mathcal{N}(-\theta\sqrt{D}) \left[ \frac{K}{m+\theta} - \frac{L_2}{m+\sigma+\theta} \right] \\ &\quad + D\theta\sqrt{2\pi D}e^{mb_2} \left[ \frac{K}{m+\theta}\mathcal{N}(m\sqrt{D})e^{\frac{m^2 D}{2}} - \frac{L_2}{m+\sigma+\theta}\mathcal{N}((m+\sigma)\sqrt{D})e^{\frac{(m+\sigma)^2 D}{2}} \right]. \end{aligned}$$

Summing  $I_{21}$ ,  $I_{22}$  and  $I_{23}$  and using the definition of  $\psi$  (see (4.2)) yield

$$I_2 = De^{mb_2}\psi(-\theta\sqrt{D}) \left[ \frac{K}{m+\theta} - \frac{L_2}{m+\sigma+\theta} \right] + 2D\theta e^{mb_2} \left[ \frac{K\psi(m\sqrt{D})}{m^2 - \theta^2} - \frac{L_2\psi((m+\sigma)\sqrt{D})}{(m+\sigma)^2 - \theta^2} \right].$$

We sum  $I_1$  and  $I_2$  to get

$$I = DK \psi(-\theta\sqrt{D})e^{(m+\theta)k}e^{-\theta b_2} \left[ \frac{1}{m+\theta} - \frac{1}{m+\sigma+\theta} \right] \\ + 2D\theta e^{mb_2} \left[ \frac{K\psi(m\sqrt{D})}{m^2-\theta^2} - \frac{L_2\psi((m+\sigma)\sqrt{D})}{(m+\sigma)^2-\theta^2} \right].$$

From the definitions of  $\widehat{A}_1$  and  $I$  (see (4.3), (4.7)), we complete the proof of Theorem 4.2.  $\square$

## 5. THE INVERSION OF LAPLACE TRANSFORMS

This section is devoted to the numerical inversion of the Laplace transforms computed previously. We recall that the Laplace transforms are computed with respect to the maturity time. We explain how to recover a function from its Laplace transform using a contour integral. The real problem is how to numerically evaluate this complex integral. This is done in two separate steps involving two different errors. First, as explained in Section 5.1 we replace the integral by a series. The first step creates a discretisation error, which is handled by Proposition 5.1. Secondly, one has to compute a non-finite series. This can be achieved by simply truncating the series but it leads to a tremendously slow convergence. Here, we prefer to use the Euler acceleration as presented in Section 5.2. Proposition 5.2 states an upper-bound for the error due to the accelerated computation of the non finite series. Theorem 5.2 gives a bound for the global error.

**5.1. The Fourier series representation.** Thanks to Widder (1941, Theorem 9.2), we know how to recover a function from its Laplace transform.

**Theorem 5.1.** *Let  $f$  be a continuous function defined on  $\mathbb{R}^+$  and  $\alpha$  a positive number. If the function  $f(t)e^{-\alpha t}$  is integrable, then given the Laplace transform  $\hat{f}$ ,  $f$  can be recovered from the contour integral*

$$(5.1) \quad f(t) = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} e^{st} \hat{f}(s) ds, \quad t > 0.$$

The variable  $\alpha$  has to be chosen greater than the abscissa of convergence of  $\hat{f}$ . The abscissa of convergence of the Laplace transforms of the double barrier Parisian option prices computed previously is smaller than  $(m+\sigma)^2/2$ . Hence,  $\alpha$  must be chosen strictly greater than  $(m+\sigma)^2/2$ .

For any real valued function satisfying the hypotheses of Theorem 5.1, we introduce a trapezoidal discretisation of Equation (5.1)

$$(5.2) \quad f_{\pi/t}(t) = \frac{e^{\alpha t}}{2t} \hat{f}(\alpha) + \frac{e^{\alpha t}}{t} \sum_{k=1}^{\infty} (-1)^k \operatorname{Re} \left( \hat{f} \left( \alpha + i \frac{k\pi}{t} \right) \right).$$

**Proposition 5.1.** *If  $f$  is a continuous bounded function satisfying  $f(t) = 0$  for  $t < 0$ , we have*

$$(5.3) \quad |e_{\pi/t}(t)| = |f(t) - f_{\pi/t}(t)| \leq \|f\|_{\infty} \frac{e^{-2\alpha t}}{1 - e^{-2\alpha t}}.$$

To prove Proposition 5.1, we need the following result adapted from Abate et al. (1999, Theorem 5)

**Lemma 5.1.** *For any continuous and bounded function  $f$  such that  $f(t) = 0$  for  $t < 0$ , we have*

$$(5.4) \quad e_{\pi/t}(t) = f_{\pi/t}(t) - f(t) = \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} f(t(1+2k)) e^{-2k\alpha t}.$$

*Proof of Proposition 5.1.* By performing a change of variables  $s = \alpha + iu$  in the integral in (5.1), we can easily obtain an integral of a real variable.

$$f(t) = \frac{e^{\alpha t}}{2\pi} \int_{-\infty}^{+\infty} \widehat{f}(\alpha + iu) (\cos(ut) + i \sin(ut)) du.$$

Moreover, since  $f$  is a real valued function, the imaginary part of the integral vanishes

$$f(t) = \frac{e^{\alpha t}}{2\pi} \int_{-\infty}^{+\infty} \operatorname{Re}(\widehat{f}(\alpha + iu)) \cos(ut) - \operatorname{Im}(\widehat{f}(\alpha + iu)) \sin(ut) du.$$

We notice that

$$\operatorname{Im}(\widehat{f}(\alpha + iu)) = -\operatorname{Im}(\widehat{f}(\alpha - iu)), \quad \operatorname{Re}(\widehat{f}(\alpha + iu)) = \operatorname{Re}(\widehat{f}(\alpha - iu)).$$

So,

$$(5.5) \quad f(t) = \frac{e^{\alpha t}}{\pi} \int_0^{+\infty} \operatorname{Re}(\widehat{f}(\alpha + iu)) \cos(ut) - \operatorname{Im}(\widehat{f}(\alpha + iu)) \sin(ut) du.$$

Using a trapezoidal integral with a step  $h = \frac{\pi}{t}$  leads to Equation (5.2). Remembering that  $f(t) = 0$  for  $t < 0$ , we can easily deduce from Lemma 5.1 that

$$e_{\pi/t}(t) = \sum_{k=0}^{\infty} f(t(1+2k)) e^{-2k\alpha t}.$$

Taking the upper bound of  $f$  yields (5.3).  $\square$

**Remark 5.1.** *For the upper bound in Proposition 5.1 to be smaller than  $10^{-8} \|f\|_{\infty}$ , one has to choose  $2\alpha t = 18.4$ . In fact, this bound holds for any choice of the discretisation step  $h$  satisfying  $h < 2\pi/t$ .*

Simply truncating the summation in the definition of  $f_{\pi/t}$  to compute the trapezoidal integral is far too rough to provide a fast and accurate numerical inversion. One way to improve the convergence of the series is to use the Euler summation.

**5.2. The Euler summation.** To improve the convergence of a series  $S$ , we use the Euler summation technique as described by Abate et al. (1999), which consists in computing the binomial average of  $q$  terms from the  $p$ -th term of the series  $S$ . The binomial average obviously converges to  $S$  as  $p$  goes to infinity. The following proposition describes the convergence rate of the binomial average to the infinite series  $f_{\pi/t}(t)$  when  $p$  goes to  $\infty$ .

**Proposition 5.2.** *Let  $f$  be a function of class  $C^{q+4}$  such that there exists  $\epsilon > 0$  s.t.  $\forall k \leq q+4$ ,  $f^{(k)}(s) = \mathcal{O}(e^{(\alpha-\epsilon)s})$ . We define  $s_p(t)$  as the approximation of  $f_{\pi/t}(t)$  when truncating the non-finite series in (5.2) to  $p$  terms*

$$(5.6) \quad s_p(t) = \frac{e^{\alpha t}}{2t} \widehat{f}(\alpha) + \frac{e^{\alpha t}}{t} \sum_{k=1}^p (-1)^k \operatorname{Re} \left( \widehat{f} \left( \alpha + i \frac{\pi k}{t} \right) \right),$$

and  $E(q, p, t) = \sum_{k=0}^q C_q^k 2^{-q} s_{p+k}(t)$ . Then,

$$|f_{\pi/t}(t) - E(q, p, t)| \leq \frac{te^{\alpha t} |f'(0) - \alpha f(0)|}{\pi^2} \frac{(p+1)! q!}{2^{q-2} (p+q+2)!} + \mathcal{O} \left( \frac{1}{p^{q+3}} \right)$$

when  $p$  goes to infinity.

Using Propositions 5.1 and 5.2, we get the following result concerning the global error on the numerical computation of the price of a double barrier Parisian call option

**Theorem 5.2.** *Let  $f$  be the price of a double barrier Parisian call option. Using the notations of Proposition 5.2, we have*

$$(5.7) \quad |f(t) - E(q, p, t)| \leq S_0 \frac{e^{-2\alpha t}}{1 - e^{-2\alpha t}} + \frac{e^{\alpha t} |f'(0) - \alpha f(0)| (p+1)! q!}{\pi^2 2^{q-2} (p+q+2)!} + \mathcal{O}\left(\frac{1}{p^{q+3}}\right)$$

where  $\alpha$  is defined in Theorem 5.1.

*Proof of Theorem 5.2.*  $f$  being the price of a double barrier Parisian call option, we know that  $f$  is bounded by  $S_0$ . Moreover,  $f$  is continuous (actually of class  $\mathcal{C}^\infty$ , see Appendix A). Hence, Proposition 5.1 yields the first term on the right-hand side of (5.7).

Relying on Proposition A.1, we know that  $\star f$  is of class  $\mathcal{C}^\infty$  and  $\star f^{(k)}(t) = \mathcal{O}(e^{\frac{(m+\sigma)^2}{2}t})$ ,  $\forall k \geq 0$ . Since  $f(t) = e^{-(r+m^2/2)t} \star f(t)$ , it is quite obvious that  $f$  is also of class  $\mathcal{C}^\infty$  and  $f^{(k)}(t) = \mathcal{O}(e^{((m+\sigma)^2/2 - (r+m^2/2))t})$ ,  $\forall k \geq 0$ . Since  $\alpha > \frac{(m+\sigma)^2}{2}$ , we can apply Proposition 5.2 to get the result.  $\square$

*Proof of Proposition 5.2.* We compute the difference between two successive terms.

$$E(q, p+1, t) - E(q, p, t) = \frac{e^{\alpha t}}{2qt} \sum_{k=0}^q C_q^k (-1)^{p+1+k} a_{p+k+1},$$

where

$$(5.8) \quad a_p = \int_0^{+\infty} e^{-\alpha s} \cos\left(\frac{p}{t}\pi s\right) f(s) ds.$$

Let  $g(s) = e^{-\alpha s} f(s)$ . Since  $g^{(k)}(\infty) = 0$  for  $k \leq q+3$  and  $g^{(q+4)}$  is integrable, we can perform  $(q+3)$  integrations by parts in (5.8) to obtain a Taylor expansion when  $p$  goes to infinity

$$(5.9) \quad a_p = \frac{c_2}{p^2} + \frac{c_4}{p^4} + \dots + \frac{c_q}{p^{2[(q+3)/2]}} + \mathcal{O}\left(\frac{1}{p^{q+4}}\right)$$

with  $c_2 = \frac{4t^2(f'(0) - \alpha f(0))}{\pi^2}$ .

We can rewrite (5.9)

$$a_p = \frac{c_2}{p(p+1)} + \frac{c'_3}{p(p+1)(p+2)} + \dots + \frac{c'_q}{p(p+1)\dots(p+q+2)} + \mathcal{O}\left(\frac{1}{p^{q+4}}\right).$$

Some elementary computations show that for  $j \geq 1$

$$\sum_{k=0}^q C_q^k (-1)^{p+1+k} \frac{1}{(p+k)(p+k+1)\dots(p+k+j)} = (-1)^{p+1} \frac{p!(q+j)!}{j!(p+q+j+1)!}.$$

Computing  $\sum_{k=0}^q C_q^k (-1)^{p+1+k} a_{p+k+1}$  leads to

$$E(q, p+1, t) - E(q, p, t) = (-1)^{p+1} c_2 \frac{e^{\alpha t} p!(q+1)!}{2qt (p+q+2)!} + \mathcal{O}\left(\frac{1}{p^{q+4}}\right).$$

Moreover,  $\frac{p!(q+1)!}{(p+q+2)!}$  is decreasing w.r.t  $p$ , so

$$|E(q, \infty, t) - E(q, p, t)| \leq c_2 \frac{e^{\alpha t} p!(q+1)!}{2qt (p+q+2)!} + \mathcal{O}\left(\frac{1}{p^{q+3}}\right).$$

$\square$

**Remark 5.2.** Whereas Proposition 5.1 in fact holds for any  $h < 2\pi/t$ , the proof of Proposition 5.2 is essentially based on the choice of  $h = \pi/t$  since the key point is to be able to write  $E(q, p+1, t) - E(q, p, t)$  as the general term of an alternating series. The impressive convergence rate of  $E(q, p, t)$  definitely relies on the choice of this particular discretisation step. For a general step  $h$ , it is much more difficult to study the convergence rate and one cannot give an explicit upper-bound.

**Remark 5.3.** For  $2\alpha t = 18.4$  and  $q = p = 15$ , the global error is bounded by  $S_0 10^{-8} + t |f'(0) - \alpha f(0)| 10^{-11}$ . As one can see, the method we use to invert Laplace transforms provides a very good accuracy with few computations.

**Remark 5.4.** Considering the case of call options in Theorem 5.2 is sufficient since put prices are computed using parity relations and their accuracy is hung up to the one of call prices. Theorem 5.2 also holds for single barrier Parisian options.

## 6. NUMERICAL EXAMPLES

In this section, we present some results obtained using the numerical inversion developed in Section 5. We have implemented our method in C and used the function `erfc` from the *Octave* library to compute the function  $\mathcal{N}$  at a complex point. In the examples, we choose  $p = 15$ ,  $q = 15$  and  $\alpha = 18.4/2T$ . Hence, when the spot is of order 100 the accuracy of our method is ensured up to  $10^{-6}$ .

In Table 1, we compare the prices of a double barrier Parisian out call with  $S_0 = K = 100$ ,  $L_1 = 90$ ,  $L_2 = 110$ ,  $r = 0.095$ ,  $\delta = 0$  and  $T = 1$  obtained with our method and the corrected Monte Carlo method of Baldi et al. (2000) with 10000 samples. For the results obtained by the corrected Monte Carlo method, we precise the width of the confidence interval at level 95%. The accuracy showed by this approach decreases as the delay of the option increases. Our method is far more accurate and incredibly faster. For instance, if we consider the option described above with  $D = 0.2$  and 250 time steps for the Monte Carlo, our algorithm takes 1.5 ms (CPU time) whereas the corrected Monte Carlo algorithm runs in 1.2 sec (CPU time).

Delay	MC Price	Price CI	Laplace
0.0500	0.545	0.0840	0.522
0.1000	1.142	0.1359	1.102
0.1500	1.774	0.1763	1.725
0.2000	2.241	0.2049	2.375
0.2500	3.044	0.2492	3.037
0.3000	3.681	0.2781	3.722
0.3500	4.530	0.3231	4.411
0.4000	4.933	0.3362	5.109

TABLE 1. Comparison corrected Monte Carlo and Laplace Transform

Figure 2 shows the evolution of the price of a double Parisian knock out call w.r.t. the delay when using the Laplace transform method or the corrected Monte Carlo one. We can see that the price given by the Laplace transform method is in the confidence interval given by the corrected Monte Carlo method. Figures 3 and 4 show the evolution of the price and the delta of a double barrier Parisian in call with respect to the spot and the strike. The delta is computed using a finite difference scheme.

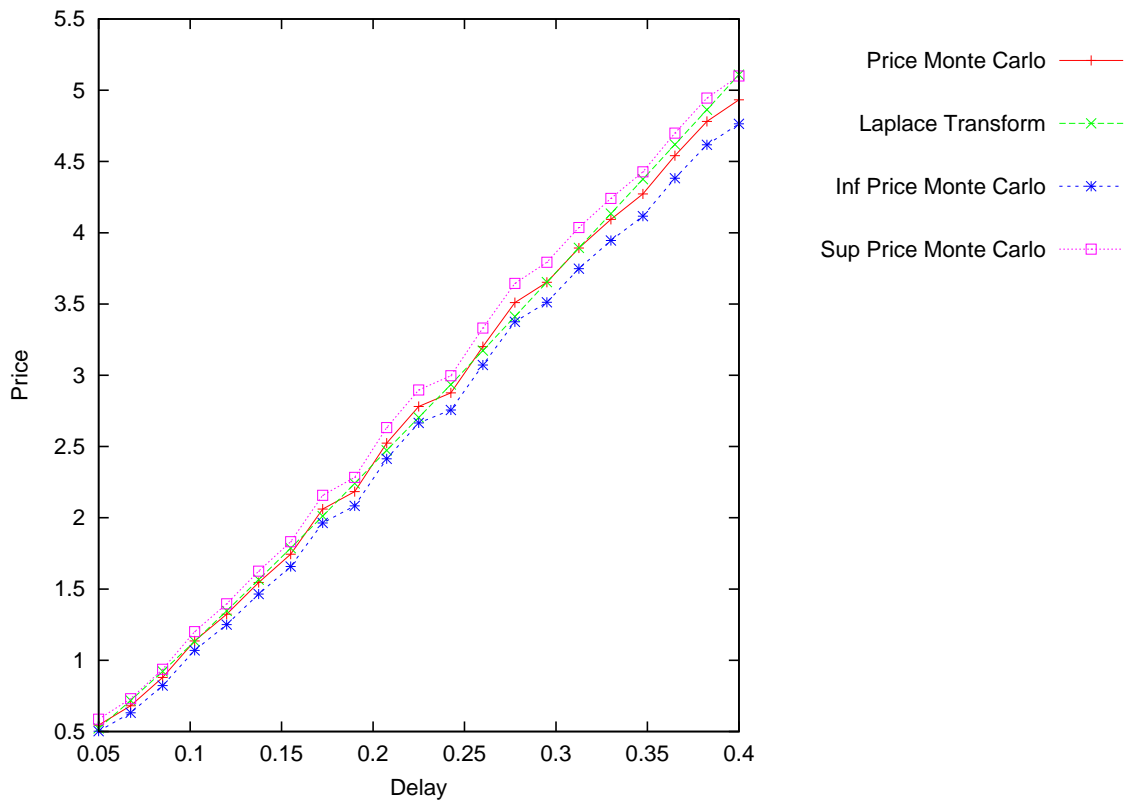


FIGURE 2. Comparison of corrected Monte Carlo and Laplace Transform

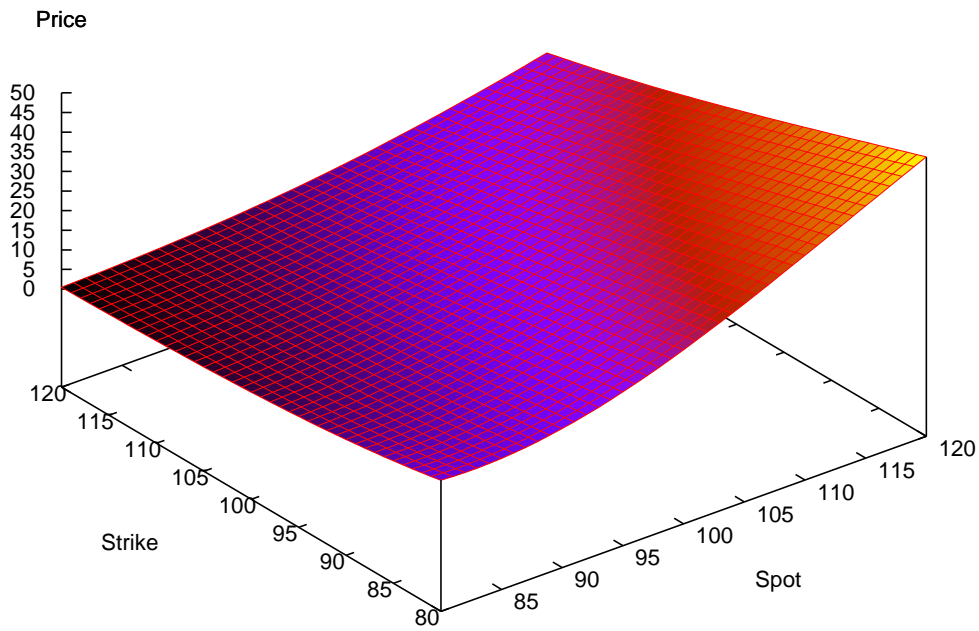


FIGURE 3. Price of a Double barrier Parisian In Call ( $\sigma = 0.2$ ,  $r = 0.02$ ,  $\delta = 0$ ,  $L = 80$ ,  $U = 120$ )

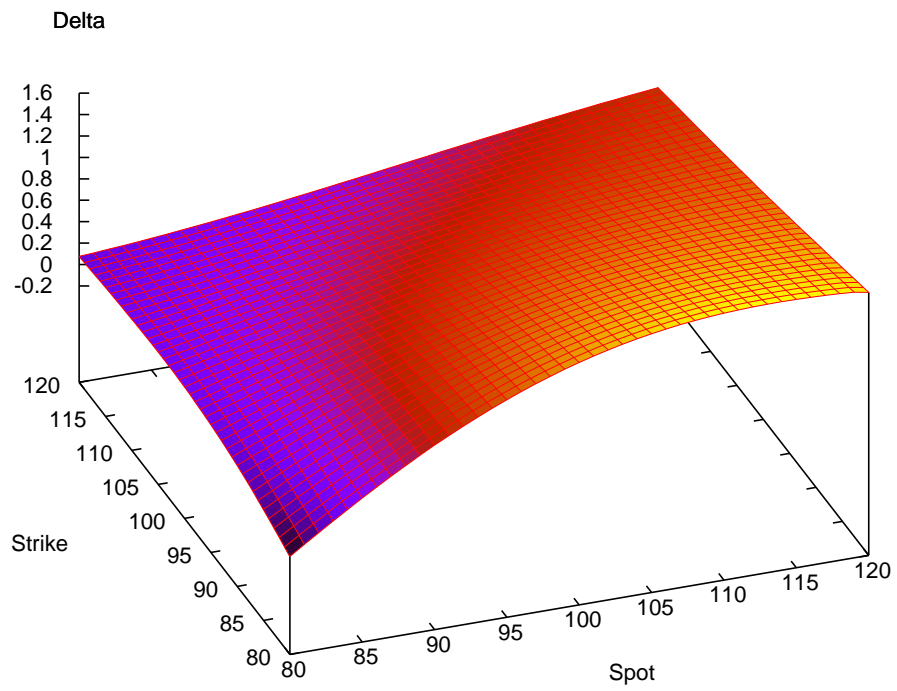


FIGURE 4. Delta of a Double barrier Parisian In Call



## APPENDIX A. REGULARITY OF OPTION PRICES

**Proposition A.1.** *Let  $f(t)$  be the “star” price of a double barrier Parisian option of maturity  $t$ . If  $b_1 < 0$  and  $b_2 > 0$ ,  $f$  is of class  $C^\infty$  and for all  $k \geq 0$ ,  $f^{(k)}(t) = \mathcal{O}\left(e^{\frac{(m+\sigma)^2}{2}t}\right)$  when  $t$  goes to infinity.*

For the sake of clearness, we will only prove Proposition A.1 for single barrier Parisian options as the scheme of the proof is still valid for double barrier Parisian options. Once again, we can restrict to calls. Let  $f(t) = PDIC(x, t; K, L; r, \delta)$ .

$$f(t) = \mathbb{E} \left[ e^{mZ_t} (S_t - K)_+ \mathbf{1}_{\{T_b^- < t\}} \right].$$

Let  $W_t$  denote  $Z_{t+T_b^-} - Z_{T_b^-}$ . Relying on the strong Markov property,

$$(A.1) \quad f(t) = \mathbb{E} \left( \mathbf{1}_{\{T_b^- < t\}} \mathbb{E} \left[ (xe^{\sigma(W_{t-\tau+z})} - K)_+ e^{m(W_{t-\tau+z})} \right]_{|z=Z_{T_b^-}, \tau=T_b^-} \right).$$

Let  $\nu$  denote the density of  $Z_{T_b^-}$  (see Chesney et al. (1997) for its expression) and  $\mu$  the density of  $T_b^-$  (see Proposition B.1 for a proof of existence). Since  $Z_{T_b^-}$  and  $T_b^-$  are independent, Equation (A.1) can be written

$$f(t) = \int_0^t d\tau \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dw (xe^{\sigma(w\sqrt{t-\tau+z})} - K)_+ e^{m(w\sqrt{t-\tau+z})} p(w) \nu(z) \mu(\tau)$$

where  $p(w) = \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}}$ . A change of variable on  $\tau$  gives

$$f(t) = \int_0^t d\tau \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dw (xe^{\sigma(w\sqrt{\tau+z})} - K)_+ e^{m(w\sqrt{\tau+z})} p(w) \nu(z) \mu(t - \tau).$$

Since  $\mu$  is of class  $C^\infty$  and all its derivatives are null at 0 and bounded on any interval  $[0, T]$  (see Appendix B), one can easily prove that  $f$  is of class  $C^\infty$  and that for all  $k \geq 0$

$$f^{(k)}(t) = \int_0^t d\tau \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dw (xe^{\sigma(w\sqrt{\tau+z})} - K)_+ e^{m(w\sqrt{\tau+z})} p(w) \nu(z) \mu^{(k)}(t - \tau).$$

This proves the first part of Proposition A.1. From Proposition B.1, we know that  $\mu$  and all its derivatives are bounded. Then, we can bound  $f^{(k)}$

$$\begin{aligned} \left| f^{(k)}(t) \right| &\leq \int_0^t d\tau \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dw x e^{(m+\sigma)(w\sqrt{\tau+z})} p(w) \nu(z) \left\| \mu^{(k)} \right\|_{\infty}, \\ &\leq \int_{-\infty}^{\infty} x e^{(m+\sigma)z} \nu(z) dz \left\| \mu^{(k)} \right\|_{\infty} \int_0^t e^{\frac{(m+\sigma)^2}{2}\tau} d\tau, \\ &\leq e^{\frac{(m+\sigma)^2}{2}t} \frac{2x}{(m+\sigma)^2} \left\| \mu^{(k)} \right\|_{\infty} \int_{-\infty}^{\infty} e^{(m+\sigma)z} \nu(z) dz. \end{aligned}$$

Relying on one more use of the strong Markov property, the same kind of computations can be reproduced for double barrier Parisian options.

APPENDIX B. REGULARITY OF THE DENSITY OF  $T_b^-$ 

In this section, we assume  $b < 0$ .

**Proposition B.1.** *The r.v.  $T_b^-$  has a density  $\mu$  w.r.t to Lebesgue's measure.  $\mu$  is of class  $C^\infty$  and for all  $k \geq 0$ ,  $\mu^{(k)}(0) = \mu^{(k)}(\infty) = 0$ .*

To prove this proposition, we need the two following lemmas.

**Lemma B.1.** *Let  $\mathcal{N}$  be the analytic prolongation of the cumulative normal distribution function on the complex plane. The following equivalent holds*

$$\mathcal{N}(r(1+i)) \sim 1 \text{ when } r \rightarrow \infty.$$

**Lemma B.2.** *For  $b < 0$ , we have for  $u \in \mathbb{R}$*

$$\mathbb{E} \left( e^{-iuT_b^-} \right) = \mathcal{O} \left( e^{-|b|\sqrt{|u|}} \right) \text{ when } |u| \rightarrow \infty.$$

*Proof of Proposition B.1.* We recall that

$$(B.1) \quad \mathbb{E} \left( e^{-\frac{\lambda^2}{2}T_b^-} \right) = \frac{e^{\lambda b}}{\psi(\lambda\sqrt{D})}.$$

We define  $\mathcal{O} = \{z \in \mathbb{C}; -\frac{\pi}{4} < \arg(z) < \frac{\pi}{4}\}$ . One can easily prove that the function  $z \mapsto \mathbb{E} \left( e^{-\frac{z^2}{2}T_b^-} \right)$  is holomorphic on the open set  $\mathcal{O}$  and hence analytic. Moreover,  $z \mapsto \frac{e^{zb}}{\psi(z\sqrt{D})}$  is also analytic on  $\mathcal{O}$  except perhaps in a countable number of isolated points. These two functions coincide on  $\mathbb{R}^+$ , so they are equal on  $\mathcal{O}$ . Consequently, we can derive the following equality. For all  $z \in \mathbb{C}$  with positive real part, we have

$$(B.2) \quad \mathbb{E} \left( e^{-zT_b^-} \right) = \frac{e^{\sqrt{2z}b}}{\psi(\sqrt{2z}D)}.$$

We use the following convention: for any  $z \in \mathbb{C}$  with positive real part,  $\sqrt{z}$  is the only complex number  $z' \in \mathcal{O}$  such that  $z = z'^2$ .

Thanks to the continuity of both terms in (B.2), the equality also holds for pure imaginary numbers. Hence, by setting  $z = iu$  for  $u \in \mathbb{R}$  in Equation (B.2), we obtain the Fourier transform of  $T_b^-$

$$\mathbb{E} \left( e^{-iuT_b^-} \right) = \frac{e^{\sqrt{2uib}}}{\psi(\sqrt{2iu}D)}.$$

From Lemma B.2, we know that the Fourier transform of  $T_b^-$  is integrable on  $\mathbb{R}$ , thus the r.v.  $T_b^-$  has a density  $\mu$  w.r.t. the Lebesgue measure given by

$$\mu(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\sqrt{2uib}}}{\psi(\sqrt{2iu}D)} e^{-iut} du.$$

Moreover, thanks to Lemma B.2,  $u \mapsto u^k \frac{e^{\sqrt{2uib}}}{\psi(\sqrt{2iu}D)}$  is integrable and continuous. Hence,  $\mu$  is of class  $C^\infty$ . Since  $\mu(t) = 0$  for  $t < D$ , for all  $k \geq 0$ ,  $\mu^{(k)}(0) = 0$ . Lemma B.3 yields that for all  $k \geq 0$ ,  $\lim_{t \rightarrow \infty} \mu^{(k)}(t) = 0$ .  $\square$

*Proof of Lemma B.1.*

$$\mathcal{N}(x+iy) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{(v+iy)^2}{2}} dv.$$

It is easy to check that  $\partial_x \mathcal{N}(x + iy) - \partial_y \mathcal{N}(x + iy) = 0$  and this definition coincides with the cumulative normal distribution function on the real axis, so it is the unique analytic prolongation. We write  $\mathcal{N}(x + iy) = \mathcal{N}(x) + \int_0^y \partial_y \mathcal{N}(x + iy)$ , to get

$$\begin{aligned} \mathcal{N}(x + iy) &= \mathcal{N}(x) - i \frac{1}{\sqrt{2\pi}} \int_0^y \int_{-\infty}^x (v + iu) e^{-\frac{(v+iu)^2}{2}} dv du, \\ &= \mathcal{N}(x) + i \frac{1}{\sqrt{2\pi}} \int_0^y e^{-\frac{(x+iu)^2}{2}} du. \end{aligned}$$

Taking  $x + iy = r(1 + i)$  gives

$$\begin{aligned} \mathcal{N}(r(1 + i)) &= \mathcal{N}(r) + i \frac{1}{\sqrt{2\pi}} \int_0^r e^{-\frac{(r+iu)^2}{2}} du, \\ (B.3) \quad &= \mathcal{N}(r) + i \frac{1}{\sqrt{2\pi}} \int_0^1 e^{\frac{r^2}{2}(t^2-1)} e^{-itr^2} r dt. \end{aligned}$$

For  $t \in [0, 1)$ ,  $e^{\frac{r^2}{2}(t^2-1)} r$  tends to 0 when  $r$  goes to infinity. The function  $r \mapsto e^{\frac{r^2}{2}(t^2-1)} r$  is maximum for  $r = \frac{1}{1-t^2}$ , hence the following upper bound holds

$$e^{\frac{r^2}{2}(t^2-1)} r \leq \frac{1}{1-t^2} e^{\frac{1}{2(t^2-1)}} \quad \text{for all } t \in [0, 1).$$

The upper bound is integrable on  $[0, 1)$ , so by using the bounded convergence theorem, we can assert that the integral on the right hand side of (B.3) tends to 0 when  $r$  goes to infinity.  $\square$

*Proof of Lemma B.2.* We only do the proof for  $u > 0$ . For  $r > 0$ ,

$$\psi(r(1 + i)) = 1 + r(1 + i)\sqrt{2\pi} e^{r^2 i} \mathcal{N}(r(1 + i)).$$

Using the equivalent of  $\mathcal{N}(r(1 + i))$  when  $r$  goes to infinity (see Lemma B.1) enables to establish that  $|\psi(r(1 + i))| \sim 2r\sqrt{\pi}$  when  $r$  goes to infinity. Noticing that  $\sqrt{iu} = \frac{\sqrt{2u}}{2}(1 + i)$  ends the proof.  $\square$

Here is a quite obvious lemma we used in the proof of Proposition B.1.

**Lemma B.3.** *Let  $g$  be an integrable function on  $\mathbb{R}$ , then*

$$\lim_{t \rightarrow \infty} \int_{-\infty}^{\infty} g(u) e^{iut} du = 0.$$

## APPENDIX C. FORMULAE OF $\widehat{A}_1, \widehat{A}_2$

Let us recall the definitions of  $A_1, A_2$

$$\begin{aligned} A_1 &= \mathbb{E} \left[ \mathbf{1}_{\{T_{b_1}^- < T\}} \mathbb{E} \left[ \mathbf{1}_{\{T_{b_1}^- \leq T_{b_2}^+ < T\}} e^{mZ_T} (xe^{\sigma Z_T} - K)_+ | \mathcal{F}_{T_{b_1}^-} \right] \right], \\ A_2 &= \mathbb{E} \left[ \mathbf{1}_{\{T_{b_2}^+ < T\}} \mathbb{E} \left[ \mathbf{1}_{\{T_{b_2}^+ \leq T_{b_1}^- < T\}} e^{mZ_T} (xe^{\sigma Z_T} - K)_+ | \mathcal{F}_{T_{b_2}^+} \right] \right]. \end{aligned}$$

**C.1. Formula of  $\widehat{A}_1$ . Case  $L_2 \geq K$**

$$\begin{aligned} \widehat{A}_1 &= \frac{K e^{2(b_1 - b_2)\theta}}{\theta \psi^2(\theta\sqrt{D})} \psi^2(-\theta\sqrt{D}) e^{(m+\theta)k} \left[ \frac{1}{m+\theta} - \frac{1}{m+\theta+\sigma} \right] \\ &\quad + \frac{2e^{2b_1 - b_2\theta}}{\psi^2(\theta\sqrt{D})} \psi(-\theta\sqrt{D}) e^{mb_2} \left[ \frac{K\psi(m\sqrt{D})}{m^2 - \theta^2} - \frac{L_2\psi((m+\sigma)\sqrt{D})}{(m+\sigma)^2 - \theta^2} \right]. \end{aligned}$$

**Case  $L_2 < K$**

$$\begin{aligned} \widehat{A}_1 &= K \frac{e^{2b_1\theta}}{\theta\psi(\theta\sqrt{D})} \psi(-\theta\sqrt{D}) e^{(m-\theta)k} \left[ \frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta} \right] \\ &- K \frac{e^{(2b_1-b_2)\theta}}{\psi^2(\theta\sqrt{D})} \psi(-\theta\sqrt{D}) \sqrt{2\pi D} e^{\lambda D} e^{mk} \left[ e^{\theta(b_2-k)} \mathcal{N}\left(\theta\sqrt{D} + \frac{b_2-k}{\sqrt{D}}\right) \left( \frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta} \right) \right. \\ &\quad \left. + e^{\theta(k-b_2)} \mathcal{N}\left(-\theta\sqrt{D} + \frac{b_2-k}{\sqrt{D}}\right) \left( \frac{1}{m+\theta} - \frac{1}{m+\sigma+\theta} \right) \right] \\ &+ 2 \frac{e^{(2b_1-b_2)\theta}}{\psi^2(\theta\sqrt{D})} \psi(-\theta\sqrt{D}) \sqrt{2\pi D} e^{mb_2} \left[ \frac{mK}{m^2-\theta^2} e^{\frac{m^2 D}{2}} \mathcal{N}\left(m\sqrt{D} + \frac{b_2-k}{\sqrt{D}}\right) \right. \\ &\quad \left. - \frac{L_2(m+\sigma)}{(m+\sigma)^2-\theta^2} e^{\frac{(m+\sigma)^2 D}{2}} \mathcal{N}\left((m+\sigma)\sqrt{D} + \frac{b_2-k}{\sqrt{D}}\right) \right]. \end{aligned}$$

**C.2. Formula of  $\widehat{A}_2$ .** Instead of computing  $\widehat{A}_2$  directly, which would mean doing again the same type of computations we did to evaluate  $\widehat{A}_1$ , we first rewrite  $A_2$  to use as much as possible the computations we have already done in the valuation of  $\widehat{A}_1$ . As  $Z$  and  $-Z$  have the same law, introducing a new Brownian motion  $\tilde{Z} = -Z$  leads to

$$\begin{aligned} A_2 &= \mathbb{E} \left[ \mathbf{1}_{\{T_{-b_1}^+ \leq \tilde{Z}\}} \mathbf{1}_{\{T_{-b_2}^- \leq T_{-b_1}^+ \leq \tilde{Z}\}} e^{-m\tilde{Z}_T} (xe^{\sigma\tilde{Z}_T} - K)_+ \right] \\ &= \mathbb{E} \left[ \mathbf{1}_{\{T_{-b_1}^+ \leq \tilde{Z}\}} \mathbf{1}_{\{T_{-b_2}^- \leq T_{-b_1}^+ \leq \tilde{Z}\}} e^{-(m+\sigma)\tilde{Z}_T} (x - Ke^{\sigma\tilde{Z}_T})_+ \right]. \end{aligned}$$

Let  $A_3$  be defined as  $\mathbb{E} \left[ \mathbf{1}_{\{T_{b_1}^- \leq T\}} \mathbf{1}_{\{T_{b_1}^- \leq T_{b_2}^+ \leq T\}} e^{mZ_T} (K - xe^{\sigma Z_T})_+ \right]$ . Analogously with Theorem 4.1, we can write, for  $L_1 \leq x \leq L_2$ ,  $A_3 = \int_{-\infty}^k dy e^{my} (K - xe^{\sigma y}) h(T, y)$ , where the Laplace transform of  $h$  is still given by Equation (4.1). Then, we compute  $\widehat{A}_3$ , and we get  $\widehat{A}_2$  by replacing in  $\widehat{A}_3$   $m$  by  $-(m+\sigma)$ ,  $x$  by  $K$ ,  $K$  by  $x$ ,  $b_1$  by  $-b_2$  and  $b_2$  by  $-b_1$  (which means we replace  $L_2$  by  $\frac{xK}{L_1}$  and  $L_1$  by  $\frac{xK}{L_2}$ ).

**Case  $K \geq L_1$**

$$\widehat{A}_2 = \frac{xe^{2(b_1-b_2)\theta}}{\theta\psi^2(\theta\sqrt{D})} \psi^2(-\theta\sqrt{D}) e^{(m+\sigma-\theta)k} \left[ \frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta} \right].$$

**Case  $K \leq L_1$**

$$\begin{aligned} \widehat{A}_2 &= \frac{xe^{-2b_2\theta}}{\theta\psi(\theta\sqrt{D})} \psi(-\theta\sqrt{D}) e^{(m+\sigma+\theta)k} \left[ \frac{1}{m+\theta} - \frac{1}{m+\sigma+\theta} \right] \\ &- \frac{xe^{(b_1-2b_2)\theta}}{\psi^2(\theta\sqrt{D})} \psi(-\theta\sqrt{D}) \sqrt{2\pi D} e^{\lambda D} e^{(m+\sigma)k} \left[ e^{(k-b_1)\theta} \mathcal{N}\left(\theta\sqrt{D} + \frac{k-b_1}{\sqrt{D}}\right) \left( \frac{1}{m+\theta} - \frac{1}{m+\sigma+\theta} \right) \right. \\ &\quad \left. + e^{(b_1-k)\theta} \mathcal{N}\left(-\theta\sqrt{D} + \frac{k-b_1}{\sqrt{D}}\right) \left( \frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta} \right) \right] \\ &+ \frac{2xe^{(b_1-2b_2)\theta}}{L_1\psi^2(\theta\sqrt{D})} \psi(-\theta\sqrt{D}) e^{(m+\sigma)b_1} \left[ \frac{K}{m^2-\theta^2} \left( \psi(-m\sqrt{D}) + m\sqrt{2\pi D} e^{\frac{m^2 D}{2}} \mathcal{N}\left(-m\sqrt{D} + \frac{k-b_1}{\sqrt{D}}\right) \right) \right. \\ &\quad \left. - \frac{L_1}{(m+\sigma)^2-\theta^2} \left( \psi(-(m+\sigma)\sqrt{D}) + (m+\sigma)\sqrt{2\pi D} e^{\frac{(m+\sigma)^2 D}{2}} \mathcal{N}\left(-(m+\sigma)\sqrt{D} + \frac{k-b_1}{\sqrt{D}}\right) \right) \right]. \end{aligned}$$

## APPENDIX D. LAPLACE TRANSFORMS OF SINGLE BARRIER PARISIAN OPTION PRICES

In this section, we only recall the prices of single barrier Parisian options that are required to compute the double barrier Parisian option prices. In the following,  $d$  denotes  $\frac{b-k}{\sqrt{D}}$ .

## D.1. Standard call option.

$$*\widehat{BSC}(x, \lambda; K; r, \delta) = \begin{cases} \frac{K}{\theta} e^{(m-\theta)k} \left( \frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta} \right) & \text{for } K \geq x, \\ \frac{2K}{m^2-\theta^2} - \frac{2x}{(m+\sigma)^2-\theta^2} + \frac{Ke^{(m+\theta)k}}{\theta} \left( \frac{1}{m+\theta} - \frac{1}{m+\sigma+\theta} \right) & \text{for } K \leq x. \end{cases}$$

## D.2. Parisian down in call.

$$*\widehat{PDIC}(x, \lambda; K, L; r, \delta) = \frac{\psi(-\theta\sqrt{D})e^{2b\theta}}{\theta\psi(\theta\sqrt{D})} Ke^{(m-\theta)k} \left( \frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta} \right),$$

for  $K > L$  and  $x \geq L$ .

$$\begin{aligned} *\widehat{PDIC}(x, \lambda; K, L) &= \frac{e^{(m+\theta)b}}{\psi(\theta\sqrt{D})} \left( \frac{2K}{m^2-\theta^2} \left[ \psi(-\sqrt{D}m) + \sqrt{2\pi D} e^{\frac{Dm^2}{2}} m \mathcal{N}(-d - \sqrt{D}m) \right] \right. \\ &\quad \left. - \frac{2L}{(m+\sigma)^2-\theta^2} \left[ \psi(-\sqrt{D}(m+\sigma)) + \sqrt{2\pi D} e^{\frac{D(m+\sigma)^2}{2}} (m+\sigma) \mathcal{N}(-d - \sqrt{D}(m+\sigma)) \right] \right) \\ &\quad + \frac{Ke^{(m+\theta)k}}{\theta\psi(\theta\sqrt{D})} \left( \frac{1}{m+\theta} - \frac{1}{m+\sigma+\theta} \right) \left[ \psi(-\theta\sqrt{D}) + \theta e^{\lambda D} \sqrt{2\pi D} \mathcal{N}(d - \theta\sqrt{D}) \right] \\ &\quad + \frac{e^{\lambda D} \sqrt{2\pi D}}{\psi(\theta\sqrt{D})} Ke^{2b\theta} e^{(m-\theta)k} \mathcal{N}(-d - \theta\sqrt{D}) \left( \frac{1}{m+\sigma-\theta} - \frac{1}{m-\theta} \right) \end{aligned}$$

for  $K \leq L \leq x$ .

## D.3. Parisian up in call.

$$\begin{aligned} *\widehat{PUIC}(x, \lambda; K, L; r, \delta) &= e^{(m-\theta)b} \frac{\sqrt{2\pi D}}{\psi(\theta\sqrt{D})} \left[ \frac{2K}{m^2-\theta^2} e^{\frac{Dm^2}{2}} m \mathcal{N}(d + \sqrt{D}m) \right. \\ &\quad \left. - \frac{2L}{(m+\sigma)^2-\theta^2} e^{\frac{D(m+\sigma)^2}{2}} (m+\sigma) \mathcal{N}(d + \sqrt{D}(m+\sigma)) \right] \\ &\quad + \frac{e^{-2b\theta}}{\psi(\theta\sqrt{D})} Ke^{(m+\theta)k} e^{\lambda D} \sqrt{2\pi D} \mathcal{N}(d - \theta\sqrt{D}) \left( \frac{1}{m+\sigma+\theta} - \frac{1}{m+\theta} \right) \\ &\quad + \frac{e^{(m-\theta)k}}{\theta\psi(\theta\sqrt{D})} K \left( \frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta} \right) \left( \psi(-\theta\sqrt{D}) + \theta\sqrt{2\pi D} e^{\lambda D} \mathcal{N}(d - \theta\sqrt{D}) \right) \end{aligned}$$

for  $x \leq L \leq K$ .

$$\begin{aligned} *\widehat{PUIC}(x, \lambda; K, L; r, \delta) &= \frac{e^{(m-\theta)b}}{\psi(\theta\sqrt{D})} \left[ \frac{2K}{m^2-\theta^2} \psi(\sqrt{D}m) - \frac{2L}{(m+\sigma)^2-\theta^2} \psi(\sqrt{D}(m+\sigma)) \right] \\ &\quad + \frac{e^{-2b\theta} \psi(-\theta\sqrt{D})}{\theta\psi(\theta\sqrt{D})} Ke^{(m+\theta)k} \left( \frac{1}{m+\theta} - \frac{1}{m+\theta+\sigma} \right) \end{aligned}$$

for  $K \leq L$  and  $x \leq L$ .

## REFERENCES

- J. Abate, L. Choudhury, and G. Whitt. An introduction to numerical transform inversion and its application to probability models. *Computing Probability*, pages 257 – 323, 1999.
- L. Andersen and R. Brotherton-Ratcliffe. Exact exotics. *Risk*, 9(10):85–89, 1996.
- M. Avellaneda and L. Wu. Pricing parisian-style options with a lattice method. *International Journal of Theoretical and Applied Finance*, 2(1):1–16, 1999.
- P. Baldi, L. Caramellino, and M. G. Iovino. Pricing complex barrier options with general features using sharp large deviation estimates. In *Monte Carlo and quasi-Monte Carlo methods 1998 (Claremont, CA)*, pages 149–162. Springer, Berlin, 2000.
- M. Chesney, M. Jeanblanc-Picqué, and M. Yor. Brownian excursions and Parisian barrier options. *Adv. in Appl. Probab.*, 29(1):165–184, 1997. ISSN 0001-8678.
- R. Haber, P. Schonbucher, and P. Wilmott. An american in paris. OFRC Working Papers Series 1999mf14, Oxford Financial Research Centre, 1999. available at <http://ideas.repec.org/p/sbs/wpsefe/1999mf14.html>.
- P. Hartley. Pricing parisian options by laplace inversion. *Decisions in Economics & Finance*, 2002.
- C. Labart and J. Lelong. Pricing parisian options. Technical report, ENPC, <http://cermics.enpc.fr/reports/CERMICS-2005/CERMICS-2005-294.pdf>, December 2005.
- D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999. ISBN 3-540-64325-7.
- M. Schröder. Brownian excursions and Parisian barrier options: a note. *J. Appl. Probab.*, 40(4):855–864, 2003. ISSN 0021-9002.
- D. V. Widder. *The Laplace Transform*. Princeton Mathematical Series, v. 6. Princeton University Press, Princeton, N. J., 1941.
- P. Wilmott. *Derivatives*. University Edition, 1998.