Rapport de recherche du CERMICS 2006-329 Novembre 2006

Height process for super-critical continuous state branching process

Jean-François Delmas

CERMICS, École Nationale des Ponts et Chaussées, ParisTech, 6-8 avenue Blaise Pascal, Champs sur Marne, 77455 Marne La Vallée Cédex 2, FRANCE

CERMICS — ENPC
6 et 8 avenue Blaise Pascal
Cité Descartes - Champs sur Marne
77455 Marne la Vallée Cedex 2

http://cermics.enpc.fr

HEIGHT PROCESS FOR SUPER-CRITICAL CONTINUOUS STATE BRANCHING PROCESS

JEAN-FRANÇOIS DELMAS

Abstract. We define the height process for super-critical continuous state branching processes with quadratic branching mechanism. It appears as a projective limit of Brownian motions with positive drift reflected at 0 and $a > 0$ as a goes to infinity. Then we extend the pruning procedure of branching processes to the super-critical case. This give a complete duality picture between pruning and size proportional immigration for quadratic continuous state branching processes.

1. Introduction

Continuous state branching process (CB) appears as the limit of Galton-Watson processes, see [8] for the quadratic branching mechanism and [11] in the general case. We shall be interested in a CB, $Z^\theta = (Z^\theta_r, r \geq 0)$, with quadratic branching mechanism, ψ_θ,

$$\psi_\theta(u) = 2u^2 + 4\theta u, \quad u \geq 0, \tag{1}$$

for a given parameter $\theta \in \mathbb{R}$. The process Z^θ is a continuous Markov process taking values in $\mathbb{R}_+$ such that for all $r \geq 0$, $\lambda \geq 0$, $x \geq 0$,

$$\mathbb{E}[\mathrm{e}^{-\lambda Z^\theta_r} | Z^\theta_0 = x] = \mathrm{e}^{-x u^\theta(\lambda, r)}, \tag{2}$$

where u^θ is the only non-negative solution of the differential equation

$$u'(r) + \psi_\theta(u(r)) = 0, \quad r \geq 0, \quad \text{and} \quad u(0) = \lambda. \tag{3}$$

(In fact the general quadratic branching is of the form $\psi(u) = 2\alpha u^2 + 4\alpha\theta u$, with $\alpha > 0$. The corresponding CB is distributed as $(Z^\theta_{\alpha r}, r \geq 0)$. Up to this time scaling, we see it is enough to consider the case $\alpha = 1$.)

The quantity Z^θ_r can be thought as the "size" at time r of a population of individual with infinitesimal mass and whose reproduction mechanism is characterized by ψ_θ. The process Z^θ is called critical (i.e. constant in mean) if $\theta = 0$, sub-critical (i.e. with exponential decay for the mean) if $\theta > 0$ and super-critical (i.e. with exponential growth for the mean) if $\theta < 0$. In the critical or sub-critical case, one can code the genealogy associated to Z^θ using the so-called height process, $H^\theta = (H^\theta_t, t \geq 0)$, see [12] for $\theta = 0$ and [7] in a more general setting. The height process is the limit of the contour processes associated to sequence of Galton-Watson trees which converge to Z^θ. Intuitively H^θ_t is the genealogy of individual with label t in a continuous branching process. The "size" of the population of individuals with label less than t and which are alive at "generation" r is given by the local time of H^θ at level r up to time t:

Date: November 7, 2006.

2000 *Mathematics Subject Classification.* 60J55, 60J65, 60J80.

Key words and phrases. Brownian snake, branching process, height process, Ray-Knight theorem, local time, reflected Brownian motion, Brownian motion with drift.

This research was partially supported by NSERC Discovery Grants of the Probability group at Univ. of British Columbia.

$L^\theta_r(t)$. To consider an initial population with "size" $x > 0$, it is enough to look at the height process up to $T^\theta = \inf\{t > 0; L^\theta_0(t) = x\}$. Intuitively $L^\theta_r(T^\theta)$ gives the "size" of the population at generation r coming from an initial population with "size" x. In particular, one expect the height process $L^\theta = (L^\theta_r(T^\theta), r \geq 0)$ to be a CB started at x with branching mechanism ψ_θ.

For $\theta = 0$, H^θ is distributed as the absolute value of a Brownian motion and the second Ray-Knight theorem gives the process L^0 is indeed a CB with branching mechanism ψ_0 started from x. For $\theta > 0$, H^θ is distributed as the Brownian motion with drift -2θ and reflected above 0. Using Girsanov theorem or the more general framework developed in [7], it is easy to check that for $\theta \geq 0$, the process L^θ is indeed a CB with branching mechanism ψ_θ started from x.

Our aim is to extend those results to the case $\theta < 0$. One would like to consider a reflected Brownian motion with positive drift -2θ. However, in this case T^θ might be infinite. In fact we have that $\mathbb{P}(T^\theta < \infty)$ is equal to the probability that a CB with branching mechanism ψ_θ become extinct that is 1 if $\theta \geq 0$ or $\mathrm{e}^{2x\theta}$ if $\theta < 0$, see [9]. Intuitively, if $T^\theta = \infty$ it means there are individuals alive at generation $r = \infty$, and the height process H^θ describes only (part of) the lineage of one of the individuals alive at time ∞. To circumvent this problem, we chose to consider the height process associated not to the whole CB process but only up to a generation a. In a discrete setting, we would consider a (super-critical) Galton-Watson process and its corresponding discrete tree, and would cut the tree above a given genealogy, and would look at the discrete height process of this finite tree. Following the procedure in [7], one would expect the height process of the discrete tree to converge to a Brownian motion (with drift) reflected at 0 and a.

This intuition lead us to consider for $a > 0$ a Brownian motion with drift -2θ and reflected above 0 and below a, $H^{\theta,a} = (H^{\theta,a}_t, t \geq 0)$. In section 2, we first check that the family $(H^{\theta,a}, a > 0)$ can be built in a consistent way. Let $\mathcal{C}^c$ be the set of continuous function defined on $\mathbb{R}_+$ taking values in $[0, c]$. We define the projection from $\mathcal{C}^a$ to $\mathcal{C}^b$, $\pi_{a,b}$, by $\pi_{a,b}(\varphi)(t) = \varphi(C_\varphi(t))$ for $t \geq 0$, where $\varphi \in \mathcal{C}^a$ and $C_\varphi(t) = \inf\{r \geq 0; \int_0^r \mathbf{1}_{\{\varphi(s) \leq b\}}\, ds > t\}$ is the inverse of the time spent by φ below b. By construction, we have for $a > b > c$ that $\pi_{a,c} = \pi_{a,b} \circ \pi_{b,c}$. As the process $\pi_{a,b}(H^{\theta,a})$ is distributed as $H^{\theta,b}$ (Lemma (2.1)), this compatibility relation implies the existence of a projective limit $\mathcal{H}^\theta = (\mathcal{H}^{\theta,a}, a \geq 0)$ such that $\mathcal{H}^{\theta,a}$ is distributed as $H^{\theta,a}$ and $\pi_{a,b}(\mathcal{H}^{\theta,a}) = \mathcal{H}^{\theta,b}$. We shall call $\mathcal{H}^\theta$ the height process of the quadratic branching process. It is defined for $\theta \in \mathbb{R}$. We can consider Z^θ_r the local time of $\mathcal{H}^{\theta,a}$ at level r up to the hitting time of x for the local time of $\mathcal{H}^{\theta,a}$ at level 0. Because of the compatibility relation, we shall see that Z^θ_r does not depend on a, as soon as $a \geq r$. We prove a Ray-Knight theorem for $\mathcal{H}^\theta$: $(Z^\theta_r, r \geq 0)$ is a CB with branching mechanism ψ_θ (see Theorem 3.1). Of course we recover the critical cases and sub-critical cases, see comments of Remark 3.2. The proof relies on Girsanov theorem and the Ray-Knight theorem for $\theta = 0$.

Following [14], we can add a spatial motion to the individuals to get a super-critical Brownian snake. Taking a Poisson process as a spatial motion, this allows to adapt the pruning procedure developed in [4, 3] (see also [1] for more general critical or sub-critical branching mechanism) for the critical case to the super-critical case. This procedure gives a nice path transformation to get $\mathcal{H}^\theta$ from $\mathcal{H}^{\theta'}$ when $\theta > \theta'$ belong to $\mathbb{R}$, see Proposition 5.3. Using this pruning transformation and the Ray-Knight theorem, we can get Z^θ from $Z^{\theta'}$ for any $\theta > \theta'$ (this result is new for $\theta' < 0$). Notice that a size proportional immigration procedure, introduced in [2] in a more general setting, allows to reconstruct $Z^{\theta'}$ from Z^θ. Our result complete the description of the duality between size proportional immigration and pruning for quadratic branching mechanisms.

The paper is organized as follows. In Section 2, we check the compatibility relation in order to define the height process in the super-critical case. Section 3 is devoted to the proof of the Ray-Knight theorem. In Section 4, following [14, 13] we define the Brownian snake for super-critical branching mechanism. The pruning procedure is developed is Section 5.

2. Height process for quadratic branching process

We assume that $\theta \in \mathbb{R}$. Let $H^{\theta,a} = (H_t^{\theta,a}, t \geq 0)$ be a Brownian motion with drift -2θ reflected in $[0,a]$ and started at 0. This process can be constructed using a version of Skorohod's equation (see [18]). This is the unique solution of the stochastic differential equation:

$$dY_t = d\beta_t - 2\theta dt + \frac{1}{2}\, dL_0(t) - \frac{1}{2}\, dL_a(t), \quad Y_0 = 0, \tag{4}$$

where $L_y(t)$ is the local time of Y at level y up to time t and $(\beta_t, t \geq 0)$ is a standard Brownian motion.

We first check that the family $(H^{\theta,a}, a > 0)$ can be built in a consistent way. Let $\mathcal{C}^c$ be the set of continuous function defined on $\mathbb{R}_+$ taking values in $[0,c]$. Let $a > b > 0$. For $\varphi \in \mathcal{C}^a$, we consider the time spent below level b up to time t: $A_t = \int_0^t \mathbf{1}_{\{\varphi(s) \leq b\}}\, ds$ and its right continuous inverse $C_\varphi(t) = \inf\{r \geq 0; A_r > t\}$, with the convention that $\inf \emptyset = \infty$ and $\varphi(\infty) = b$. We define the projection from $\mathcal{C}^a$ to $\mathcal{C}^b$, $\pi_{a,b}$, by $\pi_{a,b}(\varphi) = \varphi \circ C_\varphi$.

Lemma 2.1. *Let $a > b > 0$. The process $\pi_{a,b}(H^{\theta,a})$ is distributed as $H^{\theta,b}$.*

Proof. For convenience we shall write H instead of $H^{\theta,a}$. Notice that H solves (4). Let $(\mathcal{G}_t, t \geq 0)$ be the filtration generated by the Brownian motion β, completed the usual way. Let $A_t = \displaystyle\int_0^t \mathbf{1}_{\{H_s \leq b\}}\, ds$ and $C(t) = \inf\{r \geq 0; A_r > t\}$. Notice that a.s. the stopping time $C(t)$ is finite. Let $L_r(t)$ be the local time of H at level r up to time t. We set $\tilde{H}_t = H_{C(t)}$. Using (4), we get

$$\begin{aligned}\tilde{H}_t &= \int_0^{C(t)} (d\beta_s - 2\theta ds) + \frac{1}{2}\, L_0(C(t)) - \frac{1}{2}\, L_a(C(t)) \\ &= \int_0^{C(t)} \mathbf{1}_{\{H_s \leq b\}} (d\beta_s - 2\theta ds) + \frac{1}{2}\, L_0(C(t)) - \frac{1}{2}\, L_a(C(t)) + \int_0^{C(t)} \mathbf{1}_{\{H_s > b\}} (d\beta_s - 2\theta ds).\end{aligned}$$

Since A is continuous, , by construction we have $\displaystyle\int_0^{C(t)} \mathbf{1}_{\{H_s \leq b\}} ds = \int_0^{C(t)} dA_s = A_{C(t)} = t$. Notice that $\beta' = (\beta'_t, t \geq 0)$, where $\beta'_t = \displaystyle\int_0^{C(t)} \mathbf{1}_{\{H_s \leq b\}} d\beta_s$, is a continuous martingale (with respect to the filtration $(\mathcal{F}_{C(t)}, t \geq 0)$). Its bracket is given by $\langle \beta' \rangle_t = \displaystyle\int_0^{C(t)} \mathbf{1}_{\{H_s \leq b\}} ds = t$. Therefore, β' is a Brownian motion. On the other hand, by Tanaka formula, we have a.s. for $r \geq 0$,

$$(H_r - b)^+ = (H_0 - b)^+ + \int_0^r \mathbf{1}_{\{H_s > b\}} dH_s + \frac{1}{2}\, L_b(r),$$

where $x^+ = \max(x, 0)$. Since $H_0 = 0$ and $H_{C(t)} \in [0, b]$, we get with $r = C(t)$ that

$$\int_0^{C(t)} \mathbf{1}_{\{H_s > b\}} dH_s + \frac{1}{2}\, L_b(C(t)) = 0.$$

Use

$$\int_0^r \mathbf{1}_{\{H_s>b\}} dH_s = \int_0^r \mathbf{1}_{\{H_s>b\}} (d\beta_s - 2\theta ds) - \frac{1}{2} L_a(r)$$

to get

$$-\frac{1}{2} L_a(C(t)) + \int_0^{C(t)} \mathbf{1}_{\{H_s>b\}} (d\beta_s - 2\theta ds) = -\frac{1}{2} L_b(C(t)).$$

Therefore, we have

$$\tilde{H}_t = \beta'_t - 2\theta t + \frac{1}{2} L_0(C(t)) - \frac{1}{2} L_b(C(t)),$$

where β' is a Brownian motion. Notice the function K defined for $t \in \mathbb{R}_+$ by $K(t) = \frac{1}{2} L_0(C(t)) - \frac{1}{2} L_b(C(t))$ is continuous with bounded variation such that $K(0) = 0$. Furthermore we have

$$\int_0^\infty \mathbf{1}_{\{\tilde{H}_s \not\in \{0,b\}\}} d|K|(t) = 0$$

and

$$dK(t) = \mathbf{1}_{\{\tilde{H}_t=0\}} d|K|(t) - \mathbf{1}_{\{\tilde{H}_t=b\}} d|K|(t).$$

Since $\tilde{H}$ is a continuous function taking values in $[0, b]$, we deduce from theorem 2.1 in [18], that $\tilde{H}$ is a Brownian motion with drift -2θ reflected in $[0, b]$ started at 0. Henceforth, it is distributed as $H^{\theta,b}$.

□

Notice that by construction we have for $a > b > c$ that $\pi_{a,c} = \pi_{a,b} \circ \pi_{b,c}$. Let μ_r denotes the law of $H^{\theta,r}$ for $r \geq 0$. Lemma 2.1 entails that $\mu_a \circ (\pi_{a,b})^{-1} = \mu_b$. This compatibility relation implies the existence of a projective limit $\mathcal{H}^\theta = (\mathcal{H}^{\theta,a}, a \geq 0)$ such that $\mathcal{H}^{\theta,a}$ is distributed as $H^{\theta,a}$ and

$$\pi_{a,b}(\mathcal{H}^{\theta,a}) = \mathcal{H}^{\theta,b}. \tag{5}$$

We will call $\mathcal{H}^\theta$ the height process of the quadratic branching process.

Remark 2.2. If there exists $a \geq b > 0$ s.t. $\mathcal{H}^{\theta,a}$ does not reach b on $[0, t]$, then we have that a.s. $\mathcal{H}^{\theta,c}$ coincide on $[0, t]$ for all $c \geq b$.

3. Ray-Knight theorem for reflected Brownian motion with drift

Let $L_r^{\theta,a}(t)$ be the local time of $\mathcal{H}^{\theta,a}$ at level r up to time t. For $x > 0$ we define

$$T_x^{\theta,a} = \inf\{t \geq 0; L_0^{\theta,a}(t) > x\}, \tag{6}$$

with the convention $\inf \emptyset = \infty$. Let $r \geq 0$. Notice that equation (5) implies that for all $a \geq b \geq r$,

$$L_r^{\theta,a}(T_x^{\theta,a}) = L_r^{\theta,b}(T_x^{\theta,b}).$$

We shall denote this common value by Z_r^θ. We write $\mathcal{H}^{\theta,a,(x)} = (\mathcal{H}_t^{\theta,a}, t \in [0, T_x^{\theta,a}])$ and we call $\mathcal{H}^{\theta,(x)} = (\mathcal{H}^{\theta,a,(x)}, a \geq 0)$ the height process associated to $Z^\theta = (Z_r^\theta, r \geq 0)$.

We can now formulate the Ray-Knight theorem.

Theorem 3.1. *The process Z^θ is a CB with branching mechanism ψ_θ.*

Remark 3.2. On the event that Z^θ become extinct, there exists a level r s.t. $Z^\theta_r = 0$, that is $L^{\theta,b}_r(T^{\theta,b}_x) = 0$ for $b \geq r$. From Remark 2.2, we deduce that $(\mathcal{H}^{\theta,a}_t, t \in [0, T^{\theta,a}_x])$ does not depend on $a > r$. In the sub-critical or critical case (i.e. $\theta \geq 0$), the extinction is almost sure. Thus the process $(\mathcal{H}^{\theta,a}_t, t \in [0, T^{\theta,a}_x])$ is constant for a large enough. It is distributed as a Brownian motion with drift -2θ reflected above 0 stopped when its local time at level 0 reaches x. In this case, Theorem 3.1 correspond to the usual Ray-Knight theorem (see [16], chap. XI.2 for $\theta = 0$ and [19] for $\theta > 0$).

Proof. Let $a > 0$ and $x > 0$ be fixed. To be concise, we write H^θ for $H^{\theta,a}$ and $T^\theta = \inf\{t \geq 0; L^\theta_0(t) > x\}$, where $L^\theta_r(t)$ is the local time of H^θ at level r up to time t. Notice T^θ is finite a.s. Let g be a continuous function taking values in $\mathbb{R}_+$. By monotone convergence, we have

$$\mathbb{E}\left[\mathrm{e}^{-\int_0^{T^\theta} g(H^\theta_s)ds}\right] = \lim_{n\to\infty} \mathbb{E}\left[\mathrm{e}^{-\int_0^{T^\theta\wedge n} g(H^\theta_s)ds}\right].$$

Using Girsanov theorem and the fact that H^θ solves (4), where L_0 and L_a are continuous adapted functionals of $(\beta_t - 2\theta t, t \geq 0)$ (this is a consequence of Theorem 2.1 in [18]), we get that

$$\mathbb{E}\left[\mathrm{e}^{-\int_0^{T^\theta\wedge n} g(H^\theta_s)ds}\right] = \mathbb{E}\left[\mathrm{e}^{-2\theta\beta_{T^0\wedge n} - 2\theta^2(T^0\wedge n)}\, \mathrm{e}^{-\int_0^{T^0\wedge n} g(H^0_s)ds}\right].$$

Since H^0 solves equation (4) with $\theta = 0$, we deduce that

$$\beta_{T^0\wedge n} = H^0_{T^0\wedge n} - \frac{1}{2} L^0_0(T^0\wedge n) + \frac{1}{2} L^0_a(T^0\wedge n). \tag{7}$$

Since $L^0_0(T^0) = x$, we have $\beta_{T^0\wedge n} \geq -\frac{1}{2} L^0_0(T^0\wedge n) \geq -\frac{1}{2} L^0_0(T^0) = -\frac{x}{2}$. By monotone convergence, we get

$$\lim_{n\to\infty} \mathbb{E}\left[\mathrm{e}^{-2\theta\beta_{T^0\wedge n} - 2\theta^2(T^0\wedge n)}\, \mathrm{e}^{-\int_0^{T^0\wedge n} g(H^0_s)ds}\right] = \mathbb{E}\left[\mathrm{e}^{-2\theta\beta_{T^0} - 2\theta^2 T^0}\, \mathrm{e}^{-\int_0^{T^0} g(H^0_s)ds}\right].$$

We write $Z_r = L^0_r(T^0)$ for $r \in [0, a]$. Notice that (7) implies

$$\beta_{T^0} = \frac{1}{2} Z_a - \frac{x}{2}.$$

This and the occupation time formula for H^0 implies that $T^0 = \int_0^a Z_r\, dr$ and

$$\mathbb{E}\left[\mathrm{e}^{-2\theta\beta_{T^0} - 2\theta^2 T^0}\, \mathrm{e}^{-\int_0^{T^0} g(H^0_s)ds}\right] = \mathbb{E}\left[\mathrm{e}^{\theta x - \theta Z_a - 2\theta^2 \int_0^a Z_r\, dr}\, \mathrm{e}^{-\int_0^a g(r) Z_r\, dr}\right].$$

This leads to

$$\mathbb{E}\left[\mathrm{e}^{-\int_0^{T^\theta} g(H^\theta_s)ds}\right] = \mathbb{E}\left[\mathrm{e}^{\theta x - \theta Z_a - 2\theta^2 \int_0^a Z_r\, dr}\, \mathrm{e}^{-\int_0^a g(r) Z_r\, dr}\right]. \tag{8}$$

Use the time occupation formula for $\mathcal{H}^{\theta,a,(x)}$ (which is distributed as $(H^\theta_s, s \in [0, T^\theta])$) to get

$$\mathbb{E}\left[\mathrm{e}^{-\int_0^a g(r) Z^\theta_r\, dr}\right] = \mathbb{E}\left[\mathrm{e}^{\theta x - \theta Z_a - 2\theta^2 \int_0^a Z_r\, dr}\, \mathrm{e}^{-\int_0^a g(r) Z_r\, dr}\right]. \tag{9}$$

The Ray-Knight theorem implies that $Z = (Z_r = L^0_r(T^0), r \in [0, a])$ is distributed as the square of 0-dimensional Bessel process started at x up to time a. In particular it is the unique strong solution of

$$d\hat{Y}_t = 2\sqrt{\hat{Y}_t}\, dW_t \quad t \in [0, a], \quad \hat{Y}_0 = x,$$

where $(W_t, t \geq 0)$ is a standard Brownian motion in $\mathbb{R}$. We deduce

$$\mathbb{E}\left[e^{\theta x - \theta Z_a - 2\theta^2 \int_0^a Z_r\, dr} e^{-\int_0^a g(r) Z_r\, dr} \right] = \mathbb{E}\left[e^{-2\theta \int_0^a \sqrt{\hat{Y}_t} dW_t - 2\theta^2 \int_0^a \hat{Y}_r\, dr} e^{-\int_0^a g(r)\hat{Y}_r\, dr} \right].$$

Notice that $M = (M_t, t \geq 0)$, where $M_t = e^{-2\theta \int_0^t \sqrt{\hat{Y}_r} dW_r - 2\theta^2 \int_0^t \hat{Y}_r\, dr}$, define a local martingale. It is in fact a martingale (see section 6 in [15]). Using Girsanov theorem again, we get that

$$\mathbb{E}\left[e^{-2\theta \int_0^a \sqrt{\hat{Y}_t} dW_t - 2\theta^2 \int_0^a \hat{Y}_r\, dr} e^{-\int_0^a g(r)\hat{Y}_r\, dr} \right] = \mathbb{E}\left[e^{-\int_0^a g(r)\hat{Y}^\theta_r\, dr} \right],$$

where $\hat{Y}^\theta$ is the unique strong solution of the stochastic differential equation

$$d\hat{Y}^\theta_t = 2\sqrt{\hat{Y}_t}\, dW^\theta_t - 4\theta \hat{Y}^\theta_t\, dt \quad t \in [0,a], \quad \hat{Y}^\theta_0 = x,$$

where W^θ is a standard Brownian motion. In conclusion we get

$$\mathbb{E}\left[e^{-\int_0^a g(r) Z^\theta_r\, dr} \right] = \mathbb{E}\left[e^{-\int_0^a g(r)\hat{Y}^\theta_r\, dr} \right].$$

We deduce that $(Z^\theta_r, r \in [0,a])$ is distributed as $(\hat{Y}^\theta_r, r \in [0,a])$ for all $a > 0$. In particular, Z^θ is a continuous Markov process. Recall u^θ defined by (3). We have $u^0(\lambda, s) = \lambda/(1 + 2\lambda s)$ and for $\theta \neq 0$

$$u^\theta(\lambda, s) = \frac{\lambda\, e^{-4\theta t}}{1 + \lambda (2\theta)^{-1}(1 - e^{-4\theta t})}. \tag{10}$$

From (9) we deduce that for $\lambda \geq 0$

$$\mathbb{E}\left[e^{-\lambda Z^\theta_a} \right] = \mathbb{E}\left[e^{\theta x - (\theta + \lambda) Z_a - 2\theta^2 \int_0^a Z_r\, dr} \right].$$

Thanks to formula (2.k) in [15], we check the right hand side is equal to $e^{-x u^\theta(\lambda, a)}$. This implies that Z^θ is a CB with branching mechanism ψ_θ. □

4. The Brownian snake

4.1. **Definition.** We refer to [7], section 4.1.1, for the construction of the snake with a fixed lifetime process. Let ξ a Markov process with càdlàg paths and values in a Polish space E, whose topology is defined by a metric δ. We assume ξ has no fixed discontinuities. Let P_y denote the law of ξ started at $y \in E$. The law of ξ is called the spatial motion. For $y \in E$, let $\mathbb{W}_y$ be the space of all E-valued killed paths started at y. An element of $\mathbb{W}_y$ is a càdlàg mapping $w : [0, \zeta) \to E$ s.t. $w(0) = y$. $\zeta \in (0, \infty)$ is called the lifetime of w. By convention the point y is considered as the path with zero lifetime, and is added to $\mathbb{W}_y$. The space $\mathbb{W} = \cup_{y \in E} \mathbb{W}_y$, equipped with the distance defined in [7], section 4.1.1, is Polish. For $w \in \mathcal{W}$, we define $\hat{w} = w(\zeta-)$ if the limit exists and $\hat{w} = \Delta$ otherwise, where Δ is a cemetery point added to E.

Mimicking the proof of proposition 4.1.1 in [7], for $a > 0$, $\theta \in \mathbb{R}$, there exists a càdlàg Markov process $W^{\theta,a} = (W^{\theta,a}_s, s \geq 0)$ taking values in $\mathbb{W}_y$ s.t.

- If $\zeta^{\theta,a}_s$ denotes the lifetime of $W^{\theta,a}_s$, then $\zeta^{\theta,a} = (\zeta^{\theta,a}_s, s \geq 0)$ is distributed as $H^{\theta,a}$.
- Let $s \geq 0$. Conditionally on $\zeta^{\theta,a}$, $W^{\theta,a}_s$ is distributed as ξ under P_y on $[0, \zeta^{\theta,a}_s)$. Notice that a.s. $\hat{W}^{\theta,a}_s = W^{\theta,a}_s(\zeta^{\theta,a}_s -)$ exists (i.e. is not equal to the cemetery point).
- Let $r > s \geq 0$. Conditionally on $\zeta^{\theta,a}$ and $W^{\theta,a}_s$, we have $W^{\theta,a}_r$ is equal to $W^{\theta,a}_s$ on $[0, m)$, where $m = \min(\zeta^{\theta,a}_s, \zeta^{\theta,a}_r)$ and is distributed as ξ under $\mathrm{P}_{\hat{W}^{\theta,a}_s}$ on $[m, \zeta^{\theta,a}_r)$.

We recall the snake property: a.s. for all s, s', we have $W_s^{\theta,a}(r) = W_{s'}^{\theta,a}(r)$, for all $r < \min(\zeta_s^{\theta,a}, \zeta_{s'}^{\theta,a})$.

We first check that the family $(W^{\theta,a}, a > 0)$ can be built in a consistent way. Let $\bar{\mathcal{C}}^c$ be the set of càdlàg function defined on $\mathbb{R}_+$ taking values in $\mathbb{W}_y$ and where the life time process is continuous and lies in $[0,c]$. Let $a > b > 0$. For $\bar{\varphi} \in \bar{\mathcal{C}}^a$, with lifetime process φ. Recall the function C_φ defined in Section 2. We define the projection, $\Pi_{a,b}$, from $\bar{\mathcal{C}}^a$ to $\bar{\mathcal{C}}^b$ by $\Pi_{a,b}(\bar{\varphi}) = \bar{\varphi} \circ C_\varphi$.

Recall (6) and define $W^{\theta,a,(x)} = (W_s^{\theta,a}, s \in [0, T_x^{\theta,a}])$.

Lemma 4.1. *Let $a > b > 0$. The process $\Pi_{a,b}(W^{\theta,a})$ (resp. $\Pi_{a,b}(W^{\theta,a,(x)})$) is distributed as $W^{\theta,b}$ (resp. $W^{\theta,b,(x)}$).*

The proof is similar to the proof of Lemma 2.1, see also Remark 3.2. Notice that by construction we have for $a > b > c$, $\Pi_{a,c} = \Pi_{a,b} \circ \Pi_{b,c}$. The compatibility relation of Lemma 4.1 implies the existence of a projective limit $\mathcal{W}^\theta = (\mathcal{W}^{\theta,a}, a \geq 0)$ such that $\mathcal{W}^{\theta,a}$ is distributed as $W^{\theta,a}$ and

$$\Pi_{a,b}(\mathcal{W}^{\theta,a}) = \mathcal{W}^{\theta,b}. \tag{11}$$

Similarly, we can define a projective limit $\mathcal{W}^{\theta,(x)} = (\mathcal{W}^{\theta,a,(x)}, a \geq 0)$ s.t. $\mathcal{W}^{\theta,a,(x)}$ is distributed as $W^{\theta,a,(x)}$ and (11) holds with $\mathcal{W}^{\theta,(x)}$ instead of $\mathcal{W}^\theta$. The family of lifetime processes of $(\mathcal{W}^{\theta,a}, a \geq 0)$ is distributed as $\mathcal{H}^\theta$. Therefore, we shall denote $\mathcal{H}^\theta$ (resp. $\mathcal{H}^{\theta,a}$) the lifetime process of $\mathcal{W}^\theta$ (resp. $\mathcal{W}^{\theta,a}$). This notation is consistent with Section 3. We call the process $\mathcal{W}^\theta$ the Brownian snake. From Remark 3.2, notice that for $\theta \geq 0$, the process $\mathcal{W}^{\theta,a,(x)}$ is independent of a for a large enough. We shall identify the projective limit $\mathcal{W}^{\theta,(x)}$ to this common value. It correspond to the usual Brownian snake in [14] when $\theta = 0$ (stopped when the local time at 0 of its lifetime reaches x).

4.2. **Excursion and special Markov property.** We denote by $\mathbb{N}_y^{\theta,a}$ the excursion measure of $\mathcal{W}^{\theta,a}$ away from the trivial path y, with lifetime 0. We assume $\mathbb{N}_y^{\theta,a}$ is normalized so that the corresponding local time at y (as defined in [5] Chap. 3) is the local time at 0 of the lifetime process: $L_0^{\theta,a}$. Let $\sigma^{\theta,a} = \inf\{s > 0; \mathcal{H}_s^{\theta,a} = 0\}$. Under $\mathbb{N}_y^{\theta,a}$, $\sigma^{\theta,a}$ is the length of the lifetime excursion.

Lemma 4.2. *We have the first moment formula: for F any non-negative measurable function defined on the space of càdlàg E-valued function*

$$\mathbb{N}_y^{\theta,a}\left[\int_0^{\sigma^{\theta,a}} ds\ F(\mathcal{W}_s^{\theta,a})\right] = \int_0^a dr\ \mathrm{e}^{-4\theta r}\, \mathrm{P}_y\left[F((\xi_t, t \in [0,r]))\right]. \tag{12}$$

This result is known for $\theta \geq 0$ (see [7] proposition 1.2.5).

Proof. Let $a > 0$ and $x > 0$ be fixed. Notice it is enough to prove establish (12) with $W^{\theta,a}$ instead of $\mathcal{W}^{\theta,a}$. To be concise, we shall omit a and x, so that we write for example H^θ for $H^{\theta,a}$ or T^θ for $T_x^{\theta,a}$.

Note that $\{s; H_s^\theta > 0, s \in [0, T^\theta]\}$ is open, and consider (α_i, β_i), $i \in I$, its connected component. Let G be any non-negative measurable function defined on the space of càdlàg $\mathbb{W}_y$-valued function and set $G_i = G(W_t^\theta, t \in (\alpha_i, \beta_i))$.

Similar arguments used for the proof of (8) relying on Girsanov theorem implies that

$$\mathbb{E}_y\left[\mathrm{e}^{-\sum_{i\in I} G_i}\right] = \mathbb{E}_y\left[\mathrm{e}^{\theta x - \theta L_a^0(T^0) - 2\theta^2 T^0}\,\mathrm{e}^{-\sum_{i\in I} G_i}\right].$$

Excursion theory gives that

$$\mathbb{E}_y\left[\mathrm{e}^{-\sum_{i\in I} G_i}\right] = \exp\left\{-x\mathbb{N}_y^\theta[1-\mathrm{e}^{-G(W^\theta)}]\right\}$$

and since $T^0 = \sum_{i\in I}\beta_i - \alpha_i$ and $L_a^0(T^0) = \sum_{i\in I} L_a^0(\beta_i) - L_a^0(\alpha_i)$,

$$\mathbb{E}_y\left[\mathrm{e}^{\theta x - \theta L_a^0(T^0) - 2\theta^2 T^0}\,\mathrm{e}^{-\sum_{i\in I} G_i}\right] = \exp\left\{\theta x - x\mathbb{N}_y^0[1-\mathrm{e}^{-\theta L_a^0(\sigma^0)-2\theta^2\sigma^0 - G(W^0)}]\right\}.$$

Thus, we get

$$\mathbb{N}_y^\theta[1-\mathrm{e}^{-G(W^\theta)}] = -\theta + \mathbb{N}_y^0[1-\mathrm{e}^{-\theta L_a^0(\sigma^0)-2\theta^2\sigma^0-G(W^0)}].$$

First moment computation implies that

$$\mathbb{N}_y^\theta[G(W^\theta)] = \mathbb{N}_y^0[G(W^0)\,\mathrm{e}^{-\theta L_a^0(\sigma^0)-2\theta^2\sigma^0}]. \tag{13}$$

Now we specialize to the case $G(W^\theta) = \int_0^{\sigma^\theta} F(W_s^\theta)\,ds$, where F is a non-negative measurable function defined on the space of càdlàg E-valued function. We have

$$\mathbb{N}_y^\theta\left[\int_0^{\sigma^\theta} F(W_s^\theta)\,ds\right] = \mathbb{N}_y^0\left[\int_0^{\sigma^0} F(W_s^0)\,\mathrm{e}^{-\theta L_a^0(\sigma^0)-2\theta^2\sigma^0}\,ds\right].$$

We can replace $\mathrm{e}^{-\theta L_a^0(\sigma^0)-2\theta^2\sigma^0}$ in the right side member by its predictable projection

$$\mathrm{e}^{-\theta L_a^0(s)-2\theta^2 s}\,\mathbb{E}\left[\mathrm{e}^{-\theta L_a^0(\sigma^0)-2\theta^2\sigma^0} \mid H_0^0 = r\right]_{|r=H_s^0}.$$

Using time reversibility (see Corollary 3.1.6 of [7] in a more general case) for the first equality and predictable projection for the second equality, we get that

$$\begin{aligned}
&\mathbb{N}_y^0\left[\int_0^{\sigma^0} F(W_s^0)\,\mathrm{e}^{-\theta L_a^0(s)-2\theta^2 s}\,\mathbb{E}\left[\mathrm{e}^{-\theta L_a^0(\sigma^0)-2\theta^2\sigma^0} \mid H_0^0 = r\right]_{|r=H_s^0} ds\right]\\
&\quad= \mathbb{N}_y^0\left[\int_0^{\sigma^0} F(W_s^0)\,\mathrm{e}^{-\theta(L_a^0(\sigma)-L_a^0(s))-2\theta^2(\sigma-s)}\,\mathbb{E}\left[\mathrm{e}^{-\theta L_a^0(\sigma^0)-2\theta^2\sigma^0} \mid H_0^0 = r\right]_{|r=H_s^0} ds\right]\\
&\quad= \mathbb{N}_y^0\left[\int_0^{\sigma^0} F(W_s^0)\mathbb{E}\left[\mathrm{e}^{-\theta L_a^0(\sigma^0)-2\theta^2\sigma^0} \mid H_0^0 = r\right]^2_{|r=H_s^0} ds\right].
\end{aligned}$$

Notice from (4) that

$$\frac{1}{2}L_a^0(\sigma^0) = H_{\sigma^0} - \beta_{\sigma^0} - \frac{1}{2}L_0^0(\sigma^0) - H_0 = -\beta_{\sigma^0} - H_0.$$

Stopping time theorem for exponential martingale implies that

$$\mathbb{E}\left[\mathrm{e}^{-\theta L_a^0(\sigma^0)-2\theta^2\sigma^0} \mid H_0^0 = r\right] = \mathbb{E}\left[\mathrm{e}^{-2\theta\beta_{\sigma^0}-2\theta H_0^0-2\theta^2\sigma^0} \mid H_0^0 = r\right] = \mathrm{e}^{-2\theta r}.$$

We deduce that

$$\mathbb{N}_y^\theta\left[\int_0^{\sigma^\theta} F(W_s^\theta)\,ds\right] = \mathbb{N}_y^0\left[\int_0^{\sigma^0} F(W_s^0)\,\mathrm{e}^{-4\theta H_s^0}\,ds\right].$$

The result is then a consequence of the first moment formula for the Brownian snake (see formula (4.2) in [7]).

□

We can define the exit local time of an open subset of D of E. For $w \in \mathbb{W}$, let $\tau(w) = \inf\{t > 0; w(t) \notin D\}$. Let $y \in D$. We assume that $\mathrm{P}_y(\tau < \infty) > 0$. Following [14, 13], the limit

$$L_s^D = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{\{\tau(\mathcal{W}_u^{\theta,a}) < \zeta_u < \tau(\mathcal{W}_u^{\theta,a}) + \varepsilon\}} \, du$$

exists for all $s > 0$ $\mathbb{P}_y^\theta$-a.s. and $\mathbb{N}_y^{\theta,a}$-a.e. and defines a continuous non-decreasing additive functional. We deduce the first moment formula from (12) and proposition 4.3.2 in [7]:

$$\mathbb{N}_y^{\theta,a}\left[\int_0^{\sigma^{\theta,a}} dL_s^D \; F(\mathcal{W}_s^{\theta,a})\right] = \mathrm{P}_y\left[\mathrm{e}^{-4\theta\tau}\, \mathbf{1}_{\{\tau \le a\}} F((\xi_t, t \in [0,\tau]))\right]. \tag{14}$$

We consider the following hypothesis:

($\mathcal{A}$) For every $y \in D$, w is continuous at $s = \tau$, P_y-a.s. on $\{\tau < \infty\}$.

We recall the description of the excursions of $\mathcal{W}^{\theta,a}$ out of D. We consider

$$A_t = \int_0^t \mathbf{1}_{\{\mathcal{H}_s^{\theta,a} \le \tau(\mathcal{W}_s^{\theta,a})\}} \, ds, \tag{15}$$

and $\eta_s = \inf\{t; A_t > s\}$ its right continuous inverse. We define the càdlàg process $\tilde{\mathcal{W}}_s^a = \mathcal{W}_{\eta_s}^{\theta,a}$. Let $\tilde{\mathcal{F}} = (\tilde{\mathcal{F}}_t, t \ge 0)$ be the filtration generated by $\tilde{\mathcal{W}}^a$. Note the snake property implies that $\{s \in [0, \sigma^{\theta,a}]; \mathcal{H}_s^{\theta,a} > \tau(\mathcal{W}_s^{\theta,a})\}$ is open and consider (α_i, β_i), $i \in I$, its connected component. The excursions of $\mathcal{W}^{\theta,a}$ out of D, $\mathcal{W}^{\{i\}}$, $i \in I$, are defined by

$$\mathcal{W}^{\{i\}}(r) = \mathcal{W}_{(\alpha_i + t)\wedge\beta_i}^{\theta,a}(r + \mathcal{H}_{\alpha_i}^{\theta,a}), \quad r \in [0, \mathcal{H}_t^{\{i\}} = \mathcal{H}_{(\alpha_i+t)\wedge\beta_i}^{\theta,a}].$$

We denote by $\sigma^{\{i\}} = \alpha_i - \beta_i$ the duration of the excursion $\mathcal{W}^{\{i\}}$. Following the proof of theorem 2.4 of [13], one can check the next result.

Let $\mathbb{D}$ the space of càdlàg function defined on $\mathbb{R}_+$ taking values in $\mathbb{W}$ and let δ_z denote the Dirac mass at point z.

Proposition 4.3. *The random measure $\int_0^{\sigma^{\theta,a}} dL_s^D \; \delta_{(L_s^D, \mathcal{H}_s^{\theta,a})}$ is measurable w.r.t. $\tilde{\mathcal{F}}_\infty$. Let ϕ a non-negative measurable function defined on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{D}$, we have*

$$\mathbb{N}_y^{\theta,a}\left[\mathrm{e}^{-\sum_{i\in I} \phi(L_{\alpha_i}^D, \mathcal{H}_{\alpha_i}^{\theta,a}, \mathcal{W}^{\{i\}})} \mid \tilde{\mathcal{F}}_\infty\right] = \exp\left\{-\int_0^{\sigma^{\theta,a}} dL_s^D \; \mathbb{N}_{\hat{\mathcal{W}}_s^{\theta,a}}^{\theta,a-h}[1 - \mathrm{e}^{-\phi(\ell,h,\cdot)}]_{|\ell = L_s^D, h = \mathcal{H}_s^{\theta,a}}\right\}.$$

5. Pruning of the height process

We present a pruning of the genealogical tree described by the height process $\mathcal{H}^\theta$ using a method introduced in [3] in the case $\theta = 0$. The pruning gives a natural way to recover $Z^{\theta+\gamma}$ from Z^θ for $\gamma > 0$. This gives a dual procedure to [2], where the authors used immigration to reconstruct Z^θ from $Z^{\theta+\gamma}$. This pruning procedure goes back to [6], where the authors used an intensity of the killing rate which was dependent of the underlying motion.

5.1. **Poisson process as underlying motion.** We keep notations of the previous Section.

Let $\gamma > 0$, and following [4, 3] consider for the spatial motion (i.e. the law of ξ) the Poisson process distribution with intensity 4γ. We denote by L^D the exit local time out of $D = \{0\}$. The additive functional A defined in (15) can be written in the following way:

$$A_t = \int_0^t \mathbf{1}_{\{\mathcal{W}_s^{\theta,a} = 0\}} \, ds$$

Let $\tilde{\mathcal{H}}^a$ be the lifetime process of $\tilde{\mathcal{W}}^a$. Notice that $(\tilde{\mathcal{H}}^a, a \geq 0)$ is a consistent family in the sense that $\pi_{a,b}(\tilde{\mathcal{H}}^a) = \tilde{\mathcal{H}}^b$ for all $a \geq b \geq 0$. We shall denote by $\tilde{\mathcal{H}}$ its projective limit and call it the pruned height process.

Lemma 5.1. *We have a.s. $\mathbb{P}_0^{\theta,a}$-a.s. and $\mathbb{N}_0^{\theta,a}$-a.e. for all $s \geq 0$, $L_s^D = 4\gamma A_s$.*

Proof. We shall first prove the result for $\theta = 0$. We drop the notation a and $\theta = 0$ in the first part of the proof. We have

$$\begin{aligned}\mathbb{N}_0[(L_\sigma^D - L_t^D - 4\gamma A_\sigma + 4\gamma A_t)^2] &= 2\mathbb{N}_0\left[\int_t^\sigma (L_\sigma^D - L_s^D - 4\gamma A_\sigma + 4\gamma A_s) d(L_s^D - 4\gamma A_s)\right] \\ &= 2\mathbb{N}_0\left[\int_t^\sigma 2\mathcal{H}_s \mathbb{N}[L_\sigma^D - 4\gamma A_\sigma] d(L_s^D - 4\gamma A_s)\right],\end{aligned}$$

where we used the previsible projection of $(L_\sigma^D - L_s^D - 4\gamma A_\sigma + 4\gamma A_s)$ and proposition 2.1 in [13] to compute it for the second equality. Now (12) and (14) implies that $\mathbb{N}_0[L_\sigma^D] = 4\gamma \mathbb{N}_0[A_\sigma]$. This implies that $\mathbb{N}_0[(L_\sigma^D - L_t^D - 4\gamma A_\sigma + 4\gamma A_t)^2] = 0$ for all $t \geq 0$. Since L^D and A are continuous and equal to 0 at 0, this implies that $\mathbb{N}_0$-a.e. for all $s \geq 0$, $L_s^D = 4\gamma A_s$. Since $\mathbb{P}_0$-a.s. $\int_0^\sigma \mathbf{1}_{\{\mathcal{H}_s=0\}} dL_s^D = \int_0^\sigma \mathbf{1}_{\{\mathcal{H}_s=0\}} dA_s = 0$, we deduce from excursion theory that the result holds also $\mathbb{P}_0$-a.s.

Using Girsanov theorem (see (13)), since $L^D = 4\gamma A$ holds $\mathbb{N}_0^{0,a}$-a.e, we deduce that the equality also holds $\mathbb{N}_0^{\theta,a}$-a.e. (and also $\mathbb{P}_0^{\theta,a}$-a.s.).

□

Using Lemma 5.1 notice that $\mathbb{N}_0^{\theta,a}$-a.e.

$$\int_0^{\sigma^{\theta,a}} dL_s^D \; \delta_{(L_s^D, \mathcal{H}_s^{\theta,a})} = 4\gamma \int_0^{\tilde{\sigma}^a} du \; \delta_{(u, \tilde{\mathcal{H}}_u^a)},$$

where $\tilde{\sigma}^a = \inf\{s > 0; \tilde{\mathcal{H}}_s^a = 0\} = A_{\sigma^{\theta,a}}$ is the length of the excursion of $\tilde{\mathcal{W}}^a$. Let us also notice that $\hat{\mathcal{W}}_s^a = 0$ dA_s-a.e.

The Poisson process does not satisfy condition $(\mathcal{A})$ with $D = \{0\}$. However Proposition 4.3 can be extended to this particular case. (See [1] for a similar formulation in slightly different context. In [1], there is no Brownian part, and the Poisson process is only increasing at the nodes of the height process.) Using the previous remarks, Proposition 4.3 can be written as follows.

Proposition 5.2. *Let ϕ a non-negative measurable function defined on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{D}$, we have*

$$\mathbb{N}_0^{\theta,a}\left[\mathrm{e}^{-\sum_{i\in I}\phi(A_{\alpha_i}, \mathcal{H}_{\alpha_i}^{\theta,a}, \mathcal{W}^{\{i\}})} \mid \tilde{\mathcal{F}}_\infty\right] = \exp\left\{-4\gamma \int_0^{\tilde{\sigma}^a} dr \; \mathbb{N}_0^{\theta,a-h}[1 - \mathrm{e}^{-\phi(r,h,\cdot)}]_{|h=\tilde{\mathcal{H}}_r^a}\right\}.$$

5.2. **The main result.**

Proposition 5.3. *The pruned height process $\tilde{\mathcal{H}}$ is distributed as $\mathcal{H}^{\theta+\gamma}$.*

Recall that the height process $\mathcal{H}^\theta$ allows to code for the genealogy of continuous state branching process with branching mechanism ψ_θ. In fact, using the Poisson process with intensity 4γ as a spatial motion provides a way to remove individuals of continuous state branching process associated to the height process $\mathcal{H}^\theta$ in such a way as to preserve the genealogical structure. The height process corresponding to the remaining individuals is a height process associated to the branching mechanism $\psi_{\theta+\gamma}$. This Proposition is an extension to the super-critical case of [3].

Proof. Because of the consistency, it is enough to prove that $\tilde{\mathcal{H}}^a$ is distributed as $H^{\theta+\gamma,a}$. Let $\theta \in \mathbb{R}$, $a \geq 0$ be fixed. We shall omit θ and a in what follows and for example write $\mathcal{W}_s$ for $\mathcal{W}_s^{\theta,a}$.

Recall η is the right continuous inverse of A, where $A_t = \int_0^t \mathbf{1}_{\{\mathcal{W}_s^{\theta,a}=0\}}\, ds$. We shall use a sub-martingale problem, see [17], to give the law of $\tilde{\mathcal{H}}$. Recall that $\mathcal{H}$ solves (4):

$$\mathcal{H}_t = \beta_t - 2\theta t + \frac{1}{2}\, L_0(t) - \frac{1}{2}\, L_a(t),$$

where $\beta = (\beta_t, t \geq 0)$ is a Brownian motion. Let g be defined on $[0,\infty) \times \mathbb{R}$ with compact support with first derivative in the first variable and second derivative in the second variable continuous (in both variables). We shall write $g'(t,x) = \partial_x g(t,x)$, $g''(t,x) = \partial^2_{xx} g(t,x)$. We shall assume that $g'(t,0) \geq 0$ and $g'(t,a) \leq 0$ for all $t \geq 0$. We define for $t \geq 0$

$$M_t = g(0,0) + \int_0^t g'(A_s, \mathcal{H}_s)\, d\beta_s + \frac{1}{2}\int_0^t g'(A_s,0) dL_0(s) - \frac{1}{2}\int_0^t g'(A_s,a)\mathbf{1}_{\{\hat{\mathcal{W}}_s=0\}} dL_a(s).$$

Notice that $(M_t, t \geq 0)$ is a sub-martingale with respect to $\mathcal{F} = (\mathcal{F}_t, t \geq 0)$, the filtration generated by $\mathcal{H}$. We also have

$$\begin{aligned} M_t = g(A_t, \mathcal{H}_t) - \int_0^t \left(\frac{1}{2}\, g''(A_s,\mathcal{H}_s) - 2\theta g'(A_s,\mathcal{H}_s)\right)\, ds - \int_0^t \partial_t g(A_s,\mathcal{H}_s) dA_s \\ + \frac{1}{2}\int_0^t g'(A_s,a)\mathbf{1}_{\{\hat{\mathcal{W}}_s\neq 0\}} dL_a(s). \end{aligned}$$

Since η_t is an $\mathcal{F}$-stopping time, the stopping time Theorem implies the process $N = (N_t, t \geq 0)$, where $N_t = \mathbb{E}[M_{\eta_t}|\tilde{\mathcal{F}}_t]$, is an $\tilde{\mathcal{F}}$-sub-martingale. We set

$$\tilde{M}_t = \int_0^{\eta_t} \left(\frac{1}{2}\, g''(A_s,\mathcal{H}_s) - 2\theta g'(A_s,\mathcal{H}_s)\right)\mathbf{1}_{\{\hat{\mathcal{W}}_s\neq 0\}}\, ds - \frac{1}{2}\int_0^{\theta_t} g'(A_s,a)\mathbf{1}_{\{\hat{\mathcal{W}}_s\neq 0\}}\, dL_a(s).$$

Recall that a.s. $A_{\eta_t} = t$ to get

$$\begin{aligned} M_{\eta_t} &= g(t,\tilde{\mathcal{H}}_t) - \int_0^{\eta_t}\left(\frac{1}{2}\, g''(A_s,\mathcal{H}_s) - 2\theta g'(A_s,\mathcal{H}_s) + \partial_t g(A_s,\mathcal{H}_s)\right)\mathbf{1}_{\{\hat{\mathcal{W}}_s=0\}}\, ds - \tilde{M}_t \\ &= g(t,\tilde{\mathcal{H}}_t) - \int_0^t \left(\frac{1}{2}\, g''(s,\tilde{\mathcal{H}}_s) - 2\theta g'(s,\tilde{\mathcal{H}}_s) + \partial_t g(s,\mathcal{H}_s)\right)\, ds - \tilde{M}_t. \end{aligned}$$

We have, using notations of Proposition 5.2,

$$\begin{aligned} \tilde{M}_t &= \int_0^{\eta_t}\left(\frac{1}{2}\, g''(A_s,\mathcal{H}_s) - 2\theta g'(A_s,\mathcal{H}_s)\right)\mathbf{1}_{\{\hat{\mathcal{W}}_s\neq 0\}}\, ds - \frac{1}{2}\int_0^{\eta_t} g'(A_s,a)\mathbf{1}_{\{\hat{\mathcal{W}}_s\neq 0\}}\, dL_a(s) \\ &= \sum_{i\in I}\mathbf{1}_{\{A_{\alpha_i}\leq t\}}\int_0^{\sigma^i}\left(\frac{1}{2}\, g''(A_{\alpha_i},\mathcal{H}^i_s + \mathcal{H}_{\alpha_i}) - 2\theta g'(A_{\alpha_i},\mathcal{H}^i_s+\mathcal{H}_{\alpha_i})\right)\, ds \\ &\qquad - \frac{1}{2}\sum_{i\in I}\mathbf{1}_{\{A_{\alpha_i}\leq t\}} g'(A_{\alpha_i},a)(L_a(\beta_i) - L_a(\alpha_i)). \end{aligned}$$

We get

$$
\begin{aligned}
&\mathbb{E}\left[\tilde{M}_t|\tilde{\mathcal{F}}_\infty\right]\\
&= 4\gamma \int_0^t du\, \mathbb{N}_0^{a-h}\left[\int_0^\sigma \left(\frac{1}{2}\, g''(u,\mathcal{H}_s+h) - 2\theta g'(u,\mathcal{H}_s+h)\right)\, ds - \frac{g'(u,a)}{2} L_a(\sigma)\right]_{|h=\tilde{\mathcal{H}}_u}\\
&= 4\gamma \int_0^t du \left[\int_0^{a-\tilde{\mathcal{H}}_u} \mathrm{e}^{-4\theta s}\left(\frac{1}{2}\, g''(u,s+\tilde{\mathcal{H}}_u) - 2\theta g'(u,s+\tilde{\mathcal{H}}_u)\right)\, ds - \frac{g'(u,a)}{2}\,\mathrm{e}^{-4\theta(a-\tilde{\mathcal{H}}_u)}\right]\\
&= 4\gamma \int_0^t du \left[\left[\frac{1}{2}\, g'(u,s+\tilde{\mathcal{H}}_u)\,\mathrm{e}^{-4\theta s}\right]_0^{a-\tilde{\mathcal{H}}_u} - \frac{g'(u,a)}{2}\,\mathrm{e}^{-4\theta(a-\tilde{\mathcal{H}}_u)}\right]\\
&= -2\gamma \int_0^t du\, g'(u,\tilde{\mathcal{H}}_u),
\end{aligned}
$$

where we used Proposition 5.2 for the first equality and (12) and (14) for the second. This implies

$$
\mathbb{E}\left[\tilde{M}_t|\tilde{\mathcal{F}}_t\right] = -2\gamma \int_0^t du\, g'(u,\tilde{\mathcal{H}}_u),
$$

and we deduce that

$$
\begin{aligned}
N_t &= \mathbb{E}[M_{\eta_t}|\tilde{\mathcal{F}}_t]\\
&= g(t,\tilde{\mathcal{H}}_t) - \int_0^t \left(\frac{1}{2}\, g''(s,\tilde{\mathcal{H}}_s) - 2\theta g'(s,\tilde{\mathcal{H}}_s) + \partial_t g(s,\tilde{\mathcal{H}}_s)\right)\, ds - \mathbb{E}\left[\tilde{M}_t|\tilde{\mathcal{F}}_t\right]\\
&= g(t,\tilde{\mathcal{H}}_t) - \int_0^t \left(\frac{1}{2}\, g''(s,\tilde{\mathcal{H}}_s) - 2(\theta+\gamma) g'(s,\tilde{\mathcal{H}}_s) + \partial_t g(s,\tilde{\mathcal{H}}_s)\right)\, ds.
\end{aligned}
$$

Notice that a.s. $\tilde{\mathcal{H}}_t \in [0,a]$. Recall N is a $\tilde{\mathcal{F}}$-sub-martingale for any smooth function g such that $g'(t,0) \geq 0$ and $g'(t,a) \leq 0$ for all $t \geq 0$. We deduce from uniqueness of solution to the sub-martingale problem, see [17] theorem 5.5, that $\tilde{\mathcal{H}}$ is distributed as a Brownian motion in $[0,a]$ with drift $-2(\theta+\gamma)$ and reflected at 0 and a. This and the consistency property end the proof. □

References

[1] R. ABRAHAM and J.-F. DELMAS. Fragmentation associated to Lévy processes using snake. *Preprint CERMICS*, 2005.

[2] R. ABRAHAM and J.-F. DELMAS. Changing the branching mechanism of a continuous state branching process using immigration. *Preprint CERMICS*, 2006.

[3] R. ABRAHAM and L. SERLET. Poisson snake and fragmentation. *Elect. J. of Probab.*, 7, 2002.

[4] R. ABRAHAM and L. SERLET. Representations of the Brownian snake with drift. *Stoch. Stoch. Rep.*, 73(3-4):287–308, 2002.

[5] R. BLUMENTHAL. *Excursions of Markov processes.* Birkhäuser, Boston, 1992.

[6] J.-S. DHERSIN and L. SERLET. A stochastic calculus approach for the Brownian snake. *Canad. J. of Math.*, 52(1):92–118, 2000.

[7] T. DUQUESNE and J.-F. LE GALL. *Random trees, Lévy processes and spatial branching processes*, volume 281. Astérisque, 2002.

[8] W. FELLER. Diffusion processes in genetics. In *Proc. Berkeley Sympos. Math. Statist. Probability*, pages 227–246, 1951.

[9] D. R. GREY. Asymptotic behaviour of continuous time, continuous state-space branching processes. *J. Appl. Probability*, 11:669–677, 1974.

[10] J. LAMPERTI. Continuous state branching process. *Bull. Am. Math. Soc.*, 73:382–386, 1967.

[11] J. LAMPERTI. The limit of a sequence of branching processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 7:271–288, 1967.
[12] J.-F. LE GALL. Brownian excursions, trees and measure-valued branching processes. *Ann. Probab.*, 19(4):1399–1439, 1991.
[13] J.-F. LE GALL. The Brownian snake and solutions of $\Delta u = u^2$ in a domain. *Probab. Th. Rel. Fields*, 102:393–432, 1995.
[14] J.-F. LE GALL. *Spatial branching processes, random snakes and partial differential equations*. Lectures in Mathematics, ETH Zürich. Birkhäuser, 1999.
[15] J. PITMAN and M. YOR. A decomposition of Bessel bridges. *Z. Wahrsch. Verw. Gebiete*, 59(4):425–457, 1982.
[16] D. REVUZ and M. YOR. *Continuous martingales and Brownian motion*. Springer Verlag, Heidelberg, 1991.
[17] D. W. STROOCK and S. R. S. VARADHAN. Diffusion processes with boundary conditions. *Comm. Pure Appl. Math.*, 24:147–225, 1971.
[18] H. TANAKA. Stochastic differential equations with reflecting boundary condition in convex regions. *Hiroshima Math. J.*, 9(1):163–177, 1979.
[19] J. WARREN. Branching processes, the Ray-Knight theorem, and sticky Brownian motion. In *Séminaire de Probabilités, XXXI*, volume 1655 of *Lecture Notes in Math.*, pages 1–15. Springer, Berlin, 1997.

CERMICS, École Nationale des Ponts et Chaussées, ParisTech, 6-8 av. Blaise Pascal, Champs-sur-Marne, 77455 Marne La Vallée, France.
E-mail address: delmas@cermics.enpc.fr