Discontinuous Galerkin methods for anisotropic semi-definite diffusion with advection

D. A. Di Pietro[1], A. Ern[1] & J.-L. Guermond[2]

[1]*ENPC, ParisTech, Champs sur Marne, 77455 Marne la Vallée Cedex 2, France*

[2]*Dept. Math, Texas A&M, College Station, TX 77843-3368, USA and LIMSI (CNRS-UPR 3251), BP 133, 91403, Orsay, France*

CERMICS — ENPC
6 et 8 avenue Blaise Pascal
Cité Descartes - Champs sur Marne
77455 Marne la Vallée Cedex 2

http://cermics.enpc.fr

DISCONTINUOUS GALERKIN METHODS FOR ANISOTROPIC SEMI-DEFINITE DIFFUSION WITH ADVECTION

DANIELE A. DI PIETRO*, ALEXANDRE ERN*, AND JEAN-LUC GUERMOND†

Abstract. In this work we construct and analyze a Discontinuous Galerkin method to solve advection-diffusion-reaction PDEs with anisotropic and semi-definite diffusion. The method is designed so as to automatically detect the so-called elliptic-hyperbolic interface without requiring any further intervention. The key idea of the method is the use of weighted average and jump operators to ensure consistency. The error analysis provides optimal estimates in the broken graph norm and is consistent with well-known results when the problem is either hyperbolic or uniformly elliptic. The theoretical results are supported by numerical evidence.

Key words. Discontinuous Galerkin, advection-diffusion-reaction, discontinuous coefficients, anisotropic diffusion, coupled elliptic-hyperbolic, weighted averages

AMS subject classifications. 65N30, 65M60, 35G15

1. Introduction.

Discontinuous Galerkin (DG) methods were originally introduced to solve transport equations in [21, 22, 25] and later extended to problems involving second-order elliptic operators in [3, 26, 1]. For many years, the development and analysis of DG methods have followed two somewhat parallel routes according to the hyperbolic or elliptic nature of the problem at hand. A unifying viewpoint has recently been proposed in a series of papers [11, 9, 10], where the authors rely on the Friedrichs framework originally proposed in [15] to perform an abstract analysis valid for a variety of (linear) PDE systems.

The goal of this work is to further enlarge the picture by considering advection-diffusion-reaction problems with discontinuous, anisotropic, and semi-definite diffusivity. One major difficulty related to such problems is to clarify the notion of elliptic and hyperbolic subdomains and to devise suitable coupling conditions. This issue has been investigated by Gastaldi and Quarteroni in [16], where a set of interface conditions is derived through asymptotic analysis for a one-dimensional model problem. In [20], Houston and co-workers propose and analyze a DG method for PDEs with non-negative characteristic form in higher space dimensions. The problem of a possibly discontinuous solution across an elliptic-hyperbolic interface is solved by manually removing some penalty terms. This approach thus relies on the *a priori* knowledge of the interface location. In [12], Ern and Proft develop and analyze a method which also requires the *a priori* knowledge of the elliptic-hyperbolic interface.

In this work we address the multidimensional case, and we consider anisotropic tensor-valued diffusivity fields. We derive a multidimensional generalization of the one-dimensional interface condition introduced in [16]. This condition depend on the value of the diffusion in the normal direction together with the sign of the normal component of the advection field. After discussing the well-posedness of the continuous problem, we propose a DG approximation inspired by the weak formulation of the continuous problem with boundary and interface conditions weakly enforced. The bilinear form for the discrete problem is designed so that the correct set of interface conditions is automatically recovered without identifying a priori the elliptic/hyperbolic interfaces. The bilinear form is strongly consistent, continuous, and

*CERMICS, Ecole des Ponts, ParisTech, 77455 Marne la Vallée Cedex 2, France

†Dept. Math, Texas A&M, College Station, TX 77843-3368, USA and LIMSI (CNRS-UPR 3251), BP 133, 91403, Orsay, France

(in the spirit of Friedrichs) satisfies a positivity requirement (L^2-coercivity). All the stability and error estimates are robust with respect to the possible anisotropy and discontinuity of the diffusion coefficient. The convergence results are optimal in the broken graph norm and compatible with those presented in [10, 9] when the problem is either hyperbolic or uniformly elliptic.

The material is organized as follows. In §2 we analyze the continuous problem. After presenting the setting under scrutiny, we state the interface conditions and propose a continuous bilinear which enforces interface and boundary conditions weakly. A well-posedness result is proved under mild density assumptions. In §3 we focus our attention on the discrete problem. We introduce the discrete setting, discuss the design of the discrete bilinear form, and show how our design constraints shape the form of the consistency and penalty terms. The convergence of the method is proved in §4 and the main results are stated in Theorems 4.5 and 4.7. Implementation issues are addressed in §5 and variants of the method are also introduced. §6 is devoted to numerical experiments illustrating the performance of the proposed method. Concluding remarks are reported in §7.

2. The continuous problem. In this section we introduce the model problem and rewrite it as a first-order PDE system endowed with a Friedrichs-like structure. The corresponding weak formulation with boundary and interface conditions weakly enforced will serve as a base for the design of the DG method constructed in §3.

2.1. The PDE setting. Let $\Omega \subset \mathbb{R}^d$ be a bounded, open, and connected Lipschitz domain with boundary $\partial\Omega$ and outward normal n, and let $P_\Omega \stackrel{\text{def}}{=} \{\Omega_i\}_{i=1}^N$ be a partition of Ω into Lipschitz connected subdomains. The problem investigated in this work consists of the following scalar-valued PDE:

$$\nabla\cdot(-\nu\nabla u + \beta u) + \mu u = f, \tag{2.1}$$

with data $f \in L^2(\Omega)$. Suitable boundary conditions are prescribed on $\partial\Omega$, as specified later in this section. The following assumptions are made on the coefficients:

(i) $\nu \in [L^\infty(\Omega)]^{d,d}$ is a positive *semi*-definite tensor field, meaning that it is symmetric and, for all $r \in \mathbb{R}^d$ and a.e. $x \in \Omega$, $r^t\nu(x)r \geq 0$. Furthermore, we assume that ν is piecewise constant on the partition P_Ω and that the problem is normalized in order to have $\|\nu\|_{[L^\infty(\Omega)]^{d,d}} \leq 1$;

(ii) $\beta \in [\mathfrak{C}^1(\overline{\Omega})]^d$;

(iii) $\mu \in L^\infty(\Omega)$ is such that $\mu + \frac{1}{2}\nabla\cdot\beta \geq \mu_0$ with $\mu_0 > 0$.

Throughout the rest of this work, the symbols $\lesssim$ and $\gtrsim$ will be used for inequalities that hold up to a real positive multiplicative constant that is independent of ν (and discretization parameters like the meshsize) but may depend on β and μ (and regularity parameters of the mesh family considered later on).

We introduce the symbol Γ to denote the union of the inner boundaries of the subdomains Ω_i, i.e.,

$$\Gamma \stackrel{\text{def}}{=} \{x \in \Omega;\ \exists i_1, i_2 \in \{1, \ldots, N\},\ i_1 \neq i_2,\ x \in \partial\Omega_{i_1} \cap \partial\Omega_{i_2}\}. \tag{2.2}$$

The unit outward normals to Ω_{i_1} and Ω_{i_2} are denoted by n_1 and n_2. We shall also denote with n the two-valued field on Γ such that, for $x \in \partial\Omega_{i_1} \cap \partial\Omega_{i_2}$, $n|_{\Omega_{i_j}} = n_j$, $j \in \{i_1, i_2\}$. The following convention will be used throughout the rest of this work: For all $x \in \Gamma$, the two indices i_1, i_2 such that $x \in \partial\Omega_{i_1} \cap \partial\Omega_{i_2}$ are chosen such that $(n^t\nu n)(x)|_{\Omega_{i_1}} \geq (n^t\nu n)(x)|_{\Omega_{i_2}}$.

In the same spirit, for any two-valued function φ on Γ, we denote by φ_1 the value of φ which is defined on the side of Ω_{i_1} and by φ_2 the value of φ which is defined on the side of Ω_{i_2}. Mean values and jumps across Γ are defined as follows:

$$\{\varphi\} \stackrel{\text{def}}{=} \tfrac{1}{2}(\varphi_1 + \varphi_2), \qquad [\![\varphi]\!] \stackrel{\text{def}}{=} \varphi_1 - \varphi_2. \tag{2.3}$$

Since we are not requiring that ν be uniformly positive definite, the mathematical nature of the PDE can change over the domain. To account for this, we define

$$\Gamma \supset I \stackrel{\text{def}}{=} \{x \in \Gamma;\ (n^t \nu n)(x)|_{\Omega_{i_1}} > 0 \ \text{ and } \ (n^t \nu n)(x)|_{\Omega_{i_2}} = 0\}. \tag{2.4}$$

The following simple Lemma will be frequently invoked in the paper:

LEMMA 2.1. *Let ν be a $d \times d$ positive semi-definite matrix, then*

$$\forall r \in \mathbb{R}^d, \quad (\nu r = 0) \Leftrightarrow (r^t \nu r = 0).$$

For any x in I, we refer to Ω_{i_1} as the *elliptic side* of I at x and we refer to Ω_{i_2} as the *hyperbolic side* of I at x. Observe that the terms elliptic and hyperbolic are used in a broad sense. Indeed, the diffusivity may not be positive definite in the elliptic side, but still have a non-zero component in the normal direction and viceversa for a hyperbolic side.

Let $\kappa \stackrel{\text{def}}{=} \nu^{1/2}$. From the assumptions on ν it follows that also κ is bounded and positive semi-definite. We now rewrite (2.1) in mixed form by introducing the auxiliary unknown σ so that

$$\begin{cases} \sigma + \kappa \nabla u = 0, & \text{in } \Omega^\dagger, \\ \nabla \cdot (\kappa \sigma + \beta u) + \mu u = f, & \text{in } \Omega, \end{cases} \tag{2.5}$$

and we require the following continuity property to hold

$$[\![u]\!] = 0 \quad \text{on } I^+. \tag{INT1}$$

The new symbols appearing in the above equations are defined as follows:

$$\Omega^\dagger \stackrel{\text{def}}{=} \Omega \setminus I, \qquad I^+ \stackrel{\text{def}}{=} \{x \in I;\ \beta \cdot n_1 > 0\}, \qquad I^- \stackrel{\text{def}}{=} \{x \in I;\ \beta \cdot n_1 < 0\}. \tag{2.6}$$

The reader is referred to Figures 6.1(a)–6.1(b) for some examples. Observe that σ is only defined in $\Omega^\dagger$. Indeed, u may be discontinuous across I, in which case $\kappa \nabla u$ can not be defined in a distributional sense. According to (INT1), the continuity of u across I is only demanded on the portion of I where the advection field flows from the elliptic side to the hyperbolic side. Also, since the second equation in (2.5) holds in the whole domain Ω,

$$\{(\kappa \sigma + \beta u) \cdot n\} = 0 \quad \text{on } \Gamma. \tag{INT2}$$

Similarly, the first equation in (2.5) implies that, formally, $[\![u]\!] = 0$ across $\Gamma \setminus I$. By combining (INT1)–(INT2) on I^+ and using Lemma 2.1 together with the continuity of β, we observe that (INT1) amounts to enforcing $n_1^t \kappa_1 \nabla u_1 = 0$ on I^+.

2.2. Asymptotic justification. In one space dimension, (INT1)–(INT2) yield the interface conditions derived by Gastaldi and Quarteroni in [16] and used in [7, 12]. These conditions are deduced by considering the following regularized problem supplemented with suitable boundary conditions:

$$(-\nu u_\epsilon' + \beta u_\epsilon)' + \mu u_\epsilon - \epsilon u_\epsilon'' = f.$$

Under the hypothesis that β is a non-zero constant, it is proved that, as $\epsilon \to 0$, u_ϵ converges in $L^2(\Omega)$ to the so-called viscosity solution of (2.5) which satisfies (INT1)–(INT2).

As an example, consider $\Omega = (0,1)$ partitioned into $\Omega_1 \stackrel{\text{def}}{=} (0, \frac{1}{3})$, $\Omega_2 \stackrel{\text{def}}{=} (\frac{1}{3}, \frac{2}{3})$, $\Omega_3 \stackrel{\text{def}}{=} (\frac{2}{3}, 1)$. Take $f = 0$, $\mu = 0$, $\beta = 1$, and set $\nu|_{\Omega_1 \cup \Omega_3} = 1$ and $\nu|_{\Omega_2} = \epsilon$. The viscosity solution of (2.5) corresponding to the Dirichlet boundary conditions $u(0) = 1$, $u(1) = 0$ is

$$u|_{\Omega_1} = 1, \quad u|_{\Omega_2} = 1, \quad u|_{\Omega_3} = 1 - e^{(x-1)}. \tag{2.7}$$

It can be verified that this solution satisfies (INT1)-(INT2), so that u is continuous at $x = \frac{1}{3}$ and discontinuous at $x = \frac{2}{3}$.

Let us mention at this point that there is a theoretical difficulty in the above regularization process if the advection field is zero and $\mu = 0$. In this case, the limit solution can be shown to be

$$u|_{\Omega_1} = 1, \quad u|_{\Omega_2} = 1, \quad u|_{\Omega_3} = 3(1-x). \tag{2.8}$$

Comparing (2.8) with (2.7), we conclude that the limit process $\lim_{\epsilon\to 0, \beta\to 0}$ is not uniform.

In higher space dimensions, we assume that (INT1)-(INT2) can be obtained by means of a regularization process and there is no ambiguity on the limit provided $\mu + \frac{1}{2}\nabla{\cdot}\beta \geq \mu_0 > 0$. The goal of the present paper is *not* to justify (INT1)-(INT2) but to show that these conditions yield a well-posed problem which we propose to solve approximately using a DG method.

2.3. The functional setting. We now cast the above problem in an appropriate functional setting. To this end, we set

$$L_u \stackrel{\text{def}}{=} L^2(\Omega), \quad L_\sigma \stackrel{\text{def}}{=} [L^2(\Omega^\dagger)]^d, \quad L \stackrel{\text{def}}{=} L_\sigma \times L_u.$$

For every element $z \in L$, we denote by $z^\sigma \in L_\sigma$ and $z^u \in L_u$ the two components of z induced by the decomposition $L = L_\sigma \times L_u$. We additionally require the following density assumption to hold:

$$\mathfrak{S} \stackrel{\text{def}}{=} \{\tau \in L_\sigma;\ \kappa\tau \in [\mathfrak{D}(\Omega^\dagger)]^d\} \text{ is dense in } L_\sigma. \tag{2.9}$$

Many relevant problems satisfy this hypothesis. Let

$$W \stackrel{\text{def}}{=} \{z \in L;\ \kappa\nabla z^u \in L_\sigma, \nabla{\cdot}(\kappa z^\sigma + \beta z^u) \in L_u\},$$

where all the derivatives are understood in the weak sense, and consider the following operators:

$$\begin{aligned} K &: L \ni z \mapsto (z^\sigma, \mu z^u) \in L, \\ A &: W \ni z \mapsto (\kappa\nabla z^u, \nabla{\cdot}\Phi(z)) \in L, \end{aligned}$$

where, for all $y \in W$, $\Phi(y) \stackrel{\text{def}}{=} \kappa y^\sigma + \beta y^u$. When equipped with the following norm:

$$\|y\|_W^2 \stackrel{\text{def}}{=} \|y\|_L^2 + \|Ay\|_L^2,$$

W is clearly a Hilbert space, $K \in \mathcal{L}(L; L)$ and $A \in \mathcal{L}(W; L)$. We refer to W as the graph space of A and the norm of W is called the graph norm. Note that functions in W satisfy (INT2) but not necessarily (INT1). We shall also make use of the formal adjoint of A, say $\tilde{A} \in \mathcal{L}(W; L)$:

$$\tilde{A} : W \ni z \mapsto (-\kappa\nabla z^u, (\nabla\cdot\beta)z^u - \nabla\cdot\Phi(z)) \in L.$$

2.4. Boundary operators. Following [9, 11], we consider the operator $D : W \longrightarrow W'$ defined by

$$\langle Dz, y\rangle_{W',W} \stackrel{\text{def}}{=} (Az, y)_L - (z, \tilde{A}y)_L.$$

Clearly, $D \in \mathcal{L}(W; W')$ and D is a boundary operator in the following sense:

LEMMA 2.2. *For all $(z, y) \in W \times W$ smooth enough for the integrals to make sense,*

$$\langle Dz, y\rangle_{W',W} = \int_{\partial\Omega} [\Phi(z)\cdot n y^u + \Phi(y)\cdot n z^u - (\beta\cdot n) z^u y^u] - \int_I (\beta\cdot n_1)[\![z^u]\!][\![y^u]\!] \quad (2.10)$$

Proof. Integrating by parts over $\Omega^\dagger$ yields

$$\langle Dz, y\rangle_{W',W} = \int_I 2\left\{z^u n^t\kappa y^\sigma + y^u n^t\kappa z^\sigma + (\beta\cdot n) z^u y^u\right\} + \int_{\partial\Omega} \left[z^u n^t\kappa y^\sigma + y^u n^t\kappa z^\sigma + (\beta\cdot n) z^u y^u\right].$$

We conclude using the fact that on I, $n_1^t\kappa_1 z_1^\sigma = -\beta\cdot n_1[\![z^u]\!]$ and $n_2^t\kappa_2 = 0$, so that $2\{z^u n^t\kappa y^\sigma + y^u n^t\kappa z^\sigma + \beta\cdot n z^u y^u\} = -(\beta\cdot n_1)[\![z^u]\!]y_1^u - (\beta\cdot n_1)[\![z^u]\!]y_1^u + (\beta\cdot n_1)y_1^u z_1^u + (\beta\cdot n_2)y_2^u z_2^u = -(\beta\cdot n_1)[\![z^u]\!][\![y^u]\!]$. ∎

In other words, if z and y are smooth enough, D admits the following integral representation:

$$\langle Dz, y\rangle_{W',W} = \int_{\partial\Omega} z^t\mathcal{D}y - \int_I (\beta\cdot n_1)[\![z^u]\!][\![y^u]\!], \quad \mathcal{D} \stackrel{\text{def}}{=} \left[\begin{array}{c|c} 0 & \kappa n \\ \hline (\kappa n)^t & \beta\cdot n \end{array}\right],$$

When the traces of z and y are not in $L^2(\partial\Omega \cup I)$, the above integrals have to be understood in some duality sense that we do not try to identify here.

Still following [9, 11], we assume that there is a second boundary operator M defined as follows for all $(z, y) \in W \times W$ smooth enough for the integrals to make sense:

$$\langle Mz, y\rangle_{W',W} \stackrel{\text{def}}{=} \int_{\partial\Omega} [\alpha\Phi(z)\cdot n y^u - \alpha\Phi(y)\cdot n z^u + |\beta\cdot n| z^u y^u] + \int_I |\beta\cdot n|[\![z^u]\!][\![y^u]\!], \quad (2.11)$$

with $\alpha \in \{-1, +1\}$. The choice $\alpha = +1$ (resp., $\alpha = -1$) is used to enforce Dirichlet (resp., Neumann) boundary conditions. The operator M is also used to enforce (INT1); see Lemma 2.3 below. Furthermore, (2.11) can be rewritten as

$$\langle Mz, y\rangle_{W',W} = \int_{\partial\Omega} z^t\mathcal{M}y + \int_I |\beta\cdot n|[\![z^u]\!][\![y^u]\!], \quad \mathcal{M} \stackrel{\text{def}}{=} \left[\begin{array}{c|c} 0 & -\alpha\kappa n \\ \hline \alpha(\kappa n)^t & |\beta\cdot n| \end{array}\right],$$

The adjoint of M is defined by

$$\langle M^* y, z\rangle_{W',W} \stackrel{\text{def}}{=} \langle Mz, y\rangle_{W',W}, \quad \forall (y,z) \in W \times W. \tag{2.12}$$

2.5. Boundary conditions strongly enforced. We are now in a suitable position to introduce the following two subspaces of W, which we equip with the graph norm $\|\cdot\|_W$:

$$V \stackrel{\text{def}}{=} \operatorname{Ker}(M - D), \quad V^* \stackrel{\text{def}}{=} \operatorname{Ker}(M^* + D).$$

We henceforth assume that V and V^* are such that

$$D(V)^\perp = V^*, \qquad D(V^*)^\perp = V,$$

where for all $E \subset W'$, $E^\perp$ is the polar set of E composed of all the elements of W that are in the kernel of all the linear forms in E. This hypothesis essentially asserts the existence of surjective trace operators on I and $\partial\Omega$ and allows to prove the following

Lemma 2.3. *The following characterizations of V and V^* hold:*

(i) If $\alpha = +1$,

$$V = \{w \in W;\ [\![w^u]\!]|_{I^+} = 0,\ w^u|_{\{x \in \partial\Omega;\ \kappa n \neq 0 \text{ or } \beta\cdot n < 0\}} = 0\},$$
$$V^* = \{w \in W;\ [\![w^u]\!]|_{I^-} = 0,\ w^u|_{\{x \in \partial\Omega;\ \kappa n \neq 0 \text{ or } \beta\cdot n > 0\}} = 0\};$$

(ii) If $\alpha = -1$,

$$V = \{w \in W;\ [\![w^u]\!]|_{I^+} = 0,\ \Phi(w)\cdot n = \tfrac{1}{2}(\beta\cdot n + |\beta\cdot n|) w^u\},$$
$$V^* = \{w \in W;\ [\![w^u]\!]|_{I^-} = 0,\ \Phi(w)\cdot n = \tfrac{1}{2}(\beta\cdot n - |\beta\cdot n|) w^u\}.$$

Consider the bilinear forms $a_0 \in \mathcal{L}(W \times L; \mathbb{R})$, $a_0^* \in \mathcal{L}(W \times L; \mathbb{R})$ such that

$$a_0(z,y) \stackrel{\text{def}}{=} (Kz, y)_L + (Az, y)_L, \qquad \forall (z,y) \in W \times L, \tag{2.13}$$
$$a_0^*(z,y) \stackrel{\text{def}}{=} (Kz, y)_L + (\tilde{A}z, y)_L, \qquad \forall (z,y) \in W \times L. \tag{2.14}$$

Lemma 2.4 (L-coercivity). *a_0 and a_0^* are L-coercive in the following sense:*

$$\forall y \in V, \quad a_0(y,y) \geq \|y^\sigma\|^2_{L_\sigma} + \mu_0 \|y^u\|^2_{L_u} + \tfrac{1}{2}\|[\![y^u]\!]\|^2_{L^2(|\beta\cdot n|; I \cup \partial\Omega)}. \tag{2.15}$$
$$\forall y \in V^*, \quad a_0^*(y,y) \geq \|y^\sigma\|^2_{L_\sigma} + \mu_0 \|y^u\|^2_{L_u} + \tfrac{1}{2}\|[\![y^u]\!]\|^2_{L^2(|\beta\cdot n|; I \cup \partial\Omega)}. \tag{2.16}$$

Proof. Using the definition of D and V, we infer, for all $y \in V$,

$$\begin{aligned}((K + A)y, y)_L &= ((K + \tfrac{1}{2}(A + \tilde{A})y, y)_L + (\tfrac{1}{2}(A - \tilde{A})y, y)_L \\ &= \|y^\sigma\|^2_{L_\sigma} + ((\mu + \tfrac{1}{2}\nabla\cdot\beta) y^u, y^u)_{L_u} + \tfrac{1}{2}\langle My, y\rangle_{W',W}.\end{aligned}$$

The desired result then follows from the construction of M. Proceed similarly to prove (2.16). □

Consider the following problem: For $f \in L_u$,

$$\begin{cases} \text{Find } \tilde{z} \in V \text{ such that, for all } y \in L, \\ a_0(\tilde{z}, y) = (f, y^u)_{L_u}. \end{cases} \tag{2.17}$$

THEOREM 2.5. *The problem* (2.17) *is well-posed.*

Proof. According to the so-called Banach-Nečas-Babuška (BNB) theorem stated in [8, §2.1.3], the statement amounts to proving that the following conditions hold:

$$\forall z \in V, \qquad \sup_{y\in L\setminus\{0\}} \frac{a_0(z,y)}{\|y\|_L} \gtrsim \|z\|_V, \tag{BNB1}$$

$$\forall z \in V, \quad (\forall y \in L, \quad a_0(z,y) = 0) \Longrightarrow (y = 0). \tag{BNB2}$$

(i) Let us prove (BNB1). Let $z \in V$ and set $\mathbb{S} \stackrel{\text{def}}{=} \sup_{y\in L\setminus\{0\}} \frac{a_0(z,y)}{\|y\|_L}$. Using the definition of the L^2-norm, we deduce

$$\mathbb{S} \gtrsim \sup_{y\in L\setminus\{0\}} \frac{(Az,y)_L}{\|y\|_L} - \|z\|_L \gtrsim \|Az\|_L - \|z\|_L.$$

Then Lemma 2.4 gives

$$\|z\|_L \lesssim \frac{a_0(z,z)}{\|z\|_L} \lesssim \mathbb{S} \Longrightarrow \|z\|_L + \|Az\|_L \lesssim \mathbb{S}.$$

i.e., $\|z\|_V \lesssim \mathbb{S}$, which proves (BNB1).

(ii) Let us prove (BNB2). Assume that $y \in L$ is such that $a_0(z,y) = 0$ for all $z \in V$. (1) Take $z = (z^\sigma, 0)$ with $z^\sigma \in \mathfrak{S}$ and observe that z is a member of V. Then using z to test (2.13) we obtain

$$a_0((z^\sigma,0),y) = \langle y^\sigma - \kappa\nabla y^u, z^\sigma\rangle_{[\mathfrak{D}(\Omega^\dagger)]^d} = 0, \quad \forall z^\sigma \in \mathfrak{S},$$

meaning that $y^\sigma - \kappa\nabla y^u = 0$ in $\Omega^\dagger$ owing to the density hypothesis (2.9). This equality implies that $\kappa\nabla y^u \in L_\sigma$. (2) Use $z = (0, z^u)$ with $z^u \in \mathfrak{D}(\Omega)$ as a test function in (2.13) and observe that again z is a member of V. A distributional argument gives

$$\langle(\mu + \nabla{\cdot}\beta)y^u - \nabla{\cdot}(\kappa y^\sigma + \beta y^u), z^u\rangle_{\mathfrak{D}(\Omega)} = 0.$$

Owing to the regularity assumptions on μ and β listed in §2.1 we conclude that $\nabla{\cdot}(\kappa y^\sigma + \beta y^u) \in L^2(\Omega)$, i.e., y is a member of W and

$$(K + \tilde{A})y = 0.$$

(3) We then deduce that, for all $z \in V$,

$$\langle Dz, y\rangle_{W',W} = ((K + A)z, y)_L - ((K + \tilde{A})y, z)_L = 0,$$

i.e., y is a member of $D(V)^\perp = V^*$. In conclusion $a_0^*(y,w) = 0$ for all $w \in L$ and $y \in V^*$. Finally, the L-coercivity of a_0^* (see Lemma 2.4) implies that $y = 0$. ∎

2.6. Boundary and interface conditions weakly enforced. Having in mind that in DG methods boundary conditions are weakly enforced, we introduce the following bilinear form:

$$a(z,y) \stackrel{\text{def}}{=} a_0(z,y) + \tfrac{1}{2}\langle(M - D)z, y\rangle_{W',W}, \qquad \forall (z,y) \in W \times W. \tag{2.18}$$

Clearly, all the terms above are well-defined and $a \in \mathcal{L}(W \times W; \mathbb{R})$. In what follows we shall consider the following problem: For $f \in L_u$,

$$\begin{cases} \text{Find } z \in W \text{ such that, for all } y \in W, \\ a(z,y) = (f, y^u)_{L_u}. \end{cases} \tag{2.19}$$

THEOREM 2.6 (Well-posedness). *Problem* (2.19) *is well-posed and the solutions to* (2.17) *and* (2.19) *coincide.*

Proof. It is clear that the unique solution to (2.17) solves (2.19). Moreover, for all $y \in W$,

$$\begin{aligned} a(y,y) &= ((K+A)y,y)_L + \tfrac{1}{2}\langle (M-D)y,y\rangle_{W',W} \\ &= ((K+\tfrac{1}{2}(A+\tilde{A}))y,y)_L + \tfrac{1}{2}((A-\tilde{A})y,y)_L + \tfrac{1}{2}\langle (M-D)y,y\rangle_{W',W} \\ &\geq \|z^\sigma\|^2_{L_\sigma} + \mu_0\|z^u\|^2_{L_u} + \tfrac{1}{2}\|[\![z^u]\!]\|^2_{L^2(|\beta\cdot n|;I\cup\partial\Omega)}, \end{aligned}$$

that is to say, a is L-coercive. This immediately implies that the solution to (2.19) is unique. □

3. The discrete problem. In this section we develop the DG approximation of our model problem following a constructive approach.

3.1. The discrete setting. Let $\{\mathcal{T}_h\}_{h>0}$ be a family of affine meshes of Ω compatible with the partition P_Ω, which, for simplicity of exposition, is supposed to be made up of polyhedra. Elements are not necessarily simplices and the matching of interfaces is not required. We denote by $\mathcal{F}_h^i$ the set of element interfaces, i.e., $F \in \mathcal{F}_h^i$ if F is a $(d-1)$-manifold and there are T_1, $T_2 \in \mathcal{T}_h$ such that $F = \partial T_1 \cap \partial T_2$. The set of the faces that separate the mesh from the exterior of Ω is denoted with $\mathcal{F}_h^\partial$, i.e., $F \in \mathcal{F}_h^\partial$ if F is a $(d-1)$-manifold and there is $T \in \mathcal{T}_h$ such that $F = \partial T \cap \partial\Omega$. The set of all the faces is denoted with $\mathcal{F}_h$, i.e., $\mathcal{F}_h \stackrel{\text{def}}{=} \mathcal{F}_h^i \cup \mathcal{F}_h^\partial$. Moreover, for a given face $F \in \mathcal{F}_h$, we introduce the set $\mathcal{T}_h(F) \stackrel{\text{def}}{=} \{T \in \mathcal{T}_h;\ F \subset \partial T\}$. The diameters of $T \in \mathcal{T}_h$, $F \in \mathcal{F}_h$, and $\mathcal{T}_h(F)$ are denoted by h_T, h_F, and $h_{\mathcal{T}_h(F)}$, respectively. Without loss of generality, we assume that $h \leq 1$.

For a non-negative integer p, we define the space of scalar-valued polynomial functions possibly discontinuous across element faces, that is

$$P_{h,p} \stackrel{\text{def}}{=} \{v_h \in L^2(\Omega);\ \forall T \in \mathcal{T}_h,\ v_h|_T \in \mathbb{P}_p(T)\}, \tag{3.1}$$

where $\mathbb{P}_p(T)$ denotes the set of d-variate polynomials of total degree at most p on T. Let p_u and p_σ two non-negative integers such that $p_u - 1 \leq p_\sigma \leq p_u$ and define the following spaces:

$$\Sigma_h = [P_{h,p_\sigma}]^d, \qquad U_h = P_{h,p_u}, \qquad W_h = \Sigma_h \times U_h.$$

According to the assumptions listed in §2.1 and since $\{\mathcal{T}_h\}_{h>0}$ is compatible with P_Ω, we have that

$$\nu \in [P_{h,0}]^{d,d} \text{ and } \kappa \in [P_{h,0}]^{d,d}. \tag{3.2}$$

As in the continuous case, the behavior of the solution across an interface is determined by the diffusion in the normal direction. For $\mathcal{F}_h^i \ni F = \partial T_1 \cap \partial T_2$ we then define the two-valued field

$$\lambda_i \stackrel{\text{def}}{=} \sqrt{n^t \nu n|_{T_i}}, \quad i \in \{1,2\}, \tag{3.3}$$

where we denote with n the two-valued field on F such that $n|_{T_j} = n_j$, $j \in \{1,2\}$. Without loss of generality, we shall always assume that the index $i \in \{i_1, i_2\}$ is chosen so that $\lambda_1 \geq \lambda_2$. Similarly, for a boundary face $F \in \mathcal{F}_h^\partial$, we let $\lambda \stackrel{\text{def}}{=} \sqrt{n^t \nu n}$.

The mesh family $\{\mathcal{T}_h\}_{h>0}$ is assumed regular in the sense that

$$h_{\mathcal{T}_h(F)} \lesssim h_F, \tag{3.4}$$

$$\|\nabla v_h\|_{[L^2(T)]^d} \lesssim h_T^{-1} \|v_h\|_{L^2(T)}, \quad \forall T \in \mathcal{T}_h,\ \forall v_h \in P_{h,p} \tag{3.5}$$

$$\|v_h\|_{L^2(F)} \lesssim h_F^{-1/2} \|v_h\|_{L^2(\mathcal{T}_h(F))}, \quad \forall F \in \mathcal{F}_h,\ \forall v_h \in P_{h,p}. \tag{3.6}$$

The inverse and trace inequalities (3.5) and (3.6) can be applied component-wise to the functions in Σ_h.

3.2. Design of the DG bilinear form. The goal of this section is to construct a discrete DG counterpart of the bilinear form a defined in (2.18) matching the following constraints: (i) it should satisfy a discrete version of Lemma 2.4 (L-coercivity) and be strongly consistent. Moreover, (ii) it should *not* require the elliptic-hyperbolic interface I to be identified *a priori*. Indeed, since computers work in finite precision arithmetic, it may happen in practice that $n^t \nu n$ takes a small value instead of being exactly zero, so that I is possibly difficult to identify; (iii) it should include suitable stabilizing terms to weakly enforce boundary and interface conditions.

Let $H^s(\mathcal{T}_h) \stackrel{\text{def}}{=} \{v \in L^2(\Omega);\ v \in H^s(T),\ \forall T \in \mathcal{T}_h\}$ equipped with the usual broken Sobolev norm denoted by $\|\cdot\|_{H^s(\mathcal{T}_h)}$ and define

$$W(h) \stackrel{\text{def}}{=} W \cap [H^1(\mathcal{T}_h)]^{d+1} + W_h.$$

Let $\Gamma_h \stackrel{\text{def}}{=} \bigcup_{F \in \mathcal{F}_h^i} F$. We introduce a two-valued weight function ω such that

$$[L^2(\Gamma_h)]^2 \ni \omega = (\omega_1, \omega_2), \quad \omega_1 + \omega_2 = 1 \text{ for a.e. } x \in \Gamma_h. \tag{3.7}$$

For all $y \in W(h)$, y^u admits a (possibly two-valued) trace on every $F \in \mathcal{F}_h^i$. Then, for all $\mathcal{F}_h^i \ni F = \partial T_1 \cap \partial T_2$ and for a.e. $x \in F$ we define the weighted average and weighted jump as follows:

$$\{y^u\}_\omega \stackrel{\text{def}}{=} \omega_1 y_1^u + \omega_2 y_2^u, \quad [\![y^u]\!]_\omega \stackrel{\text{def}}{=} 2(\omega_2 y_1^u - \omega_1 y_2^u), \tag{3.8}$$

where, for a.e. $x \in F$, $y_i^u(x) = \lim_{y \to x} y^u(y)|_{T_i}$, $i \in \{1, 2\}$. When $\omega = (\frac{1}{2}, \frac{1}{2})$, the usual average and jump operators are recovered and subscripts are omitted. The normal trace of $\Phi(y)$ is also well-defined on all $F \in \mathcal{F}_h^i$, and similar definitions for $\{\Phi(y)\cdot n\}_\omega$ and $[\![\Phi(y)\cdot n]\!]_\omega$ can be introduced. Furthermore, the following algebraic formula holds:

$$\{ab\} = \{a\}\,\{b\}_\omega + \tfrac{1}{4}[\![a]\!]_\omega [\![b]\!]. \tag{3.9}$$

Let $\mathcal{I}_h$ denote the discrete counterpart of the manifold I, i.e.,

$$\mathcal{I}_h \stackrel{\text{def}}{=} \{F \in \mathcal{F}_h^i;\ \lambda_1 > 0 \text{ and } \lambda_2 = 0\}.$$

To facilitate the discussion, we suppose that the sign of $\beta\cdot n$ is constant on every interface F belonging to $\mathcal{F}_h^i$. In such a case, we can identify two subsets of $\mathcal{I}_h$, say $\mathcal{I}_h^+$ and $\mathcal{I}_h^-$, which represent discrete versions of I^+ and I^-. The sets $\mathcal{I}_h$, $\mathcal{I}_h^\pm$ and the above assumption will eventually turn out to be unnecessary. They are introduced to help the reader follow the design of the DG bilinear form.

Let us now introduce the discrete bilinear form associated with the boundary contributions of M. For all $F \in \mathcal{F}_h^\partial$, define a self-adjoint operator M_F such that, for all $(z, y) \in W(h) \times W(h)$,

$$(M_F(z), y)_{L,F} \stackrel{\text{def}}{=} -\alpha(z^u, \kappa y^\sigma\cdot n)_{L_u,F} + \alpha(\kappa z^\sigma\cdot n, y^u)_{L_u,F} + (M_F^{uu}(z^u), y^u)_{L_u,F}.$$

Moreover, M_F is assumed to satisfy the following consistency conditions:

$$\mathrm{Ker}(\mathcal{M}-\mathcal{D}) \subset \mathrm{Ker}(M_F-\mathcal{D}), \qquad \mathrm{Ker}(\mathcal{M}+\mathcal{D}) \subset \mathrm{Ker}(M_F+\mathcal{D}). \tag{3.10}$$

The correct form for M_F^{uu} will be discussed later. For the moment being, we only require that M_F^{uu} be non-negative so that it makes sense to define

$$|y^u|_{M,F}^2 \stackrel{\text{def}}{=} (M_F^{uu}(y^u), y^u)_{L_u,F}, \qquad |y^u|_M^2 \stackrel{\text{def}}{=} \textstyle\sum_{F\in\mathcal{F}_h^\partial} |y^u|_{M,F}^2,$$

The $|\cdot|_M$ seminorm will be used to measure the error due to the weak enforcement of boundary conditions.

As a starting point, let us consider the bilinear form that should be used if the diffusivity were strictly positive definite and if H^1-conforming elements were used (see [8]): For all $(z,y)\in W(h)\times W(h)$, set

$$a_h^{(-1)}(z,y) \stackrel{\text{def}}{=} \sum_{T\in\mathcal{T}_h} [(Kz,y)_{L,T} + (Az,y)_{L,T}] + \tfrac{1}{2}\sum_{F\in\mathcal{F}_h^\partial} ((M_F-\mathcal{D})z,y)_{L,F}. \tag{3.11}$$

Obviously, this bilinear form is not suitable, since the diffusivity is semi-definite and non-conforming elements are used. Instead we should consider the discrete counterpart of the bilinear form a defined in (2.18). Accordingly, we define $a_h^{(0)}$ such that, for all $(z,y)\in W(h)\times W(h)$,

$$a_h^{(0)}(z,y) \stackrel{\text{def}}{=} a_h^{(-1)}(z,y) + \sum_{F\in\mathcal{I}_h^+} ((\beta\cdot n)[\![z^u]\!], [\![y^u]\!])_{L_u,F}. \tag{3.12}$$

This bilinear form is not suitable either. In particular, it is not L-coercive. To regain L-coercivity we have to account for jumps across element interfaces.

Let us assume for the time being that we dispose of a weight function ω satisfying (3.7), and, for all $F\in\mathcal{F}_h^i$, define the bilinear $\chi_{F,\omega}$ such that, for all $F\in\mathcal{F}_h^i$,

$$\chi_{F,\omega}(z,y) \stackrel{\text{def}}{=} (\{\Phi(z)\cdot n\}, \{y^u\}_\omega)_{L_u,F} + ([\![z^u]\!], \tfrac{1}{4}[\![\Phi(y)\cdot n]\!]_\omega - \tfrac{\beta\cdot n_1}{2}\{y^u\})_{L_u,F}. \tag{3.13}$$

Then the following discrete analogous of the integration by parts formula proved in Lemma 2.2 holds:

$$\sum_{T\in\mathcal{T}_h}[(Az,y)_{L,T}-(z,\tilde{A}y)_{L,T}]=\sum_{F\in\mathcal{F}_h^\partial}(\mathcal{D}z,y)_{L,F}+\sum_{F\in\mathcal{F}_h^i}2\,[\chi_{F,\omega}(z,y)+\chi_{F,\omega}(y,z)]\,. \tag{3.14}$$

Indeed, let LHS the left-hand side of (3.14) and observe that

$$LHS = \sum_{F\in\mathcal{F}_h^\partial}(\mathcal{D}z,y)_{L,F} + 2\sum_{F\in\mathcal{F}_h^i}\int_F [\{\Phi(z)\cdot n y^u\} + \{\Phi(y)\cdot n z^u\} - \{(\beta\cdot n) z^u y^u\}]\,.$$

Apply (3.9) to the averages involving $\Phi(z)$ and $\Phi(y)$, and observe that, owing to the definition of the unweighted jump and average operators, we have $\{(\beta\cdot n)z^u y^u\} = \frac{\beta\cdot n_1}{2}[\![z^u]\!]\,\{y^u\} + \frac{\beta\cdot n_1}{2}[\![y^u]\!]\,\{z^u\}$.

To get a hint at what should be done to regain L-coercivity, let $y\in W(h)$ and proceed as in the proof of Lemma 2.4 to obtain the following expression for $a_h^{(0)}(y,y)$:

$$\begin{aligned} a_h^{(0)}(y,y) &= \sum_{T\in\mathcal{T}_h}(K+\tfrac{1}{2}(A+\tilde{A})y,y)_{L,T} + \tfrac{1}{2}\sum_{F\in\mathcal{F}_h^\partial}(M_F y,y)_{L,F} \\ &\quad + \sum_{F\in\mathcal{I}_h^+}(\beta\cdot n[\![y^u]\!],[\![y^u]\!])_{L_u,F} + \sum_{F\in\mathcal{F}_h^i}[\chi_{F,\omega}(y,y)+\chi_{F,\omega}(y,y)]. \end{aligned}$$

This expression suggests that, in order to ensure L-coercivity in a consistent way, the following bilinear form should be considered:

$$a_h^{(1)}(z,y) \stackrel{\text{def}}{=} a_h^{(0)}(z,y) - 2\sum_{F\in\mathcal{F}_h^i} \chi_{F,\omega}(z,y). \tag{3.15}$$

Indeed, in the expression of $\chi_{F,\omega}(z,y)$, the unknown z appears in terms that are consistent on all $F\in\mathcal{F}_h^i\backslash\mathcal{I}_h$ according to (INT1)–(INT2). As we shall see, an additional condition on the weighting function ω will ensure consistency on all $F\in\mathcal{I}_h$. With such a choice, it is clear that, for all $y\in W(h)$ and for all ω satisfying (3.7),

$$a_h^{(1)}(y,y) \geq \|y^\sigma\|_{L_\sigma}^2 + \mu_0\|y^u\|_{L_u}^2 + \tfrac{1}{2}|y^u|_M^2 + \tfrac{1}{4}\|[\![y^u]\!]\|_{L^2(|\beta\cdot n|;\mathcal{I}_h^+)}^2.$$

We now turn our attention to the strong consistency requirement. We want to make sure that whenever the first argument of the discrete form is a member of $V\cap W(h)$, all the terms that are not related to $a_h^{(0)}$ disappear. From this point on, we shall suppose that the weight function is designed so that

$$\forall F\in\mathcal{I}_h,\ \forall x\in F,\quad \omega(x)=(1,0). \tag{3.16}$$

This implies that, for all F in $\mathcal{I}_h$ and all $y^u\in U_h$, $\{y^u\}_\omega = y_1^u$, i.e.,

$$\tfrac{1}{4}[\![\Phi(y)\cdot n]\!]_\omega - \tfrac{\beta\cdot n_1}{2}\{y^u\} = \tfrac{\beta\cdot n_1}{2}y^u|_{T_2} - \tfrac{\beta\cdot n_1}{2}\{y^u\} = -\tfrac{\beta\cdot n_1}{2}[\![y^u]\!].$$

Then the following simplification occurs: For all $(z,y)\in W(h)\times W_h$,

$$\chi_{F,\omega}(z,y) = (\{\Phi(z)\cdot n\}, y^u|_{T_1(F)})_{L_u,F} - (\tfrac{\beta\cdot n_1}{4}[\![z^u]\!],[\![y^u]\!])_{L_u,F}.$$

As a result, whenever z is a member of $V\cap W(h)$, we obtain that for all $y_h\in W_h$,

$$a_h^{(1)}(z,y_h) = a_0(z,y_h) + \sum_{F\in\mathcal{I}_h^-}(\tfrac{\beta\cdot n_1}{2}[\![z^u]\!],[\![y^u]\!])_{L_u,F}.$$

Since z may possibly jump across $\mathcal{I}_h^-$, the last term in the right-hand side is clearly inconsistent. To remedy this, consider

$$a_h^{(2)}(z,y) \stackrel{\text{def}}{=} a_h^{(1)}(z,y) - \sum_{F\in\mathcal{I}_h^-}(\tfrac{\beta\cdot n_1}{2}[\![z^u]\!],[\![y^u]\!])_{L_u,F}. \tag{3.17}$$

Since $\beta\cdot n_1 \leq 0$ on $\mathcal{I}_h^-$, the extra term reinforces the L-coercivity of $a_h^{(1)}$. As a consequence, $a_h^{(2)}$ inherits the L-coercivity property of $a_h^{(1)}$. Moreover, it is strongly consistent in the sense that, for all $(z,y_h)\in(V\cap W(h))\times W_h$, $a_h^{(2)}(z,y_h)=a_0(z,y_h)$.

Note that $a_h^{(2)}$ requires $\mathcal{I}_h$ be *a priori* identified, which is contrary to our second design requirement. To remedy this, observe that $a_h^{(2)}$ can be rewritten as

$$a_h^{(2)}(z,y) = a_h^{(-1)}(z,y) - 2\sum_{F\in\mathcal{F}_h^i}\chi_{F,\omega}(z,y) + \sum_{F\in\mathcal{I}_h}(\tfrac{|\beta\cdot n|}{2}[\![z^u]\!],[\![y^u]\!])_{L_u,F},$$

where we recognize upwind penalty terms on $\mathcal{I}_h$. This remark suggests to consider the following bilinear form instead:

$$a_h^{(3)}(z,y) \stackrel{\text{def}}{=} a_h^{(-1)}(z,y) - 2\sum_{F\in\mathcal{F}_h^i}\chi_{F,\omega}(z,y) + \sum_{F\in\mathcal{F}_h^i}(\tfrac{|\beta\cdot n|}{2}[\![z^u]\!],[\![y^u]\!])_{L_u,F}, \tag{3.18}$$

where the sole difference with respect to (3.17) is that upwind stabilization terms are now present on all the interfaces. The terms we have added are non-negative and consistent, so that $a_h^{(3)}$ is L-coercive and strongly consistent in the sense precised above. Moreover, $\mathcal{I}_h$ *does not* appear in the definition of $a_h^{(3)}$, thus fulfilling the second design requirement.

To complete the design, it remains only to add stabilizing terms to weakly enforce boundary and interface conditions. For this purpouse we finally modify $a_h^{(3)}$ as follows:

$$a_h(z,y) \stackrel{\text{def}}{=} a_h^{(-1)}(z,y) - 2\sum_{F\in\mathcal{F}_h^i} \chi_{F,\omega}(z,y) + \sum_{F\in\mathcal{F}_h^i} (S_F([\![z^u]\!]),[\![y^u]\!])_{L_u,F},$$

where the upwind stabilization terms appearing in (3.18) are replaced by an interface operator S_F such that

$$S_F = \frac{|\beta\cdot n|}{2}, \qquad \text{for all } F\in\mathcal{F}_h \text{ such that } \lambda_2 = 0. \tag{3.19}$$

We henceforth assume that M_F and S_F are defined as follows: For all $F\in\mathcal{F}_h^\partial$ and for all $F\in\mathcal{F}_h^i$, respectively,

$$M_F^{uu} \stackrel{\text{def}}{=} \frac{|\beta\cdot n|}{2} + \frac{\alpha+1}{2}\frac{\lambda^2}{h_F}, \qquad S_F \stackrel{\text{def}}{=} \frac{|\beta\cdot n|}{2} + \frac{\lambda_2^2}{h_F}, \tag{3.20}$$

where $\alpha\in\{-1,+1\}$. Observe that λ_2 is by definition *the minimum* of λ_1 and λ_2. following the reasoning in [9, §2.5]. The choice (3.20) is clearly compatible with the design constraint (3.19). Moreover, the definition of M_F is consistent with its continuous counterpart, i.e., (3.10) holds.

To summarize, the expression of the final discrete bilinear form is

$$\begin{aligned} a_h(z,y) \stackrel{\text{def}}{=} &\sum_{T\in\mathcal{T}_h} \left[(Kz,y)_{L,T} + (Az,y)_{L,T}\right] + \tfrac{1}{2}\sum_{F\in\mathcal{F}_h^\partial} (M_F(z) - \mathcal{D}z, y)_{L,F} \\ &- 2\sum_{F\in\mathcal{F}_h^i} \Big[(\{\Phi(z)\cdot n\}, \{y^u\}_\omega)_{L_u,F} + ([\![z^u]\!], \tfrac14 [\![\Phi(y)\cdot n]\!]_\omega - \tfrac{\beta\cdot n_1}{2}\{y^u\})_{L_u,F}\Big] \\ &+ \sum_{F\in\mathcal{F}_h^i} (S_F([\![z^u]\!]),[\![y^u]\!])_{L,F}, \end{aligned} \tag{3.21}$$

with M_F and S_F defined by (3.20). To satisfy condition (3.16), the weighting function ω is chosen as follows:

$$\omega \stackrel{\text{def}}{=} \begin{cases} \left(\frac{\lambda_1}{2\{\lambda\}}, \frac{\lambda_2}{2\{\lambda\}}\right), & \text{if } \lambda_1 > 0, \\ \left(\frac12, \frac12\right), & \text{otherwise.} \end{cases} \tag{3.22}$$

Although other expressions for ω are possible, this one has the advantage of being simple and ensuring robustness of the estimates. A similar choice is made in [5, 13].

The discrete problem is now formulated as follows:

$$\begin{cases} \text{Seek } z_h\in W_h \text{ such that} \\ a_h(z_h,y_h) = (f, y_h^u)_{L_u}, \quad \forall y_h\in W_h. \end{cases} \tag{3.23}$$

Observe that the σ-component of the unknown can be eliminated locally since the jumps of this quantity across element interfaces are not penalized; see, e.g., [10, §4.4].

Remark 3.1. The use of non-symmetric weights in DG methods has been highlighted in several articles (see e.g., [14, 18, 17, 19]). Although some of the above cited works point out that the use of weights may lead to higher performance in terms of accuracy, they do not consider any connection between the weights and the coefficients of the problem. This dependency has recently been investigated in [5, 13], where the authors show that the use of a particular weighted average improves the stability of the numerical scheme in the semi-definite diffusivity limit. In the present case, resorting to weighted average and jump operators is required by our asking the method to select the proper interface conditions automatically .

4. Convergence analysis. In this section we carry out the convergence analysis of the discrete problem (3.23). The main results are Theorem 4.5 and Theorem 4.7.

4.1. Basic convergence estimates. For all $y \in W(h)$, we introduce the following seminorm:

$$|y^u|^2_{J,F} \stackrel{\text{def}}{=} (S_F([\![y^u]\!]), [\![y^u]\!])_{L_u,F}, \quad |y^u|^2_J \stackrel{\text{def}}{=} \sum_{F\in\mathcal{F}_h^i} |y^u|^2_{J,F}. \tag{4.1}$$

The space $W(h)$ is equipped with the following discrete norm:

$$\|y\|^2_{h,\kappa} \stackrel{\text{def}}{=} \|y\|^2_L + |y^u|^2_J + |y^u|^2_M + \sum_{T\in\mathcal{T}_h} \|\kappa\nabla y^u\|^2_{L_\sigma,T}. \tag{4.2}$$

The following two lemmata follow from the design procedure outlined in §3.2.

LEMMA 4.1 (Consistency). *Let z solve* (2.5) *and z_h solve* (3.23). *Assume, moreover, that $z \in [H^1(\mathcal{T}_h)]^{d+1}$. Then,*

$$\forall y_h \in W_h, \quad a_h(z - z_h, y_h) = 0.$$

LEMMA 4.2 (L-coercivity). *For all h and for all y in $W(h)$,*

$$a_h(y, y) \gtrsim \|y\|^2_L + |y^u|^2_J + |y^u|^2_M.$$

In order to estimate the L^2-norm of the diffusive derivative $\kappa\nabla z^u$ we need the following

LEMMA 4.3 (Stability). *The following bound holds:*

$$\forall z_h \in W_h, \quad \|z_h\|_{h,\kappa} \lesssim \sup_{y_h\in W_h\setminus\{0\}} \frac{a_h(z_h, y_h)}{\|y_h\|_{h,\kappa}}.$$

Proof. Let $z_h \in W_h$ and set $\mathbb{S} \stackrel{\text{def}}{=} \sup_{y_h\in W_h\setminus\{0\}} \frac{a_h(z_h,y_h)}{\|y_h\|_{h,\kappa}}$.

(1) Owing to Lemma 4.2,

$$\|z_h\|^2_L + |z_h^u|^2_M + |z_h^u|^2_J \lesssim a_h(z_h, z_h) \lesssim \mathbb{S}\|z_h\|_{h,\kappa}. \tag{4.3}$$

(2) Control of $\mathtt{B} \stackrel{\text{def}}{=} \sum_{T\in\mathcal{T}_h} \|\kappa\nabla z_h^u\|^2_{L_\sigma,T}$. Let $\pi_h^\sigma \in \Sigma_h$ be the field such that, for all $T \in \mathcal{T}_h$, $\pi_h^\sigma|_T \stackrel{\text{def}}{=} \kappa\nabla z_h^u|_T$. From the definition of a_h it follows that

$$\mathtt{B} = a_h(z_h, (\pi_h^\sigma, 0)) - (z_h^\sigma, \pi_h^\sigma)_{L_\sigma} + \sum_{F\in\mathcal{F}_h^\partial} \tfrac{1+\alpha}{2}(\kappa n z_h^u, \pi_h^\sigma)_{L_\sigma,F} + \tfrac{1}{2} \sum_{F\in\mathcal{F}_h^i} ([\![z_h^u]\!], [\![n^t\kappa\pi_h^\sigma]\!]_\omega)_{L_u,F}.$$

Let R_i, $i \in \{1,2,3\}$ denote the last three terms in the right-hand side. The first term is bounded from above as follows

$$|R_1| \leq \|z_h^\sigma\|_{L_\sigma}\|\pi_h^\sigma\|_{L_\sigma} \lesssim \|z_h^\sigma\|_{L_\sigma}^2 + \gamma \mathsf{B},$$

where γ can be chosen as small as needed.
The second term vanishes if $\alpha = -1$. If $\alpha = +1$, use trace inequality (3.6) together with (3.20) to get

$$|R_2| \lesssim \sum_{F\in\mathcal{F}_h^\partial} h_F^{-\frac{1}{2}} |(n^t\nu n z_h^u, z_h^u)_{L_u,F}|^{\frac{1}{2}} \|\pi_h^\sigma\|_{L_\sigma,\mathcal{T}_h(F)} \lesssim \sum_{F\in\mathcal{F}_h^\partial} |z_h^u|_{M,F}\|\pi_h^\sigma\|_{L_\sigma,\mathcal{T}_h(F)}.$$

Consequently, $|R_2| \lesssim |z_h^u|_M^2 + \gamma\mathsf{B}$. According to (3.22), for all $\mathcal{F}_h^i \ni F = \partial T_1 \cap \partial T_2$,

$$\|[\![n^t\kappa\pi_h^\sigma]\!]_\omega\|_{L_u,F} = \frac{\lambda_1\lambda_2}{\{\lambda\}}\|[\![(n^t\kappa/\lambda)\pi_h^\sigma]\!]\|_{L_u,F} \lesssim h_F^{-\frac{1}{2}}\frac{\lambda_1\lambda_2}{\{\lambda\}}\|\pi_h^\sigma\|_{L_\sigma,\mathcal{T}_h(F)}.$$

Using the above relation together with (3.20) yields

$$|R_3| \lesssim \sum_{F\in\mathcal{F}_h^i} \frac{\lambda_1}{\{\lambda\}}\left(\frac{\lambda_2^2}{h_F}\|[\![z_h^u]\!]\|_{L_u,F}^2\right)^{\frac{1}{2}} \|\pi_h^\sigma\|_{L_\sigma,\mathcal{T}_h(F)} \lesssim |z_h^u|_J^2 + \gamma\mathsf{B}.$$

The above bounds with $\gamma = \frac{1}{6}$ together with Lemma 4.2 give

$$\tfrac{1}{2}\sum_{T\in\mathcal{T}_h}\|\kappa\nabla z_h^\sigma\|_{L_\sigma,T}^2 \lesssim a_h(z_h,(\pi_h^\sigma,0)) + a_h(z_h,z_h) \lesssim \mathbb{S}\|z_h\|_{h,\kappa}, \tag{4.4}$$

where we used the fact that, by definition, $\|(\pi_h^\sigma,0)\|_{h,\kappa} = \|\pi_h^\sigma\|_{L_\sigma} \leq \|z_h\|_{h,\kappa}$. Observe that, owing to the choice of the weight function ω, the above estimate is robust with respect to the possible discontinuity and anisotropy of ν.

(3) Equations (4.3)–(4.4) yield $\|z_h\|_{h,\kappa}^2 \lesssim \mathbb{S}\|z_h\|_{h,\kappa}$, i.e., the desired result.
□

Let us now introduce

$$W_h^\perp \stackrel{\text{def}}{=} \{y \in W(h);\ \forall w_h \in W_h,\ (y,w_h)_L = 0\}. \tag{4.5}$$

Moreover, we define the following norm on $W(h)$:

$$\|y\|^2 \stackrel{\text{def}}{=} \|y\|_{h,\kappa}^2 + \sum_{T\in\mathcal{T}_h}\left[\frac{\mathfrak{h}_T}{h_T^2}\|y^u\|_{L_u,T}^2 + h_T\|y^\sigma\|_{L_\sigma,\partial T}^2 + \sum_{F\in\partial T}\frac{\mathfrak{h}_F}{h_F}\|y^u\|_{L_u,F}^2\right], \tag{4.6}$$

where, for all $T \in \mathcal{T}_h$ and for all $F \in \mathcal{F}_h$, we have defined

$$\mathfrak{h}_T \stackrel{\text{def}}{=} \max(\|\nu\|_{[L^\infty(T)]^{d,d}}, h_T), \quad \mathfrak{h}_F \stackrel{\text{def}}{=} \begin{cases}\max(\lambda_1^2, h_F), & \text{if } F \in \mathcal{F}_h^i,\\ \max(\lambda^2, h_F), & \text{if } F \in \mathcal{F}_h^\partial.\end{cases} \tag{4.7}$$

The last property needed to prove convergence is stated in the following
Lemma 4.4 (Continuity). *The following holds:*

$$\forall (z, y_h) \in W_h^\perp \times W_h, \quad a_h(z,y_h) \lesssim \|z\| \|y_h\|_{h,\kappa}.$$

Proof. Let $(z, y_h) \in W_h^{\perp} \times W_h$. Using the integration by parts formula (3.14), we obtain

$$a_h(z, y_h) = \sum_{T\in\mathcal{T}_h} (z, (K+\tilde{A})y_h)_{L,T} + 2\sum_{F\in\mathcal{F}_h^i} \chi_{F,\omega}(y,z) + \tfrac{1}{2}\sum_{F\in\mathcal{F}_h^\partial} (M_F(z)+\mathcal{D}z, y_h)_{L,F} + \sum_{F\in\mathcal{F}_h^i} (S_F([\![z^u]\!]), [\![y_h^u]\!])_{L_u,F}. \quad (4.8)$$

We now derive bounds for the four terms in the right-hand side, say $R_1, \ldots, R_4$. For the first one we have

$$|(z, (K+\tilde{A})y_h)_{L,T}| \lesssim (z^\sigma, y_h^\sigma - \kappa\nabla y_h^u)_{L_\sigma,T} + (z^u, \mu y_h^u - \nabla\cdot(\kappa y_h^\sigma) - \overline{\beta}\cdot\nabla y_h^u - (\beta-\overline{\beta})\cdot\nabla y_h^u)_{L_u,T}$$

where, for all $T\in\mathcal{T}_h$, $\overline{\beta}|_T$ is the mean value of the field β over T. Observe that, since $\kappa\nabla y_h^u \in \Sigma_h$, $\overline{\beta}\cdot\nabla y_h^u \in U_h$ and $z\in W_h^\perp$, $(z^\sigma, \kappa\nabla y_h^u)_{L_\sigma} = 0$ and $(z^u, \overline{\beta}\cdot\nabla y_h^u)_{L_u} = 0$. As a result,

$$\begin{aligned} |R_1| &\lesssim \|z\|_L\|y_h\|_L + \sum_{T\in\mathcal{T}_h}\left[\|z^u\|_{L_u,T} h_T^{-1}\|\kappa y_h^\sigma\|_{L_\sigma,T} + (z^u, (\beta-\overline{\beta})\cdot\nabla y_h^u)_{L_u,T}\right] \\ &\lesssim \|z\|_L\|y_h\|_L + \sum_{T\in\mathcal{T}_h}\left[\frac{\mathfrak{h}_T^{1/2}}{h_T}\|z^u\|_{L_u,T}\|y_h^\sigma\|_{L_\sigma,T} + \|\beta\|_{[\mathfrak{C}^1(\Omega)]^d}\|y_h^u\|_{L_u,T}\|z^u\|_{L_u,T}\right], \end{aligned}$$

and, therefore, $|R_1| \lesssim |\!|\!| z |\!|\!| \|y_h\|_L$. The second term R_2 can be simplified as follows:

$$|R_2| = 2\sum_{F\in\mathcal{F}_h^i}\left[(\{z^u\}_\omega, \{n^t\kappa y_h^\sigma\})_{L_u,F} + \tfrac{1}{4}([\![n^t\kappa z^\sigma]\!]_\omega, [\![y_h^u]\!])_{L_u,F} + (\tfrac{\beta\cdot n_1}{2}\{z^u\}, [\![y_h^u]\!])_{L_u,F}\right]$$

Let $R_{2,i}$, $i = 1, \ldots, 3$ be the addends of R_2. Using the definition of the weight function ω, (3.22), together with the inverse trace inequality (3.6) and definition (4.7), we infer

$$\begin{aligned} |R_{2,1}| &\lesssim \sum_{F\in\mathcal{F}_h^i} \mathfrak{h}_F^{\frac{1}{2}} h_F^{-\frac{1}{2}} (\|z_1^u\|_{L_u,F} + \|z_2^u\|_{L_u,F})\,\|y_h^\sigma\|_{L_\sigma,\mathcal{T}_h(F)}, \\ |R_{2,2}| &\lesssim \sum_{F\in\mathcal{F}_h^i} \frac{\lambda_1}{\{\lambda\}} h_F^{\frac{1}{2}} (\|z_1^\sigma\|_{L_\sigma,F} + \|z_2^\sigma\|_{L_\sigma,F})\,\lambda_2 h_F^{-\frac{1}{2}}\|[\![y_h^u]\!]\|_{L_u,F} \\ &\lesssim \sum_{F\in\mathcal{F}_h^i} h_F^{\frac{1}{2}} (\|z_1^\sigma\|_{L_\sigma,F} + \|z_2^\sigma\|_{L_\sigma,F})\,|y_h^u|_{J,F}, \\ |R_{2,3}| &\lesssim (\|z_1^u\|_{L_u,F} + \|z_2^u\|_{L_u,F})\,|y_h^u|_{J,F} \lesssim \mathfrak{h}_F^{\frac{1}{2}} h_F^{-\frac{1}{2}} (\|z_1^u\|_{L_u,F} + \|z_2^u\|_{L_u,F})\,|y_h^u|_{J,F}. \end{aligned}$$

The third term is expanded as follows:

$$|R_3| = \sum_{F\in\mathcal{F}_h^\partial}\left[\tfrac{1-\alpha}{2}(z^u, n^t\kappa y_h^\sigma)_{L_u,F} + \tfrac{1+\alpha}{2}(n^t\kappa z^\sigma, y_h^u)_{L_u,F} + \tfrac{1}{2}((M_F^{uu} + \beta\cdot n)z^u, y_h^u)_{L_u,F}\right].$$

Let $R_{3,i}$, $i = 1,\dots,3$ be the addends of R_3. If $\alpha = -1$, $R_{3,2} = 0$ and using (4.7) and (3.6) we infer that

$$|R_{3,1}| \lesssim \sum_{F\in\mathcal{F}_h^\partial} \lambda h_F^{-\frac{1}{2}} \|z^u\|_{L_u,F} \|y_h^\sigma\|_{L_\sigma,\mathcal{T}_h(F)} \lesssim \sum_{F\in\mathcal{F}_h^\partial} \mathfrak{h}_F^{\frac{1}{2}} h_F^{-\frac{1}{2}} \|z^u\|_{L_u,F} \|y_h^\sigma\|_{L_\sigma,\mathcal{T}_h(F)},$$

whereas, if $\alpha = +1$, $R_{3,1} = 0$ and (3.20) implies that

$$|R_{3,2}| \lesssim \sum_{F\in\mathcal{F}_h^\partial} h_F^{\frac{1}{2}} \|z^\sigma\|_{L_\sigma,F} \lambda h_F^{-\frac{1}{2}} \|y_h^u\|_{L_u,F} \lesssim \sum_{F\in\mathcal{F}_h^\partial} h_F^{\frac{1}{2}} \|z^\sigma\|_{L_\sigma,F} |y_h^u|_{M,F}.$$

Finally, (3.20) yields $|R_{3,3}| \lesssim |z^u|_M |y_h^u|_M$.

For the fourth term we immediately have $|R_4| \le |z^u|_J |y_h^u|_J$. The desired result is obtained collecting the above bounds. ☐

Let π_h be the L^2-projection onto W_h. Upon collecting the above results (consistency, stability, and continuity) and observing that $z - \pi_h z \in W_h^\perp$, the Second Strang Lemma immediately yields the following convergence result:

THEOREM 4.5 (Convergence). *Let z solve* (2.19) *and z_h solve* (3.23)*. Assume that $z \in [H^1(\mathcal{T}_h)]^{d+1}$. Then,*

$$\|z - z_h\|_{h,\kappa} \lesssim \|z - \pi_h z\|.$$

Owing to the regularity of the mesh family $\{\mathcal{T}_h\}_{h>0}$, the following interpolation property holds: For all $z \in [H^{r_\sigma}(\mathcal{T}_h)]^d \times H^{r_u}(\mathcal{T}_h)$,

$$\begin{aligned} \|z - \pi_h z\| \lesssim & \left(\sum_{T\in\mathcal{T}_h} h_T^{2s_\sigma+2} \|z^\sigma\|^2_{[H^{s_\sigma}(T)]^d} + \mathfrak{h}_T h_T^{2s_u} \|z^u\|^2_{H^{s_u}(T)} \right)^{\frac{1}{2}} \\ & + \left(\sum_{F\in\mathcal{F}_h} \mathfrak{h}_F h_F^{2s_u-2} \|z^u\|^2_{H^{s_u}(\mathcal{T}_h(F))} \right)^{\frac{1}{2}}, \end{aligned} \tag{4.9}$$

where $s_\sigma \stackrel{\text{def}}{=} \min(r_\sigma, p_\sigma+1)$ and $s_u \stackrel{\text{def}}{=} \min(r_u, p_u+1)$. Since $p_u - 1 \le p_\sigma$ and provided $(r_\sigma, r_u) \ge (p_\sigma + 1, p_u + 1)$, the above interpolation error is of order h^{p_u}, i.e.,

$$\|z - \pi_h z\| \lesssim h^{p_u} \|z\|_{[H^{p_\sigma+1}(\mathcal{T}_h)]^d \times H^{p_u+1}(\mathcal{T}_h)}. \tag{4.10}$$

Remark 4.1. The above estimate is optimal for the $\|\cdot\|_{h,\kappa}$-norm but yields suboptimal convergence in the L^2-norm. Note, however, that if $p_\sigma = p_u - 1$, the error estimate is optimal in the L^2-norm for z_h^σ, but is still suboptimal for the L^2-norm of z_h^u.

Remark 4.2. *(Positive definite diffusivity)* If the diffusivity is such that $\nu \ge \nu_0 \mathcal{I}_d$ with $\nu_0 = \mathcal{O}(1)$, the estimate (4.10) can be improved using a duality argument. Consider the mapping $L_u \ni y^u \longmapsto \psi \in V^*$ defined by

$$(K + \tilde{A})\psi = (0, y^u),$$

and assume the following bound holds:

$$\|\psi^u\|_{H^2(\mathcal{T}_h)} + \|\psi^\sigma\|_{[H^1(\mathcal{T}_h)]^d} \lesssim \|y^u\|_{L_u}. \tag{4.11}$$

Adapting the reasoning in [10, §5.3], if $r_u \ge p_u + 1$ and $p_u \ge 1$, it can be proved that

$$\|z - z_h\|_{L_u} \lesssim h^{p_u+1} \|z\|_{[H^{p_\sigma+1}(\mathcal{T}_h)]^d \times H^{p_u+1}(\mathcal{T}_h)}.$$

4.2. Improved convergence estimates. Owing to the definition of the $\|\cdot\|_{h,\kappa}$-norm, the convergence result of Theorem 4.5 does not contain an estimate involving the advective derivative. Such an estimate can be obtained assuming that

$$\kappa \text{ is isotropic.} \tag{4.12}$$

Observe that no further assumption is made on the definiteness of κ, i.e., we still admit that κ may vanish over a portion of the domain. Define the following new discrete norm on $W(h)$:

$$\|y\|^2_{h,\kappa,\beta} \stackrel{\text{def}}{=} \|y\|^2_{h,\kappa} + \|y^u\|^2_{h,\beta} \quad \text{with} \quad \|y^u\|^2_{h,\beta} \stackrel{\text{def}}{=} \sum_{T\in\mathcal{T}_h} h_T \|\beta\cdot\nabla y^u\|^2_{L_u,T}. \tag{4.13}$$

LEMMA 4.6. *Assume that κ satisfies* (4.12). *Then the following bound holds:*

$$\forall z_h \in W_h, \quad \|z_h\|_{h,\kappa,\beta} \lesssim \sup_{y_h\in W_h\setminus\{0\}} \frac{a_h(z_h,y_h)}{\|y_h\|_{h,\kappa,\beta}}.$$

Proof. Let $z_h \in W_h$ and set $\mathbb{S} \stackrel{\text{def}}{=} \sup_{y_h\in W_h\setminus\{0\}} \frac{a_h(z_h,y_h)}{\|y_h\|_{h,\kappa,\beta}}$.

(1) Proceeding as in Lemma 4.3 and observing that $\|(\pi_h^\sigma,0)\|_{h,\kappa,\beta} = \|(\pi_h^\sigma,0)\|_{h,\kappa}$, we conclude that

$$\|z_h\|^2_{h,\kappa} \lesssim \mathbb{S}\|z_h\|_{h,\kappa,\beta}. \tag{4.14}$$

(2) We define the field $W_h \ni \pi_h \stackrel{\text{def}}{=} (0,\pi_h^u)$ in such a way that, for all $T\in\mathcal{T}_h$, $\pi_h^u|_T = h_T\overline{\beta}\cdot\nabla z_h^u$, where $\overline{\beta}$ is the mean of β over T. Using (3.5) together with the regularity of β and the fact that $h_T \le 1$, for all $T\in\mathcal{T}_h$ we have

$$\begin{aligned} h_T^{-\frac{1}{2}}\|\pi_h^u\|_{L_u,T} &\le h_T^{\frac{1}{2}}\|(\overline{\beta}-\beta)\cdot\nabla z_h^u\|_{L_u,T} + h_T^{\frac{1}{2}}\|\beta\cdot\nabla z_h^u\|_{L_u,T} \\ &\le h_T^{\frac{1}{2}}\|\beta\|_{[\mathfrak{C}^1(\Omega)]^d}\|z_h^u\|_{L_u,T} + h_T^{\frac{1}{2}}\|\beta\cdot\nabla z_h^u\|_{L_u,T}. \end{aligned} \tag{4.15}$$

(i) We first show that $\|\pi_h\|_{h,\kappa,\beta} \lesssim \|z_h\|_{h,\kappa,\beta}$. According to the above bound, it is clear that $\|\pi_h^u\|_{L_u} \lesssim \|z_h\|_{h,\kappa,\beta}$. Commuting the operators $\kappa\nabla$ and $\overline{\beta}\cdot\nabla$ and applying the inverse inequality (3.5), we infer that

$$\sum_{T\in\mathcal{T}_h}\|\kappa\nabla\pi_h^u\|^2_{L_u,T} = \sum_{T\in\mathcal{T}_h} h_T^2\|\overline{\beta}\cdot\nabla(\kappa\nabla z_h^u)\|^2_{L_u,T} \lesssim \sum_{T\in\mathcal{T}_h}\|\kappa\nabla z_h^u\|^2_{L_\sigma,T}.$$

Moreover, the regularity of β and again (3.5) yield

$$\begin{aligned} \|\pi_h^u\|^2_{h,\beta} &\lesssim \sum_{T\in\mathcal{T}_h} h_T^3\left[\|\partial_{\overline{\beta}}(\beta\cdot\nabla z_h^u)\|_{L_u,T} + \|\partial_{\overline{\beta}}\beta\|_{[L^\infty(T)]^d} h_T^{-1}\|z_h^u\|_{L_u,T}\right]^2 \\ &\lesssim \sum_{T\in\mathcal{T}_h} h_T\left[\|\beta\cdot\nabla z_h^u\|_{L_u,T} + \|z_h^u\|_{L_u,T}\right]^2. \end{aligned}$$

The term $|\pi_h^u|_J$ is treated as follows:

$$|\pi_h^u|^2_J \lesssim \sum_{F\in\mathcal{F}_h^i}\||\beta\cdot n|^{\frac{1}{2}}[\![\pi_h^u]\!]\|^2_{L_u,F} + \sum_{F\in\mathcal{F}_h^i}\|\lambda_2 h_F^{-\frac{1}{2}}[\![\pi_h^u]\!]\|^2_{L_u,F} \stackrel{\text{def}}{=} R_1 + R_2.$$

Using (3.6) together with (4.15) we immediately conclude that

$$|R_1| \lesssim \sum_{F\in\mathcal{F}_h^i} h_F^{-1}\|\pi_h^u\|_{L_u,\mathcal{T}_h(F)}^2 \lesssim \|z_h\|_{h,\kappa,\beta}^2.$$

The second term is zero if $\lambda_2 = 0$. On the other hand, by definition, if $\lambda_2 > 0$, then $\lambda_1 > 0$, i.e., κ is nonzero of both sides of the considered element interface. We proceed using the trace inequality (3.6) together with assumption (4.12) to get

$$|R_2| \lesssim \sum_{F\in\mathcal{F}_h^i} \frac{\lambda_2^2}{h_F} h_F \|\overline{\beta}\cdot\nabla z_h^u\|_{L_u,\mathcal{T}_h(F)}^2 \lesssim \sum_{F\in\mathcal{F}_h^i} \lambda_2^2 \|\nabla z_h^u\|_{L_\sigma,\mathcal{T}_h(F)}^2 \le \sum_{F\in\mathcal{F}_h^i} \|\kappa\nabla z_h^u\|_{L_\sigma,\mathcal{T}_h(F)}^2,$$

whence $|\pi_h^u|_J \lesssim \|z_h\|_{h,\kappa,\beta}$. In a similar way we can prove that $|\pi_h^u|_M \lesssim \|z_h\|_{h,\kappa,\beta}$.
(ii) Estimate for $\|z_h^u\|_{h,\beta}$. Integrating by parts only the diffusive terms and setting $\tilde{\mu} \stackrel{\text{def}}{=} \mu + \nabla\cdot\beta$, we obtain

$$\begin{aligned}
\|z_h^u\|_{h,\beta}^2 = {} & a_h(z_h,\pi_h) \\
& + \sum_{T\in\mathcal{T}_h} \left[h_T(\beta\cdot\nabla z_h^u, (\beta-\overline{\beta})\cdot\nabla z_h^u)_{L_u,T} + (z_h^\sigma, \kappa\nabla\pi_h^u)_{L_\sigma,T} - (\tilde{\mu} z_h^u, \pi_h^u)_{L_u,T} \right] \\
& + 2\sum_{F\in\mathcal{F}_h^i} \left[-\tfrac{1}{4}([\![n^t\kappa z_h^\sigma]\!]_\omega, [\![\pi_h^u]\!])_{L_u,F} + (\tfrac{\beta\cdot n_1}{2}[\![z_h^u]\!], \{\pi_h^u\})_{L_u,F} \right] \\
& - \sum_{F\in\mathcal{F}_h^\partial} \left[(1+\alpha)(n^t\kappa z_h^\sigma, \pi_h^u)_{L_u,F} + ((M_F^{uu} - \beta\cdot n) z_h^u, \pi_h^u)_{L_u,F} \right] \\
& + \sum_{F\in\mathcal{F}_h^i} (S_F([\![z_h^u]\!]), [\![\pi_h^u]\!])_{L_u,F}.
\end{aligned}$$

Let R_i, $i = 1,\ldots,9$ be the nine terms in the right-hand side and observe that

$$|R_1| \lesssim \mathbb{S}\|\pi_h\|_{h,\kappa,\beta} \lesssim \mathbb{S}\|z_h\|_{h,\kappa,\beta}.$$

Furthermore,

$$|R_2| \lesssim \sum_{T\in\mathcal{T}_h} h_T\|\beta\cdot\nabla z_h^u\|_{L_u,T}\|\overline{\beta}-\beta\|_{[L^\infty(T)]^d} h_T^{-1}\|z_h^u\|_{L_u,T} \lesssim \gamma\|z_h^u\|_{h,\beta}^2 + \|z_h\|_{h,\kappa}^2.$$

Moreover,

$$\begin{aligned}
|R_3| + |R_4| + |R_5| + |R_9| &\lesssim \|z_h\|_{h,\kappa}\|\pi\|_{h,\kappa,\beta} \lesssim \mathbb{S}^{\frac{1}{2}}\|z_h\|_{h,\kappa,\beta}^{\frac{3}{2}}, \\
|R_6| + |R_7| + |R_8| &\lesssim \gamma\|z_h\|_{h,\kappa}\|\pi\|_{h,\kappa,\beta} \lesssim \gamma\|z_h^u\|_{h,\beta}^2 + \|z_h\|_{h,\kappa}^2.
\end{aligned}$$

Hence,

$$\|z_h^u\|_{h,\beta}^2 \lesssim \mathbb{S}\|z_h\|_{h,\kappa,\beta} + \mathbb{S}^{\frac{1}{2}}\|z_h\|_{h,\kappa,\beta}^{\frac{3}{2}} + \|z_h\|_{h,\kappa}^2,$$

whence it follows, using (4.14), that $\|z_h^u\|_{h,\beta}^2 \lesssim \mathbb{S}^2$.
□

By using Lemma 4.6 and proceeding as in the proof of Theorem 4.5, we infer

Theorem 4.7 (Convergence). *Let z solve* (2.19) *and z_h solve* (3.23). *Assume that $z \in [H^1(\mathcal{T}_h)]^{d+1}$ and that κ satisfies* (4.12). *Then,*

$$\|z - z_h\|_{h,\kappa,\beta} \lesssim [\![z - \pi_h z]\!].$$

Remark 4.3. *(Purely hyperbolic case)* A special situation is obtained when the diffusivity is identically zero over the entire domain, since, for all $T \in \mathcal{T}_h$ and for all $F \in \mathcal{F}_h$, $\mathfrak{h}_T = h_T$ and $\mathfrak{h}_F = h_F$. In such a case it is readily seen that

$$\|z^u - z_h^u\|_{0,\Omega} + \|z^u - z_h^u\|_{h,\beta} \lesssim h^{p_u+\frac{1}{2}} \|z^u\|_{H^{p_u+1}(\mathcal{T}_h)} \tag{4.16}$$

which is exactly the estimate for the problem investigated in [9, §3.1].

5. Implementation issues. In this section we discuss important implementation aspects of the method. We show how it can be interpreted in terms of so-called numerical fluxes so as to compare it with other known approximations techniques that are defined in these terms in the literature. We also present two variants of the method that yield substantial computational savings.

5.1. Flux formulation. The notion of (numerical) fluxes is widely used by engineers. This concept originally introduced in the context of finite volume methods, naturally extends to discontinuous Galerkin methods. The link between DG methods and the concept of flux has been explored in [2] for the Laplace equation and in [10] for more general cases. A number of methods have originally been presented in terms of fluxes, and it is therefore interesting to recast our formulation in this framework so as to facilitate comparisons. To this purpose, let us define

$$\phi^u_{\partial T}(z^\sigma, z^u)|_F \stackrel{\text{def}}{=} \begin{cases} \frac{1+\alpha}{2} n^t \kappa z^\sigma + (\beta{\cdot}n) z^u + M_F^{uu} z^u, & \text{if } F \in \mathcal{F}_h^\partial, \\ n_T^t \{\kappa z^\sigma\}_{\overline{\omega}} + (\beta{\cdot}n_T)\{z^u\} + (n_T{\cdot}n_F) S_F([\![z^u]\!]), & \text{if } F \in \mathcal{F}_h^i, \end{cases} \tag{5.1}$$

$$\phi^\sigma_{\partial T}(z^u)|_F \stackrel{\text{def}}{=} \begin{cases} \frac{1-\alpha}{2} (\kappa n)^t z^u, & \text{if } F \in \mathcal{F}_h^\partial, \\ (\kappa n)^t|_T \{z^u\}_\omega, & \text{if } F \in \mathcal{F}_h^i, \end{cases} \tag{5.2}$$

where $\overline{\omega} \stackrel{\text{def}}{=} (1,1) - \omega$ and n_T is the outward normal to the element T. It is possible to prove (see [10, §4.3] for the details) that the discrete problem (3.23) can be equivalently reformulated in terms of the following local problems:

$$\begin{cases} \text{Seek } z_h \in W_h \text{ such that, for all } T \in \mathcal{T}_h \text{ and for all } q \in [\mathbb{P}_{p_\sigma}(T)]^d \times \mathbb{P}_{p_u}(T), \\ (z_h, (K + \tilde{A})q)_{L,T} + (\phi_{\partial T}(z_h), q|_T)_{L,\partial T} = (f, q^u)_{L_u,T}. \end{cases}$$

The above form is known as the flux formulation of (3.23). Observe that the above flux definitions lead to the use of harmonic averages of the normal component of the diffusion tensor at mesh interfaces.

5.2. IP variant. In this section we discuss a variant of the method designed in §3.2 which reduces the size of the local problems to be solved to eliminate the σ-component of the unknown. The advantages of such a variant are that it is easier to implement and that the associated matrix pattern is sparser. To this purpose we introduce the lifting operator defined as follows: For all $F \in \mathcal{F}_h$ and for all $\varphi \in L^2(F)$, $r_{F,\kappa}(\varphi) \in \Sigma_h$ is defined by

$$\forall \tau_h \in \Sigma_h, \quad (r_{F,\kappa}(\varphi), \tau_h)_{L_\sigma} \stackrel{\text{def}}{=} \begin{cases} \frac{\alpha+1}{2} (\varphi n, \kappa \tau_h)_{L_\sigma,F}, & \text{if } F \in \mathcal{F}_h^\partial, \\ (\varphi n_1, \{\kappa \tau_h\}_{\overline{\omega}})_{L_\sigma,F}, & \text{if } F \in \mathcal{F}_h^i. \end{cases} \tag{5.3}$$

Moreover, we let $R_\kappa(\varphi) \stackrel{\text{def}}{=} \sum_{F \in \mathcal{F}_h} r_{F,\kappa}(\varphi)$. Observe that, unlike in [2], the lifting operator depends on the diffusivity. Moreover, for a given face $F \in \mathcal{F}_h$, it is clear that

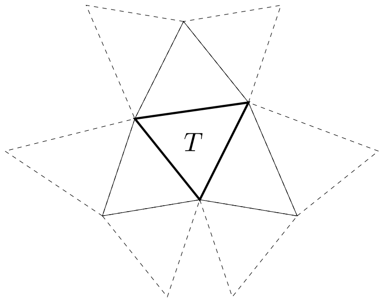


FIGURE 5.1. *Elimination of the σ-component on element T. Stencil for the IP variant of the method (solid lines) and for the LDG variant (solid and dashed lines).*

$\operatorname{supp}(r_{F,\kappa}(\varphi)) = \mathcal{T}_h(F)$. In what follows we shall extend the definition of the jump operator to boundary faces by setting

$$[\![y^u]\!] \stackrel{\text{def}}{=} y^u, \quad \forall F \in \mathcal{F}_h^\partial,\ \forall y \in W(h).$$

The following result holds:

LEMMA 5.1. *For all $F \in \mathcal{F}_h$ and for all $v_h \in U_h$,*

$$\|r_{F,\kappa}([\![v_h]\!])\|_{L_\sigma} \lesssim \begin{cases} \lambda h_F^{-\frac{1}{2}} \|v_h\|_{L_u,F}, & \text{if } F \in \mathcal{F}_h^\partial, \\ \lambda_2 h_F^{-\frac{1}{2}} \|[\![v_h]\!]\|_{L_u,F}, & \text{if } F \in \mathcal{F}_h^i. \end{cases}$$

Proof. Let $F \in \mathcal{F}_h^i$. Then, using (5.3), (3.22) and (3.6) we have that

$$\|r_{F,\kappa}([\![v_h]\!])\|_{L_\sigma}^2 = ([\![v_h]\!]n_1, \{\kappa r_{F,\kappa}([\![v_h]\!])\}_{\overline{\omega}})_{L_\sigma,F} \lesssim \|[\![v_h]\!]\|_{L_u,F} \frac{\lambda_1\lambda_2}{2\,\{\lambda\}} h_F^{-\frac{1}{2}} \|r_{F,\kappa}([\![v_h]\!])\|_{L_\sigma},$$

from which the assertion follows readily. The proof is carried out similarly for $F \in \mathcal{F}_h^\partial$. ∎

Proceeding in a similar way as in [2, §3.2] and using the fact that, owing to assumption (3.2), $\kappa\tau_h$ is in Σ_h for all $\tau_h \in \Sigma_h$, it is possible to prove that, for all $(\sigma, u) \in W(h)$ and for all $(0, v) \in W(h)$,

$$\begin{aligned} a_h((\sigma,u),(0,v)) = &\sum_{T\in\mathcal{T}_h} [(\kappa\nabla u - R_\kappa([\![u]\!]), \kappa\nabla v - R_\kappa([\![v]\!]))_{L_\sigma,T} + (\mu u, v)_{L_u,T}] \\ &- \sum_{T\in\mathcal{T}_h} (u, \beta\cdot\nabla v)_{L_u,T} + \sum_{F\in\mathcal{F}_h^\partial} (M_F^{uu}(u) + (\beta\cdot n)u, v)_{L_u,F} \\ &+ \sum_{F\in\mathcal{F}_h^i} ((\beta\cdot n_1)\{u\}, [\![v]\!])_{L_u,F} + \sum_{F\in\mathcal{F}_h^i} (S_F([\![u]\!]), [\![v]\!])_{L_u,F}. \end{aligned} \tag{5.4}$$

Notice that σ does not appear in the expression in the right-hand side, i.e., we have found a de-coupled problem for the sole primal unknown. The expression (5.4) will henceforth be referred to as *LDG variant* of the discrete bilinear form because of the similarity with the method for convection-diffusion systems proposed in [6].

One can verify that, when the basis functions are defined so that their support is restricted to one element of the triangulation, the stencil resulting from (5.4) is composed of all the elements shown in Figure 5.1 (solid and dash lines). But by having a closer look at (5.4), one realizes that the only term involving the dashed elements in Figure 5.1 is the following:

$$\sum_{T\in\mathcal{T}_h}(R_\kappa(\llbracket u\rrbracket), R_\kappa(\llbracket v\rrbracket))_{L_\sigma,T} \stackrel{\text{def}}{=} -\rho_h(u,v).$$

Hence, in order to reduce the stencil, it seems reasonable to consider the following perturbation of a_h:

$$a_h^{\mathrm{IP}}(z,y) \stackrel{\text{def}}{=} a_h(z,y) + \rho_h(z^u,y^u). \tag{5.5}$$

Let us define the following semi-norm:

$$|y^u|^2_{\mathrm{LDG}} \stackrel{\text{def}}{=} \frac{\alpha+1}{2}\sum_{F\in\mathcal{F}_h^\partial}\|\lambda h_F^{-\frac{1}{2}}y^u\|^2_{L_u,F} + \sum_{F\in\mathcal{F}_h^i}\|\lambda_2 h_F^{-\frac{1}{2}}\llbracket y^u\rrbracket\|^2_{L_u,F}, \quad \forall y\in W(h). \tag{5.6}$$

The following lemma is crucial to accommodate the proofs of Lemmata 4.2–4.4 to the new bilinear form a_h^{IP}:

LEMMA 5.2. *The following properties, uniform in h, hold:*

(i) For all $(z,y_h)\in(V\cap W(h))\times W_h$ we have

$$\forall y_h\in W_h,\quad \rho_h(z^u,y_h^u)=0.$$

(ii) For all y in $W(h)$,

$$\rho_h(y^u,y^u)\le CN_F|y^u|_{\mathrm{LDG}},$$

N_F being the maximum number of faces of one mesh element and C a positive parameter depending only on the mesh geometry and on the polynomial order of approximation.

(iii) For all $(z,y_h)\in W_h^\perp\times W_h$,

$$\rho_h(z,y_h)\lesssim |z^u|_{\mathrm{LDG}}|y_h^u|_{\mathrm{LDG}}. \tag{5.7}$$

Proof.

(i) We know that $\llbracket z^u\rrbracket=0$, and, consequently, $r_{F,\kappa}(\llbracket z^u\rrbracket)=0$, on all $F\in\mathcal{F}_h\backslash\mathcal{I}_h^-$. On the other hand, let $\mathcal{I}_h^-\ni F=\partial T_1\cap\partial T_2$ and $\tau_h\in\Sigma_h$. Then, since $n^t\kappa|_{T_2}=0$ entails $\lambda_2=0$,

$$n_1^t\{\kappa\tau_h\}_{\overline{\omega}} = \frac{\lambda_2}{2\,\{\lambda\}}n_1^t\cdot\kappa\tau_h|_{T_1} + \frac{\lambda_1}{2\,\{\lambda\}}n_1^t\cdot\kappa\tau_h|_{T_2}=0,$$

i.e., $r_{F,\kappa}(\llbracket z^u\rrbracket)=0$, which gives the desired result.

(ii) The second point can be proved as follows. Observe that

$$\|R_\kappa(\llbracket y^u\rrbracket)\|^2_{L_\sigma}\le\sum_{F\in\mathcal{F}_h}\sum_{F'\in\mathcal{F}_h}\|r_{F,\kappa}(\llbracket y^u\rrbracket)\|_{L_\sigma}\|r_{F',\kappa}(\llbracket y^u\rrbracket)\|_{L_\sigma}.$$

Let $\mathcal{F}_h^i\ni F=\partial T_1\cap\partial T_2$. Since $\mathrm{supp}(r_{F,\kappa}(\llbracket y^u\rrbracket))=T_1\cup T_2$, only a few products in the right-hand side are non-zero. In particular, the non-zero products are those for which $F' \in \Delta_F$, where $\Delta_F \stackrel{\text{def}}{=} \{F' \in \mathcal{F}_h;\ F' \subset \partial T_1 \text{ or } F' \subset \partial T_2\}$. Therefore, the only terms involving F are

$$\sum_{F' \in \Delta_F} \|r_{F,\kappa}([\![y^u]\!])\|_{L_\sigma} \|r_{F',\kappa}([\![y^u]\!])\|_{L_\sigma} \leq \tfrac{1}{2} \sum_{F' \in \Delta_F} \left(\|r_{F,\kappa}([\![y^u]\!])\|^2_{L_\sigma} + \|r_{F',\kappa}([\![y^u]\!])\|^2_{L_\sigma} \right).$$

We realize that $\|r_{F,\kappa}([\![y^u]\!])\|^2_{L_\sigma}$ is added at most N_F times. The desired result follows by repeating this argument for the other faces and using Lemma 5.1.

(iii) Deriving (5.7) is a simple application of Lemma 5.1.

□

Modifying Lemmata 4.2–4.4 so as to hold for a_h^{IP} instead of a_h is now simple in view of the above result. However, observe that, according to the second point of Lemma 5.2, in order to preserve the L-coercivity, (3.20) should be modified as follows:

$$M_F^{uu} \stackrel{\text{def}}{=} \frac{|\beta{\cdot}n|}{2} + N_F \eta \frac{\alpha+1}{2} \frac{\lambda^2}{h_F}, \qquad S_F \stackrel{\text{def}}{=} \frac{|\beta{\cdot}n|}{2} + N_F \eta \frac{\lambda_2^2}{h_F}, \tag{5.8}$$

where the multiplicative factor η must be strictly greater than the constant C appearing in (5.7). The term ρ_h that has been added to simplify the elimination of the σ-component is thus counterbalanced by adding "more stabilization". The resulting method is termed the *IP variant* because of the similarity with the IP method proposed in [3]. The method recently proposed in [13] also belongs to this class, although some modifications are introduced in the definition of the penalty parameter.

5.3. BRMPS variant. The parameter C in (5.7), and, consequently, η in (5.8), is possibly difficult to estimate in practical applications. To solve this problem, we consider the following alternative expression for the boundary and interface operators:

$$M_F^{uu}(v) \stackrel{\text{def}}{=} \frac{|\beta{\cdot}n|}{2} v + N_F \eta r_{F,\kappa}(v), \qquad S_F(v) \stackrel{\text{def}}{=} \frac{|\beta{\cdot}n|}{2} v + N_F \eta \{r_{F,\kappa}(v)\}_{\overline{\omega}}. \tag{5.9}$$

A closer look at the proof of the second point of Lemma 5.2 shows that it is sufficient to take $\eta > 1$ to preserve L-coercivity. Owing to the similarities with the approach first presented in [4], the resulting numerical method is termed *BRMPS variant*.

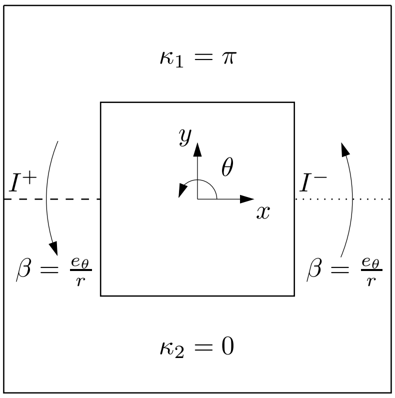
6. Numerical results. In this section we evaluate the performance of the proposed method. The simulations were run using the cheaper variant discussed in §5.3.

6.1. Convergence. In order to assess the theoretical convergence estimates, we consider the problem described in Figure 6.1(a). Here (r, θ) denote the standard cylindrical coordinates with the angle θ measured in anti-clockwise sense starting from the positive x-axis. The domain is taken to be $(-1, 1)^2 \setminus [-0.5, 0.5]^2$, while the coefficients are set to

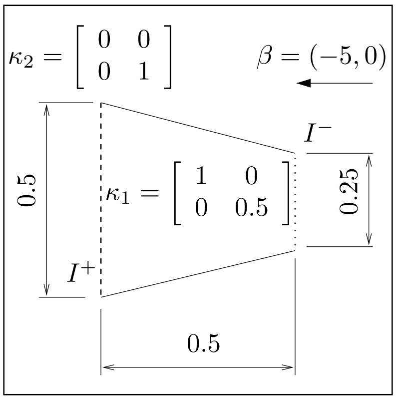
$$\kappa = \begin{cases} \pi, & \text{if } 0 < \theta < \pi, \\ 0, & \text{if } \pi < \theta < 2\pi, \end{cases} \qquad \beta = \frac{e_\theta}{r}, \qquad \mu = 10^{-3},$$

where e_θ is the unit azimuthal vector. The exact solution for a suitable right-hand side f is

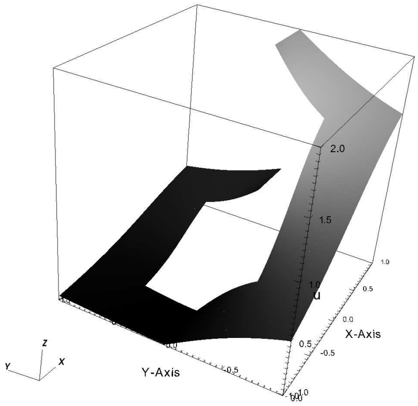
$$u = \begin{cases} (\theta - \pi)^2, & \text{if } 0 \leq \theta \leq \pi, \\ 3\pi(\theta - \pi), & \text{if } \pi < \theta < 2\pi. \end{cases}$$



(a) Description of the test case of §6.1.



(b) Description of the test case of §6.2



(c) Exact solution of the test case of §6.1.

FIGURE 6.1. *Problem setting for the numerical test cases. I^+ and I^- are plotted in dashed and dotted line respectively.*

Observe that, although piecewise polynomial in θ, the above solution does *not* belong to the discrete space U_h since we are solving the problem in cartesian coordinates. Moreover, according to the interface condition (INT1), the solution is continuous across I^+, while only (INT2) is verified on I^-. We introduce the following norm:

$$\|u\|^2_{h,\mathrm{BRMPS}} \stackrel{\mathrm{def}}{=} \|u\|^2_{L_u} + |u|^2_J + |u|^2_M + \sum_{T\in\mathcal{T}_h} \|\kappa\nabla u\|^2_{L_\sigma,T}.$$

Let (σ_h, u_h) solve the discrete problem associated with the BRMPS variant. Then, observing that $\sigma_h = \kappa\nabla u_h + R([\![u_h]\!])$, it can be proved that $\|u - u_h\|_{h,\mathrm{BRMPS}}$ is equivalent to $\|(\sigma, u) - (\sigma_h, u_h)\|_{h,\kappa}$. Coherently with the desire to avoid the additional cost coming from the computation of σ_h, $\|u - u_h\|_{h,\mathrm{BRMPS}}$ was reported in Table 6.1. The convergence results confirm the sharpness of the estimates derived in §4.1 and in §4.2. The L^2-norm is also reported for completeness, showing that convergence at order $p_u + 1$ can be expected.

TABLE 6.1
Convergence results.

h	$P_{h,1}$		$P_{h,2}$		$P_{h,3}$		$P_{h,4}$	
	err	ord	err	ord	err	ord	err	ord
				$\lVert u-u_h\rVert_{h,\mathrm{BRMPS}}$				
1/2	$3.15e+0$		$7.27e-1$		$1.74e-1$		$3.99e-2$	
1/4	$1.63e+0$	0.95	$2.05e-1$	1.83	$2.69e-2$	2.70	$3.51e-3$	3.51
1/8	$8.19e-1$	0.99	$5.32e-2$	1.94	$3.59e-3$	2.91	$2.51e-4$	3.81
1/16	$4.08e-1$	1.00	$1.34e-2$	1.99	$4.54e-4$	2.98	$1.63e-5$	3.95
1/32	$2.04e-1$	1.00	$3.36e-3$	2.00				
				$\lVert u-u_h\rVert_{h,\beta}$				
1/2	$1.97e-0$		$4.50e-1$		$1.13e-1$		$2.65e-2$	
1/4	$7.46e-1$	1.40	$9.87e-2$	2.18	$1.40e-2$	3.01	$1.92e-3$	3.79
1/8	$2.73e-1$	1.45	$1.90e-2$	2.38	$1.44e-3$	3.29	$1.06e-4$	4.18
1/16	$9.82e-2$	1.48	$3.44e-3$	2.46	$1.34e-4$	3.43	$5.03e-6$	4.40
1/32	$3.50e-2$	1.49	$6.08e-4$	2.50				
				$\lVert u-u_h\rVert_{L_u}$				
1/2	$2.92e-1$		$3.30e-2$		$5.79e-3$		$1.17e-3$	
1/4	$7.49e-2$	1.96	$4.75e-3$	2.80	$4.62e-4$	3.65	$5.50e-5$	4.41
1/8	$1.91e-2$	1.97	$6.09e-4$	2.96	$3.26e-5$	3.83	$2.01e-6$	4.77
1/16	$4.86e-3$	1.97	$7.76e-5$	2.97	$2.10e-6$	3.96	$6.32e-8$	4.99
1/32	$1.23e-3$	1.98	$9.82e-6$	2.98				

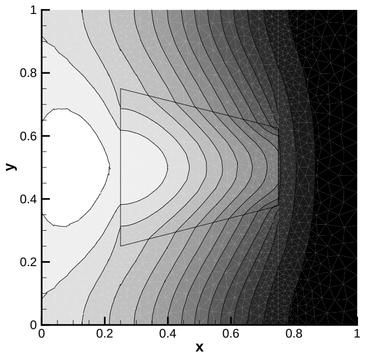
6.2. Strongly anisotropic diffusivity. To demonstrate the behaviour of the method in the presence of strongly anisotropic diffusivity we consider the test of Figure 6.1(b). The domain $\Omega = (0,1)^2$ is partitioned into two subdomains where the diffusivity takes different values; it is definite positive in one region and semi-definite positive in the other region. The advection field is $\beta = (-5,0)^t$ and the reaction coefficient is $\mu = 1$. The solution is discontinuous across the interface $I^- = \{x = 0.75;\ 0.375 \le y \le 0.625\}$. The solutions obtained for different polynomial degrees are displayed in Figure 6.2, showing that the predicted behaviour is captured accurately.

7. Conclusion. In this work we developed and analyzed a DG method for advection-diffusion-reaction equations with discontinuous, anisotropic, and semi-definite diffusivity. The proposed method is capable of treating the semi-definite diffusivity case owing to our design of the boundary and penalty terms. This is achieved by resorting to weighted average and jump operators. The convergence analysis yields estimates that are uniform with respect to the diffusivity. The theoretical results are supported by numerical evidence.

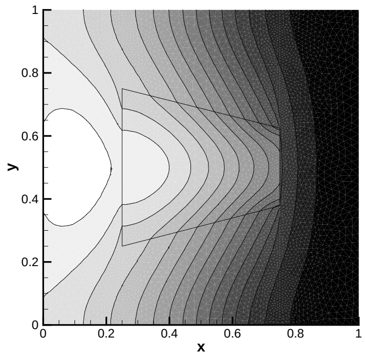
Acknowledgements. This work was partially supported by the GdR MoMaS (CNRS-2439, ANDRA, BRGM, CEA, EDF, IRSN).

REFERENCES

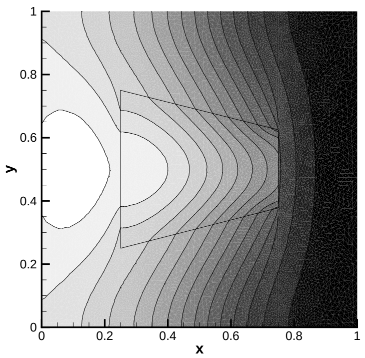
[1] D. N. ARNOLD, *An interior penalty finite element method with discontinuous elements*, SIAM J. Numer. Anal., 19 (1982), pp. 742–760.
[2] D. N. ARNOLD, F. BREZZI, B. COCKBURN, AND D. MARINI, *Unified analysis of discontinuous Galerkin methods for elliptic problems*, SIAM J. Numer. Anal., 39 (2002), pp. 1749–1779.



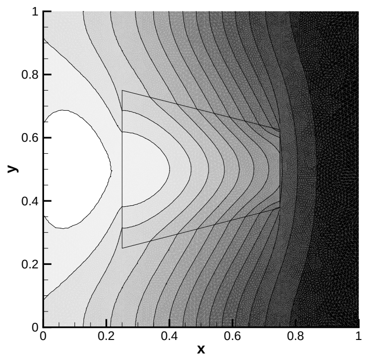
(a) $U_h = P_{h,1}$



(b) $U_h = P_{h,2}$



(c) $U_h = P_{h,3}$



(d) $U_h = P_{h,4}$

FIGURE 6.2. *Numerical results for the test case of §6.2.*

[3] G. A. BAKER, *Finite element methods for elliptic equations using nonconforming elements*, Math. Comp., 31 (1977), pp. 45–49.

[4] F. BASSI, S. REBAY, G. MARIOTTI, S. PEDINOTTI, AND M. SAVINI, *A high-order accurate discontinuous finite element method for inviscid and viscous turbomachinery flows*, in Proceedings of the 2^{nd} European Conference on Turbomachinery Fluid Dynamics and Thermodynamics, R. Decuypere and G. Dibelius, eds., 1997, pp. 99–109.

[5] E. BURMAN AND P. ZUNINO, *A domain decomposition method based on weighted interior penalties for advection-diffusion-reaction problems*, SIAM J. Numer. Anal., (2006). To appear.

[6] B. COCKBURN AND C.-W. SHU, *The local discontinuous Galerkin finite element method for convection-diffusion systems*, SIAM J. Numer. Anal., 35 (1998), pp. 2440–2463.

[7] J.-P. CROISILLE, A. ERN, T. LELIÈVRE, AND J. PROFT, *Analysis and simulation of a coupled hyperbolic/parabolic model problem*, J. Numer. Math., 13 (2005), pp. 81–103.

[8] A. ERN AND J.-L. GUERMOND, *Theory and Practice of Finite Elements*, vol. 159 of Applied Mathematical Sciences, Springer-Verlag, New York, NY, 2004.

[9] ———, *Discontinuous Galerkin methods for Friedrichs systems. I. General theory*, SIAM J. Numer. Anal., 44 (2006), pp. 753–778.

[10] ———, *Discontinuous Galerkin methods for Friedrichs systems. II. Second-order elliptic PDE's*, SIAM J. Numer. Anal., (2006). To appear.

[11] A. ERN, J.-L. GUERMOND, AND G. CAPLAIN, *An intrinsic criterion for the bijectivity of Hilbert operators related to Friedrichs systems*, Comm. Partial Differ. Eq., (2006). To appear.

[12] A. ERN AND J. PROFT, *Multi-algorithmic methods for coupled hyperbolic-parabolic problems*,

Int. J. Numer. Anal. Model., 1 (2006), pp. 94–114.
[13] A. Ern, A. F. Stephansen, and P. Zunino, *A discontinuous Galerkin method with weighted averages for advection-diffusion equations with locally vanishing and anisotropic diffusivity*, IMA J. Num. Anal., (2006). Submitted.
[14] J. Freund and R. Stenberg, *On weakly imposed boundary conditions for second order problems*, in Proceedings of the International Conference on Finite Elements in Fluids - New trends and applications, Venice, Italy, 1995.
[15] K.O. Friedrichs, *Symmetric positive linear differential equations*, Comm. Pure and Appl. Math., 11 (1958), pp. 333–418.
[16] F. Gastaldi and A. Quarteroni, *On the coupling of hyperbolic and parabolic systems: analytical and numerical approach*, App. Num. Math., 6 (1989), pp. 3–31.
[17] B. Heinrich and S. Nicaise, *The Nitsche mortar finite-element method for transmission problems with singularities*, IMA J. Numer. Anal., 23 (2003), pp. 331–358.
[18] B. Heinrich and K. Pietsch, *Nitsche type mortaring for some elliptic problem with corner singularities*, Computing, 68 (2002), pp. 217–238.
[19] B. Heinrich and K. Pönitz, *Nitsche type mortaring for singularly perturbed reaction-diffusion problems*, Computing, 75 (2005), pp. 257–279.
[20] P. Houston, C. Schwab, and E. Süli, *Discontinuous hp-finite element methods for advection-diffusion-reaction problems*, SIAM J. Numer. Anal., 39 (2002), pp. 2133–2163.
[21] P. Lesaint, *Finite element methods for symmetric hyperbolic equations*, Numer. Math., 21 (1973/74), pp. 244–255.
[22] P. Lesaint and P.-A. Raviart, *On a finite element method for solving the neutron transport equation*, in Mathematical Aspects of Finite Elements in Partial Differential Equations, Math. Res. Center, Univ. of Wisconsin-Madison, Academic Press, New York, 1974, pp. 89–123. Publication No. 33.
[23] W.H. Reed and T.R. Hill, *Triangular mesh methods for the neutron transport equation*, Tech. Report LA-UR-73-479, Los Alamos Scientific Laboratory, Los Alamos, NM, 1973.
[24] M. Wheeler, *An elliptic collocation-finite element method with interior penalties*, SIAM J. Numer. Anal., (1978), pp. 152–161.