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General Duality for Perpetual American Options

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General Duality for Perpetual American Options

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Abstract

In this paper, we investigate the generalization of the Call-Put duality equality obtained in [1] for perpetual American options when the Call-Put payoff $(y-x)^+$ is replaced by $\phi(x,y)$. It turns out that the duality still holds under monotonicity and concavity assumptions on ϕ. The specific analytical form of the Call-Put payoff only makes calculations easier but is not crucial unlike in the derivation of the Call-Put duality equality for European options. Last, we give some examples for which the optimal strategy is known explicitly.

Introduction

In [1], we have obtained a Call-Put duality equality for perpetual American options. More precisely, for an interest rate $r>0$, a dividend rate $\delta \geq 0$ and a time-homogeneous local volatility function

$$(\mathcal{H}_{\mathbf{vol}}) \quad \sigma : \mathbb{R}_+^* \to \mathbb{R}_+^* \text{ continuous and such that } \exists \underline{\sigma}, \overline{\sigma} \in \mathbb{R}_+^*,\ \forall x>0,\ \underline{\sigma} \leq \sigma(x) \leq \overline{\sigma},$$

let $(S_t^x, t \geq 0)$ denote the unique weak solution of the stochastic differential equation

$$dS_t^x = S_t^x((r-\delta)dt + \sigma(S_t^x)dW_t),\ S_0^x = x \tag{1}$$

where $(W_t, t\geq 0)$ is a standard Brownian motion. We have proved the existence of another volatility function η satisfying $(\mathcal{H}_{\mathbf{vol}})$ such that

$$\forall x,y>0,\ \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}\left[e^{-r\tau}(y-S_\tau^x)^+\right] = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}\left[e^{-\delta\tau}(\overline{S}_\tau^y - x)^+\right],$$

where $(\overline{S}^y_t, t \geq 0)$ denotes the weak solution to

$$d\overline{S}^y_t = \overline{S}^y_t((\delta - r)dt + \eta(\overline{S}^y_t)dW_t),\ \overline{S}^y_0 = y. \tag{2}$$

Here, $\mathcal{T}_{0,\infty}$ denotes the set of stopping times with respect to the usual natural filtration of the underlying.
Our primal goal was to generalize to American derivatives the Call-Put duality equality

$$\text{for } \eta \equiv \sigma,\ \forall T, x, y > 0,\ \mathbb{E}\left[e^{-rT}(y - S^x_T)^+\right] = \mathbb{E}\left[e^{-\delta T}(\overline{S}^y_T - x)^+\right] \tag{3}$$

which holds in the European case. In the perpetual American case, unless σ is a constant (usual Black-Scholes model), then η is different from σ. Then it is natural to wonder whether the European and the perpetual American Call-Put dualities are similar in nature. The European equality is equivalent to Dupire's formula [3] and, to our knowledge, the existing proofs of both results rely heavily on the following specifity of the payoff function $(y - x)^+$: for fixed y (resp. fixed x), in the distribution sense, $\partial^2_x(y - x)^+ = \delta_y(x)$ (resp. $\partial^2_y(x - y)^+ = \delta_x(y)$). This property implies for instance that the second order derivative of the European Put price with respect to the strike variable y is the actualized density of the underlying asset S^x_T. It is not obvious at all that the equality (3) could be generalized by replacing respectively $(y - S^x_T)^+$ and $(\overline{S}^y_T - x)^+$ by $\phi(S^x_T, y)$ and $\phi(x, \overline{S}^y_T)$.
In contrast, we are going to show that it is possible to generalize the perpetual American duality to payoff functions $\phi(x, y)$ which only share global properties with $(x - y)^+$. The loss of the specific analytical expression only makes calculations more complicated.

More precisely, from now on, we assume that $\phi : \mathbb{R}^*_+ \times \mathbb{R}^*_+ \to \mathbb{R}_+$ is a continuous function such $\Phi = \{(x, y) : \phi(x, y) > 0\} \neq \emptyset$, ϕ is C^2 on Φ and such that

$$\forall x, y \in \Phi,\ \partial_x\phi(x, y) < 0,\ \partial_y\phi(x, y) > 0,\ \partial^2_x\phi(x, y) \leq 0 \text{ and } \partial^2_y\phi(x, y) \leq 0. \tag{4}$$

Of course the function $(y - x)^+$ satisfies these assumptions. More general examples are given in Section 2.
For $y > 0$, let us define $X(y) = \inf\{x > 0,\ \phi(x, y) = 0\}$ with the convention $\inf\emptyset = +\infty$. Thanks to (4), we have $\{x > 0,\ \phi(x, y) = 0\} = \{x > 0, x \geq X(y)\}$ and $0 \leq X(y) < \infty$. Moreover, the function $y \mapsto X(y)$ is nondecreasing. Let us also define $Y(x) = \inf\{y > 0, \phi(x, y) > 0\} = \inf\{y > 0, x < X(y)\}$. As the pseudo-inverse of the nondecreasing function X, the function Y is nondecreasing. Finally,

$$\Phi = \{(x, y),\ \phi(x, y) > 0\} = \{(x, y),\ x < X(y)\} = \{(x, y),\ y > Y(x)\}.$$

We also make the following assumption weaker than $(\mathcal{H}_{\mathbf{vol}})$ on the volatility functions :

$$(\mathcal{H}'_{\mathbf{vol}}) \quad \sigma : \mathbb{R}^*_+ \to \mathbb{R}^*_+ \text{ continuous and such that } \exists\overline{\sigma} < +\infty,\ \forall x > 0,\ \sigma(x) < \overline{\sigma}.$$

When σ and η satisfy $(\mathcal{H}'_{\mathbf{vol}})$, then weak existence and uniqueness hold for (1) and (2) (see for example Theorem 5.15 in [4], using a log transformation). Let

$$P_\sigma(x, y) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}\left[e^{-r\tau}\phi(S^x_\tau, y)\right] \text{ and } c_\eta(y, x) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}\left[e^{-\delta\tau}\phi(x, \overline{S}^y_\tau)\right].$$

where the notations P and c standing respectively for "Put" and "Call" are slightly abusive.

The paper is structured as follows. The first section is devoted to the pricing of perpetual American options with payoff ϕ. It turns out that, as in the Call-Put case, for fixed strike $y>0$ (resp. $x>0$), there is a unique $x^*(y)$ (resp. $y^*(x)$) such that

$$\{x : P_\sigma(x,y) > \phi(x,y)\} = (x^*(y), +\infty) \text{ resp. } \{y : c_\eta(y,x) > \phi(x,y)\} = (0, y^*(x)).$$

These exercise boundaries $x^*(y)$ and $y^*(x)$ are characterized by some implicit equations involving ϕ, and we prove that they solve explicit ODEs. The second section deals with the duality result. We state a general result and, for two specific families of payoff functions, we are able to find an explicit relation between dual volatilities, as in the call-put case.

1 Pricing of the perpetual American options

1.1 Pricing formulas and exercise boundaries

In this section we will use the approach of Beibel and Lerche [2] to explicit the pricing functions P_σ and c_η. As in [1], we will denote by f (resp. g) the unique, up to a multiplicative constant, positive nonincreasing (resp. nondecreasing) solution of

$$\frac{1}{2}\sigma^2(x)x^2 f''(x) + (r-\delta)xf'(x) - rf(x) = 0, \; x>0 \tag{5}$$

$$\text{(resp. } \frac{1}{2}\eta^2(x)x^2 g''(x) + (\delta-r)xg'(x) - \delta g(x) = 0, \; x>0 \text{).} \tag{6}$$

Let us also recall that

$$\forall x>0, \; f''(x)>0 \text{ and } g''(x)>0. \tag{7}$$

This has been checked for example in [1] (Lemma 3.1) where σ is assumed to satisfy $(\mathcal{H}_{\mathbf{vol}})$, but the boundedness from below is not used in the proof.

Proposition 1.1. *Let us fix a strike $y>0$. If $X(y)=0$, then $\forall x>0, P_\sigma(x,y)=0$. Otherwise there is a unique $x^*_\sigma(y) \in (0, X(y))$ such that $\tau^P_x = \inf\{t \geq 0, S^x_t \leq x^*_\sigma(y)\}$ (convention $\inf \emptyset = +\infty$) is an optimal stopping time for P_σ and:*

$$\forall x \leq x^*_\sigma(y), P_\sigma(x,y) = \phi(x,y), \; \forall x > x^*_\sigma(y), P_\sigma(x,y) = \frac{\phi(x^*_\sigma(y),y)}{f(x^*_\sigma(y))} f(x) > \phi(x,y). \tag{8}$$

In addition, we have

$$\frac{\phi(x^*_\sigma(y),y)}{\partial_x\phi(x^*_\sigma(y),y)} = \frac{f(x^*_\sigma(y))}{f'(x^*_\sigma(y))} \tag{9}$$

which implies the smooth-fit principle. Last, the function $y \in \{z : X(z)>0\} \mapsto x^*_\sigma(y)$ is $\mathcal{C}^1$ and satisfies the following ODE:

$$x^*_\sigma(y)' = \left[\partial^2_{xy}\phi(x^*_\sigma(y),y) - \frac{\partial_x\phi(x^*_\sigma(y),y)\partial_y\phi(x^*_\sigma(y),y)}{\phi(x^*_\sigma(y),y)}\right] \times \frac{x^*_\sigma(y)^2\sigma^2(x^*_\sigma(y))}{2[r\phi(x^*_\sigma(y),y) + (\delta-r)x^*_\sigma(y)\partial_x\phi(x^*_\sigma(y),y)] - x^*_\sigma(y)^2\sigma^2(x^*_\sigma(y))\partial^2_x\phi(x^*_\sigma(y),y)}. \tag{10}$$

It is strictly increasing if one assumes moreover $\phi\partial^2_{xy}\phi > \partial_x\phi\partial_y\phi$ on Φ.

Proof. For $x > 0$, let $h(x) = \frac{\phi(x,y)}{f(x)}$. The function h is nonnegative and we have $h(0^+) = 0$ because $\lim_{x\to 0^+} f(x) = +\infty$ (see [1]) and $\lim_{x\to 0^+} \phi(x,y)$ exists thanks to the monotonicity assumption and is finite thanks to the concavity assumption made in (4). We have also $h(x) = 0$ for $x \geq X(y)$. Therefore the function h reaches its maximum at some $x^*_\sigma(y) \in (0, X(y))$. In particular we have $h'(x^*_\sigma(y)) = 0$ which also writes $F(x^*_\sigma(y), y) = 0$ where the function $F(x,y) = \frac{\phi(x,y)}{\partial_x\phi(x,y)} - \frac{f(x)}{f'(x)}$ is defined on Φ. This proves (9).
Now since $\partial^2_x\phi(x,y) \leq 0$ on $\{x < X(y)\}$ and $\frac{f(x)f''(x)}{f'(x)^2}$ is a positive function (see (7) for the convexity of f), $\partial_x F(x,y) = -\frac{\phi(x,y)\partial^2_x\phi(x,y)}{(\partial_x\phi(x,y))^2} + \frac{f(x)f''(x)}{f'(x)^2}$ is positive on $(0, X(y))$ which ensures uniqueness of $x^*_\sigma(y)$. The implicit function theorem, yields that x^*_σ is $\mathcal{C}^1$ in the neighborhood of y and $x^*_\sigma(y)' = -\frac{\partial_y F(x^*_\sigma(y),y)}{\partial_x F(x^*_\sigma(y),y)}$. Since $\partial_y F(x^*_\sigma(y), y) = \frac{\partial_x\phi(x^*_\sigma(y),y)\partial_y\phi(x^*_\sigma(y),y) - \phi(x^*_\sigma(y),y)\partial^2_{xy}\phi(x^*_\sigma(y),y)}{(\partial_x\phi(x^*_\sigma(y),y))^2}$, $x^*_\sigma(y)'$ is positive if $\phi\partial^2_{xy}\phi > \partial_x\phi\partial_y\phi$ on Φ. From (9) and the ODE (5) satisfied by f, one gets

$$\frac{f''(x^*_\sigma(y))}{f'(x^*_\sigma(y))} = \frac{2}{x^*_\sigma(y)^2\sigma^2(x^*_\sigma(y))}\left[r\frac{\phi(x^*_\sigma(y),y)}{\partial_x\phi(x^*_\sigma(y),y)} + (\delta - r)x^*_\sigma(y)\right]$$

so that we can express $\partial_x F(x^*_\sigma(y), y)$ only with the derivatives of ϕ and deduce (10).

When $x \geq x^*_\sigma(y)$, the optimality of τ^P_x follows from the arguments given in the proof of Theorem 1.4 [1]. Let us now assume that $x \in (0, x^*_\sigma(y))$ and set $\tau^x_z = \inf\{t \geq 0 : S^x_t = z\}$ for $z > 0$. Using the strong Markov property and the optimality result when the initial spot is $x^*_\sigma(y)$, then Fatou Lemma, we get for $\tau \in \mathcal{T}_{0,\infty}$,

$$\mathbb{E}[e^{-r\tau}\phi(S^x_\tau, y)] \leq \mathbb{E}[e^{-r\tau\wedge\tau^x_{x^*_\sigma(y)}}\phi(S^x_{\tau\wedge\tau^x_{x^*_\sigma(y)}}, y)] \leq \liminf_{t\to+\infty}\mathbb{E}[e^{-r\tau\wedge\tau^x_{x^*_\sigma(y)}\wedge t}\phi(S^x_{\tau\wedge\tau^x_{x^*_\sigma(y)}\wedge t}, y)].$$

By Itô's formula,

$$\begin{aligned} de^{-rt}\phi(S^x_t, y) =& e^{-rt}\left[\frac{1}{2}\sigma^2(S^x_t)(S^x_t)^2\partial^2_x\phi(S^x_t, y) + (r-\delta)S^x_t\partial_x\phi(S^x_t, y) - r\phi(S^x_t, y)\right]dt \\ &+ e^{-rt}\partial_x\phi(S^x_t, y)\sigma(S^x_t)S^x_t dW_t \end{aligned}$$

On $\{t < \tau^x_{x^*_\sigma(y)}\}$, we have $S^x_t < x^*_\sigma(y) < X(y)$. In case $r \geq \delta$, it is obvious that the drift term is nonpositive since it is a sum of three nonpositive terms. This ensures that for $t \geq 0$, $\mathbb{E}[e^{-r\tau\wedge\tau^x_{x^*_\sigma(y)}\wedge t}\phi(S^x_{\tau\wedge\tau^x_{x^*_\sigma(y)}\wedge t}, y)] \leq \phi(x,y)$ and $\tau^P_x = 0$ is optimal.
In case $r \leq \delta$, let us check that the drift term is still nonpositive by proving that the sum of the two last terms is nonpositive.
The function is $x \mapsto (r-\delta)x\partial_x\phi(x,y) - r\phi(x,y)$ is nondecreasing on $(0, X(y))$ since its derivative is equal to $-\delta\partial_x\phi(x,y) + (r-\delta)x\partial^2_x\phi(x,y)$. Thus, using (5) for the second

equality, we have for $x \leq x^*(y)$

$$\begin{aligned}
(r-\delta)x\partial_x\phi(x,y) - r\phi(x,y) &\leq (r-\delta)x^*_\sigma(y)\partial_x\phi(x^*_\sigma(y),y) - r\phi(x^*_\sigma(y),y) \\
&= \phi(x^*_\sigma(y),y)\left[(r-\delta)x^*_\sigma(y)\frac{f'(x^*_\sigma(y))}{f(x^*_\sigma(y))} - r\right] \\
&= -\frac{1}{2}\phi(x^*_\sigma(y),y)\sigma^2(x^*_\sigma(y))x^*_\sigma(y)^2\frac{f''(x^*_\sigma(y))}{f(x^*_\sigma(y))} < 0.
\end{aligned}$$

□

Proposition 1.2. *Let us fix a strike $x > 0$ and assume $Y(x) > 0$. If $Y(x) = +\infty$, then $\forall y > 0, c_\eta(y,x) = 0$. Otherwise there is a unique $y^*_\eta(x) \in (Y(x), +\infty)$ such that $\tau^c_x = \inf\{t \geq 0, \bar{S}^y_t \geq y^*_\eta(x)\}$ (convention $\inf \emptyset = +\infty$) is an optimal stopping time for c_η and:*

$$\forall y \geq y^*_\eta(x), c_\eta(y,x) = \phi(x,y), \ \forall y < y^*_\eta(x), c_\eta(y,x) = \frac{\phi(x,y^*_\eta(x))}{g(y^*_\eta(x))}g(y) > \phi(x,y). \tag{11}$$

In addition, we have $\frac{\phi(x,y^*_\eta(x))}{\partial_y\phi(x,y^*_\eta(x))} = \frac{g(y^*_\eta(x))}{g'(y^*_\eta(x))}$ and the smooth-fit principle holds. Last, $x \in \{z : 0 < Y(z) < \infty\} \mapsto y^*_\eta(x)$ is $\mathcal{C}^1$ and satisfies the following ODE:

$$\begin{aligned}
y^*_\eta(x)' &= \left[\partial^2_{xy}\phi(x,y^*_\eta(x)) - \frac{\partial_x\phi(x,y^*_\eta(x))\partial_y\phi(x,y^*_\eta(x))}{\phi(x,y^*_\eta(x))}\right] \\
&\times \frac{y^*_\eta(x)^2\eta^2(y^*_\eta(x))}{2[\delta\phi(x,y^*_\eta(x)) + (r-\delta)y^*_\eta(x)\partial_y\phi(x,y^*_\eta(x))] - y^*_\eta(x)^2\eta^2(y^*_\eta(x))\partial^2_y\phi(x,y^*_\eta(x))}.
\end{aligned} \tag{12}$$

It is strictly increasing if one assume moreover $\phi\partial_x\partial_y\phi > \partial_x\phi\partial_y\phi$ on $\{\phi(x,y) > 0\}$.

Proof. We introduce $h(y) = \frac{\phi(x,y)}{g(y)}$ which vanishes for $y \leq Y(x)$ and for $y = +\infty$. Indeed, the concavity ensures that $y \mapsto \phi(x,y)$ is bounded from above by some linear function and we have already shown in [1] that $g(y) \underset{y \to +\infty}{\geq} cy^{1+a}$ for some $a, c > 0$. We then obtain easily $\frac{\partial_y\phi(x,y^*_\eta(x))}{\phi(x,y^*_\eta(x))} = \frac{g'(y^*_\eta(x))}{g(y^*_\eta(x))}$. The uniqueness of $y^*_\eta(x) \in (Y(x), +\infty)$ and the optimality of τ^c_x can be checked by arguments similar to the ones given in the proof of Proposition 1.1. To obtain the ODE (12) satisfied by y^*_η, the calculations are the same as for (10) in Proposition 1.1 exchanging $r \leftrightarrow \delta$, $\sigma \leftrightarrow \eta$ and $\partial_x \leftrightarrow \partial_y$. □

Remark 1.3. *We incidentally obtain in the proof of Proposition 1.1 that*

$$\forall x \leq x^*_\sigma(y), (r-\delta)x\partial_x\phi(x,y) - r\phi(x,y) < 0.$$

Similarly,

$$\forall y \geq y^*_\eta(x), (\delta - r)y\partial_y\phi(x,y) - \delta\phi(x,y) < 0.$$

In particular, thanks to (4), the denominator of the second term in the r.h.s. of (10) (resp. (12)) is positive.

1.2 Estimates on the exercise boundaries

Now, we would like to get also estimations on the exercise boundaries. As in [1], we use a comparison to the Black-Scholes model with constant volatility for which estimations are easier to get.

Proposition 1.4. *Let us consider two volatility functions σ_1 and σ_2 (resp.η_1 and η_2) satisfying $(\mathcal{H}'_{\mathbf{vol}})$ such that $\forall x > 0, \sigma_1(x) \leq \sigma_2(x)$ (resp. $\forall x > 0, \eta_1(x) \leq \eta_2(x)$). Then, we have:*

$$\forall x, y > 0, P_{\sigma_1}(x, y) \leq P_{\sigma_2}(x, y) \text{ (resp. } \forall x, y > 0, c_{\eta_1}(y, x) \leq c_{\eta_2}(y, x) \text{)}$$

and we can compare the exercise boundaries:

$$\forall y > 0, x^*_{\sigma_2}(y) \leq x^*_{\sigma_1}(y) \text{ (resp. } \forall x > 0, y^*_{\eta_1}(x) \leq y^*_{\eta_2}(x) \text{).}$$

Proof. Let us focus on the put case. If $P_{\sigma_1}(x, y) = \phi(x, y)$, we have clearly $P_{\sigma_1}(x, y) \leq P_{\sigma_2}(x, y)$. Otherwise we have $P_{\sigma_1}(x, y) = \phi(x^*_{\sigma_1}(y), y)\mathbb{E}[e^{-r\tau^{x,\sigma_1}_{x^*_{\sigma_1}(y)}}]$ where for $i \in \{1, 2\}$, $\tau^{x,\sigma_i}_z = \inf\{t \geq 0 : S^{x,i}_t = z\}$ with $S^{x,i}_t$ solving (1) for the volatility function σ_i. Thanks to (7), we know that f_{σ_1} is a convex function. According to the proof of Proposition 1.9 [1], $\mathbb{E}[e^{-r\tau^{x,\sigma_1}_{x^*_{\sigma_1}(y)}}] \leq \mathbb{E}[e^{-r\tau^{x,\sigma_2}_{x^*_{\sigma_1}(y)}}]$. Therefore,

$$P_{\sigma_1}(x, y) \leq \phi(x^*_{\sigma_1}(y), y)\mathbb{E}[e^{-r\tau^{x,\sigma_2}_{x^*_{\sigma_1}(y)}}] \leq P_{\sigma_2}(x, y).$$

□

Proposition 1.5. *Let $\overline{\sigma}$ (resp. $\overline{\eta}$) denote an upper bound of the function $\sigma(.)$ (resp. $\eta(.)$). Then,*

$$\forall y > 0 \ s.t. \ X(y) > 0, \ \frac{a(\overline{\sigma})}{a(\overline{\sigma}) - 1} X(y) \leq x^*_\sigma(y) < X(y)$$

$$\left(resp. \ \forall x > 0 \ s.t. \ 0 < Y(x) < +\infty, \ Y(x) < y^*_\eta(x) \leq \frac{b(\overline{\eta})}{b(\overline{\eta}) - 1} Y(x) \right)$$

where $a(\varsigma) = \frac{\delta - r + \varsigma^2/2 - \sqrt{(\delta - r + \varsigma^2/2)^2 + 2r\varsigma^2}}{\varsigma^2}$ is an increasing function on $(0, +\infty)$ such that $\lim_{\varsigma \to +\infty} a(\varsigma) = 0$ *and*

$$\lim_{\varsigma \to 0} a(\varsigma) = \begin{cases} -\frac{r}{\delta - r} \ \textit{if } \delta > r \\ -\infty \ \textit{otherwise} \end{cases}$$

(resp. $b(\varsigma) = 1 - a(\varsigma) > 1$).

Proof. When $\delta = r$, the properties of $a(\varsigma) = \frac{1}{2} - \sqrt{\frac{1}{4} + \frac{2r}{\varsigma^2}}$ are obvious. Otherwise, $a(\varsigma) = A(\frac{\delta - r}{\varsigma^2})$ with $A(x) = x + \frac{1}{2} - \sqrt{(x + \frac{1}{2})^2 + \frac{2rx}{\delta - r}}$. Remarking that $\lim_{x \to -\infty} A(x) = -\infty$, $A(0) = 0$ and $\lim_{x \to +\infty} A(x) = -\frac{r}{\delta - r}$, one easily deduces the limits of $a(\varsigma)$ as ς tends to 0 or

$+\infty$. Since $A'(x) = \frac{\sqrt{(x+\frac{1}{2})^2+\frac{2rx}{\delta-r}}-\left(x+\frac{1}{2}+\frac{r}{\delta-r}\right)}{\sqrt{(x+\frac{1}{2})^2+\frac{2rx}{\delta-r}}}$ and $(x+\frac{1}{2})^2+\frac{2rx}{\delta-r}-\left(x+\frac{1}{2}+\frac{r}{\delta-r}\right)^2 = \frac{-r\delta}{(\delta-r)^2} \leq 0$, $A'(x)$ has the same sign as $-\left(x+\frac{\delta+r}{2(\delta-r)}\right)$. In particular A' is negative on $(0,+\infty)$ when $\delta > r$ and positive on $(-\infty, 0)$ when $\delta < r$. One easily deduces the monotonicity properties of a.

Let us deduce the estimation for the put case. Thanks to Proposition 1.4, we have $x^*_\sigma(y) \geq x^*_{\overline{\sigma}}(y)$. The solution of the EDO (5) with a volatility function constant equal to $\overline{\sigma}$ is $f_{\overline{\sigma}}(x) = x^{a(\overline{\sigma})}$. Let us consider the function $x \in (0, X(y)) \mapsto \frac{\phi(x,y)}{\partial_x\phi(x,y)} - \frac{f_{\overline{\sigma}}(x)}{f'_{\overline{\sigma}}(x)}$. Its derivative $\frac{-\phi(x,y)\partial^2_x\phi(x,y)}{(\partial_x\phi(x,y))^2} + \frac{f_{\overline{\sigma}}(x)f''_{\overline{\sigma}}(x)}{(f'_{\overline{\sigma}}(x))^2}$ is greater than $\frac{a(\overline{\sigma})-1}{a(\overline{\sigma})}$ since $\partial^2_x\phi \leq 0$ and $\frac{f_{\overline{\sigma}}(x)f''_{\overline{\sigma}}(x)}{(f'_{\overline{\sigma}}(x))^2} = \frac{a(\overline{\sigma})-1}{a(\overline{\sigma})}$. Integrating this inequality between $x^*_{\overline{\sigma}}(y)$ and $X(y)$ then using (9) and remarking that by (4), $\partial_x\phi(X(y)^-, y) < 0$ and $\frac{\phi}{\partial_x\phi}(X(y)^-, y) = 0$, we get $-\frac{1}{a(\overline{\sigma})}X(y) \geq \frac{a(\overline{\sigma})-1}{a(\overline{\sigma})}(X(y) - x^*_{\overline{\sigma}}(y))$ and thus:

$$x^*_{\overline{\sigma}}(y) \geq \frac{a(\overline{\sigma})}{a(\overline{\sigma})-1}X(y).$$

The proof for y^*_η works in the same way considering the function $y \mapsto \frac{\phi(x,y)}{\partial_y\phi(x,y)} - \frac{g_{\overline{\eta}}(y)}{g'_{\overline{\eta}}(y)}$.

□

Remark 1.6. *In the Call-Put case $\phi(x,y) = (y-x)^+$, since $\partial^2_x\phi(x,y) = 0$ for $x < X(y) = y$, under $(\mathcal{H}_{\mathbf{vol}})$ one obtains $x^*_\sigma(y) \leq \frac{a(\underline{\sigma})}{a(\underline{\sigma})-1}y$ by an easy adaptation of the arguments given in the proof of Proposition 1.5. In [1], this estimate combined with the ODE (10) derived below allowed us to characterise explicitly the set of exercise boundaries x^*_σ and get a one-to-one correspondence between the volatility functions satisfying $(\mathcal{H}_{\mathbf{vol}})$ and the exercise boundaries.*
For general payoff functions ϕ, because $\partial^2_x\phi$ does not vanish, we were not able to get under $(\mathcal{H}_{\mathbf{vol}})$ an upper-bound for x^*_σ better than $x^*_\sigma(y) < X(y)$ which already holds under $(\mathcal{H}'_{\mathbf{vol}})$. That is why we work with hypothesis $(\mathcal{H}'_{\mathbf{vol}})$ in the present paper.

2 Duality

Let us now investigate conditions ensuring

$$\forall x, y > 0,\ P_\sigma(x,y) = c_\eta(y,x). \tag{13}$$

First, in order to use the pricing formulas given in Propositions 1.1 and 1.2, we assume that for all $x > 0$, $Y(x) > 0$ condition which implies $X(0^+) = 0$.
Since $\Phi \neq \emptyset$, there exists $(x,y) \in \mathbb{R}^*_+ \times \mathbb{R}^*_+$ such that $\phi(x,y) > 0$. Then $X(y) > 0$ and $Y(x) < +\infty$, and by Propositions 1.1 and 1.2, the functions $z \mapsto P_\sigma(z,y)$ and $z \mapsto c_\eta(z,x)$ do not vanish on $(0,+\infty)$. If for some $y' \in (0,y)$, one had $X(y') = 0$, then ϕ and therefore P_σ would vanish on $(0,+\infty) \times (0,y']$. In particular P_σ would vanish on $\{x\} \times (0,y']$ preventing (13). In the same way, if one had $X(+\infty) < +\infty$, then c_η would vanish on

$(0,+\infty) \times [X(+\infty),+\infty)$ preventing (13). That is why we make the following assumption on X:

$$\forall y > 0, X(y) > 0, \ X(0^+) = 0 \text{ and } X(+\infty) = +\infty. \tag{14}$$

This assumption automatically ensures $Y(0^+) = 0$, $0 < Y(x) < +\infty$ for $x > 0$ and $Y(+\infty) = +\infty$. We are now able give a necessary and sufficient condition for (13) to hold.

Theorem 2.1. *Assume that ϕ satisfy (14) and that σ and η satisfy $(\mathcal{H}'_{\mathbf{vol}})$. Then,* (13) *holds if and only if x^*_σ and y^*_η are increasing reciprocal functions.*

Proof. This result can be checked by an immediate adaptation of the proof of Theorem 4.1 [1], except for the increasing property in the necessary condition that we explain here. The equality of the exercise regions writes

$$\{(x,y) \in (\mathbb{R}^*_+)^2 : x \le x^*_\sigma(y)\} = \{(x,y) \in (\mathbb{R}^*_+)^2 : y^*_\eta(x) \le y\}, \tag{15}$$

and thus $x \le x^*_\sigma(y^*_\eta(x))$. Therefore $(x', y^*_\eta(x))$ belongs to the exercise region for $x' \le x$ and we get $y^*_\eta(x') \le y^*_\eta(x)$. Similarly, x^*_σ is nondecreasing. Therefore, using Propositions 1.1, 1.2 and 1.5 , we get that x^*_σ and y^*_η are continuous nondecreasing functions from $\mathbb{R}^*_+$ onto $\mathbb{R}^*_+$. From (15), they are reciprocal functions. Since they are both continuous, they are increasing. $\square$

Let us recall here that under the following assumption on ϕ

$$\phi \partial^2_{xy}\phi > \partial_x\phi\partial_y\phi \text{ on } \Phi, \tag{16}$$

Propositions 1.1 and 1.2 ensure that the exercise boundaries are automatically increasing. We give a general class of functions ϕ that satisfy all the required assumptions.

Example 2.2. *Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing concave function $\mathcal{C}^2$ on $(0,+\infty)$ and such that $\psi(0) = \psi(0^+) = 0$, $\psi_x : \mathbb{R}^*_+ \to \mathbb{R}^*_+$ (resp. $\psi_y : \mathbb{R}^*_+ \to \mathbb{R}^*_+$) be a C^2 increasing convex (resp. concave) function such that $\psi_x(0^+) = 0$ (resp. $\psi_y(0^+) = 0$ and $\psi_y(+\infty) = +\infty$). Then the function $\phi(x,y) = \psi((\psi_y(y) - \psi_x(x))^+)$ satisfies* (4). *It is such that $X(y) = \psi_x^{-1}(\psi_y(y))$, $Y(x) = \psi_y^{-1}(\psi_x(x))$ and (14) and (16) hold.*

For some specific payoff functions of this family, we are now going to state conditions on σ and η such that x^*_σ and y^*_η are reciprocal functions. We first recall results obtained in [1] in the call-put case $\phi(x,y) = (y-x)^+$. Then, we address two generalizations : $\phi(x,y) = (\psi_y(y) - \psi_x(x))^+$ and $\phi(x,y) = (y-x)^{+\gamma}$.

2.1 The call-put case $\phi(x,y) = (y-x)^+$

Let us recall here the main result obtained in [1]:

Theorem 2.3. *Let us consider two volatility functions satisfying* $(\mathcal{H}_{\mathbf{vol}})$. *The following conditions are equivalent:*
(1) $\forall x, y > 0, \ P_\sigma(x,y) = c_\eta(y,x)$.
(2) $\eta \equiv \tilde{\sigma}$ *where* $\tilde{\sigma}(y) = 2(y - x^*_\sigma(y))(ry - \delta x^*_\sigma(y))/[y x^*_\sigma(y)\sigma(x^*_\sigma(y))]$.
(3) $\sigma \equiv \underset{\sim}{\eta}$ *where* $\underset{\sim}{\eta}(x) = 2(y^*_\eta(x) - x)(r y^*_\eta(x) - \delta x)/[y^*_\eta(x) x \eta(y^*_\eta(x))]$.

As proved in [1], if σ (resp. η) satisfies ($\mathcal{H}_{\mathbf{vol}}$), then $\tilde{\sigma}$ (resp. $\underset{\sim}{\eta}$) also satisfies ($\mathcal{H}_{\mathbf{vol}}$). This very convenient property ensures that for a given volatility function σ satisfying ($\mathcal{H}_{\mathbf{vol}}$), there always exists a dual volatility function η also satisfying ($\mathcal{H}_{\mathbf{vol}}$) such that condition (1) above holds. Unfortunately, for the more general payoff functions that we consider in the sequel, stability of the Hypotheses ($\mathcal{H}_{\mathbf{vol}}$) or ($\mathcal{H}'_{\mathbf{vol}}$) is no longer straightforward. And it may happen that no dual volatility function η can be associated with σ.

2.2 The case $\phi(x, y) = (\psi_y(y) - \psi_x(x))^+$

In this section, we will focus on the case $\phi(x, y) = (\psi_y(y) - \psi_x(x))^+$ where $\psi_y : \mathbb{R}^*_+ \to \mathbb{R}^*_+$ (resp. $\psi_x : \mathbb{R}^*_+ \to \mathbb{R}^*_+$) is a C^2 increasing concave (resp. convex) function such that $\psi_y(0^+) = 0$ and $\psi_y(+\infty) = +\infty$ (resp. $\psi_x(0^+) = 0$). Then one has $X(y) = \psi_x^{-1}(\psi_y(y))$ and $Y(x) = \psi_y^{-1}(\psi_x(x))$.

Let us first give an example of application of Theorem 2.1 when ψ_x and ψ_y are power functions and the local volatility functions σ and η are constant.

Example 2.4. *Let us suppose that $\phi(x, y) = (y^{\gamma'} - x^{\gamma})^+$ where $\gamma' \in (0, 1]$ and $\gamma \geq 1$. When the local volatility function σ is a constant and equal to ς, $f(x) = x^{a(\varsigma)}$ with $a(\varsigma)$ given in Proposition 1.5. The equality $\frac{\partial_x \phi(x^*_\sigma(y), y)}{\phi(x^*_\sigma(y), y)} = \frac{f'(x^*_\sigma(y))}{f(x^*_\sigma(y))}$ then yields $x^*_\sigma(y) = \left(\frac{a(\varsigma)}{a(\varsigma)-\gamma}\right)^{1/\gamma} y^{\gamma'/\gamma}$. In the same way, for η constant equal to ν, as $g(x) = x^{b(\nu)}$ with $b(\nu) = 1 - a(\nu)$, $y^*_\eta(x) = \left(\frac{b(\nu)}{b(\nu)-\gamma'}\right)^{1/\gamma'} x^{\gamma/\gamma'}$. These boundaries are reciprocal functions as soon as*

$$\gamma' a(\varsigma) + \gamma b(\nu) = \gamma\gamma'.$$

According to Proposition 1.5, when $r \geq \delta$, for fixed $\varsigma \in (0, +\infty)$ this equation admits a solution $\nu \in (0, +\infty)$ iff $\varsigma < a^{-1}(\gamma(1 - \frac{1}{\gamma'}))$ and it admits a solution $\varsigma \in (0, +\infty)$ for any fixed $\nu \in (0, +\infty)$. When $\delta > r$, there is no solution if $\gamma(1 - \frac{1}{\gamma'}) \leq -\frac{r}{\delta - r}$ and otherwise it admits a solution ν for fixed $\varsigma < a^{-1}(\gamma(1 - \frac{1}{\gamma'}))$ and a solution ς for fixed $\nu > b^{-1}(\gamma'(1 + \frac{r}{\gamma(\delta-r)}))$.

For general functions ψ_x and ψ_y, we are able to investigate uniqueness for the ODEs (10) and (12) which respectively write :

$$\begin{aligned} x^*_\sigma(y)' &= \frac{\psi'_x(x^*_\sigma(y))\psi'_y(y)}{\psi_y(y) - \psi_x(x^*_\sigma(y))} && (17)\\ &\times \frac{x^*_\sigma(y)^2\sigma^2(x^*_\sigma(y))}{2[r(\psi_y(y) - \psi_x(x^*_\sigma(y))) + (r-\delta)x^*_\sigma(y)\psi'_x(x^*_\sigma(y))] + x^*_\sigma(y)^2\sigma^2(x^*_\sigma(y))\psi''_x(x^*_\sigma(y))} \\ y^*_\eta(x)' &= \frac{\psi'_x(x)\psi'_y(y^*_\eta(x))}{\psi_y(y^*_\eta(x)) - \psi_x(x)} && (18)\\ &\times \frac{y^*_\eta(x)^2\eta^2(y^*_\eta(x))}{2[\delta(\psi_y(y^*_\eta(x)) - \psi_x(x)) + (r-\delta)y^*_\eta(x)\psi'_y(y^*_\eta(x))] - y^*_\eta(x)^2\eta^2(y^*_\eta(x))\psi''_y(y^*_\eta(x))}. \end{aligned}$$

Proposition 2.5. *When η satisfies $(\mathcal{H}'_{\mathbf{vol}})$, the boundary $y^*_\eta(x)$ is the unique solution $y(x)$ of (18) on $\mathbb{R}^*_+$ that is increasing, such that $Y(x) < y(x)$ and $y(0^+) = 0$.*

Proof. Let us consider $y_1(x)$ and $y_2(x)$, two solutions of (18) on $\mathbb{R}^*_+$ that are increasing and such that $y_i(x) > Y(x)$ and $y_i(0^+) = 0$ for $i \in \{1, 2\}$. In particular, y_1 and y_2 are bijections on $\mathbb{R}^*_+$ and we may define $\tilde{I}(x) = y_1^{-1}(y_2(x))/x$. By an easy computation, one checks

$$\tilde{I}'(x) = \frac{1}{x}\left(\frac{\psi'_x(x)}{\psi'_x(x\tilde{I}(x))} \times \left[1 + \frac{\psi_x(x) - \psi_x(x\tilde{I}(x))}{\psi_y(y_2(x)) - \psi_x(x)}\right]\right.$$
$$\left.\times \left[1 + \frac{2\delta[\psi_x(x) - \psi_x(x\tilde{I}(x))]}{2[\delta(\psi_y(y_2(x)) - \psi_x(x)) + (r-\delta)y_2(x)\psi'_y(y_2(x))] - y_2(x)^2\eta^2(y_2(x))\psi''_y(y_2(x))}\right] - \tilde{I}(x)\right).$$

The constant 1 is clearly solution to this equation and we want to check that $\tilde{I} \equiv 1$. Let us suppose that $\tilde{I} \not\equiv 1$. Thanks to the Cauchy-Lipschitz theorem, it induces that either $\forall x > 0, \tilde{I}(x) > 1$ or $\forall x > 0, \tilde{I}(x) < 1$. Let us suppose $\forall x > 0, \tilde{I}(x) < 1$. Then, it is easy to see from the last expression that

$$\forall x > 0,\ \tilde{I}'(x) \geq \frac{1}{x}(1 - \tilde{I}(x)).$$

Indeed, since ψ'_x is non decreasing, we have $\psi'_x(x) \geq \psi'_x(x\tilde{I}(x))$ and the terms into brackets are also greater than 1 because ψ_x is increasing and both denominators are nonnegative as $\psi_y(y_2(x)) > \psi_x(x)$, $y'_2(x) \geq 0$ and y_2 solves (18). In particular we have shown that $\tilde{I}'(x) > 0$ and therefore,

$$\forall x \in (0, 1),\ \tilde{I}'(x) \geq \frac{1}{x}(1 - \tilde{I}(1)).$$

Thus, we get $\tilde{I}(1) - \tilde{I}(x) \geq (\tilde{I}(1) - 1)\ln(x) \underset{x\to 0^+}{\to} +\infty$ and so $\tilde{I}(x) \underset{x\to 0^+}{\to} -\infty$ which is contradictory since $\forall x > 0, \tilde{I}(x) > 0$.

When $\forall x > 0, \tilde{I}(x) > 1$, considering $y_2^{-1}(y_1(x))/x$ instead of $\tilde{I}(x)$, we get the same contradiction as previously. □

Now let us turn to the uniqueness result on the boundary $x^*_\sigma(y)$.

Proposition 2.6. *Let σ satisfy $(\mathcal{H}'_{\mathbf{vol}})$ and ψ_x be such that :*

$$\forall \alpha \in (0, 1), \exists C_\alpha > 0, \forall x > 0,\ \psi_x(\alpha x) \geq C_\alpha \psi_x(x). \tag{19}$$

The boundary $x^*_\sigma(y)$ is the unique solution $x(y)$ of (17) on $\mathbb{R}^*_+$ that is increasing and such that $\exists \alpha \in (0, 1)$,

$$\forall y > 0,\ \alpha X(y) \leq x(y) < X(y). \tag{20}$$

Hypothesis (19) is satisfied by the function x^a with $a \geq 1$ but not by the function $\exp(bx) - 1$ with $b > 0$.

Proof. The boundary $x^*_\sigma(y)$ satisfies (20) with $\alpha = \frac{a(\overline{\sigma})}{a(\overline{\sigma})-1}$ according to Proposition 1.5. Let x_1 and x_2 denote two solutions of (17) satisfying (20) with respective constants $\alpha_1, \alpha_2 \in (0,1)$ and $\hat{I}(y) = \psi_y(x_1^{-1}(x_2(y)))/\psi_y(y)$. One has

$$\hat{I}'(y) = \frac{\psi_y'(y)}{\psi_y(y)} \times \left(\left[\frac{\hat{I}(y) - \psi_x(x_2(y))/\psi_y(y)}{1-\psi_x(x_2(y))/\psi_y(y)} \right] \right.$$
$$\left. \times \left[1 + \frac{2r\psi_y(y)(\hat{I}(y)-1)}{2[r(\psi_y(y)-\psi_x(x_2(y))) + (r-\delta)x_2(y)\psi_x'(x_2(y))] + x_2(y)^2\sigma^2(x_2(y))\psi_x''(x_2(y))} \right] - \hat{I}(y) \right).$$

Let us suppose that $\hat{I}(y) \not\equiv 1$. Thanks to the Cauchy-Lipschitz theorem, we have either $\forall y > 0, \hat{I}(y) > 1$ or $\forall y > 0, \hat{I}(y) < 1$. Let us suppose that $\forall y > 0, \hat{I}(y) > 1$. As in the last proof, the second bracket is greater than 1, and we get

$$\forall y > 0,\ \hat{I}'(y) \geq \frac{\psi_y'(y)}{\psi_y(y)}(\hat{I}(y)-1)\frac{\psi_x(x_2(y))/\psi_y(y)}{1-\psi_x(x_2(y))/\psi_y(y)}.$$

Since $x_2(y) < X(y) = (\psi_x)^{-1}(\psi_y(y))$, we have $0 < \psi_x(x_2(y))/\psi_y(y) < 1$ and therefore $\hat{I}'(y) > 0$. Since x_2 satisfies (20) with constant α_2, we get by (19)

$$\forall y > 0,\ \psi_x(x_2(y)) \geq \psi_x(\alpha_2 X(y)) \geq C_{\alpha_2}\psi_y(y).$$

Since $z \mapsto \frac{z}{1-z} = -1 + \frac{1}{1-z}$ is increasing on $(0,1)$ and $\hat{I}$ is increasing, we deduce that

$$\forall y \geq 1,\ \hat{I}'(y) \geq \frac{\psi_y'(y)}{\psi_y(y)}(\hat{I}(1)-1)\frac{C_{\alpha_2}}{1-C_{\alpha_2}}.$$

As a consequence,

$$\hat{I}(y) - \hat{I}(1) \geq (\hat{I}(1)-1)\frac{C_{\alpha_2}}{1-C_{\alpha_2}} \ln\left(\frac{\psi_y(y)}{\psi_y(1)}\right) \underset{y\to+\infty}{\longrightarrow} +\infty.$$

In the same time, since x_1 satisfies (20) with constant α_1, we have $X(y) = (\psi_x)^{-1}(\psi_y(y)) \leq \frac{1}{\alpha_1}x_1(y)$ and therefore $x_1^{-1}(x) \leq (\psi_y)^{-1}(\psi_x(\frac{x}{\alpha_1}))$. We get $\psi_y(x_1^{-1}(x_2(y))) \leq \psi_x(\frac{x_2(y)}{\alpha_1}) \leq \frac{\psi_x(x_2(y))}{C_{\alpha_1}} < \frac{\psi_y(y)}{C_{\alpha_1}}$ and thus $\hat{I}(y) \leq \frac{1}{C_{\alpha_1}}$ which is contradictory with $\hat{I}(+\infty) = +\infty$.

When $\forall y > 0, \hat{I}(y) < 1$, considering $\psi_y(x_2^{-1}(x_1(y)))/\psi_y(y)$ instead of $\hat{I}(y)$, we reach the same contradiction as previously. □

Like in the call-put case, we are now able to state a more precise duality result.

Theorem 2.7. *Let us assume that σ and η satisfy $(\mathcal{H}'_{\mathbf{vol}})$ and set*
$A(y) = \left[\frac{\psi_y(y)-\psi_x(x^*_\sigma(y))}{\psi_x'(x^*_\sigma(y))\psi_y'(y)}\right]^2 \times \frac{2[r(\psi_y(y)-\psi_x(x^*_\sigma(y)))+(r-\delta)x^*_\sigma(y)\psi_x'(x^*_\sigma(y))]+x^*_\sigma(y)^2\sigma^2(x^*_\sigma(y))\psi_x''(x^*_\sigma(y))}{x^*_\sigma(y)^2\sigma^2(x^*_\sigma(y))}$,
$B(x) = \left[\frac{\psi_y(y^*_\eta(x))-\psi_x(x)}{\psi_x'(x)\psi_y'(y^*_\eta(x))}\right]^2 \times \frac{2[\delta(\psi_y(y^*_\eta(x))-\psi_x(x))+(r-\delta)y^*_\eta(x)\psi_y'(y^*_\eta(x))]-y^*_\eta(x)^2\eta^2(y^*_\eta(x))\psi_y''(y^*_\eta(x))}{y^*_\eta(x)^2\eta^2(y^*_\eta(x))}$ *which*

are positive functions according to Remark 1.3. Then, the following assertions are equivalent :
(1) $\forall x, y > 0,\ P_\sigma(x,y) = c_\eta(y,x)$.
(2) $\forall y > 0$, $\min[1+\psi_y''(y)A(y), \delta(\psi_y(y)-\psi_x(x_\sigma^*(y))) + (r-\delta)y\psi_y'(y)] > 0$ *and* $\eta \equiv \tilde{\sigma}$ *where*

$$\tilde{\sigma}(y) = \frac{1}{y}\sqrt{2[\delta(\psi_y(y)-\psi_x(x_\sigma^*(y))) + (r-\delta)y\psi_y'(y)]\frac{A(y)}{1+\psi_y''(y)A(y)}}. \tag{21}$$

If one assumes moreover that ψ_x *satisfies (19), they are also equivalent to*
(3) $\forall x > 0$, $\min[1-\psi_x''(x)B(x), r(\psi_y(y_\eta^*(x)) - \psi_x(x)) + (r-\delta)x\psi_x'(x)] > 0$ *and* $\sigma \equiv \underset{\sim}{\eta}$ *where*

$$\underset{\sim}{\eta}(x) = \frac{1}{x}\sqrt{2[r(\psi_y(y_\eta^*(x)) - \psi_x(x)) + (r-\delta)x\psi_x'(x)]\frac{B(x)}{1-\psi_x''(x)B(x)}}. \tag{22}$$

Notice that one easily recovers the call-put formulas given in Theorem 2.3 if one takes $\psi_x(x) = x$ and $\psi_y(y) = y$.

Proof. Since the payoff function satisfies (16), by Theorem 2.1 the assertion (1) is equivalent to the reciprocity of the functions x_σ^* and y_η^*. Therefore the implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are obtained by combining respectively $(y_\eta^*)'(x_\sigma^*(y))x_\sigma^*(y)' = 1$ and $(x_\sigma^*)'(y_\eta^*(x))y_\eta^*(x)' = 1$ with (17) and (18), the positivity of the terms between brackets in (21) and (22) coming from Remark 1.3.
Let us prove (2) $\Rightarrow$ (1). Computing $(x_\sigma^{*-1})'(x)$ thanks to (17), then using (21) written at the point $y = x_\sigma^{*-1}(x)$, we check that x_σ^{*-1} solves the same ODE as y_η^*. Since x_σ^{*-1} is increasing and $x_\sigma^{*-1}(0^+) = 0$, we conclude by Proposition 2.5 that $x_\sigma^{*-1} \equiv y_\eta^*$.

To prove (3) $\Rightarrow$ (1), we check in the same manner that $y_\eta^{*-1}(y)$ solves the same ODE as $x_\sigma^*(y)$. The function $y_\eta^*(x)$ is increasing and according to Proposition 1.5, $y_\eta^*(x) \le \beta Y(x)$ for $\beta = \frac{b(\overline{\eta})}{b(\overline{\eta})-1} > 1$. With the concavity of ψ_y and ψ_x^{-1} combined with $\psi_y(0^+) = \psi_x^{-1}(0^+) = 0$, this ensures

$$\forall y > 0,\ (y_\eta^*)^{-1}(y) \ge X(y/\beta) = (\psi_x)^{-1}(\psi_y(y/\beta)) \ge (\psi_x)^{-1}(\psi_y(y)/\beta) \ge X(y)/\beta.$$

By Proposition 2.6, we conclude that $(y_\eta^*)^{-1} \equiv x_\sigma^*$. $\square$

To give an analytical example of non constant dual volatility functions, we now assume that $\boxed{\phi(x,y) = (\alpha y - x^\gamma)^+}$ with $\alpha > 0$ and $\gamma \ge 1$. For $a, b, c > 0$, we introduce the reciprocal functions

$$y^*(x) = \frac{1}{\alpha}x^\gamma \frac{x^\gamma + a}{bx^\gamma + c} \text{ and } x^*(y) = \left[\frac{1}{2}\left(b\alpha y - a + \sqrt{(b\alpha y - a)^2 + 4c\alpha y}\right)\right]^{1/\gamma}.$$

Under some assumptions on the coefficients a, b and c, these functions are the exercise boundaries associated with explicit dual volatility functions.

Proposition 2.8. *Let us assume that either* $r \geq \delta$ *and* $\max(c/a, b) \leq 1$ *with* $\min(c/a, b) < 1$ *or* $r < \delta$ *and* $\max(c/a, b) \leq \frac{1}{1+(\delta/r-1)\gamma}$ *with* $\min(c/a, b) < \frac{1}{1+(\delta/r-1)\gamma}$*. Let us also assume* $(\gamma-1)b(2c-a)+c(\gamma+1) \geq 0$*. Then, the volatility functions*

$$\begin{aligned}
\sigma(x) &= \frac{1}{x}\sqrt{2[r(\alpha y^*(x)-x^\gamma)+(r-\delta)\gamma x^\gamma]\frac{B(x)}{1-\gamma(\gamma-1)x^{\gamma-2}B(x)}} \\
& \quad \text{with } B(x) = \frac{1}{y^*(x)'}\left[\frac{\alpha y^*(x)-x^\gamma}{\alpha\gamma x^{\gamma-1}}\right], \text{ and} \\
\eta(y) &= \frac{1}{y}\sqrt{2[r\alpha y-\delta x^*(y)^\gamma]A(y)} \text{ with } A(y) = \frac{1}{x^*(y)'}\left[\frac{\alpha y - x^*(y)^\gamma}{\alpha\gamma x^*(y)^{\gamma-1}}\right]
\end{aligned}$$

are well defined and satisfy $(\mathcal{H}'_{\mathbf{vol}})$*. Moreover, we have* $y^*_\eta \equiv y^*$ *and* $x^*_\sigma \equiv x^*$ *and thus the duality holds:* $\forall x, y > 0,\ P_\sigma(x, y) = c_\eta(y, x)$*.*

When $r \geq \delta$, it is easy to fulfill the required assumptions by taking for example a, b and c such that $b < 1$ and $1/2 \leq c/a < 1$. When $r < \delta$, the first condition is satisfied if $\max(c/a, b) < \frac{1}{1+(\delta/r-1)\gamma}$ and the second condition can be rewritten $2 \geq \frac{a}{c} - \frac{1}{b}\frac{\gamma+1}{\gamma-1}$. Thus taking for example $b < \frac{1}{1+(\delta/r-1)\gamma}$ and then $\frac{c}{a} = b\frac{\gamma-1}{\gamma+1}$, one can get dual volatility functions.

Proof. First step: let us check that the functions σ *and* η *are well defined and satisfy* $(\mathcal{H}'_{\mathbf{vol}})$. Since we have $\max(c/a, b) \leq 1$ and $\min(c/a, b) < 1$, we get $y^*(x) > \frac{1}{\alpha}x^\gamma = Y(x)$ (and thus $x^*(y) < (\alpha y)^{1/\gamma} = X(y)$). Since $y^*(x)' > 0$ (and thus $x^*(y)' = 1/y^{*\prime}(x^*(y)) > 0$), this ensures $B(x) > 0$ and $A(y) > 0$. For $r \geq \delta$, it is then clear that $r(\alpha y^*(x) - x^\gamma) + (r-\delta)\gamma x^\gamma > 0$ and $r\alpha y - \delta x^*(y)^\gamma > 0$. For $\delta > r$, the condition $\max(c/a, b) \leq \frac{1}{1+(\delta/r-1)\gamma}$ and $\min(c/a, b) < \frac{1}{1+(\delta/r-1)\gamma}$ ensures that $r(\alpha y^*(x) - x^\gamma) + (r-\delta)\gamma x^\gamma > 0$, but also $r\alpha y - \delta x^*(y)^\gamma > 0$ (or equivalently $r\alpha y^*(x) - \delta x^\gamma > 0$) since $\frac{1}{1+(\delta/r-1)\gamma} \leq r/\delta$ for $\gamma \geq 1$. Thus, η is well defined and positive. Since

$$y^*(x)' = \frac{1}{\alpha}\gamma x^{\gamma-1}\frac{bx^{2\gamma}+2cx^\gamma+ac}{(bx^\gamma+c)^2},$$

we get after some calculations that $\frac{B(x)}{1-\gamma(\gamma-1)x^{\gamma-2}B(x)}$ is equal to

$$\frac{1}{\gamma}x^{2-\gamma}\frac{(bx^\gamma+c)((1-b)x^\gamma+a-c)}{b(1+(\gamma-1)b)x^{2\gamma}+((\gamma-1)b(2c-a)+c(\gamma+1))x^\gamma+c(\gamma c+a-c)}$$

and is positive because we have assumed $(\gamma-1)b(2c-a)+c(\gamma+1) \geq 0$ (all other terms are positive). Thus σ is well defined and we have

$$\sigma(x) = \sqrt{\frac{2}{\gamma}\frac{\left[r(\frac{x^\gamma+a}{bx^\gamma+c}-1)+(r-\delta)\gamma\right](bx^\gamma+c)((1-b)x^\gamma+a-c)}{b(1+(\gamma-1)b)x^{2\gamma}+((\gamma-1)b(2c-a)+c(\gamma+1))x^\gamma+c(\gamma c+a-c)}}. \quad (23)$$

that is clearly bounded from above. To see that η is also bounded from above we calculate

$$\eta(y) = \sqrt{2\frac{r\alpha y - \delta x^*(y)^\gamma}{y} \times \frac{\alpha y - x^*(y)^\gamma}{\alpha^2 y} \times \frac{bx^*(y)^{2\gamma} + 2cx^*(y)^\gamma + ac}{(bx^*(y)^\gamma + c)^2}}$$

using that $1/x^{*\prime}(y) = y^{*\prime}(x^*(y))$.

Second step: We have $\frac{1}{\alpha}x^\gamma < y^*(x) \leq \max(\frac{1}{b}, \frac{a}{c})\frac{1}{\alpha}x^\gamma$ and thus $\max(\frac{1}{b}, \frac{a}{c})^{-1/\gamma}(\alpha y)^{1/\gamma} \leq x^*(y) < (\alpha y)^{1/\gamma}$. From the definition of σ, we get $B(x) = \frac{x^2\sigma^2(x)}{2[r(\alpha y^*(x) - x^\gamma) + (r-\delta)\gamma x^\gamma] + \gamma(\gamma-1)x^\gamma\sigma^2(x)}$. Combining this equality for $x = x^*(y)$ with the definition of B, we deduce that x^* solves the ODE (17). In the same manner, we show that y^* solves the ODE (18). Thanks to Propositions 2.5 and 2.6, we conclude that $y^* \equiv y^*_\eta$ and $x^* \equiv x^*_\sigma$. □

Remark 2.9. *For $b = 1$, we get cases where σ and η satisfy $(\mathcal{H}'_{\mathbf{vol}})$ but not $(\mathcal{H}_{\mathbf{vol}})$ since $\eta(y^*(x)) = \sqrt{\frac{2}{\alpha}\frac{(r-\delta b)x^\gamma + ra - \delta c}{x^\gamma + a} \times \frac{(bx^{2\gamma} + 2cx^\gamma + ac)((1-b)x^\gamma + a - c)}{(x^\gamma + a)(bx^\gamma + c)^2}}$. For $\delta > r$, $b = \frac{1}{1 + (\delta/r - 1)\gamma}$ and $\gamma > 1$ we get cases where η satisfies $(\mathcal{H}_{\mathbf{vol}})$ and σ satisfies $(\mathcal{H}'_{\mathbf{vol}})$ but not $(\mathcal{H}_{\mathbf{vol}})$ (see (23)). If we have $\max(c/a, b) < 1$ when $r \geq \delta$ or $\max(c/a, b) < \frac{1}{1+(\delta/r-1)\gamma}$ when $r < \delta$, one can check that σ and η satisfy $(\mathcal{H}_{\mathbf{vol}})$.*

We have plotted in Figure 1 an example that illustrates the duality. We have computed prices of American options with finite maturity T, precisely $\sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}\left[e^{-r\tau}\phi(S^x_\tau, y)^+\right]$ and $c_{\tilde{\sigma}}(T, y, x) = \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}\left[e^{-\delta\tau}\phi(x, \overline{S}^y_\tau)\right]$ where the supremum is taken over $\mathcal{T}_{0,T}$, the set of stopping times almost surely smaller than T. We see that both converge to the same limit when T is large.

2.3 The case $\phi(x, y) = (y - x)^{+\gamma}$, $\gamma \in (0, 1]$

Let us first give an example of application of Theorem 2.1 for this payoff when the local volatility functions σ and η are constant.

Example 2.10. *When the local volatility function σ is a constant and equal to ς, $f(x) = x^{a(\varsigma)}$ with $a(\varsigma)$ given in Proposition 1.5. The equality $\frac{\partial_x\phi(x^*_\sigma(y), y)}{\phi(x^*_\sigma(y), y)} = \frac{f'(x^*_\sigma(y))}{f(x^*_\sigma(y))}$ then yields $x^*_\sigma(y) = \frac{a(\varsigma)}{a(\varsigma) - \gamma}y$. In the same way, for η constant equal to ν, as $g(x) = x^{b(\nu)}$ with $b(\nu) = 1 - a(\nu)$, $y^*_\eta(x) = \frac{b(\nu)}{b(\nu) - \gamma}x$. These boundaries are reciprocal functions as soon as*

$$a(\varsigma) + b(\nu) = \gamma.$$

According to Proposition 1.5, when $r \geq \delta$, for fixed $\varsigma \in (0, +\infty)$ this equation admits a solution $\nu \in (0, +\infty)$ iff $\varsigma < a^{-1}(\gamma - 1)$ and it admits a solution $\varsigma \in (0, +\infty)$ for any fixed $\nu \in (0, +\infty)$. When $\delta > r$, there is no solution if $\gamma - 1 \leq -\frac{r}{\delta - r}$ and otherwise it admits a solution ν for fixed $\varsigma < a^{-1}(\gamma - 1)$ and a solution ς for fixed $\nu > b^{-1}(\gamma + \frac{r}{\delta - r})$.

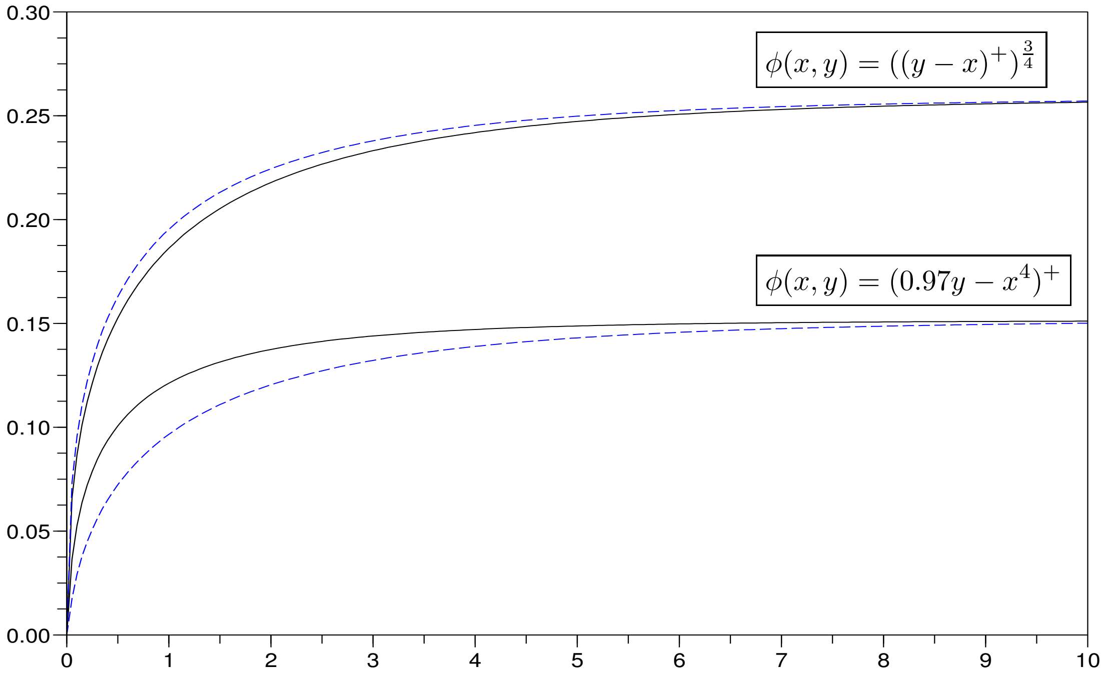

Figure 1: $P_\sigma(T,x,y)$ (solid line) and $c_{\tilde{\sigma}}(T,y,x)$ (dashed line) in function of the time T for $a = 1.5$, $b = 5/9$, $c = 1$, $x = 1$, $y = 0.99$, $r = 0.2$ and $\delta = 0.1$.

For the particular choice $\phi(x,y) = (y-x)^{+\gamma}$ the ODEs (10) and (12) write

$$x^*_\sigma(y)' = \frac{\gamma x^*_\sigma(y)^2\sigma^2(x^*_\sigma(y))}{2[r(y-x^*_\sigma(y))^2 + \gamma(r-\delta)x^*_\sigma(y)(y-x^*_\sigma(y))] + \gamma(1-\gamma)x^*_\sigma(y)^2\sigma^2(x^*_\sigma(y))} \qquad (24)$$

$$y^*_\eta(x)' = \frac{\gamma y^*_\eta(x)^2\eta^2(y^*_\eta(x))}{2[\delta(y^*_\eta(x)-x)^2 + \gamma(r-\delta)y^*_\eta(x)(y^*_\eta(x)-x)] + \gamma(1-\gamma)y^*_\eta(x)^2\eta^2(y^*_\eta(x))}. \qquad (25)$$

Since $Y(x) = x$, $y^*_\eta(x) > x$ and by Remark 1.3, $\delta(y^*_\eta(x) - x) + \gamma(r-\delta)y^*_\eta(x) > 0$. It turns out that uniqueness holds for the ODE (25) under these conditions.

Proposition 2.11. *When η satisfies $(\mathcal{H}'_{\mathbf{vol}})$, the boundary $y^*_\eta(x)$ is the unique solution $y(x)$ of (25) on $\mathbb{R}^*_+$ that is increasing, such that $y(0^+) = 0$ and $\min[y(x) - x, \delta(y(x)-x) + \gamma(r-\delta)y(x)] > 0$ for all $x > 0$.*

Proof. Let $y_1(x)$ and $y_2(x)$ denote two solutions of (25) satisfying the above hypotheses and $\tilde{I}(x) = \frac{y_1^{-1}(y_2(x))}{x}$. We have $\tilde{I}'(x) = \frac{1}{x}\left(\frac{F(x\tilde{I}(x), y_2(x))}{F(x, y_2(x))} - \tilde{I}(x)\right)$ where

$$F(z,y) = 2(y-z)[\delta(y-z) + \gamma(r-\delta)y] + \gamma(1-\gamma)y^2\eta^2(y).$$

Writing the estimations satisfied by y_1 (resp. y_2) at $y_1^{-1}(y_2(x))$ (resp. x) one obtains $F(x\tilde{I}(x), y_2(x)) > 0$ (resp. $F(x, y_2(x)) > 0$). Moreover, since $\partial_z F(z,y) = -2[2\delta(y-z) + \gamma(r-\delta)y]$ both $x\tilde{I}(x)$ and x belong to the interval $(0, \frac{(2\delta+\gamma(r-\delta))y_2(x)}{2\delta})$ on which $z \mapsto F(z, y_2(x))$ is decreasing. One easily concludes by the same argument as in the proof of Proposition 2.5. □

Proposition 2.12. *If σ satisfies $(\mathcal{H}'_{\mathbf{vol}})$ and $\max\left(r-\delta, \frac{(\delta-r)(\gamma\delta+(1-\gamma)r)}{(1-\gamma)\delta+\gamma r}\right) > \frac{(1-\gamma)\bar{\sigma}^2}{2}$, then $x^*_\sigma(y)$ is the unique solution $x(y)$ of (24) on $\mathbb{R}^*_+$ that is increasing and such that $\exists \varepsilon > 0$, $\forall y > 0$, $\varepsilon y < x(y) < \min\left(1, \frac{(1-\gamma)\delta+\gamma r}{\delta}\right) y$.*

Proof. By the convexity of $x \mapsto 1/x$, one has $r/(\gamma\delta + (1-\gamma)r) \le r(\gamma/\delta + (1-\gamma)/r) = (\gamma r + (1-\gamma)\delta)/\delta$. Therefore, using Remark 1.3 for the first inequality, one deduces

$$x^*_\sigma(y) < \frac{r}{\gamma\delta + (1-\gamma)r} y \le \frac{\gamma r + (1-\gamma)\delta}{\delta} y. \tag{26}$$

Let $x(y)$ denote a solution of (24) and $\tilde{I}(y) = \frac{x^{-1}(x^*_\sigma(y))}{y}$, one has $\tilde{I}'(y) = \frac{\tilde{I}(y)-1}{y} G(y)$ with

$$G(y) = \frac{2[r(\tilde{I}(y)y^2 - x^*_\sigma(y)^2) + \gamma(r-\delta)x^*_\sigma(y)^2] - \gamma(1-\gamma)x^*_\sigma(y)^2\sigma^2(x^*_\sigma(y))}{2[r(y - x^*_\sigma(y))^2 + \gamma(r-\delta)x^*_\sigma(y)(y - x^*_\sigma(y))] + \gamma(1-\gamma)x^*_\sigma(y)^2\sigma^2(x^*_\sigma(y))}.$$

By Proposition 1.4 and Example 2.10, $\frac{a(\bar{\sigma})}{a(\bar{\sigma})-\gamma} y = x_{\bar{\sigma}}(y) \le x^*_\sigma(y) < y$, which implies that the denominator in the definition of G is not greater than $\left(2\gamma^2[\frac{r}{a^2(\bar{\sigma})} - \frac{(r-\delta)^+}{a(\bar{\sigma})}] + \gamma(1-\gamma)\bar{\sigma}^2\right) x^*_\sigma(y)^2$. If $r - \delta > \frac{(1-\gamma)\bar{\sigma}^2}{2}$ and $x(y) < y$, then $x^*_\sigma(y) < x^{-1}(x^*_\sigma(y)) = y\tilde{I}(y)$ and

$$\forall y > 0,\ G(y) > \frac{\gamma a^2(\bar{\sigma})(2(r-\delta) - (1-\gamma)\bar{\sigma}^2)}{2\gamma^2[r - a(\bar{\sigma})(r-\delta)] + \gamma(1-\gamma)a^2(\bar{\sigma})\bar{\sigma}^2} > 0.$$

If $\frac{(\delta-r)(\gamma\delta+(1-\gamma)r)}{(1-\gamma)\delta+\gamma r} > \frac{(1-\gamma)\bar{\sigma}^2}{2}$ and $x(y) < \frac{(1-\gamma)\delta+\gamma r}{\delta} y$ then $\frac{\delta}{(1-\gamma)\delta+\gamma r} x^*_\sigma(y) < x^{-1}(x^*_\sigma(y)) = y\tilde{I}(y)$ and using the first inequality in (26), we get

$$\forall y > 0,\ G(y) > \frac{\gamma a^2(\bar{\sigma})\left(\frac{2(\delta-r)(\gamma\delta+(1-\gamma)r)}{(1-\gamma)\delta+\gamma r} - (1-\gamma)\bar{\sigma}^2\right)}{2\gamma^2 r + \gamma(1-\gamma)a^2(\bar{\sigma})\bar{\sigma}^2} > 0.$$

In both cases, when $\tilde{I}(1) > 1$ then $\forall y > 0$, $\tilde{I}(y) > 1$ and for $y > 1$, $\tilde{I}(y) - \tilde{I}(1) \ge c(\tilde{I}(1)-1)\log(y)$ for some positive constant c. This contradicts the inequality $x^{-1}(x^*_\sigma(y)) < x^{-1}(y) < \frac{y}{\varepsilon}$ which holds as soon as for all $y > 0$, $x(y) > \varepsilon y$. When $\tilde{I}(1) < 1$, then for $y > 1$, $\tilde{I}(y) - \tilde{I}(1) \le c(\tilde{I}(1)-1)\log(y)$ which contradicts the positivity of $\tilde{I}$. □

Theorem 2.13. *Let us assume that σ and η satisfy $(\mathcal{H}'_{\mathbf{vol}})$. Then, the following assertions are equivalent:*

(1) $\forall x, y > 0,\ P_\sigma(x,y) = c_\eta(y,x)$.
(2) $\forall y > 0,\ x^*_\sigma(y)^2\sigma^2(x^*_\sigma(y)) > \frac{2(1-\gamma)}{\gamma^2(2-\gamma)}[r(y - x^*_\sigma(y))^2 + \gamma(r-\delta)x^*_\sigma(y)(y - x^*_\sigma(y))]$ *and* $\eta \equiv \tilde{\sigma}$ *where*

$$\tilde{\sigma}^2(y) = \frac{2[\delta(y - x^*_\sigma(y))^2 + \gamma(r-\delta)y(y - x^*_\sigma(y))]}{\gamma y^2} \times \frac{2[r(y - x^*_\sigma(y))^2 + \gamma(r-\delta)x^*_\sigma(y)(y - x^*_\sigma(y))] + \gamma(1-\gamma)x^*_\sigma(y)^2\sigma^2(x^*_\sigma(y))}{\gamma^2(2-\gamma)x^*_\sigma(y)^2\sigma^2(x^*_\sigma(y)) - 2(1-\gamma)[r(y - x^*_\sigma(y))^2 + \gamma(r-\delta)x^*_\sigma(y)(y - x^*_\sigma(y))]}.$$

If moreover $\max\left(r - \delta, \frac{(\delta - r)(\gamma\delta + (1-\gamma)r)}{(1-\gamma)\delta + \gamma r}\right) > \frac{(1-\gamma)\bar{\sigma}^2}{2}$*, they are also equivalent to*
(3) $\forall x > 0, y^*_\eta(x)^2\eta^2(y^*_\eta(x)) > \frac{2(1-\gamma)}{\gamma^2(2-\gamma)}[\delta(y^*_\eta(x) - x)^2 + \gamma(r-\delta)y^*_\eta(x)(y^*_\eta(x) - x)],\ y^*_\eta(x) > \frac{\gamma\delta + (1-\gamma)r}{r}x$ *and* $\sigma(x) = \underset{\sim}{\eta}(x)$ *where*

$$\underset{\sim}{\eta}^2(x) = \frac{2[r(y^*_\eta(x) - x)^2 + \gamma(r-\delta)x(y^*_\eta(x) - x)]}{\gamma x^2} \times \frac{2[\delta(y^*_\eta(x) - x)^2 + \gamma(r-\delta)y^*_\eta(x)(y^*_\eta(x) - x)] + \gamma(1-\gamma)y^*_\eta(x)^2\eta^2(y^*_\eta(x))}{\gamma^2(2-\gamma)y^*_\eta(x)^2\eta^2(y^*_\eta(x)) - 2(1-\gamma)[\delta(y^*_\eta(x) - x)^2 + \gamma(r-\delta)y^*_\eta(x)(y^*_\eta(x) - x)]}.$$

Using (26), the numerator in the first term of the r.h.s. of the equation giving $\tilde{\sigma}^2(y)$ is positive. Notice that in (3), the condition $y^*_\eta(x) > \frac{\gamma\delta + (1-\gamma)r}{r}x$ that ensures the positivity of the analogous term is satisfied as soon as $\delta \leq r$.

Proof. The proof is similar to the one of Theorem 2.7. For (1) $\Rightarrow$ (3), $y^*_\eta(x) > \frac{\gamma\delta + (1-\gamma)r}{r}x$ comes from $y^*_\eta \equiv x^{*\,-1}_\sigma$ and (26). For (2) $\Rightarrow$ (1), one remarks that according to (26) and Proposition 1.5, $\forall x > 0,\ x^{*\,-1}_\sigma(x) > \max(\frac{\delta x}{\gamma r + (1-\gamma)\delta}, x)$ which, combined with Proposition 2.11, ensures that $x^{*\,-1}_\sigma = y_{\tilde{\sigma}}$.
For (3) $\Rightarrow$ (1), since according to Proposition 1.4, Example 2.10 and Remark 1.3, $\frac{b(\bar{\eta})}{b(\bar{\eta}) - \gamma}x \geq y^*_\eta(x) > \max\left(1, \frac{\delta}{\gamma r + (1-\gamma)\delta}\right)x$, one has $\frac{(b(\bar{\eta}) - \gamma)}{b(\bar{\eta})}y \leq y^{*\,-1}_\eta(y) < \min\left(1, \frac{\gamma r + (1-\gamma)\delta}{\delta}\right)y$. $\square$

To obtain an analytical example of non-constant dual volatility functions, we consider the same reciprocal boundaries as in [1] :

$$y^*(x) = x\frac{x+a}{bx+c} \quad \text{and} \quad x^*(y) = \frac{1}{2}\left(by - a + \sqrt{(by-a)^2 + 4cy}\right) \quad \text{with} \quad a, b, c > 0.$$

Under some assumptions on a, b, c, these functions are the exercise boundaries associated with explicit dual volatility functions.

Proposition 2.14. *Let us assume that* $\max(c/a, b) < \min(1, \frac{r}{(1-\gamma)r + \gamma\delta})$, $\min(c/a, b) > 1 - \gamma$ *and* $\frac{1-\gamma}{\gamma^2}[\max(\frac{1}{b}, \frac{a}{c}) - 1][r\max(\frac{1}{b}, \frac{a}{c}) - ((1-\gamma)r + \gamma\delta)] < \max\left(r - \delta, \frac{(\delta - r)(\gamma\delta + (1-\gamma)r)}{(1-\gamma)\delta + \gamma r}\right)$*. Then* *the volatility functions*

$$\begin{aligned}\sigma(x) &= \sqrt{\frac{2}{\gamma}\frac{\left[x(1-b)+a-c\right]\left[x[r-b((1-\gamma)r+\gamma\delta)]+a[r-\frac{c}{a}((1-\gamma)r+\gamma\delta)]\right]}{bx^2+2cx+ac+(\gamma-1)(bx+c)^2}}\\ \eta(y) &= \sqrt{\frac{2}{\gamma}\frac{[y-x^*(y)][\delta(y-x^*(y))+\gamma(r-\delta)y][bx^*(y)^2+2cx^*(y)+ac]}{y^2[b(b+\gamma-1)x^*(y)^2+2c(b+\gamma-1)x^*(y)+ca(\frac{c}{a}+\gamma-1)]}}\end{aligned}$$

are well defined and satisfy $(\mathcal{H}_{\mathbf{vol}})$. *Moreover we have* $y^*_\eta \equiv y^*$ *and* $x^*_\sigma \equiv x^*$ *and thus the duality holds:* $\forall x, y > 0,\ P_\sigma(x, y) = c_\eta(y, x)$.

We have plotted in Figure 1 an example that illustrates this duality result.

Remark 2.15. *Let us comment briefly the assumptions on the coefficients* a, b, c. *Under the second hypothesis,* $\max(1/b, c/a) < (1-\gamma)^{-1}$ *and therefore the third assumption will be automatically satisfied if*

$$\begin{aligned}&\frac{r}{\gamma^2}(\max(1/b, a/c)-1) < r-\delta \text{ when } r > \delta,\\ &\frac{r}{\gamma}(\max(1/b, a/c)-((1-\gamma)+\gamma\delta/r)) < \frac{(\delta-r)(\gamma\delta+(1-\gamma)r)}{(1-\gamma)\delta+\gamma r} \text{ when } r < \delta.\end{aligned}$$

When $r > \delta$, *it is always possible to take* b *and* a/c *close enough to* 1 *so that the equivalent condition* $\max(1/b, a/c) < 1+\gamma^2(1-\delta/r)$ *is satisfied. In the same manner, if* $\delta > r$, *the first hypothesis writes* $\min(1/b, a/c) > 1-\gamma+\gamma\delta/r$, *and the third assumption will be always satisfied if one takes parameters such that* $1/b$ *and* a/c *are close enough to* $1-\gamma+\gamma\delta/r$. *This is nonetheless compatible with the second assumption only if* $1-\gamma < \frac{r}{r(1-\gamma)+\delta\gamma}$, *i.e.* $\delta/r < \frac{2-\gamma}{1-\gamma}$. *Otherwise, there are no parameters* a, b, c *that fulfill the three assumptions. Let us remark incidentally that this condition is the same as the condition* $\gamma-1 > -\frac{r}{\delta-r}$ *which appears in the Black-Scholes case (see Example 2.10).*

Proof. First step: let us check that the functions σ *and* η *are well defined and satisfy* $(\mathcal{H}_{\mathbf{vol}})$. The denominator in the definition of σ is equal to $b(1+(\gamma-1)b)x^2+2c(1+(\gamma-1)b)x+ac(1+(\gamma-1)c/a)$: this is a second degree polynomial with positive coefficients because $\max(c/a, b) < 1$. It is then easy to check that σ is well defined and satisfy $(\mathcal{H}_{\mathbf{vol}})$ using that $\max(c/a, b) < \min(1, \frac{r}{(1-\gamma)r+\gamma\delta})$. We get after some calculations

$$\eta(y^*(x)) = \sqrt{\frac{2}{\gamma}\frac{[x(1-b)+a-c][(\gamma r+(1-\gamma)\delta)-\delta b)x+(\gamma r+(1-\gamma)\delta-\delta\frac{c}{a})a][bx^2+2cx+ac]}{(x+a)^2[b(b+\gamma-1)x^2+2c(b+\gamma-1)x+ca(\frac{c}{a}+\gamma-1)]}}$$

From the first hypothesis and the argument given at the beginning of the proof of Proposition 2.12, one obtains $\max(c/a, b) < \min(1, \frac{\gamma r+(1-\gamma)\delta)}{\delta})$. Using the second hypothesis and the one-to-one onto property of the function y^*, we deduce that η is also well defined and satisfies $(\mathcal{H}_{\mathbf{vol}})$.

Second step: We easily check that x^* and y^* respectively solve the ODEs (24) and (25). Since $\max(c/a, b) < \min(1, \frac{\gamma r+(1-\gamma)\delta)}{\delta})$, we have $y^*(x) > x$ and $\delta(y^*(x)-x)+\gamma(r-\delta)y^*(x) > 0$. Proposition 2.11 then ensures that $y^* \equiv y^*_\eta$. Since $bx^2 + 2cx + ac > (bx + c)^2$, we have

$$\begin{aligned}\sigma^2(x) &\leq \frac{2}{\gamma^2}\frac{x(1-b)+a-c}{bx+c}\frac{x[r-b((1-\gamma)r+\gamma\delta)]+a[r-\frac{c}{a}((1-\gamma)r+\gamma\delta)]}{bx+c}\\ &\leq \frac{2}{\gamma^2}\left[\max(\frac{1}{b},\frac{a}{c})-1\right]\left[r\max(\frac{1}{b},\frac{a}{c})-((1-\gamma)r+\gamma\delta)\right]\\ &\leq \frac{2}{1-\gamma}\max\left(r-\delta, \frac{(\delta-r)(\gamma\delta+(1-\gamma)r)}{(1-\gamma)\delta+\gamma r}\right).\end{aligned}$$

By Proposition 2.12, we conclude that $x^* \equiv x^*_\sigma$. □

2.4 The theoretical calibration procedure

In [1], the calibration issue was the practical motivation for our interest in Call-Put duality. Even if, in the present framework, calibration is purely theoretical since the payoff $\phi(x, y)$ is not traded, we are going to explain shortly how to recover the local volatility function from the perpetual prices of options. More precisely, let us suppose that we observe for all $K > 0$ the market price $p(K)$ of the American security with payoff $\phi(., K)$, with either $\phi(x, y) = ((y-x)^+)^\gamma$ and $\gamma \in (0, 1]$ or $\phi(x, y) = (\psi_y(y) - \psi_x(x))^+$, with ψ_y and ψ_x satisfying the assumptions mentioned before. We also assume that either $\max\left(r-\delta, \frac{(\delta-r)(\gamma\delta+(1-\gamma)r)}{(1-\gamma)\delta+\gamma r}\right) > \frac{(1-\gamma)\bar{\sigma}^2}{2}$ or ψ_x satisfy (19) so that we have equivalence between the three conditions in Theorems 2.13 or 2.7. We denote by x_0 the current value of the stock, and we suppose that there is a function σ satisfying $(\mathcal{H}'_{\textbf{vol}})$ such that $\forall K > 0$, $p(K) = P_\sigma(x_0, K)$ and that $\tilde{\sigma}$ is well defined and satisfy $(\mathcal{H}'_{\textbf{vol}})$. Within this framework, as in the call-put case, we are able to get $(\sigma(x), 0 < x \leq x_0)$.

Indeed, let us define $Y = \inf\{K > 0, p(K) = \phi(x_0, K)\}$. Since $p(K) = c_{\tilde{\sigma}}(K, x_0)$, we have $Y = y^*_{\tilde{\sigma}}(x_0)$ and

$$\forall K < Y,\ \frac{K^2\tilde{\sigma}(K)^2}{2}p''(K) + K(\delta - r)p'(K) - \delta p(K) = 0$$

We deduce then $\forall K \leq Y, \tilde{\sigma}(K) = \frac{1}{K}\sqrt{\frac{2(\delta p(K)+K(r-\delta)p'(K))}{p''(K)}}$ because $p''(K) = \partial^2_K c_{\tilde{\sigma}}(K, x_0) = \frac{\phi(x_0,Y)}{g(Y)}g''(K) > 0$ for $K < Y$ using (7). Then, we get $(y^*_{\tilde{\sigma}}(x), 0 < x \leq x_0)$ solving backward either (25) or (18) starting from $y^*_{\tilde{\sigma}}(x_0) = Y$. Finally, we get $(\sigma(x), 0 < x \leq x_0)$ thanks to Theorem 2.13 or 2.7, using that $\sigma(x) = \tilde{\underset{\sim}{\sigma}}(x)$.

Now, to formalize our calibration result, we introduce the set

$$\Sigma = \{\sigma \text{ satisfying } (\mathcal{H}'_{\textbf{vol}}) \text{ s.t. } \tilde{\sigma} \text{ is well defined and satisfies } (\mathcal{H}'_{\textbf{vol}})\}.$$

Proposition 2.16. *Under the above assumptions on ϕ, r and δ, for $\sigma_1, \sigma_2 \in \Sigma$,*

$$\forall K > 0,\ P_{\sigma_1}(x_0, K) = P_{\sigma_2}(x_0, K) \ \Leftrightarrow\ \sigma_1\big|_{(0,x_0]} \equiv \sigma_2\big|_{(0,x_0]} \text{ and } y^*_{\tilde{\sigma}_1}(x_0) = y^*_{\tilde{\sigma}_2}(x_0).$$

Proof. The necessary condition is a consequence of the above calibration procedure. To check the sufficient condition, we consider σ_1 and σ_2 in Σ such that

$$\forall x \leq x_0, \sigma_1(x) = \sigma_2(x) \text{ and } y^*_{\tilde{\sigma}_1}(x_0) = y^*_{\tilde{\sigma}_2}(x_0) = Y.$$

On the one hand, we have $x^*_{\sigma_1}(Y) = x^*_{\sigma_2}(Y) = x_0$, and thus $x^*_{\sigma_1}(y) = x^*_{\sigma_2}(y)$ for $y \leq Y$ since they solve the same ODE. Therefore, using either Theorem 2.13 or Theorem 2.7, one gets $\tilde{\sigma}_1(y) = \tilde{\sigma}_2(y)$ for $y \leq Y$. On the other hand, the smooth fit principle gives $\frac{g'_{\tilde{\sigma}_1}(Y)}{g_{\tilde{\sigma}_1}(Y)} = \frac{\partial_y \phi(x_0,Y)}{\phi(x_0,Y)} = \frac{g'_{\tilde{\sigma}_2}(Y)}{g_{\tilde{\sigma}_2}(Y)}$. The set of solutions to $\frac{1}{2}y^2\tilde{\sigma}_1^2(y)g''(y) + (\delta - r)yg'(y) - \delta g(y) = 0$ on $(0, Y]$ is a two-dimensional vectorial space, and by the previous equality, $g_{\tilde{\sigma}_1}$ and $g_{\tilde{\sigma}_2}$ are proportional on $(0, Y]$. Therefore, we have for $0 < K \leq Y$, $P_{\sigma_1}(x_0, K) = \phi(x_0, Y)\frac{g_{\tilde{\sigma}_1}(K)}{g_{\tilde{\sigma}_1}(Y)} = \phi(x_0, Y)\frac{g_{\tilde{\sigma}_2}(K)}{g_{\tilde{\sigma}_2}(Y)} = P_{\sigma_2}(x_0, K)$, and $P_{\sigma_1}(x_0, K) = \phi(x_0, K) = P_{\sigma_2}(x_0, K)$ for $K \geq Y$. $\square$

Like in Proposition 5.1 [1], we can get an analogous calibration of the complementary upper part of the local volatility function to the perpetual prices of the "Call" options with payoff $\phi(K, x_0)$ by exchanging the roles of η and σ, and of r and δ.

References

[1] Alfonsi, A. and Jourdain, B. (2006). A Call-Put Duality for Perpetual American Options, Avril 2006, *rapport CERMICS [2006-307].*

[2] Beibel, M. and Lerche, R. (1997). A New Look at Optimal Stopping Problems related to Mathematical Finance, *Statistica Sinica Vol. 7, pp. 93-108.*

[3] Dupire, B. (1994). Pricing with a smile. *Risk, Vol. 7, No. 1, January 1994, pp. 18-20.*

[4] Karatzas, I. and Shreve, S. (1991) Brownian Motion and Stochastic Calculus, *2nd ed. New York, Springer-Verlag.*