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An error estimate for a new scheme for mean curvature motion

N. Forcadel[1]

[1] *Cermics, ENPC, ParisTech, Champs-sur-Marne, F-77455 Marne-la-Vallée Cedex 2*

CERMICS — ENPC
6 et 8 avenue Blaise Pascal
Cité Descartes - Champs sur Marne
77455 Marne la Vallée Cedex 2

http://cermics.enpc.fr

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N. Forcadel *

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Abstract

In this work, we propose a new numerical scheme for the anisotropic mean curvature equation. The solution of the scheme is not unique, but for all numerical solutions, we provide an error estimate between the continuous solution and the numerical approximation. This error estimate is not optimal, but as far as we know, this is the first one for mean curvature type equation. Our scheme is also applicable to compute the solution to dislocations dynamics equation.

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1 Introduction

Mean curvature motion has been largely studied in terms of both theory and computation, in particular due to the large number of applications like front propagation, image processing, fluid dynamics...(see for instance Sethian [25] and Osher, Paragios [22]).

The level set framework has been used in both theoretical and numerical problems. Theoretically, the mean curvature equation has been well understood using the framework of viscosity solutions by Chen, Giga, Goto [10] and Evans, Spruck [15]. However, this equation has serious problems for the question of numerical approximations. Nevertheless, there are several works on this question. First, let us mention the numerical method of Osher, Sethian [23]. This method is very used in practice but, as far as we know, there is no convergence result. Another algorithm is the Merriman, Bence, Osher scheme [20] in which motion by mean curvature is viewed as singular limit of a diffusion equation with threshold. The convergence of this scheme has been proved by Barles, Georgelin [4] and Evans [14] (see also Ishii [16], Ishii, Pires, Souganidis [17], and Chambolle, Novaga [9]). A class of convergent schemes for nonlinear parabolic equations including mean curvature motion have been proposed by Crandall, Lions [12]. Let us mention also the recent work of Oberman [21]. In this last two works, two different scales are used. The first one is the space step Δx and the second one is the size of the stencil ε. As it was point out in [12], these two scales are very important to approximate degenerate equations like mean curvature equation.

The goal of our work is to propose a new scheme for mean curvature motion and to prove an error estimate between the continuous solution and its numerical approximation. This error estimate is not optimal, but as far as we know, this is the first one for complete discretized scheme for mean curvature type motion.

The idea is to use a recent work of Da Lio, Monneau and the author [13] concerning the convergence of dislocations dynamics to mean curvature motion. Dislocations are linear defects which move in crystals. Their dynamics can be represented by a non-local first order Hamilton-Jacobi equation (see Alvarez, Hoch, Le Bouar and Monneau [3]). The first goal of this work is to prove an error estimate between the solution of dislocations dynamics and the solution of mean curvature motion. To do this, we will use in a more quantitative way the definition of viscosity solutions for mean curvature motion proposed by Barles, Georgelin [4] by considering a regularization with quartics.

*CERMICS, Ecole des Ponts, Paris Tech, 6 et 8 avenue Blaise Pascal, Cité Descartes, Champs-sur-Marne, 77455 Marne-la-Vallée Cedex 2, email: forcadel@cermics.enpc.fr

The second goal of this work is to propose a numerical scheme for dislocations dynamics. The main properties of this scheme is that it is implicit (so there is no CFL condition) and uses two different scales. Moreover, this scheme is not monotone and does not admit a unique solution. It is only “almost monotone” (see Lemma 5.2). Nevertheless, the fact that it is implicit and the monotonicity of the velocity, allows us to “freeze” the velocity and to prove a Crandall Lions type [11] error estimate for any solutions (we refer to Alvarez, Carlini, Monneau and Rouy [1, 2] for error estimate for dislocations dynamics in the non monotone case). It is also possible to explicit the scheme. In this case, we are able to prove the convergence of the scheme under a CFL condition, but error estimates in this case are still open. This comes from the fact that there is no consistency error for the scheme (see Proposition 5.6).

Finally, let us mention some works on error estimate for numerical scheme for Hamilton-Jacobi equations. For the first order case, we refer to Souganidis [27]. For the second order case, when the Hamiltonian is convex, we refer to Krylov [18, 19] and Barles, Jakobsen [5, 6]. When the Hamiltonian is uniformly elliptic, we refer to Caffarelli, Souganidis [8].

Let us now explain how this paper is organized. In Section 2, we state the main results of this work. In Section 3, we prove the main result concerning mean curvature type motion. Section 4 is devoted to prove the error estimate between the solution of dislocations dynamics and the one of mean curvature motion. In Section 5, we study the numerical scheme for dislocations dynamics and we prove an error estimate between the continuous solution and its numerical approximation. Some numerical simulations are provided in Section 6.

2 Main Results

2.1 Error estimate for Mean Curvature Motion

In order to propose a numerical scheme for anisotropic mean curvature motion, we will use the work of Da Lio *et al.* [13] that we briefly recall here. Given a function g defined on the unit sphere $\mathbf{S}^{N-1}$of $\mathbb{R}^N$ by

$$g \in \mathrm{Lip}(\mathbf{S}^{N-1}), \quad g(-\theta) = g(\theta) \geq 0, \quad \forall \theta \in \mathbf{S}^{N-1} \tag{2.1}$$

we consider kernels $c_0 \in L^\infty(\mathbb{R}^N) \cap W^{1,1}(\mathbb{R}^N)$ satisfying

$$\begin{cases} c_0(x) = \dfrac{1}{|x|^{N+1}} \; g\left(\dfrac{x}{|x|}\right) & \text{if} \quad |x| \geq 1, \\ \\ c_0(-x) = c_0(x) \geq 0, \quad \forall x \in \mathbb{R}^N \end{cases} \tag{2.2}$$

and we use the following scaling for $0 < \varepsilon < e^{-1}$

$$c_0^\varepsilon(x) = \frac{1}{\varepsilon^{N+1}|\ln \varepsilon|} \; c_0\left(\frac{x}{\varepsilon}\right). \tag{2.3}$$

The term $|\ln \varepsilon|$ comes from the bad decay at infinity of the kernel c_0 (see [13, Section 4.1]).

We then consider the following auxiliary problem

$$\begin{cases} u_t^\varepsilon(x,t) = \left((c_0^\varepsilon \star 1_{\{u^\varepsilon(\cdot,t) \geq u^\varepsilon(x,t)\}})(x) - \dfrac{1}{2}\displaystyle\int_{\mathbb{R}^N} c_0^\varepsilon\right) |Du^\varepsilon(x,t)| & \text{in} \quad \mathbb{R}^N \times (0,T), \\ \\ u^\varepsilon(\cdot,0) = u_0^\varepsilon & \text{in} \quad \mathbb{R}^N \end{cases} \tag{2.4}$$

where u_t^ε denotes the derivative with respect to the time variable, Du^ε indicates the gradient of u^ε with respect to the space variables, the convolution is done in space only and $1_{\{u^\varepsilon(\cdot,t) \geq u^\varepsilon(x,t)\}}$ is the characteristic function of the set $\{u^\varepsilon(\cdot,t) \geq u^\varepsilon(x,t)\}$ (which is equal to 1 on the set and 0 outside). This equation arises in the theory of dislocations dynamics (see Alvarez, Hoch, Le Bouar and Monneau [3]) and uses the Slepčev formulation [26] to consider the simultaneous evolutions of all the level sets of the function u^ε. We refer to Da Lio *et al.* [13] for the study of this model and in particular to [13, Definition 2.1] for the definition of viscosity solutions for Problem (2.4). In particular, we recall that we take the indicatrice of $\{u^\varepsilon(\cdot,t) \geq u^\varepsilon(x,t)\}$ for sub-solution and the indicatrice of $\{u^\varepsilon(\cdot,t) > u^\varepsilon(x,t)\}$ for super-solution.

The main result of [13] is that, if $u_0^\varepsilon = u_0$, then the unique solution u^ε of (2.4) converges locally uniformly on compact sets to the solution u^0 of the following limit equation

$$(2.5)\qquad \begin{cases} u_t^0 - \text{trace}\left(D^2u^0 \cdot A\left(\dfrac{Du^0}{|Du^0|}\right)\right) = 0 & \text{in} \quad \mathbb{R}^N \times (0,T) \\ u^0(\cdot,0) = u_0 & \text{in} \quad \mathbb{R}^N \end{cases}$$

with

$$(2.6)\qquad A\left(\frac{p}{|p|}\right) = \int_{\theta \in \mathbf{S}^{N-2} = \mathbf{S}^{N-1} \cap \left\{\langle x, \frac{p}{|p|}\rangle = 0\right\}} \left(\frac{1}{2} g(\theta)\, \theta \otimes \theta\right) d\theta$$

and where $M \cdot A$ and $\langle \cdot, \cdot \rangle$ denote respectively the product between the two matrices and the usual scalar product. Our first main result is an error estimate between u^ε and u^0:

Theorem 2.1 (Error estimate for Mean Curvature Motion)
Let $N \geq 1$ and $T \leq 1$. Assume that $u_0, u_0^\varepsilon \in \text{Lip}(\mathbb{R}^N)$, $g \in \text{Lip}(\mathbf{S}^{N-1})$ and that c_0 given in (2.2) satisfies $c_0 \in W^{1,1}(\mathbb{R}^N)$. Then, there exists a constant K_1 depending on N, $\sup_{\mathbb{R}^N} c_0$, $|Dg|_{L^\infty(\mathbf{S}^{N-1})}$ and $|Du_0|_{L^\infty(\mathbb{R}^N)}$ such that the difference between the solution u^ε of (2.4) and the solution u^0 of (2.5) is given by

$$(2.7)\qquad \sup_{\mathbb{R}^N \times (0,T)} |u^\varepsilon - u^0| \leq K_1 \left(\frac{T}{|\ln \varepsilon|}\right)^{\frac{1}{6}} + \sup_{\mathbb{R}^N} |u_0^\varepsilon - u_0|.$$

2.2 Discrete-continuous error estimate for dislocations dynamics

To approximate the solution u^0 of (2.5), we then have to approximate the solution u^ε of (2.4). Up to a change of variable (see Corollary 3.1), it suffices to approximate the solution u of

$$(2.8)\qquad \begin{cases} u_t(x,t) = \left((c_0 \star 1_{\{u(\cdot,t) \geq u(x,t)\}})(x) - \dfrac{1}{2}\displaystyle\int c_0\right) |Du(x,t)| & \text{in} \quad \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = \bar{u}_0 & \text{in} \quad \mathbb{R}^N. \end{cases}$$

Given a mesh size $\Delta x_1, ..., \Delta x_N, \Delta t$ and a lattice $Q_T^\Delta = Q^\Delta \times \{0, ..., (\Delta t)N_T\}$ where $Q^\Delta = \{(i_1\Delta x_1, ..., i_N \Delta x_N),\ I = (i_1, ..., i_N) \in \mathbb{Z}^N\}$ and N_T is the integer part of $T/\Delta t$, we will denote by $(x_1, ..., x_N, t_n) = (x_I, t_n)$ the node $(i_1\Delta x_1, ..., i_N \Delta x_N, n\Delta t)$ and by v_I^n the value of a numerical approximation of the exact solution $u(x_I, t_n)$. We set $\Delta x = \sqrt{\Delta x_1^2 + ... + \Delta x_N^2}$ the space mesh size. We shall assume throughout that $\Delta x + \Delta t \leq 1$.

The discrete solution v is computed iteratively by solving the implicit scheme

$$(2.9)\qquad v_I^0 = \tilde{u}_0(x_I), \qquad \frac{v_I^{n+1} - v_I^n}{\Delta t} = c^\Delta[v]_I^{n+1} G(v^{n+1})_I$$

where $\tilde{u}_0$ is an approximation of $\bar{u}_0$ and $G(v^{n+1})_I$ is a suitable approximation of the gradient of v^{n+1} taken at point x_I. The non-local velocity is the discrete convolution

$$(2.10)\qquad c^\Delta[v]_I^{n+1} = c^\Delta[v](x_I, t_{n+1}) = \sum_{J \in \mathbb{Z}^N} \bar{c}_{I-J}^0 1_{\{v_J^{n+1} \geq v_I^{n+1}\}} \Delta x_1 ... \Delta x_N - \frac{1}{2} \sum_{J \in \mathbb{Z}^N} \bar{c}_J^0 \Delta x_1 ... \Delta x_N$$

with

$$(2.11)\qquad \bar{c}_I^0 = \frac{1}{|Q_I|} \int_{Q_I} c_0(x)\, dx$$

where Q_I is the square cell centred in x_I

$$(2.12)\qquad Q_I = [x_{i_1} - \Delta x_1/2, x_{i_1} + \Delta x_1/2] \times ... \times [x_{i_N} - \Delta x_N/2, x_{i_N} + \Delta x_N/2].$$

Finally, let us define

$$(2.13)\qquad v_\#(y, t_n) = \sum_I v(x_I, t_n) \chi_{Q_I}(y)$$

where χ_{Q_I} is the indicator function of Q_I.

The approximation of the gradient is obtained using the Osher, Sethian scheme [23] (we can also use the one proposed by Rouy, Tourin [24]). It is monotone, consistent and depends on the sign of the non-local velocity. Its precised definition is recalled in Section 5.

Since the velocity c^Δ is non-local and not continuous, we have to give a sense to the equality in the scheme (2.9). In fact, we will use the analogue of the Slepčev formulation [26] for discrete sub and super-solution (see Definition 5.1) and we will use a discrete version of the Perron's method to construct a discrete solution. The solution of the scheme is not unique but, for any solutions, we have the following Crandall-Lions type [11] error estimate:

Theorem 2.2 (Discrete-continuous error estimate for (2.8))
Let $N \geq 1$ and $T \leq 1$. Assume that $\bar{u}_0, \tilde{u}_0 \in W^{1,\infty}(\mathbb{R}^N)$ and that c_0 given in (2.2) satisfies $c_0 \in W^{1,1}(\mathbb{R}^N)$. Assume that $\Delta x + \Delta t \leq 1$. Then there exists a constant $K_2 > 0$ depending only on N, $|c_0|_{W^{1,1}(\mathbb{R}^N)}$, $|D\bar{u}_0|_{L^\infty(\mathbb{R}^N)}$ and $|D\tilde{u}_0|_{L^\infty(\mathbb{R}^N)}$ such that the error estimate between the continuous solution u of (2.8) and any discrete solution v of the finite difference scheme (2.9) is given by

$$\sup_{\mathbb{R}^N \times \{0,\dots,t_{N_T}\}} |u - v_\#| \leq K_2 \sqrt{T} \left(\Delta x + \Delta t\right)^{1/2} + \sup_{\mathbb{R}^N} |\bar{u}_0 - (\tilde{u}_0)_\#|$$

provided $\Delta x + \Delta t \leq \dfrac{1}{K_2^2}$.

Remark 2.3 *It is possible to explicit the computation of the gradient, i.e., to replace the term $G(v^{n+1})_I$ by $G(v^n)_I$ in the scheme (2.9) and to consider the solution v of*

$$v_I^0 = \tilde{u}_0(x_I), \qquad \frac{v_I^{n+1} - v_I^n}{\Delta t} = c^\Delta[v]_I^{n+1} G(v^n)_I. \tag{2.14}$$

In this case, as usual, we have to satisfy a CFL condition like for instance

$$\Delta t \leq \frac{\Delta x}{2|c_0|_{L^1(\mathbb{R}^N)}}$$

for the Osher Sethian discretisation of the gradient. Under this additional assumption, Theorem 2.2 remains true with v solution of the scheme (2.14).

2.3 Discrete continuous error estimate for Mean Curvature Motion

Using the above results, we will prove in Section 3 the following theorem:

Theorem 2.4 (Discrete-continuous error estimate for the mean curvature motion)
Let $N \geq 1$ and $T \leq 1$. Let us denote by v^ε a solution of (2.9)-(2.10)-(2.11) with initial condition $\tilde{u}_0^\varepsilon$ (which is an approximation of u_0) and with c_0^ε in the place of c_0. Assume that $u_0, \tilde{u}_0^\varepsilon \in W^{1,\infty}(\mathbb{R}^N)$, $g \in \mathrm{Lip}(\mathbf{S}^{N-1})$ and that c_0 given in (2.2) satisfies $c_0 \in W^{1,1}(\mathbb{R}^N)$. Assume also that $\Delta x + \Delta t \leq 1$. Then there exists a constant $K_3 > 0$ depending only on N, $\sup_{\mathbb{R}^N} c_0$, $|Dg|_{L^\infty(\mathbf{S}^{N-1})}$, $|c_0|_{W^{1,1}(\mathbb{R}^N)}$, $|Du_0|_{L^\infty(\mathbb{R}^N)}$ and $|D\tilde{u}_0^\varepsilon|_{L^\infty(\mathbb{R}^N)}$ such that the error estimate between the continuous solution u^0 of (2.5) and its numerical approximation v^ε is given by

$$\sup_{\mathbb{R}^N \times \{0,..,t_{N_T}\}} |u^0 - v_\#^\varepsilon| \leq K_3 \left(\frac{T}{|\ln \varepsilon|}\right)^{1/6} + \sup_{\mathbb{R}^N} |u_0 - (\tilde{u}_0^\varepsilon)_\#|$$

where $\varepsilon \geq K_3(\Delta x + \sqrt{\Delta t})$.

Remark 2.5 *If $T \geq 1$, since $|Du^0|_{L^\infty(\mathbb{R}^N \times (0,T))} \leq |Du_0|_{L^\infty(\mathbb{R}^N)}$ and "$|Dv^\varepsilon|_{L^\infty(\mathbb{R}^N)}$"$\leq |D\tilde{u}_0^\varepsilon|_{L^\infty(\mathbb{R}^N)}$ (see Proposition 5.4 for the exact setting), we can iterate the process and get a linear estimate in T, i.e*

$$\sup_{\mathbb{R}^N \times \{0,..,t_{N_T}\}} |u^0 - v_\#^\varepsilon| \leq \frac{K_3}{|\ln \varepsilon|^{1/6}} T + \sup_{\mathbb{R}^N} |u_0 - (\tilde{u}_0^\varepsilon)_\#|.$$

Remark 2.6 *We can truncate the kernel c_0 at infinity and consider*

$$\tilde{c}_0^R(x) = \begin{cases} c_0(x) & \text{if } |x| \le R, \\ 0 & \text{else} \end{cases}$$

In this case, we make an error of order $\displaystyle\int_{\mathbb{R}^N \setminus B_R(0)} c_0 \le \frac{K}{R}$ *and we can make the computation on a finite stencil, even if Δx goes to zero. This is possible, if we choose ε of the same order of Δx. The condition $\varepsilon \ge K(\Delta x + \sqrt{\Delta t})$ in Theorem 2.4 then implies that we need to impose a CFL condition $\Delta t \le K\Delta x^2$ (which is classical for second order equation).*

At the opposite, if we do not impose any CFL condition, we can choose Δt larger than Δx^2, but we have to choose ε of the same order of $\sqrt{\Delta t}$ and so to make the convolution on larger and larger stencils as Δx goes to zero.

As we mention in the introduction, it is also possible to explicit the computation of the velocity in the scheme (2.9) and to consider

$$v_I^0 = \tilde{u}_0(x_I), \qquad \frac{v_I^{n+1} - v_I^n}{\Delta t} = c^{\Delta}[v]_I^n G(v^n)_I, \tag{2.15}$$

with c_0^ε in the place of c_0 in the definition of c^Δ.

We can prove the convergence of the scheme (see Theorem 2.7) under the CFL condition

$$\Delta t \le \frac{\varepsilon |\ln \varepsilon|}{2|c_0|_{L^1(\mathbb{R}^N)}} \Delta x. \tag{2.16}$$

But, in this case, we are not able to prove an error estimate. The reason is that when we implicit the velocity, it "freezes" it, and then we can use the consistency error on the scheme with given velocity. At the opposite, for the explicit scheme, we have to control the consistency error of the velocity which is not possible as we point out in Proposition 5.6.
We define

$$v^{\varepsilon,\delta}(x,t) = v^\varepsilon(x_I, t_n) \quad \text{if} \quad x \in Q_I,\ t \in [t_n, t_{n+1})$$

where $\delta = (\Delta x, \Delta t)$ and v^ε is the solution of the scheme (2.15)-(2.10)-(2.11) with the kernel c_0^ε in the place of c_0. Then we have the following convergence result:

Theorem 2.7 (Convergence of the explicit scheme)
Assume (2.16)*. Under the assumptions of Theorem 2.4, the function $v^{\varepsilon,\delta}$ converges locally uniformly on compact sets as $\delta \to 0$ to u^ε solution of* (2.4)*.*

Remark 2.8 *In Theorem 2.7, if we take the limit $\delta \to 0$ and $\varepsilon \to 0$ with $\delta \ll \varepsilon$, we will approach the solution of* (2.5)*. The condition $\delta \ll \varepsilon$ implies that, in practice, we have to make the convolution in larger and larger stencil as $\delta \to 0$.*

Remark 2.9 *In the scheme* (2.15)*, it is possible to implicit the computation of the gradient and thus to withdraw the CFL condition* (2.16) *to get the same result as Theorem 2.7.*

Notation In what follows, we will denote by K a generic constant, which will then satisfy $K + K = K$, $K \cdot K = K$, and so on.

3 Numerical scheme for mean curvature motion

3.1 Proof of Theorem 2.4

Using a rescaling argument, we will prove the following corollary of Theorem 2.2:

Corollary 3.1 (Discrete-continuous error estimate for (2.4))
Let us denote by v^ε a solution of (2.9)-(2.10)-(2.11) with initial condition $\tilde{u}_0^\varepsilon$ (which is an approximation of u_0^ε) and with c_0^ε in the place of c_0. Under the assumptions of Theorem 2.2, there exists a constant $K > 0$

depending only on $|c_0|_{W^{1,1}(\mathbb{R}^N)}$, $|Du_0^\varepsilon|_{L^\infty(\mathbb{R}^N)}$ and $|D\tilde{u}_0^\varepsilon|_{L^\infty(\mathbb{R}^N)}$ such that the error estimate between the continuous solution u^ε of (2.4) and its numerical approximation v^ε is given by

$$\sup_{\mathbb{R}^N\times\{0,\dots,t_{N_T}\}} |u^\varepsilon - v^\varepsilon_\#| \le K\frac{1}{\varepsilon}\sqrt{\frac{T}{|\ln\varepsilon|}}\sqrt{\varepsilon\Delta x + \frac{\Delta t}{|\ln\varepsilon|}} + \sup_{\mathbb{R}^N}|u_0^\varepsilon - (\tilde{u}_0^\varepsilon)_\#|$$

provided $\varepsilon\Delta x + \dfrac{\Delta t}{|\ln\varepsilon|} \le \dfrac{\varepsilon^2}{K^2}$.

Proof of Corollary 3.1
First we remark that a simple change of variable gives (for u and v solutions respectively of (2.8) and (2.9))

$$u^\varepsilon(y,\tau) = \varepsilon u\left(\frac{y}{\varepsilon}, \frac{\tau}{\varepsilon^2|\ln\varepsilon|}\right), \qquad u_0^\varepsilon(y) = \varepsilon\bar{u}_0\left(\frac{y}{\varepsilon}\right)$$

$$v^\varepsilon(y,\tau) = \varepsilon v\left(\frac{y}{\varepsilon}, \frac{\tau}{\varepsilon^2|\ln\varepsilon|}\right), \qquad \tilde{u}_0^\varepsilon(y) = \varepsilon\tilde{u}_0\left(\frac{y}{\varepsilon}\right).$$

Let us denote by $x = \dfrac{y}{\varepsilon}$, $t = \dfrac{\tau}{\varepsilon^2|\ln\varepsilon|}$, $\Delta x = \dfrac{\Delta y}{\varepsilon}$, $\Delta t = \dfrac{\Delta\tau}{\varepsilon^2|\ln\varepsilon|}$, $T = \dfrac{\Gamma}{\varepsilon^2|\ln\varepsilon|}$, $s_n = n\Delta\tau$ and by N_Γ the integer part of $\Gamma/\Delta\tau$. Then, the following inequality holds:

$$\begin{aligned}\sup_{R^N\times\{0,\dots,s_{N_\Gamma}\}} |u^\varepsilon - v^\varepsilon_\#| =& \varepsilon \sup_{R^N\times\{0,\dots,t_{N_T}\}} |u - v_\#| \\ \le& K\frac{\sqrt{\Gamma}}{\sqrt{|\ln\varepsilon|}}\sqrt{\frac{\Delta y}{\varepsilon} + \frac{\Delta\tau}{\varepsilon^2|\ln\varepsilon|}} + \varepsilon\sup_{\mathbb{R}^N}|\bar{u}_0 - (\tilde{u}_0)_\#| \\ \le& K\frac{\sqrt{\Gamma}}{\varepsilon\sqrt{|\ln\varepsilon|}}\sqrt{\varepsilon\Delta y + \frac{\Delta\tau}{|\ln\varepsilon|}} + \sup_{\mathbb{R}^N}|u_0^\varepsilon - (\tilde{u}_0^\varepsilon)_\#|\end{aligned}$$

provided that

$$\frac{\Delta y}{\varepsilon} + \frac{\Delta\tau}{\varepsilon^2|\ln\varepsilon|} \le \frac{1}{K^2},$$

i.e.

$$\varepsilon\Delta y + \frac{\Delta\tau}{|\ln\varepsilon|} \le \frac{\varepsilon^2}{K^2}.$$

This ends the proof of the corollary.

Proof of Theorem 2.4
The proof of Theorem 2.4 is now very easy. Indeed, using Theorem 2.1 and Corollary 3.1, we deduce that

$$\begin{aligned}|u^0 - v^\varepsilon| \le& |u^0 - u^\varepsilon| + |u^\varepsilon - v^\varepsilon| \\ \le& K\left(\frac{T}{|\ln\varepsilon|}\right)^{\frac{1}{6}} + \frac{K}{\varepsilon\sqrt{|\ln\varepsilon|}}\sqrt{T}\sqrt{\varepsilon\Delta x + \Delta t} + \sup_{\mathbb{R}^N}|u_0^\varepsilon - (\tilde{u}_0^\varepsilon)_\#| + \sup_{\mathbb{R}^N}|u_0^\varepsilon - u_0|.\end{aligned}$$

Taking $\varepsilon \ge K(\Delta x + \sqrt{\Delta t})$ implies the prevalence of the first term with respect to the second one and so this implies the desired estimate for the choice of initial condition $u_0^\varepsilon = u_0$.

3.2 Proof of Theorem 2.7

The idea of the proof is borrowed from Barles, Souganidis [7]. Let us set

$$\overline{v}^\varepsilon = \limsup_{\delta\to 0}{}^* v^{\varepsilon,\delta}, \quad \underline{v}^\varepsilon = \liminf_{\delta\to 0}{}_* v^{\varepsilon,\delta}.$$

We now prove that $\overline{v}^\varepsilon$ and $\underline{v}^\varepsilon$ are respectively sub and super-solution of (2.4). Let $\varphi \in C^\infty(\mathbb{R}^N \times [0,T))$ such that $\overline{v}^\varepsilon - \varphi$ reaches a strict maximum at the point (x_0, t_0) with $t_0 \in (0,T)$. Then, there exists $\delta \to 0$,

$(x^\delta, t^\delta) \to (x_0, t_0)$ such that $v^{\varepsilon,\delta} - \varphi$ reaches a maximum at (x^δ, t^δ). We denote by $(x_I^\delta, t_n^\delta) \in Q_T^\Delta$ the node such that $v^{\varepsilon,\delta}(x^\delta, t^\delta) = v^{\varepsilon,\delta}(x_I^\delta, t_n^\delta)$. This implies that, for all (x,t):

$$v^{\varepsilon,\delta}(x,t) \le v^{\varepsilon,\delta}(x_I^\delta, t_n^\delta) - \varphi(x^\delta, t^\delta) + \varphi(x,t).$$

Using the monotony of the scheme for given velocity and the fact that v is solution of the scheme, we get

$$\begin{aligned} &\frac{\varphi(x^\delta, t^\delta) - \varphi(x^\delta, t^\delta - \Delta t)}{\Delta t} \\ \le & c^\varepsilon[v^{\varepsilon,\delta}](x_I^\delta, t^\delta - \Delta t) \begin{cases} G^+(D^+\varphi(x^\delta, t^\delta - \Delta t), D^-\varphi(x^\delta, t^\delta - \Delta t)) & \text{if} \quad c[v^{\varepsilon,\delta}](x_I^\delta, t^\delta - \Delta t) \ge 0 \\ G^-(D^+\varphi(x^\delta, t^\delta - \Delta t), D^-\varphi(x^\delta, t^\delta - \Delta t)) & \text{if} \quad c[v^{\varepsilon,\delta}](x_I^\delta, t^\delta - \Delta t) \le 0 \end{cases}\end{aligned}$$

with

$$c^\varepsilon[u](x,t) = c_0^\varepsilon \star 1_{\{u(\cdot,t) \ge u(x,t)\}}(x) - \frac{1}{2}\int_{\mathbb{R}^N} c_0^\varepsilon$$

and where we have used Lemma 5.5 to obtain the velocity. Sending $\delta \to 0$, using Slepčev Lemma [26, equation (5)] and the consistency of the scheme for given velocity, we get

$$\varphi_t(x_0, t_0) \le \left(c_0^\varepsilon \star 1_{\{\overline{v}^\varepsilon(\cdot, t_0) \ge \overline{v}^\varepsilon(x_0, t_0)\}}(x_0) - \frac{1}{2}\int_{\mathbb{R}^N} c_0^\varepsilon \right) |D\varphi(x_0, t_0)|.$$

This proves that $\overline{v}^\varepsilon$ is a sub-solution of (2.4). The proof for $\underline{v}^\varepsilon$ is the same and we skip it. Moreover, using a barrier argument (using the equivalent of Proposition 5.4) we get that $\overline{v}^\varepsilon(\cdot, 0) = \underline{v}^\varepsilon(\cdot, 0) = u_0^\varepsilon$. Then we have that $v^\varepsilon = \overline{v}^\varepsilon = \underline{v}^\varepsilon$ is solution of (2.4). This ends the proof of the theorem.

Remark 3.2 *In the scheme* (2.15)*, we have made the choice to take the velocity* $c^\Delta[v]_I^n$*. An another possibility is to take*

$$\tilde{c}^\Delta[v]_I^{n+1} = \sum_{J \in \mathbb{Z}^N} \bar{c}_{I-J}^0 1_{\{v_J^{n+1} > v_I^{n+1}\}} \Delta x_1 ... \Delta x_N - \frac{1}{2}\sum_{J \in \mathbb{Z}^N} \bar{c}_J^0 \Delta x_1 ... \Delta x_N.$$

This construct two discrete solution v and $\tilde{v}$. So, we can take any discrete function comprised between $\tilde{v}$ and v. In fact, this is equivalent to define a discrete solution as a sub and a super-solution as in Definition 5.1. With this definition, v will be the greater sub-solution and $\tilde{v}$ will be the lower super-solution and Theorem 2.7 will be true for every solutions.

3.3 What happens if we change the kernel?

A natural question is what happens if we change the kernel and if we take a kernel which decrease more quickly at infinity. For this kind of kernel K_0, the natural scaling is the following one (see [13, Section 4.1])

$$K_0^\varepsilon(x) = \frac{1}{\varepsilon^{n+1}} K_0\left(\frac{x}{\varepsilon}\right).$$

In fact, using the same arguments as the one we use for the proof of Theorem 2.1, we can prove the same kind of error estimate. For example for $K_0(x) = \dfrac{1}{|x|^{n+p}}$ for $|x| \ge 1$, with $p \ge 3$, we get

$$|u^\varepsilon - u^0| \le K\sqrt{\varepsilon} T^{1/4}.$$

This is the best estimate for ε small that we can obtain for general kernel, as we can see in Step 3, Case 1 of the proof of Theorem 2.1.

The main difference is that the estimate of Corollary 3.1 is replaced by

$$\sup_{\mathbb{R}^N \times \{0,...,t_{N_T}\}} |u^\varepsilon - v_\#^\varepsilon| \le K\frac{\sqrt{T}}{\varepsilon}\sqrt{\varepsilon\Delta x + \Delta t} + \sup_{\mathbb{R}^N} |u_0^\varepsilon - (\tilde{u}_0^\varepsilon)_\#| \quad \text{for } \varepsilon\Delta x + \Delta t \le \frac{\varepsilon^2}{K^2}.$$

Finally, we obtain for the choice $u_0^\varepsilon = u_0$

$$\sup_{\mathbb{R}^N \times \{0,...,t_{N_T}\}} |u^0 - (v^\varepsilon)_\#| \le K\sqrt{\varepsilon} T^{\frac{1}{4}} + \sup_{\mathbb{R}^N} |u_0 - \tilde{u}_0^\varepsilon|$$

if $\varepsilon \ge K\left(\sqrt{\Delta x} + (\Delta t)^{\frac{1}{3}}\right)$. This implies that $\dfrac{\Delta x}{\varepsilon} \to 0$ as $\Delta x \to 0$. So, when Δx goes to zero, we have to make the convolution on all the space which can be very expensive in practice. This approach can also be used for Gaussian kernel and should give the same estimates. In particular, this could give an error estimate for the equivalent version of the classical Bence, Merriman, Osher [20] algorithm.

4 Proof of Theorem 2.1

Before to prove Theorem 2.1, we need some notation. Let us define

$$F(M,p) = \text{trace}\left(M \cdot A\left(\frac{p}{|p|}\right)\right) \quad \text{and} \quad G(M,p) = \frac{-1}{|p|}F(M,p)$$

where we recall that $A(p) \cdot p = 0\ \forall p \in \mathbf{S}^{N-1}$.

Then we have the following fundamental estimate for balls:

Lemma 4.1 (Error estimate for a ball)
Let $\varphi \in C^2(\mathbb{R}^N)$ with $D\varphi(x_0) \neq 0$, be such that the set $\{\varphi(x) \geq \varphi(x_0)\}$ is a ball of radius R. For $c_0^\varepsilon(\cdot) = \frac{1}{\varepsilon^{n+1}|\ln \varepsilon|}\ c_0\left(\frac{\cdot}{\varepsilon}\right)$, let us define

$$c^\varepsilon = (c_0^\varepsilon \star 1_{\{\varphi(\cdot)>\varphi(x_0)\}})(x_0) - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon.$$

Then, there exists a constant $K = K(N, \sup_{\mathbb{R}^n} c_0, |Dg|_{L^\infty(\mathbf{S}^{N-1})}) > 0$ such that for $0 < \varepsilon < \delta$ with $0 < \delta \leq R/2$, we have

$$|c^\varepsilon + G(D^2\varphi(x_0), D\varphi(x_0))| \leq K \cdot e\,(\varepsilon, \delta, R)$$

with

$$e\,(\varepsilon, \delta, R) = \frac{1}{|\ln \varepsilon|}\left(\frac{1}{\delta} + \frac{1}{R}|\ln \delta|\right) + \frac{\delta}{R^2}.$$

This is a straightforward consequence of Da Lio *et al.* [13, Proposition 4.1] where their Lemma 4.3 is replaced by a direct estimate for an explicit function $h(x') = R - \sqrt{R^2 - |x'|^2}$ with $x' \in \mathbb{R}^{N-1}$ and $|x'| \leq R$.

Proof of Theorem 2.1
Let u^0 be the solution of the mean curvature motion (2.5). The idea of the proof is to regularise the function u^0 by a kind of sup-convolution but using quartic penalization and then to plug the regularised function $u^{0,\alpha}$ into (2.4). This regularisation allows us to control quantitatively the first and the second derivatives in space of $u^{0,\alpha}$. This is a quantitative version of the definition of Barles and Georgelin [4] for viscosity solutions of mean curvature motion where they used this kind of regularisation to prove that one can take test functions such that $D\varphi \neq 0$ or $D\varphi = 0$ and $D^2\varphi = 0$ in the definition of viscosity solutions. This kind of arguments is also used to obtained the comparison principle for mean curvature type equations.

The proof is decomposed into five steps:

Step 1. Regularisation of u^0
By classical estimates for mean curvature type equations, we have

$$\|Du^0\|_{L^\infty(\mathbb{R}^N \times (0,T))} \leq \|Du_0\|_{L^\infty(\mathbb{R}^N)}. \tag{4.17}$$

We regularise u^0 by considering a spatial sup-convolution by quartics

$$u^{0,\alpha}(x,t) = \sup_{y \in \mathbb{R}^N}\left\{u^0(x-y,t) - \frac{1}{4\alpha}|y|^4\right\} \tag{4.18}$$

$$= \sup_{z \in \mathbb{R}^N}\left\{u^0(z,t) - \frac{1}{4\alpha}|z-x|^4\right\}. \tag{4.19}$$

Since u^0 is Lipschitz continuous, the supremum is reached.

Step 2. Estimate between u^0 and $u^{0,\alpha}$
For $x \in \mathbb{R}^N$ and $t \in (0,T)$, we denote by $\bar{x} = \bar{x}(x,t)$ a point where the maximum is reached in (4.19). Then

$$u^{0,\alpha}(x,t) = u^0(\bar{x},t) - \frac{1}{4\alpha}|\bar{x}-x|^4 \geq u^0(x,t). \tag{4.20}$$

Therefore

$$\frac{1}{4\alpha}|\bar{x}-x|^4 \le u^0(\bar{x},t) - u^0(x,t) \le K|\bar{x}-x| \tag{4.21}$$

and so

$$|\bar{x}-x| \le K\alpha^{1/3}. \tag{4.22}$$

Moreover, by (4.20), we get

$$0 \le u^{0,\alpha}(x,t) - u^0(x,t) \le u^0(\bar{x},t) - u^0(x,t) \le K|\bar{x}-x| \tag{4.23}$$

Using (4.22) and (4.23), we then deduce for $x \in \mathbb{R}^N$ and $t \in (0,T)$

$$|u^{0,\alpha}(x,t) - u^0(x,t)| \le K\alpha^{1/3} \tag{4.24}$$

Before to continue the proof, we need some notation. We now want to compare u^ε and $u^{0,\alpha}$. To do that, we will prove that $u^{0,\alpha} \pm (f(\varepsilon,\alpha)t + K\alpha^{\frac{1}{3}} + \sup_{\mathbb{R}^N}|u_0^\varepsilon - u_0|)$ are respectively super and sub-solutions of Problem (2.4) for

$$f(\varepsilon,\alpha) = K\left(\frac{1}{\sqrt{|\ln\varepsilon|}} + \frac{\ln|\ln\varepsilon|}{\alpha^{1/3}|\ln\varepsilon|} + \frac{1}{\alpha^{2/3}}\left(\frac{1}{\sqrt{|\ln\varepsilon|}} + \varepsilon\right)\right).$$

Let $\psi \in C^\infty(\mathbb{R}^N \times (0,T))$ such that $u^{0,\alpha} - \psi$ reaches a global strict maximum at $(x_0,t_0) \in \mathbb{R}^N \times (0,T)$. Since $u^{0,\alpha}$ is semi-convex in the space variable, the functions $u^{0,\alpha}$ is derivable with respect to x at the point (x_0,t_0).

Let us denote by

$$\varphi_z(x) = u^0(z,t_0) - \frac{1}{4\alpha}|z-x|^4.$$

Then

$$u^{0,\alpha}(x,t_0) = \sup_{z\in\mathbb{R}^N} \varphi_z(x).$$

We denote by $\bar{x}_0$ a point where the maximum is reached for $x = x_0$ and by $\varphi = \varphi_{\bar{x}_0}$. We then deduce that

$$\psi(x,t_0) \ge u^{0,\alpha}(x,t_0) \ge \varphi(x) \tag{4.25}$$

with equality for $x = x_0$. This implies that

$$D\varphi(x_0) = Du^{0,\alpha}(x_0,t_0) = D\psi(x_0,t_0) \tag{4.26}$$

and

$$D^2\varphi(x_0) \le D^2\psi(x_0,t_0). \tag{4.27}$$

Moreover, with $h = |\bar{x}_0 - x_0|$, we have (by (4.22))

$$h \le K\alpha^{\frac{1}{3}}, \tag{4.28}$$

$$|D\varphi(x_0)| = \frac{h^3}{\alpha} \tag{4.29}$$

and the set $\{\varphi(x) \ge \varphi(x_0)\}$ is a ball of radius $R = h$. Let us define the error

$$e(\varphi) = |c^\varepsilon[\varphi](x_0)|D\varphi(x_0)| - F(D^2\varphi(x_0), D\varphi(x_0))|$$

where

$$c^\varepsilon[\varphi](x_0) = \left(c_0^\varepsilon \star 1_{\{\varphi(\cdot)\ge\varphi(x_0)\}}\right)(x_0) - \frac{1}{2}\int_{\mathbb{R}^N} c_0^\varepsilon.$$

Step 3. $e(\varphi) \le f(\varepsilon,\alpha)$

We distinguish three cases:

Case 1. $h \le 2\varepsilon$. In this case, we directly have the estimate (using (4.29))

$$|c^\varepsilon[\varphi](x_0)|D\varphi(x_0)|| \le |c_0^\varepsilon|_{L^1(\mathbb{R}^N)}\frac{h^3}{\alpha} \le \frac{h^2|c_0|_{L^1(\mathbb{R}^N)}}{\varepsilon|\ln\varepsilon|}\frac{h}{\alpha} \le \frac{K}{\alpha^{2/3}}\frac{\varepsilon}{|\ln\varepsilon|}$$

where we have also used (4.28). Using moreover the fact that

$$|F(D^2\varphi(x_0), D\varphi(x_0))| \le \frac{K}{h}|D\varphi(x_0)| \le K\frac{h^2}{\alpha},$$

we then deduce that

$$e(\varphi) \le \frac{K}{\alpha^{2/3}}\varepsilon \le f(\varepsilon,\alpha).$$

Case 2. $2\varepsilon < h \le \dfrac{2}{\sqrt{|\ln\varepsilon|}}$. Using (4.29) and Lemma 4.1, with $\delta = h/2$, we then deduce that

$$e(\varphi) \le \frac{Kh^3}{\alpha}\left(\frac{1}{|\ln\varepsilon|}\left(\frac{1}{h}+\frac{1}{h}|\ln\delta|\right)+\frac{1}{h}\right) \le \frac{Kh}{\alpha^{2/3}} \le \frac{K}{\alpha^{2/3}}\frac{1}{\sqrt{|\ln\varepsilon|}} \le f(\varepsilon,\alpha).$$

Case 3. $h > \dfrac{2}{\sqrt{|\ln\varepsilon|}}$. We set $\delta = \dfrac{1}{\sqrt{|\ln\varepsilon|}}$. We then have $\varepsilon < \delta < h/2$. So by Lemma 4.1, we deduce that

$$\begin{aligned}
e(\varphi) \le& \frac{Kh^3}{\alpha}\left(\frac{1}{\sqrt{|\ln\varepsilon|}} + \frac{1}{h}\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|} + \frac{1}{h^2}\frac{1}{\sqrt{|\ln\varepsilon|}}\right)\\
\le& K\left(\frac{1}{\sqrt{|\ln\varepsilon|}} + \frac{1}{\alpha^{1/3}}\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|} + \frac{1}{\alpha^{2/3}}\frac{1}{\sqrt{|\ln\varepsilon|}}\right)\\
\le& f(\varepsilon,\alpha).
\end{aligned}$$

Step 4. Estimate between $u^{0,\alpha}$ and u^ε

By construction, we have, since $u^{0,\alpha}$ is a sub-solution of (2.5) (but does not satisfy the initial condition)

$$\begin{aligned}
\psi_t(x_0,t_0) \le& F(D^2\psi(x_0,t_0), D\psi(x_0,t_0))\\
\le& c^\varepsilon[\varphi](x_0)|D\psi(x_0,t_0)| + F(D^2\varphi(x_0), D\varphi(x_0)) - c^\varepsilon[\varphi](x_0)|D\varphi(x_0)|\\
\le& c^\varepsilon[u^{0,\alpha}(\cdot,t_0)](x_0)|D\psi(x_0,t_0)| + f(\varepsilon,\alpha)
\end{aligned}$$

where we have used (4.25) for the last line. On the other hand, we have

$$u^{0,\alpha}(\cdot,0) - K\alpha^{1/3} - \sup_{\mathbb{R}^N}|u_0^\varepsilon - u_0| \le u_0^\varepsilon$$

where we have used (4.24) at the limit $t = 0$. We then deduce that $\tilde{u}^{0,\alpha}(x,t) = u^{0,\alpha}(x,t) - f(\varepsilon,\alpha)t - K\alpha^{1/3} - \sup_{\mathbb{R}^N}|u_0^\varepsilon - u_0|$ is sub-solution of (2.4). This implies that $\tilde{u}^{0,\alpha} \le u^\varepsilon$ and so

$$u^{0,\alpha} - u^\varepsilon \le f(\varepsilon,\alpha)t + K\alpha^{1/3} + \sup_{\mathbb{R}^N}|u_0^\varepsilon - u_0|. \tag{4.30}$$

Step 5. Estimate between u^0 and u^ε

Using (4.24) and (4.30), we then get

$$u^0 - u^\varepsilon = u^0 - u^{0,\alpha} + u^{0,\alpha} - u^\varepsilon \le f(\varepsilon,\alpha)t + K\alpha^{1/3} + \sup_{\mathbb{R}^N}|u_0^\varepsilon - u_0|.$$

But

$$\begin{aligned}
f(\varepsilon,\alpha)t + K\alpha^{1/3} =& K\left(\frac{1}{\sqrt{|\ln\varepsilon|}} + \frac{\ln|\ln\varepsilon|}{\alpha^{1/3}|\ln\varepsilon|} + \frac{1}{\alpha^{2/3}}\left(\frac{1}{\sqrt{|\ln\varepsilon|}}+\varepsilon\right)\right)t + K\alpha^{1/3}\\
\le& K\left(\frac{T}{\sqrt{|\ln\varepsilon|}}\right)^{\frac{1}{3}}
\end{aligned}$$

for $\alpha = \dfrac{Kt}{\sqrt{|\ln \varepsilon|}}$ and $t \leq T \leq 1$. We then get

$$u^0 - u^\varepsilon \leq K \left(\frac{T}{\sqrt{|\ln \varepsilon|}} \right)^{\frac{1}{3}} + \sup_{\mathbb{R}^N} |u_0^\varepsilon - u_0|.$$

A lower bound is proved similarly (using the same result as Lemma 4.1 where the set $\{\varphi(x) \geq \varphi(x_0)\}$ is replaced by $\{\varphi(x) > \varphi(x_0)\}$). This implies (2.7) for $T \leq 1$. This ends the proof of the theorem.

5 Numerical scheme for dislocations dynamics

5.1 Definitions and preliminary results

We recall here the notation used in the scheme. The discrete solution v is computed iteratively by solving the implicit scheme

$$(5.31) \qquad v_I^0 = \tilde{u}_0(x_I), \qquad \frac{v_I^{n+1} - v_I^n}{\Delta t} = c^\Delta[v]_I^{n+1} G(v^{n+1})_I$$

where the non-local velocity is defined in (2.10). The approximation of the gradient of v^{n+1} at the point x_I is given by

$$G(v^{n+1})_I = \begin{cases} G^+(D^+v_I^{n+1}, D^-v_I^{n+1}) & \text{if} \quad c^\Delta[v]_I^{n+1} \geq 0, \\ G^-(D^+v_I^{n+1}, D^-v_I^{n+1}) & \text{if} \quad c^\Delta[v]_I^{n+1} < 0 \end{cases}$$

where $G^\pm$ is a suitable approximation of the Euclidean norm and $D^\pm v^n(x_I) = (D^\pm_{x_1} v^n(x_I), ..., D^\pm_{x_N} v^n(x_I))$ are the discrete gradients. The terms $D^\pm_{x_i} v^n(x_I)$ are the standard forward and backward first order differences, *i.e.* for a general function $f(x_I)$:

$$D^+_{x_i} f(x_I) = \frac{f(x_{I^{i,+}}) - f(x_I)}{\Delta x_i},$$

$$D^-_{x_i} f(x_I) = \frac{f(x_I) - f(x_{I^{i,-}})}{\Delta x_i},$$

where

$$(5.32) \qquad I^{k,\pm} = (i_1, ..., i_{k-1}, i_k \pm 1, i_{k+1}, ..., i_N).$$

The approximations of the Euclidean norm $G^\pm$ are those proposed by Osher and Sethian [23] (we can also use the ones proposed by Rouy, Tourin [24]):

$$(5.33) \qquad G^+(P,Q) = \left(\sum_{i=1,..,N} \max(P_i, 0)^2 + \sum_{i=1,..,N} \min(Q_i, 0)^2 \right)^{\frac{1}{2}},$$

$$G^-(P,Q) = \left(\sum_{i=1,..,N} \min(P_i, 0)^2 + \sum_{i=1,..,N} \max(Q_i, 0)^2 \right)^{\frac{1}{2}}.$$

We recall that the functions $G^\pm$ are Lipschitz continuous with respect to the discrete gradients, *i.e.*

$$(5.34) \qquad |G^\pm(P_1, P_2) - G^\pm(P_1', P_2')| \leq (|P_1 - P_1'| + |P_2 - P_2'|).$$

They are consistent with the Euclidean norm

$$(5.35) \qquad G^\pm(P, P) = |P|,$$

and $G^\pm = G^\pm(P_1^+, ..., P_N^+, P_1^-, ..., P_N^-)$ enjoy suitable monotonicity with respect to each variable

$$(5.36) \qquad \frac{\partial G^+}{\partial p_i^+} \geq 0, \quad \frac{\partial G^+}{\partial p_i^-} \geq 0, \quad \frac{\partial G^-}{\partial p_i^+} \geq 0, \quad \frac{\partial G^-}{\partial p_i^-} \geq 0.$$

To define discrete solution, we need the following notation

$$\tilde{c}^{\Delta}[v]_I^{n+1} = \sum_{J\in\mathbb{Z}^N} \bar{c}^0_{I-J} 1_{\{v_J^{n+1} > v_I^{n+1}\}} \Delta x_1 ... \Delta x_N - \frac{1}{2} \sum_{J\in\mathbb{Z}^N} \bar{c}^0_J \Delta x_1 ... \Delta x_N, \tag{5.37}$$

$$\tilde{G}(v^{n+1})_I = \begin{cases} G^+(D^+ v_I^{n+1}, D^- v_I^{n+1}) & \text{if} \quad \tilde{c}^{\Delta}[v]_I^{n+1} \geq 0, \\ G^-(D^+ v_I^{n+1}, D^- v_I^{n+1}) & \text{if} \quad \tilde{c}^{\Delta}[v]_I^{n+1} < 0 \end{cases}$$

where $\bar{c}_0$ is defined in (2.11). Finally, for simplicity of presentation, let us denote by

$$s[v]_I^{n+1} = \mathrm{Sign}(c^{\Delta}[v]_I^{n+1}), \quad \tilde{s}[v]_I^{n+1} = \mathrm{Sign}(\tilde{c}^{\Delta}[v]_I^{n+1}).$$

Definition 5.1 (Numerical sub, super and solution of the scheme)
We say that v is a discrete sub-solution (resp super-solution) of the scheme (2.9) if for all $I \in \mathbb{Z}^N$, $n \in \mathbb{N}$, we have

$$v_I^{n+1} \leq v_I^n + \Delta t \, c^{\Delta}[v]_I^{n+1} G(v^{n+1})_I$$

$$\left(\text{resp. } v_I^{n+1} \geq v_I^n + \Delta t \, \tilde{c}^{\Delta}[v]_I^{n+1} \tilde{G}(v^{n+1})_I\right).$$

We say that v is a discrete solution if and only if it is a sub and a super-solution.

Lemma 5.2 (Almost monotonicity of the scheme)
The scheme (2.9) is almost monotone in the following sense. Let v, w be two discrete functions and assume that there is I such that $w_I = v_I$ and $w_J \geq v_J$ for $J \neq I$. Then

$$c^{\Delta}[v]_I G(v)_I \leq c^{\Delta}[w]_I G(w)_I$$

and

$$\tilde{c}^{\Delta}[v]_I G(v)_I \leq \tilde{c}^{\Delta}[w]_I G(w)_I.$$

Proof of Lemma 5.2
First, we remark that $c^{\Delta}[v]_I \leq c^{\Delta}[w]_I$. Then, there is three cases:

1. $c^{\Delta}[v]_I \leq 0 \leq c^{\Delta}[w]_I$. In this case, the result is trivial.
2. $0 \leq c^{\Delta}[v]_I \leq c^{\Delta}[w]_I$. Using the monotonicity of G, we get that $G(v)_I \leq G(w)_I$. This implies the result.
3. $c^{\Delta}[v]_I \leq c^{\Delta}[w]_I \leq 0$. Using the monotonicity of G, we get that $G(v)_I \geq G(w)_I$. This implies the result.

The proof for $\tilde{c}^{\Delta}$ is the same and we skip it. This ends the proof of the lemma.

The existence of a solution for the scheme is not trivial (since it is implicit and non-local). This is the subject of the following proposition:

Proposition 5.3 (Existence of solution for the scheme (2.9))
There exists, at least, one discrete solution v of the scheme (2.9) in the sense of Definition 5.1.

Proof of Proposition 5.3
We assume that there exists a solution v^n at step n and we will construct a solution v^{n+1} at step $n+1$. First, we remark that if $(v^{n+1,i})_i$ is a family indexed by i of discrete sub-solution at step $n+1$, then $v^{n+1} = \max \; v^{n+1,i}$ is still a sub-solution (it suffices to use Lemma 5.2). Moreover, $w^+ = \sup_{Q^{\Delta}} |v^n|$ and $w^- = -\sup_{Q^{\Delta}} |v^n|$ are respectively discrete super and sub-solution of the scheme (2.9). Then, let us define

$$v^{n+1} = \max\{w \text{ subsolution at step } n+1 \text{ s.t } w \leq w^+\}.$$

Then v^{n+1} is a discrete sub-solution at step $n+1$. Now let us prove that v^{n+1} is a super-solution. By contradiction, assume that there is I such that

$$\frac{v_I^{n+1} - v_I^n}{\Delta t} < \tilde{c}[v]_I^{n+1} G^{\tilde{s}[v]_I^n}(D^+ v_I^{n+1}, D^- v_I^{n+1}).$$

This implies in particular that $v_I^{n+1} < w^+$. Now, let us consider, the solution w_I (in a sense similar to Definition 5.1) of

$$\frac{w_I - v_I^n}{\Delta t} = c^{\Delta}_{v^{n+1}}[w]_I \left\{ \begin{array}{ll} G^+(D^+_{v^{n+1}} w_I, D^-_{v^{n+1}} w_I) & \text{if} \quad c^{\Delta}_{v^{n+1}}[w]_I \geq 0 \\ G^-(D^+_{v^{n+1}} w_I, D^-_{v^{n+1}} w_I) & \text{if} \quad c^{\Delta}_{v^{n+1}}[w]_I < 0 \end{array} \right. ,$$

where

$$c^{\Delta}_{v^{n+1}}[w]_I = \sum_{J\in\mathbb{Z}^N} \bar{c}^0_{I-J} 1_{\{v_J^{n+1} \geq w_I\}} \Delta x_1 ... \Delta x_N - \frac{1}{2} \sum_{J\in\mathbb{Z}^N} \bar{c}^0_J \Delta x_1 ... \Delta x_N$$

and $D^{\pm}_{v^{n+1}} w_I = (D^{\pm}_{x_1, v^{n+1}} w_I, ..., D^{\pm}_{x_N, v^{n+1}} w_I)$ with

$$D^+_{x_i, v^{n+1}} w_I = \frac{v^{n+1}_{I^{i,+}} - w_I}{\Delta x_i},$$

$$D^-_{x_i, v^{n+1}} w_I = \frac{w_I - v^{n+1}_{I^{i,-}}}{\Delta x_i},$$

where $I^{k,\pm}$ is defined in (5.32). The existence of such a solution comes from the fact that the left hand side is non-decreasing in w_I and the right hand side is non-increasing. Then, it is easy to prove (using Lemma 5.2) that $w_I > v_I^{n+1}$ and that w defined by

$$w_J = \left\{ \begin{array}{ll} w_I & \text{if } J = I \\ v_J^{n+1} & \text{otherwise} \end{array} \right.$$

is a discrete sub-solution of (2.9) at step $n+1$. This contradicts the definition of v^{n+1} and ends the proof of the proposition.

Proposition 5.4 (Properties of the discrete solutions)
Assume that $\tilde{u}_0 \in W^{1,\infty}(\mathbb{R}^N)$. Let us denote by $L = |D\tilde{u}_0|_{L^\infty(\mathbb{R}^N)}$ and

(5.38) $$\overline{v} = \sup\{v \textit{ subsolution of } (2.9) \textit{ with initial condition } v^0 = \tilde{u}_0, v \leq |\tilde{u}_0|_{L^\infty(\mathbb{R}^N)}\}.$$

For $k \in \mathbb{Z}^N$ such that $k \cdot \Delta X > 0$ (with $\Delta X = (\Delta x_1, ..., \Delta x_N)$), the following estimates hold

(5.39) $$\frac{\overline{v}(x_I, t_n) - \overline{v}(x_I + k\cdot\Delta X, t_n)}{k \cdot \Delta X} \leq L,$$

(5.40) $$\frac{\overline{v}(x_I, t_n) - \overline{v}(x_I + k\cdot\Delta X, t_n)}{k \cdot \Delta X} \geq -L,$$

(5.41) $$\left| \frac{\overline{v}_I^{n+1} - \overline{v}_I^n}{\Delta t} \right| \leq \frac{\sqrt{N}}{2} |c_0|_{L^1(\mathbb{R}^N)} \, L.$$

The same estimates hold for

(5.42) $$\underline{v} = \inf\{v \textit{ supersolution of } (2.9) \textit{ with initial condition } v^0 = \tilde{u}_0, v \geq -|u_0|_{L^\infty(\mathbb{R}^N)}\}.$$

Proof of Proposition 5.4
For $k \in \mathbb{Z}^N$, let us denote by

$$\tilde{u}_{0,k}(x_I) = \tilde{u}_0(x_I + k\cdot\Delta X) + Lk\cdot\Delta X \geq \tilde{u}_0(x_I)$$

and by $\overline{v}_k$ the greater sub-solution, with initial condition $\tilde{u}_{0,k}$, *i.e.*

$$\overline{v}_k = \sup\{v_k \text{ subsolution of (2.9) with initial condition } v_k^0 = \tilde{u}_{0,k}, v_k \leq |u_0|_{L^\infty(\mathbb{R}^N)}\}.$$

First, since the equation does not see the constants, we remark that $\overline{v} - Lk\cdot\Delta X$ is a sub-solution of (2.9) with initial condition $\overline{v}^0 - Lk\cdot\Delta X \leq \tilde{u}_{0,k}$. So, by definition of $\overline{v}_k$, we have

$$\overline{v} - Lk\cdot\Delta X \leq \overline{v}_k.$$

Moreover, using the fact that the scheme is invariant by translation in space, we deduce that $\overline{v}(x_I+k\cdot\Delta X,t_n)$ is the greater sub-solution with initial condition $\overline{v}(x_I+k\cdot\Delta X,0)=\tilde{u}_{0,k}(x_I)$. We then deduce that

$$\overline{v}_k(x_I,t_n)=\overline{v}(x_I+k\cdot\Delta X,t_n).$$

This implies that for $k\cdot\Delta X>0$

$$\frac{\overline{v}(x_I,t_n)-\overline{v}(x_I+k\cdot\Delta X,t_n)}{k\cdot\Delta X}\le L.$$

The proof of (5.40) is similar and we skip it.

To prove (5.41), we use the fact that $\overline{v}$ is a solution and the two previous estimates. This implies

$$-\frac{\sqrt{N}}{2}|c_0|_{L^1(\mathbb{R}^N)}L\le\frac{\overline{v}_I^{n+1}-\overline{v}_I^n}{\Delta t}\le\frac{\sqrt{N}}{2}|c_0|_{L^1(\mathbb{R}^N)}L$$

which is the desired result.

Before to prove Theorem 2.2, we need the following lemma:

Lemma 5.5 (Equivalent formulation for the discrete velocity)
The discrete velocity $c^{\Delta}[v]$ can be rewritten as

$$c^{\Delta}[v]_I^n=\left(c_0\star 1_{\{v_\#(.,t_n)\ge v_\#(x_I,t_n)\}}\right)(x_I)-\frac{1}{2}|c_0|_{L^1(\mathbb{R}^N)}$$

where $v_\#$ is defined in (2.13).

Proof of Lemma 5.5
The idea of the proof is borrowed from Alvarez *et al.* [2]. Using the definition of the discrete velocity and $\overline{c}^0$, we get

$$\begin{aligned}
c^{\Delta}[v]_I^n&=\sum_{J\in\mathbb{Z}^N}\overline{c}^0_{I-J}1_{\{v_J^n\ge v_I^n\}}\Delta x_1...\Delta x_N-\frac{1}{2}\sum_{J\in\mathbb{Z}^N}\overline{c}^0_J\Delta x_1...\Delta x_N\\
&=\sum_{J\in\mathbb{Z}^N}\int_{Q_{J-I}}c_0(y)\,dy\,1_{\{v_J^n\ge v_I^n\}}-\frac{1}{2}\sum_{J\in\mathbb{Z}^N}\int_{Q_J}c_0(y)\,dy\\
&=\sum_{J\in\mathbb{Z}^N}\int_{Q_J}c_0(x_I-y)\,1_{\{v_J^n\ge v_I^n\}}\,dy\;-\frac{1}{2}|c_0|_{L^1(\mathbb{R}^N)}\\
&=\sum_{J\in\mathbb{Z}^N}\int_{Q_J}c_0(x_I-y)\,1_{\{v_\#(y,t_n)\ge v_\#(x_I,t_n)\}}\,dy\;-\frac{1}{2}|c_0|_{L^1(\mathbb{R}^N)}\\
&=\left(c_0\star 1_{\{v_\#(\cdot,t_n)\ge v_\#(x_I,t_n)\}}\right)(x_I)-\frac{1}{2}|c_0|_{L^1(\mathbb{R}^N)}.
\end{aligned}$$

This ends the proof of the lemma.

We now prove Theorem 2.2:

Proof of Theorem 2.2
The proof is an adaptation of the one of Crandall Lions [11], revisited by Alvarez *et al.* [1]. Nevertheless, for the reader's convenience, we give the main steps in order to show the new difficulties due to the non-local term. The main idea of the proof is the same as the one of comparison principles, *i.e.* to consider the maximum of $u-v_\#$, to duplicate the variable and to use the viscosity inequalities to get the result.

The proof splits into four steps.

We first assume that

$$\bar{u}_0(x_I)\ge(\tilde{u}_0(x_I))_\#,\quad\text{for all } I\in\mathbb{Z}^N. \tag{5.43}$$

and we set

$$\mu^0=\sup_{\mathbb{R}^N}(\bar{u}_0-(\tilde{u}_0)_\#)\ge 0. \tag{5.44}$$

We denote throughout by K various constant depending only on N, $|c_0|_{W^{1,1}(\mathbb{R}^N)}$, $|D\bar{u}_0|_{L^\infty(\mathbb{R}^N)}$, $|D\tilde{u}_0|_{L^\infty(\mathbb{R}^N)}$ and μ^0.

Step 1: Estimate on v

We have the following estimate for the discrete solution

$$-Kt_n \le \bar{u}_0(x) - v_{\#}(x,t_n) \le Kt_n + \mu^0. \tag{5.45}$$

To show this, it suffices to use the estimate (5.41) of Proposition 5.4. This implies that, for $x \in Q_I$ (with the notation $\overline{v}$ and $\underline{v}$ defined in Proposition 5.4)

$$(\tilde{u}_0)_{\#}(x) - Kt_n = v_I^0 - Kt_n \le \underline{v}_I^n \le v_I^n = v_{\#}(x,t_n) \le \overline{v}_I^n \le Kt_n + v_I^0 = Kt_n + (\tilde{u}_0)_{\#}(x).$$

This implies the desired estimate.

Before continuing the proof, we need a few notation. We put

$$\mu = \sup_{\mathbb{R}^N}(u - v_{\#}).$$

We want to bound from above μ by μ^0 plus a constant. For every $0 < \alpha \le 1$, $0 < \gamma \le 1$ and $0 < \eta \le 1$, we set

$$M_\eta^{\alpha,\gamma} = \sup_{\mathbb{R}^N\times\mathbb{R}^N\times(0,T)\times\{0,\dots,t_{N_T}\}} \Psi_\eta^{\alpha,\gamma}(x,y,t,t_n)$$

with

$$\Psi_\eta^{\alpha,\gamma}(x,y,t,t_n) = u(x,t) - v_{\#}(y,t_n) - \frac{|x-y|^2}{2\gamma} - \frac{|t-t_n|^2}{2\gamma} - \eta t - \alpha(|x|^2+|y|^2).$$

We shall drop the super and subscripts on Ψ when no ambiguity arises as concerning the value of the parameter.

The main difference with the classical Crandall Lions proof, is to consider the function $v_{\#}$ in the place of v in the separation of variables. This allows us to treat the non-local velocity.

Since u is Lipschitz continuous and $T \le 1$, we have

$$|u(x,t)| \le K(1+|x|). \tag{5.46}$$

Moreover by Step 1, we have

$$\begin{aligned}|v(x_i,t_n)| \le& |v(x_i,t_n) - \bar{u}_0(x_i)| + |\bar{u}_0(x_i)|\\ \le& Kt_n + K(1+|x_i|)\\ \le& K(1+|x_i|).\end{aligned}$$

We then deduce that Ψ achieves its maximum at some point that we denote by (x^*,y^*,t^*,t_n^*).

Step 2: Estimates for the maximum point of Ψ

The maximum point of Ψ enjoys the following estimates

$$\alpha|x^*| + \alpha|y^*| \le K \tag{5.47}$$

and

$$|x^*-y^*| \le K\gamma, \quad |t^*-t_n^*| \le (K+2\eta)\gamma. \tag{5.48}$$

Indeed, by inequality $\Psi(x^*,y^*,t^*,t_n^*) \ge \Psi(0,0,0,0) \ge 0$, we obtain

$$\begin{aligned}\alpha|x^*|^2 + \alpha|y^*|^2 \le& u(x^*,t^*) - v_{\#}(y^*,t_n^*) \le K(1+|x^*|+|y^*|)\\ \le& K + \frac{K^2}{\alpha} + \frac{\alpha}{2}|x^*|^2 + \frac{\alpha}{2}|y^*|^2.\end{aligned}$$

This implies (5.47), since $\alpha \le 1$.

The first bound of (5.48) follows from the Lipschitz regularity in space of u, from the inequality $\Psi(x^*,y^*,t^*,t_n^*) \ge \Psi(y^*,y^*,t^*,t_n^*)$ and from (5.47). The second bound of (5.48) is obtained in the same way, using the inequality $\Psi(x^*,y^*,t^*,t_n^*) \ge \Psi(x^*,y^*,t_n^*,t_n^*)$.

Step 3: A better estimate for the maximum point of Ψ

Inequality (5.47) can be strengthened to

$$\alpha|x^*|^2 + \alpha|y^*|^2 \leq K. \tag{5.49}$$

Indeed, using the Lipschitz regularity of u, the inequality $\Psi(x^*, y^*, t^*, t_n^*) \geq \Psi(0,0,0,0)$, Step 1 and equation (5.48), yields

$$\begin{aligned}\alpha|x^*|^2 + \alpha|x_i^*|^2 \leq& u(x^*, t^*) - v_\#(y^*, t_n^*) + u_0(y^*) - u_0(y^*)\\ \leq& K(|x^* - y^*| + t^*) + K t_n^* + \mu^0 \leq K.\end{aligned}$$

Step 4 : Upper bound of μ

We have the bound $\mu \leq K\sqrt{T}\,(\Delta x + \Delta t)^{\frac{1}{2}} + \mu^0$ if $\Delta x + \Delta t \leq \frac{1}{K^2}$.

First, we claim that for η large enough, we have either $t^* = 0$ or $t_n^* = 0$. We argue by contradiction. Then the function $(x,t) \mapsto \Psi(x, y^*, t, t_n^*)$ achieves its maximum at a point of $\mathbb{R}^N \times (0,T)$. Then, using the fact that u is a sub-solution of (2.8), we deduce that

$$\eta + p_t^* \leq c[u](x^*, t^*)|p_x^* + 2\alpha x^*| \tag{5.50}$$

where

$$c[u](x,t) = c_0 \star 1_{\{u(\cdot,t)\geq u(x,t)\}}(x) - \frac{1}{2}\int_{\mathbb{R}^N} c_0$$

and

$$p_t^* = \frac{t^* - t_n^*}{\gamma}, \qquad p_x^* = \frac{x^* - y^*}{\gamma}.$$

Since $t_n^* > 0$, we also have $\Psi(x^*, y^*, t^*, t_n^*) \geq \Psi(x^*, y, t^*, t_n)$ for $t_n \geq t_n^* - \Delta t$. This implies

$$v_\#(y^*, t_n^*) - v_\#(y, t_n) \leq \varphi(y^*, t_n^*) - \varphi(y, t_n) \quad \text{for } t_n \geq t_n^* - \Delta t \tag{5.51}$$

for $\varphi(y, t_n) = -\left(\frac{|x^*-y|^2}{2\gamma} + \frac{|t^*-t_n|^2}{2\gamma} + \alpha|y|^2\right)$. We denote by x_I^* the node such that $y^* \in Q_{I^*}$ (the unit cube centred in x_I^*). We then deduce

$$\begin{aligned}\frac{\varphi(y^*, t_n^*) - \varphi(y^*, t_n^* - \Delta t)}{\Delta t} \geq& \frac{v_\#(y^*, t_n^*) - v_\#(y^*, t_n^* - \Delta t)}{\Delta t}\\ =& \frac{v(x_I^*, t_n^*) - v(x_I^*, t_n^* - \Delta t)}{\Delta t}\\ \geq& \tilde{c}^\Delta[v](x_I^*, t_n^*) G^{s[v](x_I^*, t_n^*)}(D^+ v(x_I^*, t_n^*), D^- v(x_I^*, t_n^*))\\ \geq& \tilde{c}^\Delta[v](x_I^*, t_n^*) G^{s[v](x_I^*, t_n^*)}(D^+ \varphi(y^*, t_n^*), D^- \varphi(y^*, t_n^*))\end{aligned}$$

where we have use the monotonicity of $G^\pm$ and (5.51) with $t_n = t_n^*$ for the last line. We denote by $s^* = s[v](x_I^*, t_n^*)$. Straightforward computations of the discrete derivative of φ yield

$$p_t^* + \frac{\Delta t}{2\gamma} \geq \tilde{c}^\Delta[v](x_I^*, t_n^*) G^{s*}\left(p_x^* - \frac{\Delta x}{2\gamma} - \alpha(2y^* + \Delta x), p_x^* + \frac{\Delta x}{2\gamma} - \alpha(2y^* - \Delta x)\right). \tag{5.52}$$

Subtracting the above inequality to (5.50) yields

$$\begin{aligned}\eta \leq& \frac{\Delta t}{2\gamma} + c[u](x^*, t^*)|p_x^* + 2\alpha x^*|\\ &- \tilde{c}^\Delta[v](x_I^*, t_n^*) G^{s^*}\left(p_x^* - \frac{\Delta x}{2\gamma} - \alpha(2y^* + \Delta x), p_x^* + \frac{\Delta x}{2\gamma} - \alpha(2y^* - \Delta x)\right)\\ \leq& \frac{\Delta t}{2\gamma} + (c[u](x^*, t^*) - \tilde{c}^\Delta[v](x_I^*, t_n^*))|p_x^*| + \alpha K|x^*|\\ &+ \tilde{c}^\Delta[v](x_I^*, t_n^*)\left|G^{s^*}\left(p_x^* - \frac{\Delta x}{2\gamma} - \alpha(2y^* + \Delta x), p_x^* + \frac{\Delta x}{2\gamma} - \alpha(2y^* - \Delta x)\right) - G^{s^*}(p_x^*, p_x^*)\right|\\ \leq& \frac{\Delta t}{2\gamma} + (c[u](x^*, t^*) - \tilde{c}^\Delta[v](x_I^*, t_n^*))|p_x^*| + \alpha K|x^*| + K\left(\frac{\Delta x}{\gamma} + 2\alpha|y^*| + 2\alpha\Delta x\right)\end{aligned}$$

We now have to bound the term

$$\begin{aligned}\mathcal{I} =&(c[u](x^*,t^*) - \tilde{c}^{\Delta}[v](x_I^*,t_n^*))|p_x^*|\\ =&(c_0\star 1_{\{u(\cdot,t^*)\geq u(x^*,t^*)\}}(x^*) - c_0\star 1_{\{v_\#(\cdot,t_n^*)>v_\#(y^*,t_n^*)\}}(x_I^*))|p_x^*|.\end{aligned}$$

We have to distinguish two cases:

Case 1. Assume that for $\gamma>0$ fixed $\dfrac{|x^*-y^*|^2}{\gamma}\geq C_\gamma^2$ for some constant $C_\gamma>0$ and for all α small. By inequality $\Psi(x^*,y^*,t^*,t_n^*)\geq\Psi(x,x,t^*,t_n^*)$, we deduce that

$$u(x,t^*)-u(x^*,t^*)\leq v_\#(x,t_n^*)-v_\#(y^*,t_n^*)-\frac{|x^*-y^*|^2}{2\gamma}-\alpha(|x^*|^2+|y^*|^2)+2\alpha|x|^2.$$

We then get the inclusion

$$\{u(\cdot,t^*)\geq u(x^*,t^*)\}\cap\{v_\#(\cdot,t_n^*)\leq v_\#(y^*,t_n^*)\}\subset\{|x|^2\geq R_{\alpha,\gamma}^2\}$$

where $R_{\alpha,\gamma}^2=\frac{1}{2\alpha}\left(\frac{|x^*-y^*|^2}{2\gamma}+\alpha(|x^*|^2+|y^*|^2)\right)$. We also have

$$\{|x-x^*|\geq R_{\alpha,\gamma}\}\subset\{|x|\geq\tilde{R}_{\alpha,\gamma}\}$$

where $\tilde{R}_{\alpha,\gamma}=R_{\alpha,\gamma}-|x^*|\to\infty$ as $\alpha\to 0$ (see Da Lio *et al.* [13, Lemma 2.5]). We then obtain

$$\begin{aligned}\mathcal{I}\leq&(c_0\star 1_{\{v_\#(\cdot,t_n^*)>v_\#(y^*,t_n^*)\}}(x^*)-c_0\star 1_{\{v_\#(\cdot,t_n^*)>v_\#(y^*,t_n^*)\}}(x_I^*)+c_0\star 1_{B^c(0,R_{\alpha,\gamma})}(x^*))|p_x^*|\\ \leq&K\left(|Dc_0|_{L^1(\mathbb{R}^N)}|x^*-x_I^*|+\int_{B^c(0,\tilde{R}_{\alpha,\gamma})}c_0(x)dx\right)\\ <&K(\gamma+\Delta x)+o_\alpha(1)\end{aligned}$$

where we have used (5.48) to bound $|p_x^*|$ in the second line.

Case 2. Assume that there exists a subsequence $\alpha_n>0$ which we still denote by α such that

$$\frac{|x^*-y^*|^2}{\gamma}\to 0\quad\text{as }\alpha\to 0.$$

We then get that $p_x^*\to 0$ as $\alpha\to 0$. Since the velocities are bounded, we get that $\mathcal{I}=o_\alpha(1)$.

To sum up, we have shown that

$$\mathcal{I}<K(\gamma+\Delta x)+o_\alpha(1).$$

We then get that

$$\eta<K\left(\frac{\Delta x+\Delta t}{\gamma}+\gamma\right)+o_\alpha(1).$$

Putting $\eta^*=K\left(\dfrac{\Delta x+\Delta t}{\gamma}+\gamma\right)+o_\alpha(1)$, we then conclude that either $t^*=0$ or $t_n^*=0$ for $\eta\geq\eta^*$. Whenever $t^*=0$, we get for $\eta^*\leq\eta\leq 1$

$$\begin{aligned}M=&\Psi(x^*,y^*,0,t_n^*)\leq u_0(x^*)-v_\#(y^*,t_n^*)\\ \leq&K|x^*-y^*|+Kt_n^*+\mu^0\\ \leq&K\gamma+\mu^0\end{aligned}$$

The same result holds whenever $t_n^*=0$. We then deduce that for $\eta=\eta^*$ and $(x,t_n)\in\mathbb{R}^N\times\{0,...,t_{N_T}\}$

$$u(x,t_n)-v_\#(x,t_n)-\eta^*T-2\alpha|x_I|^2\leq M\leq K\gamma+\mu^0.$$

Sending $\alpha\to 0$, taking the supremum over (x,t_n) and choosing $\gamma=T^{1/2}(\Delta x+\Delta t)^{1/2}$, we get

$$\mu\leq K(\Delta x+\Delta t)^{1/2}\sqrt{T}+\mu^0$$

provided $\Delta x+\Delta t\leq\dfrac{1}{K^2}$. Using the same arguments of Alvarez *et al.*[1, Theorem 2], we easily deduce the result in the general case. This ends the proof of Theorem 2.2.

We now point out that there is no consistency error for the scheme (2.15). If there is a consistency error, then there will be a consistency error for the velocities, *i.e.* for all $\varphi \in C^1(\mathbb{R}^N \times (0,T))$ with $\|\varphi\|_{C^1(\mathbb{R}^N\times(0,T))} \leq C$ then

$$c^{\Delta}[\varphi_{|Q_T^{\Delta}}]_I^n - c[\varphi](x_I, t_n) \leq f(\Delta x, \Delta t) \tag{5.53}$$

with $f(\Delta x, \Delta t) \to 0$ as $(\Delta x, \Delta t) \to 0$.

Proposition 5.6 (No consistency error)
There is no consistency error for the scheme (2.15), *i.e. equation* (5.53) *does not hold.*

Proof of Proposition 5.6
We have to prove that there exists a constant $C_0 > 0$ such that for all $\Delta x, \Delta t > 0$ there exists a function φ such that

$$c^{\Delta}[\varphi_{|Q_T^{\Delta}}]_I^n - c[\varphi](x_I, t_n) \geq C_0 \quad \text{and } |\varphi|_{C^1(\mathbb{R}^N\times(0,T))} \leq C.$$

To see this, it suffices to take a function φ which oscillates as shown in Figure 1.

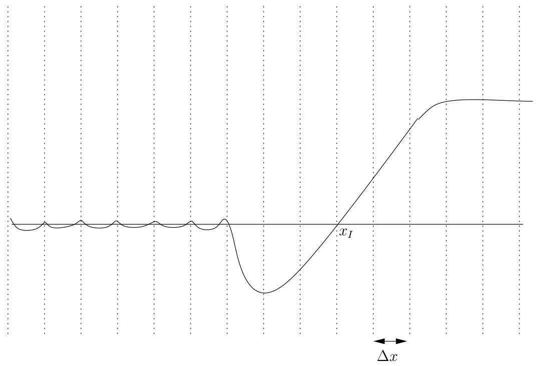

Figure 1: Graph of the function φ defined in $\mathbb{R}$.

Remark 5.7 *There is no consistency error, but the scheme is consistent. Indeed, if we fix the function φ in the previous proof, then using Slepčev Lemma [26, equation (5)], we get*

$$\lim_{\Delta x \to 0} c^{\Delta}[\varphi_{|Q_T^{\Delta}}]_I^n - c[\varphi](x_I, t_n) = 0.$$

We now point out that the two scheme (2.9) and (2.15) are not monotone:

Proposition 5.8 (Non monotonicity of the scheme)
The two scheme (2.9) *and* (2.15) *are not monotone for general $\bar{c}^0$.*

Proof of Proposition 5.8

1. **The implicit scheme** (2.9)
   We give a simple counter-example in the one dimensional case. We take $\tilde{u}_i^n = v_i^n = i\Delta x$. The goal is to construct a sub-solution u^{n+1} and a super-solution v^{n+1} such that

   $$u_i^{n+1} = i\Delta x + c_1$$
   $$v_i^{n+1} = i\Delta x + c_2.$$

   A simple computation gives

   $$c_1 = \Delta t \sum_{j\geq 0} \bar{c}_j^0 \quad \text{and} \quad c_2 = \Delta t \sum_{j>0} \bar{c}_j^0.$$

   We then deduce that $c_1 - c_2 = \Delta t \bar{c}_0^0 > 0$ (if $\bar{c}_0^0 > 0$). This implies that we can construct two different solutions (the supremum of sub-solutions and the infimum of super-solutions). This implies that the scheme (2.9) is not monotone.

2. **The explicit scheme** (2.15)
We also give a counter-example in the one dimensional case. We take

$$u_i^n = \begin{cases} -\delta & \text{if } i=0 \\ 1-\delta & \text{if } i=-1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad v_i^n = \begin{cases} \delta & \text{if } i=0 \\ 1-\delta & \text{if } i=-1 \\ 0 & \text{otherwise} \end{cases}$$

We then have $u_i^n \le v_i^n$ for all $i \in \mathbb{Z}$. We then prove that $u_0^{n+1} > v_0^{n+1}$ for δ small enough. First we remark that $c^\Delta[v^n]_0 \le 0$ (if $\sum_{i\neq 0,-1} \bar{c}_j^0 \ge \bar{c}_0^0 + \bar{c}_{-1}^0$), so $v_0^{n+1} \le \delta$. Moreover,

$$\begin{aligned} u_0^{n+1} =& -\delta + \frac{\Delta t}{\Delta x}\left(\sum_j \bar{c}_j^0 - \frac{1}{2}\sum_j \bar{c}_j^0\right)\sqrt{1+\delta^2} \\ \ge& -\delta + \frac{\Delta t}{\Delta x}\left(\frac{1}{2}|c_0|_{L^1(\mathbb{R})}\right) \\ >& \delta \end{aligned}$$

if

$$\delta < \frac{\Delta t}{2\Delta x}\left(\frac{1}{2}|c_0|_{L^1(\mathbb{R})}\right).$$

This contradicts the monotonicity of the scheme.

This ends the proof of the Proposition.

6 Numerical Simulations

In this section, we provide some numerical simulations. In a first subsection, we explain a method to solve numerically the implicit scheme (2.9)-(2.10)-(2.11). Then we provide a simple simulation concerning a collapsing circle, with the implicit scheme and an another one, to highlight the fattening phenomena, with the explicit scheme.

6.1 How to solve the implicit scheme?

To solve (2.9)-(2.10)-(2.11), we will use an iterative process. Assume that we have a solution v^n at step n. We now want to compute a solution v^{n+1} at step $n+1$. To do this, for $\tilde{w} \in \mathbb{R}^{Q^\Delta}$, we denote by $w = \Phi(\tilde{w})$ the solution of the following auxiliary scheme:

$$\text{(6.54)} \qquad \frac{w_I - v_I^n}{\Delta t} = c_{\tilde{w}}^\Delta[w]_I \begin{cases} G^+(D_{\tilde{w}}^+ w_I, D_{\tilde{w}}^- w_I) & \text{if} \quad c_{\tilde{w}}^\Delta[w]_I \ge 0 \\ G^-(D_{\tilde{w}}^+ w_I, D_{\tilde{w}}^- w_I) & \text{if} \quad c_{\tilde{w}}^\Delta[w]_I < 0 \end{cases}$$

where

$$c_{\tilde{w}}^\Delta[w]_I = \sum_{J\in\mathbb{Z}^N} \bar{c}_{I-J}^0 1_{\{\tilde{w}_J \ge w_I\}} \Delta x_1 ... \Delta x_N - \frac{1}{2}\sum_{J\in\mathbb{Z}^N} \bar{c}_J^0 \Delta x_1 ... \Delta x_N$$

and $D_{\tilde{w}}^\pm w_I = (D_{x_1,\tilde{w}}^\pm w_I, ..., D_{x_N,\tilde{w}}^\pm w_I)$ with

$$D_{x_i,\tilde{w}}^+ w_I = \frac{\tilde{w}_{I^{i,+}} - w_I}{\Delta x_i},$$

$$D_{x_i,\tilde{w}}^- w_I = \frac{w_I - \tilde{w}_{I^{i,-}}}{\Delta x_i},$$

where $I^{k,\pm}$ is defined in (5.32). Since the left hand side of (6.54) is non-decreasing in w_I and the right hand side is non-increasing, there exists a unique solution (in a sense similar of Definition 5.1) which can be computed using a dichotomy process.

The important point is that if $\tilde{w}$ is a sub-solution (resp. super-solution) then $w = (w_I)_I$ is still a sub-solution (resp. super-solution) and satisfies $w_I \ge \tilde{w}_I$ (resp. $w_I \le \tilde{w}_I$) (see the proof of Proposition 5.3).

The idea is then to define $w^0 = v^n - C$ where the constant C is such that w^0 is a sub-solution of (2.9) and then to construct iteratively $w^{k+1} = \Phi(w^k)$. Setting $v^{n+1} = \lim_{k\to\infty} w^k$, we have that v^{n+1} is a solution of (2.9).

6.2 A collapsing circle

In this subsection, we provide a simple test with the implicit scheme concerning the evolution of a circle. The goal of this simple simulation is just to check that the circle will disappear with the good time. We take a circle of radius 1. The parameters are $\Delta x = 0,05$, $\Delta t = 0,01$ and $\varepsilon = 0,3$. Moreover, we take the kernel

$$c_0^\varepsilon = \begin{cases} 1 & \text{if} \quad |x| \leq 0,05 \\ \frac{1}{|\ln \varepsilon||x|^3} & \text{if} \quad 0,05 \leq |x| \leq 2 \\ 0 & \text{if} \quad |x| \geq 2 \end{cases}$$

The initial condition is the distance to the circle. The result is shown in Figure 2.

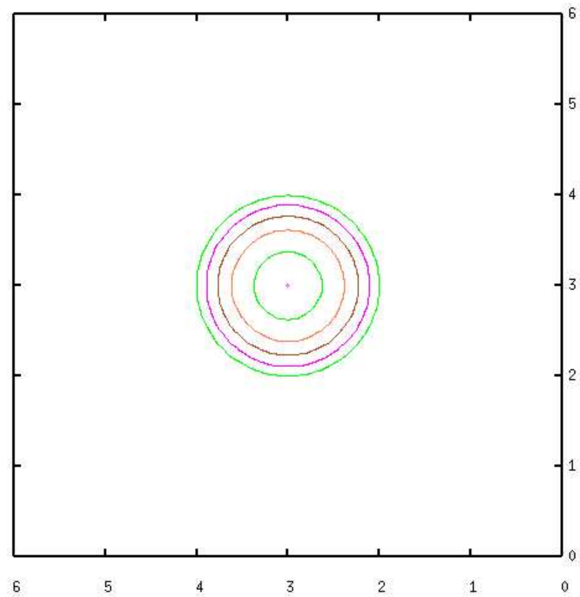

Figure 2: Evolution of a circle of radius 1 at time $0, 0.1, 0.2, 0.3, 0.4$ and 0.49

Numerically, the disappearing time is comprise between 0.49 and 0.50 which correspond to what we expect theoretically (the real time is 0.50).

6.3 Fattening phenomena

The second test is concerning with the evolution of the 8 to point out the fattening phenomena. This test have been made with the explicit scheme. We take two circles of radius $0,58$ such that they are tangent in one point and we look at the evolution of the level set 0 and $-0,06$. The parameters are $\Delta x_1 = \Delta x_2 = 0,01$, $\Delta t = 0,0001$ and $\varepsilon = 0,1$. Moreover, we take the kernel

$$c_0^\varepsilon(x) = \begin{cases} 0 & \text{if} \quad |x| \leq 0,3 \\ \frac{1}{|\ln \varepsilon||x|^3} & \text{if} \quad 0,05 \leq |x| \leq \frac{10}{3} \\ 0 & \text{if} \quad |x| \geq \frac{10}{3} \end{cases}$$

The results are provided in Figure 3.

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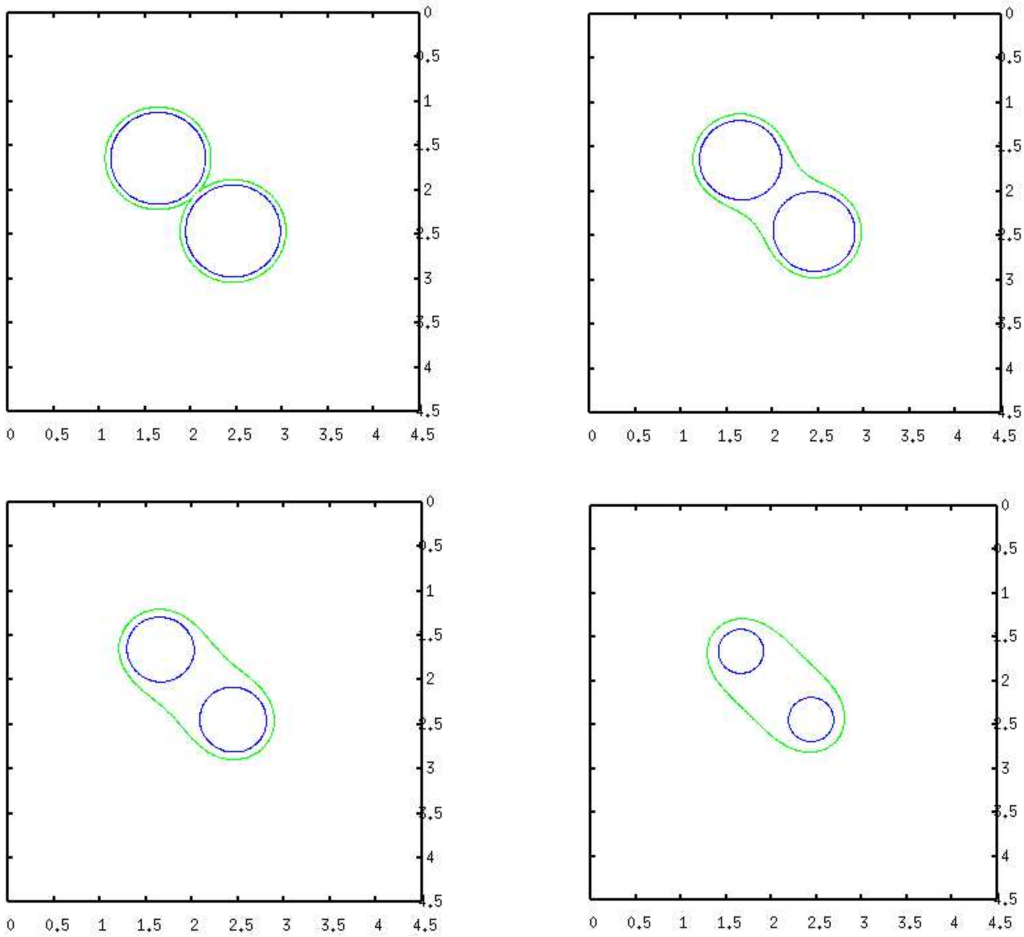

Figure 3: Evolution of the 8 at time 0, 0.05, 0.1 and 0.15.

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