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D. A. Di Pietro[1]

[1]*ENPC, ParisTech, Champs sur Marne, 77455 Marne la Vallée Cedex 2, France*

CERMICS — ENPC
6 et 8 avenue Blaise Pascal
Cité Descartes - Champs sur Marne
77455 Marne la Vallée Cedex 2

http://cermics.enpc.fr

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Daniele A. Di Pietro

CERMICS, Ecole des Ponts/ParisTech,
Champs sur Marne, 77455 Marne la Vallée Cedex 2, France
e-mail dipietro@cermics.enpc.fr
tel. 0033 (0)1 64 15 35 29

SUMMARY

In this work we propose and analyse a discontinuous Galerkin (DG) method for the Stokes problem based on an artificial compressibility numerical flux. A crucial step in the definition of a DG method is the choice of the numerical fluxes, which affect both the accuracy and the order of convergence of the method. We propose here to treat the viscous and the inviscid terms separately. The former is discretized using the well-known BRMPS method. For the latter, the problem is locally modified by adding an artificial compressibility term of the form $\frac{1}{c^2}\frac{\partial p}{\partial t}$ for the sole purpose of interface flux computation. The flux is obtained as the exact solution of a local Riemann problem. The analysis of the method extends the well-established strategies for the DG discretization of the Laplacian to the resulting partially coercive problem. Copyright © 2000 John Wiley & Sons, Ltd.

KEY WORDS: discontinuous Galerkin, artificial compressibility, Stokes problem

1. Introduction

Discontinuous Galerkin methods have been gaining an increasing interest in the scientific computing community since they have proved to be suited for the construction of robust high-order numerical schemes on arbitrary unstructured and non-conforming grids for a variety of problems. The application of the DG space discretization to incompressible flows has been recently considered as well. In a series of papers [12, 9, 10], Cockburn and co-workers introduce and analyse the LDG method applied to the Stokes, Oseen and Navier-Stokes equations. The expressions for the numerical fluxes associated to the divergence-free constraint mimic those introduced for the elliptic term in mixed formulation. A synthetic review can be found in [11]. In [20], Toselli introduces and analyses a *hp*-DG method for the Stokes problem. Finally, in [19] Girault and coworkers present and analyse a DG method with non-overlapping domains for the Stokes and Navier-Stokes problems.

*Correspondence to: Daniele A. Di Pietro, dipietro@cermics.enpc.fr

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A new formulation for the inviscid fluxes was proposed in [3, 14]. The key idea of the method is to introduce a local modification of the problem at the elementary interface level. Following the approach first presented in [5], the authors treat the viscous and inviscid terms separately. The former is discretized using the BRMPS method introduced in [6] and analysed in [1]. For the latter, a technique similar to the one used for the compressible case is adopted, i.e., the fluxes are obtained from the solution of a Riemann problem with initial datum given by the (discontinuous) solution. In order to obtain a hyperbolic problem, the mass equation is perturbed by adding an artificial compressibility term of the form

$$\frac{1}{c^2}\frac{\partial p}{\partial t},$$

where $c > 0$ is a parameter to be suitably chosen. This Riemann problem-based method can be easily extended to the more complicate Oseen and Navier-Stokes cases by simply modifying the Riemann solver. When applied to the Navier-Stokes equations, it gives raise to non-linear fluxes, which can only be defined implicitly, while an explicit expression is available in the other cases. Thorough numerical testing has been provided in [3, 4].

The local artificial compressibility flux displays some advantages with respect to other methods: (i) the approach can be easily generalized to a variety of incompressible problems, from the viscosity-dominated Stokes equations to the advection-dominated incompressible Euler equations; (ii) the approximation of the pressure seems to benefit from this physically grounded scheme, as pointed out in [3], where a comparison with the schemes proposed in [12, 9, 10] is presented; (iii) unlike some other DG methods, stability is achieved even when the same polynomial order is used for both velocity and pressure. This feature may be of some practical importance from the implementation viewpoint. Moreover, it makes the method accessible to those practitioners who already dispose of a compressible DG code and want to be able to perform incompressible computations. Besides this specific advantages, we have the usual ones associated with the use of discontinuous finite elements such as the possibility of handling non-conforming meshes in a natural way, the easy implementation of *hp*-adaptive versions, etc.

In this work we analyse the local artificial compressibility method applied to the Stokes problem extending the strategy used in [1] for the purely elliptic case to the resulting partially coercive problem. To this purpose, inspired by [17], we define two norms, one for continuity and the other for stability, and prove an inf-sup condition based on the partial coercivity of the bilinear form as well as on its continuous counterpart. Using the above results, we obtain error estimates in the energy norm and refine the estimate for the L^2-norm of the velocity error deploying a standard duality argument. A general abstract framework for the analysis of DG methods for more general three-field Friedrichs' systems with partial coercivity has recently been proposed in [18]. To pinpoint the issues related to the inviscid terms, the analysis in the present work is entirely carried out on the primal formulation of the problem and, recalling [1], it can be easily extended to other stable and completely consistent DG approximations of the Laplacian. The most relevant difference from the schemes analysed in [20, 19] is that the local artificial compressibility perturbation automatically adds a stabilising term for the pressure which makes the method suitable for equal order approximation.

The paper is organized as follows: in §2 we derive the DG discretization of the Stokes equations, pointing out the terms where the numerical fluxes appear and showing how they can be computed by means of a suitable Riemann solver; in §3 we list some preliminary results from the literature and we introduce some hypotheses that are necessary for the subsequent proofs; in §4 we analyze the discrete problem and prove optimal convergence estimates; in §5 we numerically evaluate the performance of the method and draw some conclusions in §6.

2. Formulation of the method

2.1. Discontinuous Galerkin discretization

We consider the Stokes problem

$$\begin{cases} -\Delta \mathbf{u} + \boldsymbol{\nabla} p = \mathbf{f}, & \text{in } \Omega, \\ \boldsymbol{\nabla}\cdot\mathbf{u} = 0, & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0}, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

Ω being a bounded connected Lipschitz domain in $\mathbb{R}^d$, $d \in \{2,3\}$. Since Dirichlet boundary conditions are prescribed for $\mathbf{u}$, the pressure is only defined up to a constant. In order to remove this ambiguity, we further require that

$$\int_\Omega p \, \mathrm{d}\mathbf{x} = 0. \tag{2}$$

For the proof that problem (1) together with condition (2) is well-posed see, e.g., [16, §4].

To avoid unnecessary complications, we shall henceforth assume that Ω is polygonal and that $\mathcal{T}_h$ represents a family of triangulations parametrised by h which cover it exactly. We denote by $\mathcal{F}_h^i$ the set of element interfaces, i.e., $f \in \mathcal{F}_h^i$ if f is a $(d-1)$-manifold and there are K^+, $K^- \in \mathcal{T}_h$ such that $f = \partial K^+ \cap \partial K^-$. The set of the faces that separate the mesh from the exterior of Ω is denoted with $\mathcal{F}_h^\partial$, i.e., $f \in \mathcal{F}_h^\partial$ if f is a $(d-1)$-manifold and there is $K \in \mathcal{T}_h$ such that $f = \partial K \cap \partial\Omega$. For a given interface $\mathcal{F}_h^i \ni f = \partial K^+ \cap \partial K^-$, we shall note $\mathcal{T}_h(f) \stackrel{\text{def}}{=} K^+ \cup K^-$. Similarly, for a boundary face $\mathcal{F}_h^\partial \ni f = \partial K \cap \partial\Omega$, we shall let $\mathcal{T}_h(f) \stackrel{\text{def}}{=} K$. The set of all the faces is denoted with $\mathcal{F}_h$, i.e., $\mathcal{F}_h \stackrel{\text{def}}{=} \mathcal{F}_h^i \cup \mathcal{F}_h^\partial$.

In order to derive the DG approximation, we introduce the auxiliary variable $\boldsymbol{\sigma}$ and re-write the problem as the first order system

$$\begin{cases} \boldsymbol{\sigma} - \boldsymbol{\nabla}\mathbf{u} = \mathbf{0}, & \text{in } \Omega, \\ -\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} + \boldsymbol{\nabla} p = \mathbf{f}, & \text{in } \Omega, \\ \boldsymbol{\nabla}\cdot\mathbf{u} = 0, & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0}, & \text{on } \partial\Omega. \end{cases}$$

The weak formulation on the generic element $K \in \mathcal{T}_h$ can be obtained multypling every equation by a smooth test function and integrating over K,

$$\begin{cases} \displaystyle\int_K \boldsymbol{\sigma}:\boldsymbol{\tau} \, \mathrm{d}\mathbf{x} + \int_K \boldsymbol{\nabla}\cdot\boldsymbol{\tau}\cdot\mathbf{u} \, \mathrm{d}\mathbf{x} - \int_{\partial K} \mathbf{n}_K\cdot\boldsymbol{\tau}\cdot\mathbf{u} \, \mathrm{d}\sigma = 0, & \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}, \\ \displaystyle\int_K \boldsymbol{\sigma}:\boldsymbol{\nabla}\mathbf{v} \, \mathrm{d}\mathbf{x} - \int_{\partial K} \mathbf{n}_K\cdot\boldsymbol{\sigma}\cdot\mathbf{v} \, \mathrm{d}\sigma - \int_K p\boldsymbol{\nabla}\cdot\mathbf{v} \, \mathrm{d}\mathbf{x} + \int_{\partial K} p\mathbf{v}\cdot\mathbf{n}_K \, \mathrm{d}\sigma = 0, & \forall \mathbf{v} \in \mathbf{V}, \\ \displaystyle -\int_K \mathbf{u}\cdot\boldsymbol{\nabla} q \, \mathrm{d}\mathbf{x} + \int_{\partial K} \mathbf{u}\cdot\mathbf{n}_K q \, \mathrm{d}\sigma = 0, & \forall q \in Q, \end{cases}$$

where $\mathbf{n}_K$ denotes the normal unit vector pointing out of the element K. The above system of equations makes sense for all $(\boldsymbol{\sigma}, \mathbf{u}, p)\,,\; (\boldsymbol{\tau}, \mathbf{v}, q) \in \boldsymbol{\Sigma} \times \mathbf{V} \times Q$ with

$$\boldsymbol{\Sigma} \stackrel{\text{def}}{=} [H^1(\mathcal{T}_h)]^{d^2}, \quad \mathbf{V} \stackrel{\text{def}}{=} [H^1(\mathcal{T}_h)]^d, \quad Q \stackrel{\text{def}}{=} H^1(\mathcal{T}_h)\,,$$

where, for $k \geq 1$, we have set

$$H^k(\mathcal{T}_h) \stackrel{\text{def}}{=} \{v \in L^2(\Omega)\,;\; v|_K \in H^k(K)\,,\; \forall K \in \mathcal{T}_h\}.$$

We shall consider the above space in order for weak derivatives and integrals of traces to make sense as well as to avoid technical details linked with the use of dualities.

We next introduce the finite dimensional space made up of polynomial functions possibly discontinuous across element boundaries: For $k \geq 1$,

$$V_h \equiv V_h^k \stackrel{\text{def}}{=} \{v_h \in L^2(\Omega)\,;\; v_h|_K \in \mathbb{P}_k(K),\; \forall K \in \mathcal{T}_h\}.$$

The components of the discrete solution are sought in the following spaces:

$$\boldsymbol{\Sigma}_h \stackrel{\text{def}}{=} [V_h^{k_\sigma}]^{d^2}, \quad \mathbf{V}_h \stackrel{\text{def}}{=} [V_h^{k_u}]^d, \quad Q_h \stackrel{\text{def}}{=} V_h^{k_p}, \tag{3}$$

with $k_u - 1 \leq k_\sigma \leq k_u$ and $k_u - 1 \leq k_p \leq k_u$. For future use, we shall introduce the symbols $\mathbf{W} \stackrel{\text{def}}{=} \mathbf{V} \times Q$ and $\mathbf{W}_h \stackrel{\text{def}}{=} \mathbf{V}_h \times Q_h$.

Optimal error estimates with respect to the approximation properties of the discrete spaces are obtained taking $k_\sigma = k_p = k_u - 1$. Nevertheless, a common choice is to set $k_\sigma = k_p = k_u$. This results in a slightly increased computational effort, but, in some cases, may be preferable from the implementation viewpoint. Moreover, when dealing with the complete Navier-Stokes system, the above choice ensures better convergence results in the low viscosity limit. Indeed, estimates similar to those obtained for hyperbolic problems were proved in [14] for the incompressible Euler equations provided the same polynomial degree is used for both velocity and pressure.

All the results obtained for $k_p = k_u$ extend directly to the case when $k_p = k_u - 1$, which is more favourable in terms of stability. We shall therefore focus the analysis on the former, more difficult, case.

Remark 2.1. *By definition of the space V_h,*

$$v_h \in V_h \Rightarrow \boldsymbol{\nabla} v_h \in \left[V_h^{k-1}\right]^d \subset \left[V_h^k\right]^d.$$

This inclusion property is not enjoyed by standard conforming approximations.

In order to write the discrete problem, we replace the infinite-dimensional spaces with the discrete ones. Since the test functions are possibly discontinuous across element boundaries, it is necessary to establish weak inter-element links by introducing numerical fluxes. The resulting problem reads: Find $(\boldsymbol{\sigma}_h, \mathbf{u}_h, p_h) \in \boldsymbol{\Sigma}_h \times \mathbf{V_h} \times Q_h$ such that

$$\begin{cases} \displaystyle\int_K \boldsymbol{\sigma}_h{:}\boldsymbol{\tau}_h \,\mathrm{d}\mathbf{x} + \int_K \boldsymbol{\nabla}{\cdot}\boldsymbol{\tau}_h{\cdot}\mathbf{u}_h \,\mathrm{d}\mathbf{x} - \int_{\partial K} \mathbf{n}_K{\cdot}\boldsymbol{\tau}_h{\cdot}\hat{\mathbf{u}}_\nu \,\mathrm{d}\sigma = 0, & \forall \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h, \\ \displaystyle\int_K \boldsymbol{\sigma}_h{:}\boldsymbol{\nabla}\mathbf{v}_h \,\mathrm{d}\mathbf{x} - \int_{\partial K} \mathbf{n}_K{\cdot}\hat{\boldsymbol{\sigma}}{\cdot}\mathbf{v}_h \,\mathrm{d}\sigma - \int_K p_h \boldsymbol{\nabla}{\cdot}\mathbf{v}_h \,\mathrm{d}\mathbf{x} + \int_{\partial K} \hat{p}\mathbf{v}_h{\cdot}\mathbf{n}_K \,\mathrm{d}\sigma = 0, & \forall \mathbf{v}_h \in \mathbf{V}_h, \\ \displaystyle -\int_K \mathbf{u}_h{\cdot}\boldsymbol{\nabla} q_h \,\mathrm{d}\mathbf{x} + \int_{\partial K} \hat{\mathbf{u}}_{\mathrm{div}}{\cdot}\mathbf{n}_K q_h \,\mathrm{d}\sigma = 0, & \forall q_h \in Q_h, \end{cases} \tag{4}$$

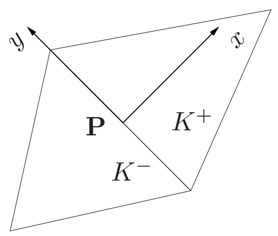

Figure 1. Frame for the computation of the inviscid numerical fluxes at point **P**.

where $\hat{\mathbf{u}}_\nu$ and $\hat{\boldsymbol{\sigma}}$ indicate the numerical fluxes associated with the viscous term, while $\hat{p}$ and $\hat{\mathbf{u}}_{\mathrm{div}}$ are the numerical fluxes associated with the incompressibility constraint.

2.2. Numerical fluxes

To complete the formulation of the method, it only remains to devise suitable expressions for the numerical fluxes. Following [3, 14], we consider the steady case as the limit of a pseudo-evolutive problem. Let $f \in \mathcal{F}_h^i$ be a generic internal face of the triangulation. In order to compute the inviscid fluxes at a point $\mathbf{P} \in f$, we consider the projection of the problem onto the normal direction to the face (see Figure 1). For brevity of notation, let $u \stackrel{\mathrm{def}}{=} \mathbf{u}{\cdot}\mathbf{n}$ be the normal component of the velocity at $\mathbf{P}$ and consider a frame such that the x-axis is aligned with $\mathbf{n}$. The expressions for the inviscid fluxes are then obtained by solving the Riemann problem associated with the perturbed system

$$\begin{cases} \boxed{\dfrac{1}{c^2}\dfrac{\partial p}{\partial t}} + \dfrac{\partial u}{\partial x} = 0, \\ \dfrac{\partial u}{\partial t} + \dfrac{\partial p}{\partial x} = 0, \end{cases} \tag{5}$$

with discontinuous initial datum given by $(\mathbf{u}^+, p^+)$ and $(\mathbf{u}^-, p^-)$ respectively. The unsteady pressure term in the box is indeed the local artificial compressibility perturbation that allows to recover the hyperbolicity of the projected problem. Suitable modifications can be introduced on boundary faces to account for the weak enforcement of Dirichlet boundary conditions on the velocity, as briefly discussed below.

The system (5) only accounts for the normal component of the velocity, which is a scalar irrespectively of the number of space dimensions of the problem, d. In [3] the authors show that the tangential component of the velocity can be computed independently once the normal component and the pressure have been determined. However, since the tangential component will not be relevant for the definition of the numerical fluxes, we omit the details here and refer the reader to the cited work.

The Riemann problem thus obtained has the structure depicted in Figure 2. The left and right states correspond to the initial datum. The star region is separated from the left and right states by two centered waves, which can be either rarefactions or shocks, and it contains a contact discontinuity, across which the sole tangential component of the velocity may vary. It was proved in [3] that the solution on the $x/t = 0$ line of this problem has the following

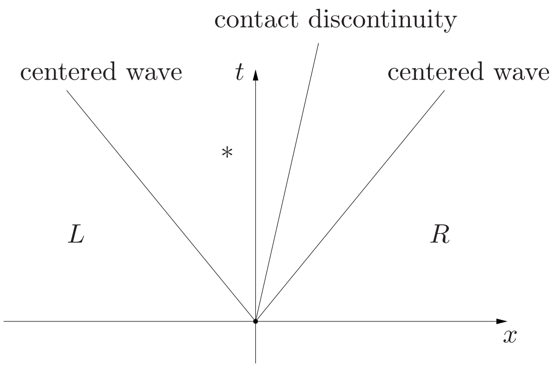

Figure 2. Structure of the Riemann problem

analytical expression:

$$u^* = \{u\} + \frac{1}{2c}[p], \quad p^* = \{p\} + \frac{c}{2}[u], \tag{6}$$

where $[\cdot]$ denotes the usual jump operator and $\{\cdot\}$ is the average operator defined below. Observe that (i) the sole normal component of $\mathbf{u}_{\text{div}}$ appears in the integral in the third line of (4), which renders the definition of its tangential component unnecessary; (ii) the expressions of the fluxes can be directly plugged into the steady problem when the tuning parameter c does not depend on time, since the fluxes themselves are not time-dependent.

We next introduce a few trace operators. Let $\mathcal{F}_h^i \ni f = K^+ \cap K^-$ be an internal face, and let $\boldsymbol{\varphi}$ be a tensor field of rank N such that a (possibly two-valued) trace is defined on f. Such an assumption is verified, e.g., by functions in $[V_h]^{d^N} \oplus [H^1(\mathcal{T}_h)]^{d^N}$. Let $\mathbf{n}^{\pm}$ denote the outward normal to $K^{\pm}$ respectively. Using Einstein's notation, we define

$$\begin{aligned}
\{\boldsymbol{\varphi}\} &\stackrel{\text{def}}{=} \frac{\boldsymbol{\varphi}^+ + \boldsymbol{\varphi}^-}{2}, \\
(\llbracket \boldsymbol{\varphi} \rrbracket)_{i_1 i_2 \ldots i_{N+1}} &\stackrel{\text{def}}{=} \varphi^+_{i_1 i_2 \ldots i_N} n^+_{i_{N+1}} + \varphi^-_{i_1 i_2 \ldots i_N} n^-_{i_{N+1}}, \\
([\boldsymbol{\varphi}]_{\mathbf{n}})_{i_1 i_2 \ldots i_{N-1}} &\stackrel{\text{def}}{=} \varphi^+_{i_1 i_2 \ldots i_{N-1} j} n^+_j + \varphi^-_{i_1 i_2 \ldots i_{N-1} j} n^-_j.
\end{aligned}$$

An example will clarify the above definitions. For the sake of simplicity, let, $\boldsymbol{\psi}_h$ be a tensor field of rank 2 belonging to Σ_h. Then, $\{\boldsymbol{\psi}_h\}$ will be the component-wise algebraic mean between its trace taken from K^+ and that taken from K^-. On the other hand, at a given point $\mathbf{P} \in f$, $\llbracket \boldsymbol{\psi}_h \rrbracket$ will be the third rank tensor whose (i, j, k)th component is given by

$$(\llbracket \boldsymbol{\psi}_h \rrbracket)_{ijk} = \psi^+_{ij} n^+_k + \psi^-_{ij} n^-_k.$$

Finally, $[\boldsymbol{\psi}_h]_{\mathbf{n}}$ will be the vector whose ith component is given by

$$([\boldsymbol{\psi}_h]_{\mathbf{n}})_i = \psi^+_{ik} n^+_k + \psi^-_{ik} n^-_k,$$

where a summation over the saturated index k is understood. These definitions generalize the ones introduced in [1] and can be extended to boundary faces to account for the weak imposition of the Dirichlet conditions on $\mathbf{u}$. In particular, for all $f \in \mathcal{F}_h^\partial$ and for all $(q, \mathbf{v}, \boldsymbol{\tau}) \in L^2(f) \times [L^2(f)]^d \times [L^2(f)]^{d^2}$, we set

$$[\![q]\!] \stackrel{\text{def}}{=} \mathbf{0}, \quad \{q\} \stackrel{\text{def}}{=} q, \quad \{\mathbf{v}\} \stackrel{\text{def}}{=} \mathbf{v}, \quad [\![\mathbf{v}]\!] \stackrel{\text{def}}{=} \mathbf{v} \otimes \mathbf{n}, \quad [\mathbf{v}]_{\mathbf{n}} \stackrel{\text{def}}{=} \mathbf{v}\cdot\mathbf{n}, \quad \{\boldsymbol{\tau}\} \stackrel{\text{def}}{=} \boldsymbol{\tau}.$$

Let now $f \in \mathcal{F}_h$. Following [3] and keeping in mind the discussion above, we define the numerical fluxes across f as:

$$\hat{\mathbf{u}}_\nu = \{\mathbf{u}_h\}, \quad \hat{\boldsymbol{\sigma}} = \{\boldsymbol{\nabla}_h \mathbf{u}_h\} + \eta\{\mathbf{r}_f([\![\mathbf{u}_h]\!])\}, \quad \hat{p} = \{p_h\} + \frac{c}{2}[\mathbf{u}_h]_{\mathbf{n}}, \quad \hat{\mathbf{u}}_{\text{div}} = \{\mathbf{u}_h\} + \frac{1}{2c}[\![p_h]\!], \quad (7)$$

wher η and c are positive parameters and $\boldsymbol{\nabla}_h$ indicates the element-wise gradient operator. For all second rank tensors $\boldsymbol{\varphi} \in [L^2(f)]^{d^2}$, the lifting operator $\mathbf{r}_f(\boldsymbol{\varphi}) \in \boldsymbol{\Sigma}_h$ is defined as the solution of the following problem: For all $f \in \mathcal{F}_h$,

$$\int_\Omega \mathbf{r}_f(\boldsymbol{\varphi}):\boldsymbol{\tau}_h \, \mathrm{d}\mathbf{x} = -\int_f \boldsymbol{\varphi}:\{\boldsymbol{\tau}_h\} \, \mathrm{d}\sigma, \quad \forall \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h.$$

Observe that we are indeed defining one lifting operator for each face. Since the test functions with non-zero mean on $f \in \mathcal{F}_h$ are only the ones supported in $\mathcal{T}_h(f)$, we conclude that the support of the associated lifting operator $\mathbf{r}_f$ coincides with $\mathcal{T}_h(f)$. This definition is perfectly coherent with the face-based definitions of numerical fluxes given in (7).

Clearly, the inviscid fluxes $\hat{p}$, $\hat{\mathbf{u}}_{\text{div}}$ are directly derived from the solution (6) of the Riemann problem with initial datum given by $(\mathbf{u}_h^+, p_h^+)$, $(\mathbf{u}_h^-, p_h^-)$.

2.3. The discrete problem

We proceed summing eq. (4) over the elements and counter-integrating by parts the viscous terms in the momentum equation and the mass equation. For the latter step, the following formula proved in [1] can be used: For all tensors fields $\boldsymbol{\varphi}$ and $\boldsymbol{\phi}$ such that a (possibly two-valued) trace is defined on all $f \in \mathcal{F}_h$,

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} \boldsymbol{\varphi}:\boldsymbol{\phi} \otimes \mathbf{n} \, \mathrm{d}\sigma = \sum_{f \in \mathcal{F}_h^i} \int_f \{\boldsymbol{\varphi}\}:[\![\boldsymbol{\phi}]\!] + \{\boldsymbol{\phi}\}\cdot[\boldsymbol{\varphi}]_{\mathbf{n}} \, \mathrm{d}\sigma + \sum_{f \in \mathcal{F}_h^\partial} \int_f \boldsymbol{\varphi}:\boldsymbol{\phi} \otimes \mathbf{n} \, \mathrm{d}\sigma. \quad (8)$$

To conclude, the auxiliary variable $\boldsymbol{\sigma}_h$ can be eliminated proceeding as in [1] to recover the primal formulation of the problem. The discrete problem then reads: Find $(\mathbf{u}_h, p_h) \in \mathbf{W}_h$ such that

$$B(\mathbf{u}_h, p_h; \mathbf{v}_h, q_h) = G(\mathbf{v}_h), \quad \forall (\mathbf{v}_h, q_h) \in \mathbf{W}_h, \quad (9)$$

where

$$B(\mathbf{u}, p; \mathbf{v}, q) \stackrel{\text{def}}{=} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) - b(\mathbf{u}, q) + j_\nu(\mathbf{u}, \mathbf{v}) + j_p(p, q) + j_{\mathbf{n}}(\mathbf{u}, \mathbf{v}),$$
$$G(\mathbf{v}) \stackrel{\text{def}}{=} \int_\Omega \mathbf{g}\cdot\mathbf{v} \, \mathrm{d}\mathbf{x},$$

and

$$
\begin{aligned}
a(\mathbf{u},\mathbf{v}) &\stackrel{\text{def}}{=} \int_\Omega \boldsymbol{\nabla}_h\mathbf{u}{:}\boldsymbol{\nabla}_h\mathbf{v}\ \mathrm{d}\mathbf{x} - \sum_{f\in\mathcal{F}_h}\int_f \{\boldsymbol{\nabla}_h\mathbf{u}\}{:}[\![\mathbf{v}]\!] + [\![\mathbf{u}]\!]{:}\{\boldsymbol{\nabla}_h\mathbf{v}\}\ \mathrm{d}\sigma \\
&= \int_\Omega \boldsymbol{\nabla}_h\mathbf{u}{:}\boldsymbol{\nabla}_h\mathbf{v}\ \mathrm{d}\mathbf{x} + \sum_{f\in\mathcal{F}_h}\int_\Omega \boldsymbol{\nabla}_h\mathbf{u}{:}\mathbf{r}_f([\![\mathbf{v}]\!]) + \mathbf{r}_f([\![\mathbf{u}]\!]){:}\boldsymbol{\nabla}_h\mathbf{v}\ \mathrm{d}\mathbf{x}, \\
b(\mathbf{v},p) &\stackrel{\text{def}}{=} -\int_\Omega p\boldsymbol{\nabla}_h{\cdot}\mathbf{v}\ \mathrm{d}\mathbf{x} + \sum_{f\in\mathcal{F}_h}\int_f \{p\}[\mathbf{v}]_{\mathbf{n}}\ \mathrm{d}\sigma \\
j_\nu(\mathbf{u},\mathbf{v}) &\stackrel{\text{def}}{=} \sum_{f\in\mathcal{F}_h}\int_\Omega \eta\mathbf{r}_f([\![\mathbf{u}]\!]){:}\mathbf{r}_f([\![\mathbf{v}]\!])\ \mathrm{d}\mathbf{x}, \\
j_{\mathbf{n}}(\mathbf{u},\mathbf{v}) &\stackrel{\text{def}}{=} \sum_{f\in\mathcal{F}_h}\int_f \frac{c}{2}[\mathbf{u}]_{\mathbf{n}}[\mathbf{v}]_{\mathbf{n}}\ \mathrm{d}\sigma, \\
j_p(p,q) &\stackrel{\text{def}}{=} \sum_{f\in\mathcal{F}_h}\int_f \frac{1}{2c}[\![p]\!]{\cdot}[\![q]\!]\ \mathrm{d}\sigma.
\end{aligned}
$$

Recalling [1], we know that a sufficient condition for the stability of the discretization of the Laplacian is that η be greater than the maximum number of faces of one element in the mesh. A similar condition can be found in the Stokes case as well (see Proposition 4.2). In addition, it turns out that also the term $j_{\mathbf{n}}(\cdot,\cdot)$ contributes to stabilise the velocity. Furthermore, in what follows it will become clear that a suitable form for the artificial compressibility parameter is $c = \gamma/h_f$ with $\gamma > 0$. For the sake of simplicity, we shall henceforth assume that both γ and η are real positive constants. Although possible, more general choices do not seem to be particularly useful in practice.

2.4. Analogies with other methods

It is worth further investigating the analogy with interior penalty and artificial compressibility methods. In eq. (9) we recognize two contributions: the first one, involving only volume integral terms, is exactly the customary mixed formulation of the Stokes problem; the second one, which gathers up all the boundary integral terms, is responsible for consistency, stability and the weak enforcement of boundary conditions. Three stabilizing contributions are present, collected in the bilinear forms $j_\nu(\cdot,\cdot)$, $j_{\mathbf{n}}(\cdot,\cdot)$ and $j_p(\cdot,\cdot)$. The first and the second terms penalize, respectively, the jumps of the velocity and of its normal component across element boundaries. Although, strictly speaking, only the $j_\nu(\cdot,\cdot)$ term is necessary for the analysis, numerical experiments show that adding $j_{\mathbf{n}}(\cdot,\cdot)$ seems to enhance the accuracy. Moreover, if divergence free bases like the ones proposed in [13] were used, this term would penalize the only divergence contribution left, i.e., the one due to discontinuous normal components. The last term is in some sense analogous to the traditional artificial compressibility contribution. As a matter of fact, the matrix $\mathbf{B}$ resulting from the discretization of the Stokes problem can be partitioned into the following four blocks:

$$
\mathbf{B}(\mathbf{u},p;\mathbf{v},q) = \left[\begin{array}{c|c} \mathbf{B}_{11}(\mathbf{u},\mathbf{v}) & \mathbf{B}_{12}(\mathbf{v},p) \\ \hline -\mathbf{B}_{21}(\mathbf{u},q) & \mathbf{B}_{22}(p,q) \end{array}\right].
$$

In traditional artificial compressibility methods, the $\mathbf{B}_{22}$ block is the pressure mass matrix scaled by the inverse of the artificial compressibility parameter c. The method considered in the present work replaces the pressure mass matrix by the matrix of pressure jumps on the union of all the faces, which constitutes a consistent perturbation of the problem.

3. Preliminary results

We introduce the notation

$$h \stackrel{\text{def}}{=} \max_{K\in\mathcal{T}_h} h_K, \quad h_K \stackrel{\text{def}}{=} \max_{f\subset\partial K} h_f,$$

where f is the generic face of K and h_f its diameter. For a given $h>0$, the triangulation $\mathcal{T}_h$ is assumed to match the following conditions:

(i) every element $K \in \mathcal{T}_h$ is affinely equivalent to one of several elements in an arbitrary but fixed set;

(ii) the triangulation can be 1-irregular but it has to satisfy the following property:

$$\exists\sigma_1 > 0 \;:\; 0 < \frac{h_K}{\rho_K} \leq \sigma_1, \quad \forall K \in \mathcal{T}_h,$$

where ρ_K denotes the diameter of the biggest ball included in K. This property implies that there exists $0<\sigma_K < 1$ such that

$$\sigma_K \leq \frac{\min_{f\subset\partial K} h_f}{\max_{f\subset\partial K} h_f} \leq 1, \quad \forall K \in \mathcal{T}_h. \tag{10}$$

Notice that σ_K is independent of the meshsize but it depends on the regularity of the mesh. Whenever possible, the symbols $\lesssim$ and $\gtrsim$ will be used for inequalities that hold up to a real positive parameter independent of the meshsize h but possibly depending on the polynomial degrees $k_{\boldsymbol{\sigma}}$, $k_{\mathbf{u}}$ and k_p as well as on the mesh regularity.

We shall consider projection operators satisfying the following

Lemma 3.1 (Projection operator) *For all $K \in \mathcal{T}_h$, let π be a linear continuous operator from $H^{s+1}(K)$, $s \geq 0$, onto $\mathbb{P}^k(K)$ such that $\pi w = w$ for all $w \in \mathbb{P}^k(K)$, $k \geq 0$. Then, for all $K \in \mathcal{T}_h$,*

$$\begin{aligned} |w-\pi w|_{r,K} &\lesssim h_K^{\min(s,k)+1-r}\|w\|_{s+1,K}, \quad r\in\{0,1,2\},\\ \|w-\pi w\|_{0,\partial K} &\lesssim h_K^{\min(s,k)+1/2}\|w\|_{s+1,K}. \end{aligned}$$

The proof is classical and can be found, e.g., in [7]. With little abuse of notation, we shall denote with the same symbol π the discontinuous projection operator obtained applying π element-wise. Among the operators that meet the requirements of Lemma 3.1 we shall select the L^2-orthogonal projector onto the discontinuous space V_h^k An operator which meets the requirements of (see e.g., [16, §1.6.3]), since L^2-orthogonality will be needed to prove the discrete inf-sup condition. When the projection operator is applied to a vector or tensor quantity, the notation has to be intended component-wise. We shall also assume that the following H^1-stability condition is satisfied:

$$\|\pi w\|_{1,K} \lesssim \|w\|_{1,K}, \quad \forall K\in\mathcal{T}_h. \tag{11}$$

In order to treat the boundary terms, we shall need the following trace inequality (see [8]):

$$\|v\|_{0,\partial K}^2 \lesssim \left(h_K^{-1}\|v\|_{0,K}^2 + h_K|v|_{1,K}^2\right), \quad \forall v \in H^1(K). \tag{12}$$

Some useful bounds for the trace operators are collected in the following

Lemma 3.2 (Trace operator bounds) *Let $\boldsymbol{\varphi} \in [H^s(\Omega)]^{d^N}$ be a tensor quantity of rank N and let $\psi \in L^2(f)$ be non-negative for all $f \in \mathcal{F}$. Then it holds*

$$\sum_{f\in\mathcal{F}_h} \|\psi^{1/2}\{\boldsymbol{\varphi}\}\|_{0,f}^2 \le \sum_{K\in\mathcal{T}_h} \|\psi^{1/2}\boldsymbol{\varphi}\|_{0,\partial K}^2,$$
$$\sum_{f\in\mathcal{F}_h} \|\psi^{1/2}[\![\boldsymbol{\varphi}]\!]\|_{0,f}^2 \le 2\sum_{K\in\mathcal{T}_h} \|\psi^{1/2}\boldsymbol{\varphi}\|_{0,\partial K}^2,$$
$$\sum_{f\in\mathcal{F}_h} \|\psi^{1/2}[\boldsymbol{\varphi}]_{\mathbf{n}}\|_{0,f}^2 \le 2\sum_{K\in\mathcal{T}_h} \|\psi^{1/2}\boldsymbol{\varphi}\|_{0,\partial K}^2.$$

Proof. We focus on a generic internal face $\mathcal{F}_h^i \ni f = \partial K^+ \cap \partial K^-$, since the assert is trivially verified for $f \in \mathcal{F}_h^\partial$. The total contribution from $K^\pm$ can be split as follows:

$$\int_{\partial K^\pm} \psi\{\boldsymbol{\varphi}\}^2 \,\mathrm{d}\sigma = \int_{f\in\partial K^\pm} \psi\{\boldsymbol{\varphi}\}^2 \,\mathrm{d}\sigma + \int_{\partial K^\pm\setminus f} \psi\{\boldsymbol{\varphi}\}^2 \,\mathrm{d}\sigma,$$

where $f \in \partial K^\pm$ means that we are regarding the face as belonging to the boundary of element $K^\pm$. The total contribution on the face f is, therefore,

$$\begin{aligned}
\int_{f\in\partial K^+} \psi\{\boldsymbol{\varphi}\}^2 \,\mathrm{d}\sigma + \int_{f\in\partial K^-} \psi\{\boldsymbol{\varphi}\}^2 \,\mathrm{d}\sigma &= \frac{1}{2}\int_f \left(\psi{\boldsymbol{\varphi}^+}^2 + \psi{\boldsymbol{\varphi}^-}^2 + 2\psi\boldsymbol{\varphi}^+\boldsymbol{\varphi}^-\right) \,\mathrm{d}\sigma \\
&\le \int_f \left(\psi{\boldsymbol{\varphi}^+}^2 + \psi{\boldsymbol{\varphi}^-}^2\right) \,\mathrm{d}\sigma \\
&= \int_{f\in\partial K^+} \psi\boldsymbol{\varphi}^2 \,\mathrm{d}\sigma + \int_{f\in\partial K^-} \psi\boldsymbol{\varphi}^2 \,\mathrm{d}\sigma,
\end{aligned}$$

where we have set $\boldsymbol{\varphi}^\pm = \boldsymbol{\varphi}|_{K^\pm}$ and the average operator was expanded according to its definition. The other formulas can be proved in a similar way. □

In what follows we shall often use the above lemma with $\psi = h_f$ or $\psi = 1/h_f$, which obviously satisfy the hypotheses. The following lemma was proved in [1]:

Lemma 3.3 (Lifting operator bounds) *Let $f \in \mathcal{F}_h$ and assume that $\boldsymbol{\varphi}_h \in [\mathbb{P}^k(f)]^{d^2}$. Then,*

$$\|\mathbf{r}_f(\boldsymbol{\varphi}_h)\|_{0,\Omega}^2 \lesssim h_f^{-1}\|\boldsymbol{\varphi}_h\|_{0,f}^2 \lesssim \|\mathbf{r}_f(\boldsymbol{\varphi}_h)\|_{0,\Omega}^2.$$

A local inverse inequality is assumed to hold for all $v_h \in V_h^k$, $K \in \mathcal{T}_h$ and $0 < h \le 1$,

$$\|v_h\|_{l,K} \lesssim h_K^{m-l}\|v_h\|_{m,K}. \tag{13}$$

For the proof we refer the reader to [16].

Finally, we define a shorthand notation for the components of the error inside and outside the discrete space. Let $(\mathbf{u}, p) \in \mathbf{V} \times Q$ and $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$. We then let

$$\begin{aligned}
\mathbf{u} - \mathbf{u}_h &= (\mathbf{u} - \pi\mathbf{u}) + (\pi\mathbf{u} - \mathbf{u}_h) \stackrel{\text{def}}{=} \mathbf{e}_{\pi,\mathbf{u}} + \mathbf{e}_{h,\mathbf{u}}, \\
p - p_h &= (p - \pi p) + (\pi p - p_h) \stackrel{\text{def}}{=} e_{\pi,p} + e_{h,p}.
\end{aligned} \tag{14}$$

4. Convergence analysis

4.1. Existence and uniqueness of the discrete solution, Galerkin orthogonality and adjoint consistency

Expressions for the fluxes that are single-valued on every face of the triangulations are called *conservative*. It is not difficult to realize that the numerical fluxes defining the method are conservative, a property which will be used in the following

Theorem 4.1 (Existence and uniqueness) *The problem defined by (9) has a unique approximate solution* $(\mathbf{u}_h, p_h) \in \mathbf{W}_h$.

Proof. The proof can be carried out by showing that the only admissible solution to the homogeneous problem with $\mathbf{f} \equiv \mathbf{0}$ in Ω and $\mathbf{u} = 0$ on $\partial\Omega$ is the trivial solution $(\mathbf{0}, 0)$. (i) Taking $(\mathbf{v}_h, q_h) = (\mathbf{u}_h, p_h)$ as a test function in (9) we have that

$$a(\mathbf{u}_h, \mathbf{u}_h) + j_\nu(\mathbf{u}_h, \mathbf{u}_h) + j_\mathbf{n}(\mathbf{u}_h, \mathbf{u}_h) + j_p(p_h, p_h) = 0,$$

which implies that $\nabla \mathbf{u}_h|_K \equiv \mathbf{0}$ on every $K \in \mathcal{T}_h$, that $[\![p_h]\!] = \mathbf{0}$ across all $f \in \mathcal{F}_h^i$ and that $[\![\mathbf{u}_h]\!] = \mathbf{0}$ across all $f \in \mathcal{F}_h$. Since $\mathbf{u} = \mathbf{0}$ on $\partial\Omega$, we conclude that $\mathbf{u}_h \equiv \mathbf{0}$. (ii) In order to prove that the approximate pressure is also zero, we substitute $\mathbf{u}_h = \mathbf{0}$ in the momentum equation, integrate by parts using (8) and deploy the fact that p_h is continuous across intefaces to write

$$\begin{aligned} 0 &= a(\mathbf{0}, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) + j_\nu(\mathbf{0}, \mathbf{v}_h) + j_\mathbf{n}(\mathbf{0}, \mathbf{v}_h) \\ &= -\int_\Omega p_h \nabla_h \cdot \mathbf{v}_h \,\mathrm{d}\mathbf{x} + \sum_{f\in\mathcal{F}_h} \int_f \{p_h\}[\mathbf{v}_h]_\mathbf{n} \,\mathrm{d}\sigma \\ &= \int_\Omega \nabla_h p_h \cdot \mathbf{v}_h \,\mathrm{d}\mathbf{x} - \sum_{f\in\mathcal{F}_h} \int_f [\![p_h]\!] \cdot \{\mathbf{v}_h\} \,\mathrm{d}\sigma = \int_\Omega \nabla_h p_h \cdot \mathbf{v}_h \,\mathrm{d}\mathbf{x}, \end{aligned}$$

for all $\mathbf{v}_h \in \mathbf{V}_h$. Since $\nabla_h p_h$ is in $\mathbf{V}_h$, this entails that $\nabla p_h|_K \equiv \mathbf{0}$ for all $K \in \mathcal{T}_h$. The proof is concluded recalling that we required $\int_\Omega p_h \,\mathrm{d}\mathbf{x} = 0$ in order for the pressure to be uniquely defined and that the jumps of the pressure across internal faces are zero. □

Theorem 4.2 (Galerkin orthogonality) *Let* $(\mathbf{u}, p) \in \left[H^2(\Omega)\right]^d \times H^1(\Omega)$ *be the solution of the Stokes problem (1) and be* $(\mathbf{u}_h, p_h) \in \mathbf{W}_h$ *its approximation obtained solving (9). Then*

$$B(\mathbf{e_u}, e_p; \mathbf{v}_h, q_h) = 0, \quad \forall\, (\mathbf{v}_h, q_q) \in \mathbf{W}_h.$$

Proof. The assert can be proved as in [1] using the consistency of the numerical fluxes and the regularity assumptions on the exact solution. By virtue of the latter, the jumps of the solution on element boundaries are all zero. We therefore have:

$$B(\mathbf{u}, p; \mathbf{v}_h, q_h) = G(\mathbf{v}_h), \quad \forall\, (\mathbf{v}_h, q_h) \in \mathbf{W}_h.$$

The proof can be completed by subtracting (9) from the previous equation and accounting for the linearity of the form B. □

Notice that the stabilization contribution $j_p(\cdot,\cdot)$ plays an important role in establishing the unicity of the approximate solution. We conclude by stating a property which will be useful to obtain an optimal error estimate for the L^2-norm of the velocity error.

Remark 4.1. *Let* $(\mathbf{w}, r)$ *solve*

$$\begin{cases} -\Delta\mathbf{w} + \boldsymbol{\nabla} r = \boldsymbol{\lambda}, & \text{in } \Omega, \\ \boldsymbol{\nabla}\cdot\mathbf{w} = 0, & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0}, & \text{on } \partial\Omega. \end{cases} \tag{15}$$

It is a simple matter to realize that the following adjoint consistency condition holds:

$$B(\mathbf{w}, r; \mathbf{v}, q) = B(\mathbf{v}, -q; \mathbf{w}, -r) = (\mathbf{v}, \boldsymbol{\lambda})_\Omega, \quad \forall \mathbf{v} \in \mathbf{V}.$$

4.2. Norms for the analysis

Let

$$\mathbf{V}(h) \stackrel{\text{def}}{=} \mathbf{V} \oplus \mathbf{V}_h, \quad Q(h) \stackrel{\text{def}}{=} Q \oplus Q_h, \quad \mathbf{W}(h) \stackrel{\text{def}}{=} \mathbf{V}(h) \times Q(h).$$

Following an established practice, we introduce two norms, one for the stability and one for the continuity. For all $(\mathbf{v}, q) \in \mathbf{W}(h)$, we let

$$\begin{aligned} |||\,(\mathbf{v}, q)\,|||^2 &\stackrel{\text{def}}{=} \sum_{K\in\mathcal{T}_h} |\mathbf{v}|_{1,K}^2 + \|q\|_{0,\Omega}^2 + |\mathbf{v}|_\nu^2 + |\mathbf{v}|_\mathbf{n}^2 + |q|_p^2, \\ |]\,(\mathbf{v}, q)\,[|^2 &\stackrel{\text{def}}{=} |||\,(\mathbf{v}, q)\,|||^2 + \sum_{K\in\mathcal{T}_h} \left[h_K^{-2}\|\mathbf{v}\|_{0,K}^2 + h_K^2|\mathbf{v}|_{2,K}^2 + h_K^2|q|_{1,K}^2\right]. \end{aligned}$$

The seminorms associated with the penalty terms are defined as follows:

$$|\mathbf{v}|_\nu^2 \stackrel{\text{def}}{=} \sum_{f\in\mathcal{F}_h} \|\mathbf{r}_f([\![\mathbf{v}]\!])\|_{0,\Omega}^2, \quad |\mathbf{v}|_\mathbf{n}^2 \stackrel{\text{def}}{=} \sum_{f\in\mathcal{F}_h} h_f^{-1}\|[\mathbf{v}]_\mathbf{n}\|_{0,f}^2, \quad |q|_p^2 \stackrel{\text{def}}{=} \sum_{f\in\mathcal{F}_h} h_f\|[\![q]\!]\|_{0,f}^2.$$

Owing to the inverse inequality (13), the above norms are equivalent on the discrete space $\mathbf{W_h}$, i.e., for all $(\mathbf{v}_h, q_h) \in \mathbf{W}_h$,

$$|||\,(\mathbf{v}_h, q_h)\,||| \lesssim |]\,(\mathbf{v}_h, q_h)\,[| \lesssim |||\,(\mathbf{v}_h, q_h)\,|||. \tag{16}$$

In what follows we shall therefore assume that $c = \gamma/h_f$ with $\gamma > 0$.

Lemma 4.1 (Interpolation) *Let* π *be a projection operator satisfying the assumptions of Lemma 3.1 and let* $(\mathbf{v}, q) \in \left[H^{s+1}(\mathcal{T}_h)\right]^d \times H^{t+1}(\mathcal{T}_h)$ *with* $s \geq 1$ *and* $t \geq 0$. *Then,*

$$|]\,(\mathbf{v} - \pi\mathbf{v}, q - \pi q)\,[| \lesssim \left(\sum_{K\in\mathcal{T}_h} h_K^{2\min(s,k)}\|\mathbf{v}\|_{s+1,K}^2 + \sum_{K\in\mathcal{T}_h} h_K^{2\min(t,k)+2}\|q\|_{t+1,K}^2\right)^{1/2}.$$

Proof. Observe that, owing to the regularity assumptions on $\mathbf{v}$, $[\![\mathbf{v}]\!]$ vanishes across all the faces. As a consequence, the bounds for the lifting operator $\mathbf{r}_f(\boldsymbol{\varphi})$ stated in Lemma 3.3 are still valid for $\boldsymbol{\varphi} = [\![\mathbf{v} - \pi\mathbf{v}]\!]$. We therefore have

$$\begin{aligned} \sum_{f\in\mathcal{F}_h} \|\mathbf{r}_f([\![\mathbf{v} - \pi\mathbf{v}]\!])\|_{0,\Omega}^2 &\lesssim \sum_{f\in\mathcal{F}_h} \|h_f^{-1/2}[\![\mathbf{v} - \pi\mathbf{v}]\!]\|_{0,f}^2 \\ &\lesssim \sum_{K\in\mathcal{T}_h} \underline{h}_K^{-1}\|\mathbf{v} - \pi\mathbf{v}\|_{0,\partial K}^2 \\ &\lesssim \sum_{K\in\mathcal{T}_h} \sigma_K h_K^{-1}\left(h_K^{-1}\|\mathbf{v} - \pi\mathbf{v}\|_{0,K}^2 + h_K|\mathbf{v} - \pi\mathbf{v}|_{1,K}^2\right) \\ &\lesssim \sum_{K\in\mathcal{T}_h} h_K^{2\min(s,k)}\|\mathbf{v}\|_{s+1,K}^2, \end{aligned}$$

where we used Lemma 3.3, Lemma 3.2, mesh regularity (10), trace inequality (12) and Lemma 3.1. The rest of the proof can be carried out in a standard way by a repeated use of Lemma 3.1 and of the trace inequality (12). □

4.3. Continuity

Proposition 4.1 (Continuity) *Let* $(\mathbf{u},p),(\mathbf{v},q)\in\mathbf{W}(h)$ *and let* π *be a projection satisfying Lemma 3.1. We have that*

$$B(\mathbf{u},p;\mathbf{v},q)\lesssim |]\,(\mathbf{u},p)\,[|\;|]\,(\mathbf{v},q)\,[|.$$

Proof. The proof can be carried out bounding each term in the bilinear form B separately. For the first term, observe that

$$a(\mathbf{u},\mathbf{v})=\sum_{K\in\mathcal{T}_h}\int_K\boldsymbol{\nabla}\mathbf{u}{:}\boldsymbol{\nabla}\mathbf{v}\;\mathrm{d}\mathbf{x}+\sum_{f\in\mathcal{F}_h}\int_f\{\boldsymbol{\nabla}_h\mathbf{u}\}{:}[\![\mathbf{v}]\!]\;\mathrm{d}\sigma+\sum_{f\in\mathcal{F}_h}\int_f[\![\mathbf{u}]\!]{:}\{\boldsymbol{\nabla}_h\mathbf{v}\}\;\mathrm{d}\sigma\stackrel{\mathrm{def}}{=}T_1+T_2+T_3.$$

Clearly, $T_1\leq\sum_{K\in\mathcal{T}_h}|\mathbf{u}|_{1,K}|\mathbf{v}|_{1,K}\lesssim |]\,(\mathbf{u},0)\,[|\;|]\,(\mathbf{v},0)\,[|$. For the second contribution, we use Lemma 3.2 together with the trace inequality (12) to write

$$T_2\leq\sum_{f\in\mathcal{F}_h}\|h_f^{1/2}\{\boldsymbol{\nabla}_h\mathbf{u}\}\|_{0,f}\|h_f^{-1/2}[\![\mathbf{v}]\!]\|_{0,f}\lesssim\left(\sum_{K\in\mathcal{T}_h}|\mathbf{u}|_{1,K}^2+\sum_{K\in\mathcal{T}_h}h_K^2|\mathbf{u}|_{2,K}^2\right)^{1/2},$$

and, finally, $T_2\lesssim |]\,(\mathbf{u},0)\,[|\;|]\,(\mathbf{v},0)\,[|$. A similar result holds for T_3.

The second term reads

$$b(\mathbf{v},p)=-\sum_{K\in\mathcal{T}_h}\int_K p\nabla{\cdot}\mathbf{v}\;\mathrm{d}\mathbf{x}+\sum_{f\in\mathcal{F}_h}\int_f\{p\}[\mathbf{v}]_{\mathbf{n}}\;\mathrm{d}\sigma=T_1+T_2.$$

For the first contribution we immediately have $T_1\leq\sum_{K\in\mathcal{T}_h}\|p\|_{0,K}\|\mathbf{v}\|_{1,K}\leq |]\,(0,p)\,[|\;|]\,(\mathbf{v},0)\,[|$. The second contribution can be treated using Lemma 3.3 together with trace inequality (12) as follows:

$$T_2\leq\sum_{f\in\mathcal{F}_h}\|h_f^{1/2}\{p\}\|_{0,f}\|h_f^{-1/2}[\mathbf{v}]_{\mathbf{n}}\|_{0,f}\lesssim\left(\sum_{K\in\mathcal{T}_h}\|p\|_{0,K}^2+\sum_{K\in\mathcal{T}_h}h_K^2|p|_{1,K}^2\right)^{1/2}|]\,(\mathbf{v},0)\,[|,$$

whence $T_2\lesssim |]\,(\mathbf{0},p)\,[|\;|]\,(\mathbf{v},0)\,[|$.

Finally, we immediately have

$$j_\nu(\mathbf{u},\mathbf{v})+j_{\mathbf{n}}(\mathbf{u},\mathbf{v})+j_p(p,q)\leq |]\,(\mathbf{u},p)\,[|\;|]\,(\mathbf{v},q)\,[|,$$

which concludes the proof. □

4.4. Partial coercivity and discrete inf-sup *condition*

In this section we prove a partial coercivity result for the velocity and a discrete equivalent of the continuous inf-sup condition for the Stokes system (1). For all $(\mathbf{v}_h,q_h)\in\mathbf{W}_h$, we define the following seminorm, with respect to which the bilinear form is coercive:

$$|\,(\mathbf{v}_h,q_h)\,|_c^2\stackrel{\mathrm{def}}{=}\sum_{K\in\mathcal{T}_h}|\mathbf{v}|_{1,K}^2+|\mathbf{v}|_\nu^2+|\mathbf{v}|_{\mathbf{n}}^2+|q|_p^2.$$

Proposition 4.2 (Partial coercivity) *The bilinear form B satisfies*

$$B(\mathbf{v}_h, q_h; \mathbf{v}_h, q_h) \gtrsim |\,(\mathbf{v}_h, q_h)\,|_c^2, \quad \forall\,(\mathbf{v}_h, q_h) \in \mathbf{W}_h.$$

Proof. Plugging $(\mathbf{v}_h, q_h)$ into the definition of the bilinear form B we obtain

$$B(\mathbf{v}_h, q_h; \mathbf{v}_h, q_h) = a(\mathbf{v}_h, \mathbf{v}_h) + j_\nu(\mathbf{v}_h, \mathbf{v}_h) + j_{\mathbf{n}}(\mathbf{v}_h, \mathbf{v}_h) + j_p(q_h, q_h).$$

From the definition of the form a we have that

$$a(\mathbf{v}_h, \mathbf{v}_h) \geq \sum_{K\in\mathcal{T}_h} |\mathbf{v}_h|_{1,K}^2 - 2\sum_{f\in\mathcal{F}_h} \left| \int_f \{\boldsymbol{\nabla}_h \mathbf{v}_h\}{:}[\![\mathbf{v}_h]\!]\ \mathrm{d}\sigma \right|.$$

Let T denote the second term in the right hand side. Using Lemma 3.2, inequalities (12) and (13) we have that

$$T \leq \sum_{f\in\mathcal{F}_h} \|h_f^{1/2}\{\boldsymbol{\nabla}_h \mathbf{v}_h\}\|_{0,f} \|h_f^{-1/2}[\![\mathbf{v}_h]\!]\|_{0,f} \leq C\left(\sum_{K\in\mathcal{T}_h} |\mathbf{v}_h|_{1,K}^2\right)^{1/2} |\mathbf{v}_h|_\nu,$$

with C a positive parameter independent of the meshsize h. Using the above result together with the arithmetic-geometric inequality $(ab \leq a^2\epsilon/2 + b^2/(2\epsilon))$ we conclude that

$$B(\mathbf{v}_h, q_h; \mathbf{v}_h, q_h) \geq \left(1 - \frac{C\epsilon}{2}\right) \sum_{K\in\mathcal{T}_h} |\mathbf{v}_h|_{1,K}^2 + \left(\eta - \frac{C}{2\epsilon}\right) |\mathbf{v}_h|_\nu^2 + |\mathbf{v}_h|_{\mathbf{n}}^2 + |q_h|_p^2,$$

which gives the desired results provided (i) ϵ is chosen so that the first term in brackets is positive and η is large enough for the second term in brackets to be positive as well; (ii) we take $C_s = \min\left(1 - \frac{C\epsilon}{2}, \eta - \frac{C}{2\epsilon}\right)$. □
Following the guidelines of the reasoning in [15, Lemma 5.2], it can be proved that a sufficient condition for the stability is that η be greater than the maximum number of faces of a mesh element.

The error estimate for the pressure is based on the following proposition, which contains a discrete equivalent of the inf-sup condition. We recall that, by the continuous inf-sup condition for the standard Stokes forms, there exists a velocity field $\mathbf{u} \in \left[H_0^1(\Omega)\right]^d$ satisfying

$$-\int_\Omega q\boldsymbol{\nabla}{\cdot}\mathbf{u}\ \mathrm{d}\mathbf{x} \geq \kappa\|q\|_{0,\Omega}^2, \quad \|\mathbf{u}\|_{1,\Omega} \leq \|q\|_{0,\Omega}, \quad \forall q \in Q. \tag{17}$$

Proposition 4.3 (Discrete inf-sup **Condition)** *There exist positive constants κ_1 and κ_2 independent of the meshsize such that for all $(\mathbf{v}_h, q_h) \in \mathbf{W}_h$ there is $\mathbf{w}_h \in \mathbf{V}_h$ such that*

$$B(\mathbf{v}_h, q_h; \mathbf{w}_h, 0) \geq \kappa_1\|q_h\|_{0,\Omega}^2 - \kappa_2|\,(\mathbf{v}_h, q_h)\,|_c^2, \quad |\!|\!|\,(\mathbf{w}_h, 0)\,|\!|\!| \leq \|q_h\|_{0,\Omega}.$$

Proof. Let $\mathbf{u}$ be the velocity field for which condition (17) is satisfied for $(\mathbf{v}, q) = (\mathbf{v}_h, q_h)$. By definition of the bilinear form B and deploying the fact that $[\![\mathbf{u}]\!] = \mathbf{0}$ across all $f \in \mathcal{F}_h$,

$$\begin{aligned} B(\mathbf{v}_h, q_h; \pi\mathbf{u}, 0) &\geq b(\mathbf{u}, q_h) - |b(\mathbf{u} - \pi\mathbf{u}, q_h)| - |a(\pi\mathbf{u}, \mathbf{v}_h)| - |j_\nu(\mathbf{v}_h, \pi\mathbf{u})| - |j_{\mathbf{n}}(\mathbf{v}_h, \pi\mathbf{u})| \\ &\stackrel{\text{def}}{=} b(\mathbf{u}, q_h) - T_1 - T_2 - T_3 - T_4. \end{aligned}$$

Thanks to (17), we immediately conclude that $b(\mathbf{u}, q_h) \geq \kappa \|q_h\|^2_{0,\Omega}$. We then proceed to bound the remaining terms. Throughout the rest of the proof, the symbols C_i, $i \in \{1, \ldots, 3\}$ will denote real positive parameters independent of the meshsize h but possibly depending on the mesh regularity and on the polynomial degrees used in the approximation. Owing to (8),

$$\begin{aligned} b(\mathbf{u} - \pi\mathbf{u}, q_h) &= \underbrace{\int_\Omega (\mathbf{u} - \pi\mathbf{u}) \cdot \boldsymbol{\nabla}_h q_h \, \mathrm{d}\mathbf{x}}_{=0} - \sum_{f \in \mathcal{F}_h} \int_f \{\mathbf{u} - \pi\mathbf{u}\} \cdot [\![q_h]\!] \, \mathrm{d}\sigma \\ &\lesssim \left(\sum_{K \in \mathcal{T}_h} h_K^{-2} \|\mathbf{u} - \pi\mathbf{u}\|^2_{0,K} + \sum_{K \in \mathcal{T}_h} |\mathbf{u} - \pi\mathbf{u}|^2_{1,K} \right)^{1/2} |q_h|_p \\ &\lesssim |]\mathbf{u} - \pi\mathbf{u}[| \; |q_h|_p \lesssim \|q_h\|_{0,\Omega} \; |q_h|_p, \end{aligned}$$

where we have used the fact that $(\mathbf{u} - \pi\mathbf{u}) \in \mathbf{V}_h^\perp$ together with Lemma 3.1 and (17) to write

$$|] \, (\mathbf{u} - \pi\mathbf{u}, 0) \, [| \lesssim \|\mathbf{u}\|_{1,\Omega} \lesssim \|q_h\|_{0,\Omega}. \tag{18}$$

We then conclude that

$$T_1 \leq \frac{C_1 \epsilon_1}{2} \|q_h\|^2_{0,\Omega} + \frac{C_1}{2\epsilon_1} | \, (\mathbf{0}, q_h) \, |^2_c.$$

The second term can be bounded as follows: since $\mathbf{u}$ is continuous across interfaces and equal to zero on boundary faces, deploying the definition of the boundary operator we have that

$$\begin{aligned} T_2 \leq & \sum_{K \in \mathcal{T}_h} \left| \int_K \boldsymbol{\nabla} \mathbf{v}_h : \boldsymbol{\nabla}(\mathbf{u} - \pi\mathbf{u}) \, \mathrm{d}\mathbf{x} \right| + \sum_{f \in \mathcal{F}_h} \left| \int_\Omega \mathbf{r}_f([\![\mathbf{v}_h]\!]) : \boldsymbol{\nabla}_h (\mathbf{u} - \pi\mathbf{u}) \, \mathrm{d}\mathbf{x} \right| \\ & \sum_{K \in \mathcal{T}_h} \left| \int_K \boldsymbol{\nabla} \mathbf{v}_h : \boldsymbol{\nabla} \mathbf{u} \, \mathrm{d}\mathbf{x} \right| + \sum_{f \in \mathcal{F}_h} \left| \int_\Omega \mathbf{r}_f([\![\mathbf{v}_h]\!]) : \boldsymbol{\nabla} \mathbf{u} \, \mathrm{d}\mathbf{x} \right| + \sum_{f \in \mathcal{F}_h} \left| \int_\Omega \mathbf{r}_f(\mathbf{u} - \pi\mathbf{u}) : \boldsymbol{\nabla}_h \mathbf{v}_h \, \mathrm{d}\mathbf{x} \right| \\ \leq & \left(\sum_{K \in \mathcal{T}_h} |\mathbf{v}_h|^2_{1,K} + |\mathbf{v}_h|^2_\nu \right)^{1/2} \left[\left(\sum_{K \in \mathcal{T}_h} |\mathbf{u} - \pi\mathbf{u}|^2_{1,K} + |\mathbf{u} - \pi\mathbf{u}|^2_\nu \right)^{1/2} + |\mathbf{u}|_{1,\Omega} \right] \\ \lesssim & | \, (\mathbf{v}_h, 0) \, |_c \, (|]\mathbf{u} - \pi\mathbf{u}[| + |\mathbf{u}|_{1,\Omega}) \leq \frac{C_2 \epsilon_2}{2} \|q_h\|^2_{0,\Omega} + \frac{C_2}{2\epsilon_2} | \, (\mathbf{v}_h, 0) \, |^2_c, \end{aligned}$$

where again we have used (18) to conclude.

Proceeding in a similar way as before we have that

$$T_3 + T_4 = |j_\nu(\mathbf{v}_h, \mathbf{u} - \pi\mathbf{u})| + |j_{\mathbf{n}}(\mathbf{v}_h, \mathbf{u} - \pi\mathbf{u})| \leq \frac{C_3 \epsilon_3}{2} \|q_h\|^2_{0,\Omega} + \frac{C_3}{2\epsilon_3} | \, (\mathbf{v}_h, 0) \, |^2_c.$$

Collecting the above results, we conclude that

$$B(\mathbf{v}_h, q_h; \mathbf{u}, 0) \geq (\kappa - C_1\epsilon_1 - C_2\epsilon_2 - C_3\epsilon_3) \, \|q\|^2_{0,\Omega} - \left(\frac{C_1}{\epsilon_1} + \frac{C_2}{\epsilon_2} + \frac{C_3}{\epsilon_3} \right) | \, (\mathbf{v}_h, q_h) \, |_c,$$

for all $\epsilon_1, \epsilon_2, \epsilon_3 > 0$. On a proper choice of the parameters ϵ_i we can find two positive values $K_1, \; K_2 > 0$ independent of the meshsize such that

$$B(\mathbf{v}_h, q_h; \mathbf{u}, 0) \geq K_1 \|q_h\|^2_{0,\Omega} - K_2 | \, (\mathbf{v}_h, q_h) \, |^2_c.$$

Finally, we notice that, by virtue of the regularity of $\mathbf{u}$ we have

$$\begin{aligned}|||\,(\pi\mathbf{u},0)\,|||^2 &= |\pi\mathbf{u}|^2_{1,\Omega} + |\mathbf{u}-\pi\mathbf{u}|^2_{\nu} + |\mathbf{u}-\pi\mathbf{u}|^2_{\mathbf{n}} \\ &\leq \|\mathbf{u}\|^2_{1,\Omega} + |]\,(\mathbf{u}-\pi\mathbf{u},0)\,[|^2 \lesssim \|\mathbf{u}\|^2_{1,\Omega} \leq K_3\|q\|^2_{0,\Omega},\end{aligned}$$

where we used eq. (11), Lemma 4.1 and the continuous inf-sup condition. The proof is completed by taking $\mathbf{w}_h = \pi\mathbf{u}/K_3$, $\kappa_1 = K_1/K_3$ and $\kappa_2 = K_2/K_3$. □

4.5. *Error estimates*

In this section we obtain *a priori* error estimates for the velocity and the pressure. In particular, we show that, when equal order approximation with polynomials of degree $k \geq 1$ is used and the solution is sufficiently regular, the error in the velocity and in the pressure scale respectively as h^{k+1} and h^k. This result is stated in the following theorem.

Theorem 4.3 (Error estimates) *Let $(\mathbf{u},p) \in \left[H^{s+1}(\Omega)\right]^d \times H^{t+1}(\Omega)$ be the solution of (1) and let $(\mathbf{u}_h,p_h)$ its approximation obtained solving (9). Assume that the hypotheses on the mesh listed in §3 are satisfied and that the space setting (3) is chosen for $k \geq 0$. Take moreover $c = \gamma/h_f$ for some $\gamma > 0$ and η large enough to ensure stability. Then we have that*

$$|]\,(\mathbf{u}-\mathbf{u}_h, p-p_h)\,[| \lesssim h^{\min(s,k)}\|\mathbf{u}\|_{s+1,\Omega} + h^{\min(t,k)+1}\|p\|_{t+1,\Omega}, \tag{19}$$

$$\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega} \lesssim h^{\min(s,k)+1}\|\mathbf{u}\|_{s+1,\Omega} + h^{\min(t,k)+2}\|p\|_{t+1,\Omega}, \tag{20}$$

Proof. We proceed to prove the estimates.

(i) Using Proposition 4.2, Theorem 4.2, Proposition 4.1 and the norm equivalence stated in (16) we have that

$$\begin{aligned}|\,(\mathbf{e}_{h,\mathbf{u}}, e_{h,p})\,|^2_c &\lesssim B(\mathbf{e}_{h,\mathbf{u}}, e_{h,p}; \mathbf{e}_{h,\mathbf{u}}, e_{h,p}) = B(-\mathbf{e}_{\pi,\mathbf{u}}, -e_{\pi,p}; \mathbf{e}_{h,\mathbf{u}}, e_{h,p}) \\ &\lesssim |]\,(\mathbf{e}_{\pi,\mathbf{u}}, e_{\pi,p})\,[|\ |||\,(\mathbf{e}_{h,\mathbf{u}}, e_{h,p})\,|||.\end{aligned}$$

It only remains to estimate the L^2-norm of the pressure. Using Proposition 4.3 and proceeding in a similar way as before, we deduce that

$$\begin{aligned}\|e_{h,p}\|^2_{0,\Omega} &\lesssim B(\mathbf{e}_{h,\mathbf{u}}, e_{h,p}; \mathbf{w}_h, 0) + |\,(\mathbf{e}_{h,\mathbf{u}}, e_{h,p})\,|^2_c \\ &\lesssim |]\,(\mathbf{e}_{\pi,\mathbf{u}}, e_{\pi,p})\,[|\ |||\,(\mathbf{e}_{h,\mathbf{u}}, e_{h,p})\,||| + |\,(\mathbf{e}_{h,\mathbf{u}}, e_{h,p})\,|^2_c.\end{aligned}$$

Summing the above equations and using Lemma 3.1 we obtain

$$|||\,(\mathbf{e}_{h,\mathbf{u}}, e_{h,p})\,||| \lesssim |]\,(\mathbf{e}_{\pi,\mathbf{u}}, e_{\pi,p})\,[|.$$

The estimate (19) is then obtained by using the error decomposition (14) together with Lemma 4.1.

(ii) In order to prove (20), we use a standard duality argument. Consider the homogeneous Stokes problem (15) with right hand side $\boldsymbol{\lambda} = \mathbf{u} - \mathbf{u}_h$. The adjoint consistency condition discussed in Remark 4.1 gives

$$B(\mathbf{v}, -q; \mathbf{w}, -r) = (\mathbf{u}-\mathbf{u}_h, \mathbf{v})_{\Omega}, \quad \forall\,(\mathbf{v},q) \in \mathbf{V} \times Q.$$

Now setting $\tilde{r} \stackrel{\text{def}}{=} -r$ and choosing $(\mathbf{v},-q) = (\mathbf{u}-\mathbf{u}_h, p-p_h)$ we conclude that

$$B(\mathbf{u}-\mathbf{u}_h, p-p_h; \mathbf{w}, \tilde{r}) = \|\mathbf{u}-\mathbf{u}_h\|^2_{0,\Omega}.$$

Let $\left(\pi^1\mathbf{w}, \pi^0\tilde{r}\right)$ be the piecewise linear and the piecewise constant interpolant of $\mathbf{w}$ and $\tilde{r}$ respectively. Then by Theorem 4.2 and the inclusion property of Remark 2.1 we have that $B(\mathbf{u}-\mathbf{u}_h, p-p_h; \pi^1\mathbf{w}, \pi^0\tilde{r}) = 0$. As a consequence,

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_h\|^2_{0,\Omega} &= B(\mathbf{u}-\mathbf{u}_h, p-p_h; \mathbf{w}-\pi^1\mathbf{w}, \tilde{r}-\pi^0\tilde{r}) \\ &\lesssim |]\,(\mathbf{u}-\mathbf{u}_h, p-p_h)\,[|\ |]\left(\mathbf{w}-\pi^1\mathbf{w}, \tilde{r}-\pi^0\tilde{r}\right)[|.\end{aligned}$$

By virtue of Lemma 4.1, we have $|]\left(\mathbf{w}-\pi^1\mathbf{w}, \tilde{r}-\pi^0\tilde{r}\right)[| \lesssim h\,(\|\mathbf{w}\|_{2,\Omega} + \|\tilde{r}\|_{1,\Omega})$. Assuming that the elliptic regularity condition $\|\mathbf{w}\|_{2,\Omega} + \|r\|_{1,\Omega} \le \|\boldsymbol{\lambda}\|_{0,\Omega}$ holds for the solution of system (15), we conclude that

$$|]\left(\mathbf{w}-\pi^1\mathbf{w}, \tilde{r}-\pi^0\tilde{r}\right)[| \lesssim h\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}.$$

Using this result together with Lemma 4.1 and (19) we obtain the sought estimate. The proof is thus concluded.
□

An estimate for the L^2-norm of the pressure can be obtained from (19), since it is part of the energy norm $|]\cdot[|$.

The proofs above fit the case when mixed order elements are used and the polynomial order for the pressure is one unit lower than that for the velocity. Notice that, in the latter case, the estimate for the pressure error is also optimal with respect to to the approximation properties of the space Q_h. Finally, the choice $c = \gamma/h$ is now fully justified by the theory.

5. Numerical results

In this section we provide numerical assessment of the theoretical results derived above. We consider the same analytical solution of the Stokes problem in the square domain $\Omega = (-1,1)^2$ used in [12], i.e.,

$$\mathbf{u} = \begin{bmatrix} -\exp(x)\,(y\cos y + \sin y) \\ \exp(x) y \sin y \end{bmatrix}, \quad p = 2\exp(x)\sin y, \quad \mathbf{f} = \mathbf{0}.$$

In order to disambiguate the pressure, it was sufficient to remove one degree of freedom from the matrix and then re-scale the solution so as to impose the zero average constraint. Even with such a naive approach no degradation of the solution in the neighbourhood of the removed degree of freedom was observed. The numerical results collected in Table I were obtained for $c = 1/h_f$ and $\eta_f = 4.1$ using uniform rectangular meshes. The resulting linear system was solved by means of the direct solver in PETSc (see [2]).

The experiments show that the error estimates are sharp, and the expected orders of convergence are observed for both the pressure and the velocity. The norm of the divergence can be estimated by simply noticing that it is smaller or equal than the H^1-seminorm of the velocity and, hence, of the triple norm. One would therefore expect to observe convergence with order k. This theoretical estimate, however, seems over-pessimistic for $\mathbb{P}^1$ elements. The extent to which the zero-divergence constraint is satisfied inside every element is measured by

$$\|\boldsymbol{\nabla}_h\cdot\mathbf{u}_h\|_{0,\Omega} \stackrel{\text{def}}{=} \left(\sum_{K\in\mathcal{T}_h} \|\boldsymbol{\nabla}_h\cdot\mathbf{u}_h\|^2_{0,K}\right)^{1/2}.$$

Table I. Convergence results for $k = 1, 2, 3$.

k	Grid size	$\|\mathbf{e_u}\|_{0,\Omega}$ error	order	$\|e_p\|_{0,\Omega}$ error	order	$\|\nabla_h \cdot \mathbf{u}_h\|_{0,\Omega}$ error	order
1	32×32	1.00e-3	2.00	7.87e-3	1.09	9.36e-4	1.44
	64×64	2.52e-4	1.99	3.76e-3	1.07	3.65e-4	1.36
	128×128	6.36e-5	1.99	1.82e-3	1.04	1.39e-4	1.39
2	16×16	9.33e-5	3.01	4.34e-4	1.91	9.63e-4	1.95
	32×32	1.16e-5	3.01	1.25e-4	1.79	2.48e-4	1.96
	64×64	1.45e-6	3.00	3.41e-5	1.88	6.29e-5	1.98
3	8×8	2.89e-5	4.00	1.18e-4	2.97	2.99e-4	3.05
	16×16	1.79e-6	4.02	1.56e-5	2.91	3.65e-5	3.03
	32×32	1.11e-7	4.01	2.12e-6	2.88	4.56e-6	3.00

6. Conclusion

In this work we have analysed a new DG approximation of the Stokes problem first presented in [3], where only numerical assessment was provided. Following the approach originally proposed in [5], the viscous and inviscid fluxes were treated separately. The former were computed using the well-established BRMPS method, while for the latter a local artificial compressibility perturbation of the problem was introduced at the elementary interface level. The inviscid fluxes were then computed by solving a suitable Riemann problem. The analysis was carried out by extending the techniques for the elliptic case presented in [1]. Unlike in [12], the problem was considered in its primal formulation. Optimal error estimates were found for both the velocity and the pressure and the results were assessed by thorough numerical testing.

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