Rapport de recherche du CERMICS 2007-338 Mars 2007

Existence and uniqueness for a nonlinear parabolic/Hamilton-Jacobi coupled system describing the dynamics of dislocation densities

H. Ibrahim[1]

[1]*ENPC, ParisTech, Champs sur Marne, 77455 Marne la Vallée Cedex 2, France*

CERMICS — ENPC
6 et 8 avenue Blaise Pascal
Cité Descartes - Champs sur Marne
77455 Marne la Vallée Cedex 2

http://cermics.enpc.fr

Abstract

We study a mathematical model describing the dynamics of dislocation densities in crystals. This model is expressed as a 1D system of a parabolic equation and a first order Hamilton-Jacobi equation that are coupled together. We show the existence and uniqueness of a viscosity solution among those assuming a lower bound on their gradient for all time including the initial data. Then when we have no such restriction on the initial data, we show the existence of a viscosity solution. We also state a result of existence and uniqueness of an entropy solution of the system obtained by spatial derivation. This uniqueness holds in the class of "bounded from below" solutions. In order to do so, we use a relation between scalar conservation laws and Hamilton-Jacobi equations, mainly to have some gradient controls. This study will take place in the whole space, and on a bounded domain with suitable boundary conditions.

AMS Classification: 70H20, 35L65, 49L25, 54C70, 74H20, 74H25.
Keywords: Hamilton-Jacobi equations, scalar conservation laws, viscosity solutions, entropy solutions, dynamics of dislocation densities.

1 Introduction

1.1 Physical motivation

A dislocation is a defect, or irregularity within a crystal structure that can be observed by electron microscopy. The theory was originally developed by Vito Volterra in 1905. Dislocations are a non-stationary phenomena and their motion is the main explanation of the plastic deformation in metallic crystals (see [28, 19] for a recent and mathematical presentation).

Geometrically, each dislocation is characterized by a physical quantity called the Burgers vector, which is responsible for its orientation and magnitude. Dislocations are classified as being positive or negative due to the orientation of its Burgers vector, and they can move in certain crystallographic directions.

Starting from the motion of individual dislocations, a continuum description can be derived by adopting a formulation of dislocation dynamics in terms of appropriately defined dislocation densities, namely the density of positive and negative dislocations. In this paper we are interested in the model described by Groma, Csikor and Zaiser [18], that sheads light on the evolution of the dynamics of the "two type" densities of a system of straight parallel dislocations, taking into consideration the influence of the short range dislocation-dislocation interactions. The model was originally presented in $\mathbb{R}^2 \times (0,T)$ as follows:

$$\begin{cases} \dfrac{\partial \theta^+}{\partial t} + \mathbf{b}\cdot \dfrac{\partial}{\partial \mathbf{r}}\left[\theta^+ \left\{(\tau_{sc}+\tau_{eff}) - AD\dfrac{\mathbf{b}}{(\theta^+ + \theta^-)}\cdot \dfrac{\partial}{\partial \mathbf{r}}\left(\theta^+ - \theta^-\right)\right\}\right] = 0, \\ \dfrac{\partial \theta^-}{\partial t} - \mathbf{b}\cdot \dfrac{\partial}{\partial \mathbf{r}}\left[\theta^- \left\{(\tau_{sc}+\tau_{eff}) - AD\dfrac{\mathbf{b}}{(\theta^+ + \theta^-)}\cdot \dfrac{\partial}{\partial \mathbf{r}}\left(\theta^+ - \theta^-\right)\right\}\right] = 0. \end{cases} \tag{1.1}$$

Where $T > 0$, $\mathbf{r} = (x, y)$ represents the spatial variable, $\mathbf{b}$ is the burger's vector, $\theta^+(\mathbf{r}, t)$ and $\theta^-(\mathbf{r}, t)$ denote the densities of the positive and negative dislocations respectively. The quantity A is defined by the formula $A = \mu/[2\pi(1-\nu)]$, where μ is the shear modulus and ν is the Poisson ratio. D is a non-dimensional constant. Stress fields are represented through the self-consistent stress $\tau_{sc}(\mathbf{r}, t)$, and the effective stress $\tau_{eff}(\mathbf{r}, t)$. $\frac{\partial}{\partial \mathbf{r}}$ denotes the gradient with respect to the coordinate vector $\mathbf{r}$. An earlier investigation of the continuum description of the dynamics of dislocation densities has been done in [17]. However, a major drawback of these investigations is that the short range dislocation-dislocation correlations have been neglected and dislocation-dislocation interactions were described only by the long-range term which is the self-consistent stress field. Moreover, for the model described in [17], we refer the reader to [11, 12] for a one-dimensional mathematical and numerical study, and to [4] for a two-dimensional existence result.

In our work, we are interested in a particular setting of (1.1) where we make the following assumptions:

(a1) the quantities in equations (1.1) are independent of y,

(a2) $\mathbf{b} = (1, 0)$, and the constants A and D are set to be 1,

(a3) the effective stress is assumed to be zero.

Remark 1.1 *(a1) gives that the self-consistent stress τ_{sc} is null; this is a consequence of the definition of τ_{sc} (see [18]).*

Assumptions (a1)-(a2)-(a3) permit rewriting the original model as a **1D** problem in $\mathbb{R} \times (0, T)$:

$$\begin{cases} \theta_t^+(x,t) - \left(\theta^+(x,t)\left(\dfrac{\theta_x^+(x,t) - \theta_x^-(x,t)}{\theta^+(x,t) + \theta^-(x,t)}\right)\right)_x = 0, \\ \theta_t^-(x,t) + \left(\theta^-(x,t)\left(\dfrac{\theta_x^+(x,t) - \theta_x^-(x,t)}{\theta^+(x,t) + \theta^-(x,t)}\right)\right)_x = 0. \end{cases} \tag{1.2}$$

We consider an integrated form of (1.2) and we let:

$$\rho_x^\pm = \theta^\pm, \quad \theta = \theta^+ + \theta^-, \quad \rho = \rho^+ - \rho^- \quad \text{and} \quad \kappa = \rho^+ + \rho^-, \tag{1.3}$$

in order to obtain, for special values of the constants of integration, the following system of PDEs in terms of ρ and κ :

$$\begin{cases} \kappa_t \kappa_x = \rho_t \rho_x & \text{in} \quad Q_T = \mathbb{R} \times (0, T), \\ \kappa(x, 0) = \kappa^0(x) & \text{in} \quad \mathbb{R}, \end{cases} \tag{1.4}$$

and

$$\begin{cases} \rho_t = \rho_{xx} & \text{in} \quad Q_T, \\ \rho(x, 0) = \rho^0(x) & \text{in} \quad \mathbb{R}, \end{cases} \tag{1.5}$$

where $T > 0$ is a fixed constant. Enough regularity on the initial data will be given in order to impose the physically relevant condition,

$$\kappa_x^0 \geq |\rho_x^0| \,. \tag{1.6}$$

This condition is natural: it indicates nothing but the positivity of the dislocation densities $\theta^\pm(x, 0)$ at the initial time (see (1.3)).

1.2 Main results

In this paper, we show the existence and uniqueness of a viscosity solution κ of (1.4) in the class of all Lipschitz continuousviscosity solutions having special "bounded from below" spatial gradients. However, we show the existence of a Lipschitz continuousviscosity solution of (1.4) when this restriction is relaxed. A relation between scalar conservation laws and Hamilton-Jacobi equations will be exploited to get almost all our gradient controls of κ. This relation, that will be made precise later, will also lead to a result of existence and uniqueness of a bounded entropy solution of the following equation:

$$\begin{cases} \theta_t = \left(\frac{\rho_x \rho_{xx}}{\theta}\right)_x & \text{in} \quad Q_T, \\ \theta(x,0) = \theta^0(x) & \text{in} \quad \mathbb{R}, \end{cases} \tag{1.7}$$

which is deduced formally by taking a spatial derivation of (1.4). The uniqueness of this entropy solution is always restricted to the class of bounded entropy solutions with a special lower-bound.

Let $Lip(\mathbb{R})$ denotes:

$$Lip(\mathbb{R}) = \{f : \mathbb{R} \mapsto \mathbb{R};\ \ f \text{ is a Lipschitz continuous function}\}.$$

We prove the following theorems:

Theorem 1.2 (Existence and uniqueness of a viscosity solution)
Let $T > 0$. Take $\kappa^0 \in Lip(\mathbb{R})$ and $\rho^0 \in C_0^\infty(\mathbb{R})$ as initial data that satisfy:

$$\kappa_x^0 \geq \sqrt{(\rho_x^0)^2 + \epsilon^2} \quad a.e. \ in \quad \mathbb{R}, \tag{1.8}$$

for some constant $\epsilon > 0$. Then, given the solution ρ of (1.5), there exists a viscosity solution $\kappa \in Lip(\bar{Q}_T)$ of (1.4), unique among the viscosity solutions satisfying:

$$\kappa_x \geq \sqrt{\rho_x^2 + \epsilon^2} \quad a.e. \ in \quad \bar{Q}_T.$$

Theorem 1.3 (Existence and uniqueness of an entropy solution)
Let $T > 0$. Take $\theta^0 \in L^\infty(\mathbb{R})$ and $\rho^0 \in C_0^\infty(\mathbb{R})$ such that,

$$\theta^0 \geq \sqrt{(\rho_x^0)^2 + \epsilon^2} \quad a.e. \ in \quad \mathbb{R},$$

for some constant $\epsilon > 0$. Then, there exists an entropy solution $\theta \in L^\infty(\bar{Q}_T)$ of (1.7), unique among the entropy solutions satisfying:

$$\theta \geq \sqrt{\rho_x^2 + \epsilon^2} \quad a.e. \ in \quad \bar{Q}_T.$$

Moreover, we have $\theta = \kappa_x$, where κ is the solution given by Theorem 1.2.

The notion of viscosity solutions and entropy solutions will be recalled in Section 2. We now relate these results to our one-dimensional problem (1.2). Remarking that $\rho_x = \theta^+ - \theta^-$ and $\kappa_x = \theta^+ + \theta^-$, we have as a consequence:

Corollary 1.4 (Existence and uniqueness for problem (1.2))
Let $T > 0$. Let θ_0^+ and θ_0^- be two given functions representing the initial positive and negative dislocation densities respectively. If the following conditions are satisfied:

(1) $\theta_0^+ - \theta_0^- \in C_0^\infty(\mathbb{R})$,

(2) $\theta_0^+,\ \theta_0^- \in L^\infty(\mathbb{R})$,

together with,

$$\theta_0^+ + \theta_0^- \geq \sqrt{(\theta_0^+ - \theta_0^-)^2 + \epsilon^2} \qquad a.e.\ in \quad \mathbb{R},$$

then there exists a solution $(\theta^+, \theta^-) \in (L^\infty(Q_T))^2$ to the system (1.2), in the sense of Theorems 1.2 and 1.3, unique among those satisfying:

$$\theta^+ + \theta^- \geq \sqrt{(\theta^+ - \theta^-)^2 + \epsilon^2} \quad a.e. \quad in \quad \bar{Q}_T.$$

Remark 1.5 *Conditions (1) and (2) are sufficient requirements for the compatibility with the regularity of ρ^0 and κ^0 previously stated.*

Theorem 1.6 (Existence of a viscosity solution, case $\epsilon = 0$)
Let $T > 0$, $\kappa^0 \in Lip(\mathbb{R})$ and $\rho^0 \in C_0^\infty(\mathbb{R})$. If the condition (1.6) is satisfied a.e. in $\mathbb{R}$, then there exists a viscosity solution $\kappa \in Lip(\bar{Q}_T)$ of (1.4) satisfying:

$$\kappa_x \geq |\rho_x| \quad a.e.\ in \quad \bar{Q}_T. \tag{1.9}$$

Remark 1.7 *In the limit case where $\epsilon = 0$, we remark that having (1.9) was intuitively expected due to the positivity of the dislocation densities θ^+ and θ^-. This reflects in some way the well-posedness of the model (1.2) of the dynamics of dislocation densities. We also remark that our result of existence of a solution of (1.4) under (1.9) still holds if we start with $\kappa_x^0 = \rho_x^0 = 0$ on some interval of the real line. In other words, we can imagine that we start with the probability of the formation of no dislocation zones.*

Problem with boundary conditions.
We consider once again problem (1.4), similar results to that announced above will be shown on a bounded interval of the real line with Dirichlet boundary conditions (see Section 5). This problem corresponds physically to the study of the dynamics of dislocation densities in a part of a material with the geometry of a slab (see [18]).

1.3 Organization of the paper

The paper is organized as follows. In Section 2, we start by stating the definition of viscosity and entropy solutions with some of their properties. In Section 3, we prove the existence and uniqueness of a viscosity solution to an approximated problem of (1.4), namely Proposition 3.1, and we move on, giving additional properties of our approximated solution (Proposition 3.2) and consequently proving Theorems 1.2 and 1.3. In Section 4, we present the proof of Theorem 1.6. Section 5 is devoted to the study of problem (1.4) on a bounded domain with suitable boundary conditions. Finally, Section 6 is an appendix containing a sketch of the proof to the classical comparison principle of scalar conservation laws adapted to our equation with low regularity.

2 Notations and Preliminaries

We first fix some notations. If Ω is an open subset of $\mathbb{R}^n$, k is a positive integer, we denote by $C^k(\Omega)$ the space of all real valued k times continuously differentiable functions. $C_0^k(\Omega)$ is the subspace of $C^k(\Omega)$ consisting of function of compact support in Ω, and $C_b^k(\Omega) = C^k(\Omega) \cap W^{k,\infty}(\Omega)$ where $W^{k,\infty}(\Omega)$ is defined below. Furthermore, let $UC(\Omega)$ and $Lip(\Omega)$ denote the spaces of uniformly continuous functions and Lipschitz continuousfunctions on Ω respectively. The sobolev space $W^{m,p}(\Omega)$ with $m \geq 1$ an integer and $p : 1 \leq p \leq \infty$ a real, is defined by

$$W^{n,p}(\Omega) = \left\{ u \in L^p(\Omega) \;\middle|\; \begin{array}{l} \forall \alpha \text{ with } |\alpha| \leq n \ \exists f_\alpha \in L^p(\Omega) \quad \text{such that} \\ \displaystyle\int_\Omega u D^\alpha \phi = (-1)^{|\alpha|} \int_\Omega f_\alpha \phi \ \ \forall \phi \in C_0^\infty(\Omega) \end{array} \right\},$$

where we denote $D^\alpha u = f_\alpha$. This space equipped with the norm

$$||u||_{W^{n,p}} = \sum_{0 \leq |\alpha| \leq n} ||D^\alpha u||_{L^p}$$

is a Banach space. In what follows, $T > 0$. A map $m : [0, \infty) \mapsto [0, \infty)$ that satisfy

- m is continuous and non-decreasing;
- $\lim_{x \to 0^+} m(x) = 0$;
- $m(a + b) \leq m(a) + m(b)$ for $a, b \geq 0$;

is said to be "a modulus", and $UC_x(\Omega \times [0, T])$ denotes the space of those $u \in C(\Omega \times [0, T])$ for which there is a modulus m and $r > 0$ such that

$$|u(x, t) - u(y, t)| \leq m(|x - y|) \text{ for } x, y \in \Omega,\ |x - y| \leq r \text{ and } t \in [0, T].$$

We will deal with two types of equations:

1. Hamilton-Jacobi equation:

$$\begin{cases} u_t + F(x, t, u_x) = 0 & \text{in} \quad Q_T, \\ u(x, 0) = u^0(x) & \text{in} \quad \mathbb{R}, \end{cases} \tag{2.1}$$

2. Scalar conservation laws:

$$\begin{cases} v_t + (F(x, t, v))_x = 0 & \text{in} \quad Q_T, \\ v(x, 0) = v^0(x) & \text{in} \quad \mathbb{R}, \end{cases} \tag{2.2}$$

where

$$\begin{array}{rcl} F : \mathbb{R} \times [0, T] \times \mathbb{R} & \to & \mathbb{R} \\ (x, t,\ u) & \mapsto & F(x, t, u) \end{array}$$

is called the Hamiltonian in the Hamilton-Jacobi equations and the flux function in the scalar conservation laws. We will agree on the continuity of this function, while additional and specific regularity will be given when it is needed.

Remark 2.1 *We will use the function F as a notation for the Hamiltonian/flux function. Although F might differ from one equation to another, it will be clarified in all what follows.*

Remark 2.2 *The major part of this work concerns a Hamiltonian/flux function of a special form, namely:*

$$F(x,t,u) = g(x,t)f(u), \tag{2.3}$$

where such forms often arise in problems of physical interest including traffic flow [31] and two-phase flow in porous media [16].

We start by defining the notion of viscosity solution to Hamilton-Jacobi equations (2.1), and entropy solution to scalar conservation laws (2.2) with a flux function given by Remark 2.2, as well as some results about existence, uniqueness, and regularity properties of these solutions. We will end by a classical relation between these two problems. These results will be needed throughout this paper, precise references for the proofs will be mentioned later on.

2.1 Viscosity solution: definition and properties

Definition 2.3 ([10], Viscosity solution: non-stationary case)
1) A function $u \in C(Q_T;\mathbb{R})$ is a viscosity sub-solution of

$$u_t + F(x,t,u_x) = 0 \quad in \quad Q_T, \tag{2.4}$$

if for every $\phi \in C^1(Q_T)$, whenever $u-\phi$ attains a local maximum at $(x_0,t_0) \in Q_T$, then

$$\phi_t(x_0,t_0) + F(x_0,t_0,\phi_x(x_0,t_0)) \leq 0.$$

2) A function $u \in C(Q_T;\mathbb{R})$ is a viscosity super-solution of (2.4) if for every $\phi \in C^1(Q_T)$, whenever $u-\phi$ attains a local minimum at $(x_0,t_0) \in Q_T$, then

$$\phi_t(x_0,t_0) + F(x_0,t_0,\phi_x(x_0,t_0)) \geq 0.$$

3) A function $u \in C(Q_T;\mathbb{R})$ is a viscosity solution of (2.4) if it is both a viscosity sub- and super-solution of (2.4).
4) A function $u \in C(\bar{Q}_T;\mathbb{R})$ is a viscosity solution of the initial value problem (2.1) if u is a viscosity solution of (2.4) and $u(x,0) = u^0(x)$ in $\mathbb{R}$.

It is worth mentioning here that if a viscosity solution of a Hamilton-Jacobi equation is differentiable at a certain point, then it solves the equation there (see [10, Corollary I.6]). An equivalent definition depending on the sub- and super-differential of a continuous function is now presented. This definition will be used for the demonstration of Proposition 2.10. Let us recall that the sub- and the super-differential of a continuous function $u \in C(\mathbb{R}^n \times (0,T))$, at a point $(x,t) \in \mathbb{R}^n \times (0,T)$, are defined as the closed convex sets:

$$D^{1,-}u(x,t) = \Big\{(p,\alpha) \in \mathbb{R}^n \times \mathbb{R} : \\ \liminf_{(y,s)\to(x,t)} \frac{u(y,s) - u(x,t) - (p\cdot(y-x) + \alpha\cdot(s-t))}{|y-x|+|s-t|} \geq 0\Big\},$$

and

$$D^{1,+}u(x,t) = \Big\{(p,\alpha) \in \mathbb{R}^n \times \mathbb{R} :$$

$$\limsup_{(y,s)\to(x,t)} \frac{u(y,s) - u(x,t) - (p\cdot(y-x) + \alpha\cdot(s-t))}{|y-x| + |s-t|} \leq 0\Big\},$$

respectively.

Definition 2.4 (Equivalent definition of viscosity solution)
1) A function $u \in C(\mathbb{R}^n \times (0,T))$ is a viscosity super-solution of (2.1) if and only if, for every $(x,t) \in \mathbb{R}^n \times (0,T)$:

$$\forall (p,\alpha) \in D^{1,-}u(x,t), \qquad \alpha + F(x,t,p) \geq 0. \tag{2.5}$$

2) A function $u \in C(\mathbb{R}^n \times (0,T))$ is a viscosity sub-solution of (2.1) if and only if, for every $(x,t) \in \mathbb{R}^n \times (0,T)$:

$$\forall (p,\alpha) \in D^{1,+}u(x,t), \qquad \alpha + F(x,t,p) \leq 0. \tag{2.6}$$

This definition is more local, for it permits verification that a given explicit function is a viscosity solution in a more classical way, i.e. using the derivative calculus. A similar definition, that will be used later, could be given in the stationary case. Let $\Omega \subset \mathbb{R}^n$ be an open domain, and consider the PDE

$$F(x, u(x), \nabla u(x)) = 0, \quad \forall x \in \Omega, \tag{2.7}$$

where $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}$ is a continuous mapping.

Definition 2.5 (Viscosity solution: stationary case)
A continuous function $u : \Omega \mapsto \mathbb{R}$ is a viscosity sub-solution of the PDE (2.7) if for any continuously differentiable function $\phi : \Omega \mapsto \mathbb{R}$ and any local maximum $x_0 \in \Omega$ of $u - \phi$, one has

$$F(x_0, u(x_0), \nabla\phi(x_0)) \leq 0.$$

Similarly, if at any local minimum point $x_0 \in \Omega$ of $u - \phi$, one has

$$F(x_0, u(x_0), \nabla\phi(x_0)) \geq 0,$$

then u is a viscosity super-solution. Finally, if u is both a viscosity sub-solution and a viscosity super-solution, then u is called a viscosity solution.

In fact, this definition is used for interpreting solutions of (1.4) in the viscosity sense. Furthermore, we say that u is a viscosity solution of the Dirichlet problem (2.7) with $u = \zeta \in C(\partial\Omega)$ if:
(1) $u \in C(\bar{\Omega})$,
(2) u is a viscosity solution of (2.7) in Ω,
(3) $u = \zeta$ on $\partial\Omega$.
For a better understanding of the viscosity interpretation of boundary conditions of Hamilton-Jacobi equations, we refer the reader to [2, Section 4.2].

Now, we will proceed by giving the main results concerning viscosity solutions of (2.1). In order to have existence and uniqueness, the Hamiltonian F will be restricted by the following conditions :

(F0) $F \in C(\mathbb{R} \times [0,T] \times \mathbb{R})$;

(F1) for each $R > 0$ there is a constant C_R such that for all (x,t,p), $(y,t,q) \in \mathbb{R} \times [0,T] \times [-R,R]$,

$$|\, F(x,t,p) - F(y,t,q)\,| \leq C_R(\,|p-q| + |x-y|);$$

(F2) there is a constant C_F such that for all $(t,p) \in [0,T] \times \mathbb{R}$ and all $x,\ y \in \mathbb{R}$,

$$|\, F(x,t,p) - F(y,t,p)\,| \leq C_F|x-y|(1+|p|).$$

We use these conditions to write down some results on viscosity solutions.

Theorem 2.6 (Comparison, [9, Theorem 1])
Let F satisfy **(F0)**-**(F1)**-**(F2)**. *If u, $\bar{u} \in UC_x(\bar{Q}_T)$ are two viscosity sub- and super-solution of the Hamilton-Jacobi equation (2.1) respectively, with*

$$u(x,0) \leq \bar{u}(x,0) \quad \text{in} \quad \mathbb{R},$$

then $u \leq \bar{u}$ in $\bar{Q}_T$.

Theorem 2.7 (Existence, [9, Theorem 1])
Let F satisfy **(F0)**-**(F1)**-**(F2)**. *If $u^0 \in UC(\mathbb{R})$, then (2.1) has a viscosity solution $u \in UC_x(\bar{Q}_T)$.*

Remark 2.8 *The "comparison" theorem stated above gives the uniqueness of the viscosity solution.*

Remark 2.9 *In the case where the Hamiltonian has the form*

$$F(x,t,u) = g(x,t)f(u),$$

the following conditions:

(V0) $f \in C^1_b(\mathbb{R};\mathbb{R})$,

(V1) $g \in C_b(\bar{Q}_T;\mathbb{R})$,

(V2) $g_x \in L^\infty(\bar{Q}_T)$,

imply **(F0)**-**(F1)**-**(F2)** *together with the boundedness of the Hamiltonian.*

The next proposition reflects the behavior of viscosity solutions under additional regularity assumptions on u^0 and F.

Proposition 2.10 (Additional regularity of the viscosity solution)
Let $F = gf$ satisfy **(V0)**-**(V1)**-**(V2)**. *If $u^0 \in Lip(\mathbb{R})$ and $u \in UC_x(\bar{Q}_T)$ is the unique viscosity solution of (2.1), then $u \in Lip(\bar{Q}_T)$.*

Proof. Consider the function u^ε defined on $\mathbb{R} \times [0, T]$ by:

$$u^\varepsilon(x,t) = \sup_{y \in \mathbb{R}} \left\{ u(y,t) - e^{kt} \frac{|x-y|^2}{2\varepsilon} \right\}.$$

By [20, Theorem 3], the function u satisfies,

$$|u(x,t)| \le c^*(|x|+1) \quad \text{for } (x,t) \in \mathbb{R} \times [0,T],$$

where c and c^* are two positive constants. Therefore, u is a sublinear function for every time $t \in [0,T]$. The function u^ε is defined via a supremum which is attained because of the sublinearity of the function u (a quadratic function always control a linear one); the supremum can be achieved at several points; let x_ε be one of them, so we can write

$$u^\varepsilon(x,t) = u(x_\varepsilon, t) - e^{kt} \frac{|x-x_\varepsilon|^2}{2\varepsilon}.$$

We are going to prove that for $(p,\alpha) \in \mathbb{R} \times \mathbb{R}$, we have:

$$(p,\alpha) \in D^{1,+} u^\varepsilon(x,t) \Rightarrow \left(p, \alpha + k e^{kt} \frac{|x-x_\varepsilon|^2}{2\varepsilon} \right) \in D^{1,+} u(x_\varepsilon, t). \tag{2.8}$$

Since $(p,\alpha) \in D^{1,+} u^\varepsilon(x,t)$, then we can write for $(y,s) \sim (x,t)$ that,

$$L = u^\varepsilon(y,s) \le u^\varepsilon(x,t) + \alpha(s-t) + p(y-x) + o(|s-t| + |y-x|) = R, \tag{2.9}$$

where the left side L of (2.9) satisfies,

$$L \ge u(z,s) - e^{ks} \frac{|z-y|^2}{2\varepsilon}, \quad z \in \mathbb{R}, \tag{2.10}$$

and the right side R of (2.9) satisfies,

$$R \le u(x_\varepsilon, t) - e^{kt} \frac{|x-x_\varepsilon|^2}{2\varepsilon} + \alpha(s-t) + p(y-x) + o(|s-t| + |y-x|). \tag{2.11}$$

Choose z such that $z - y = x_\varepsilon - x$, then

$$z = x_\varepsilon + (y-x) \sim x_\varepsilon, \text{ since } y \sim x. \tag{2.12}$$

Combining (2.9), (2.10), (2.11) and (2.12) together, we get

$$\begin{aligned} u(x_\varepsilon + (y-x), s) - e^{ks} \frac{|x-x_\varepsilon|^2}{2\varepsilon} &\le \\ u(x_\varepsilon, t) - e^{kt} \frac{|x-x_\varepsilon|^2}{2\varepsilon} &+ \alpha(s-t) + p(z-x_\varepsilon) + o(|s-t| + |z-x_\varepsilon|), \end{aligned}$$

and hence,

$$\begin{aligned} u(z,s) \le u(x_\varepsilon, t) &+ (e^{ks} - e^{kt}) \frac{|x-x_\varepsilon|^2}{2\varepsilon} \\ &+ \alpha(s-t) + p(z-x_\varepsilon) + o(|s-t| + |z-x_\varepsilon|). \end{aligned} \tag{2.13}$$

We have

$$(e^{ks}-e^{kt})\frac{|x-x_\varepsilon|^2}{2\varepsilon}=ke^{kt}\frac{|x-x_\varepsilon|^2}{2\varepsilon}(s-t)+o(|s-t|),$$

then using inequality (2.13), we get

$$\begin{aligned}u(z,s)\le u(x_\varepsilon,t)+\left(\alpha+ke^{kt}\frac{|x-x_\varepsilon|^2}{2\varepsilon}\right)(s-t)\\ +p(z-x_\varepsilon)+o(|s-t|+|z-x_\varepsilon|),\end{aligned}$$

which proves that

$$\left(\alpha+ke^{kt}\frac{|x-x_\varepsilon|^2}{2\varepsilon},p\right)\in D^{1,+}u(x_\varepsilon,t),$$

and hence statement (2.8) is true. Since u is a viscosity sub-solution of (2.1), we have

$$\alpha+ke^{kt}\frac{|x-x_\varepsilon|^2}{2\varepsilon}+F(x_\varepsilon,t,p)\le 0.$$

We use condition (**F1**) with $p=q$, to get

$$\begin{aligned}\alpha+ke^{kt}\frac{|x-x_\varepsilon|^2}{2\varepsilon}+F(x,t,p)&\le F(x,t,p)-F(x_\varepsilon,t,p),\\ &\le C|x-x_\varepsilon|,\end{aligned}$$

therefore,

$$\begin{aligned}\alpha+F(x,t,p)&\le C|x-x_\varepsilon|-ke^{kt}\frac{|x-x_\varepsilon|^2}{2\varepsilon},\\ &\le Cr_\varepsilon-k\frac{r_\varepsilon^2}{2\varepsilon},\\ &\le \sup_{r>0}\left(Cr-\frac{kr^2}{2\varepsilon}\right),\end{aligned}$$

where $r_\varepsilon=|x-x_\varepsilon|$. At the maximum $\bar r$, we have $C=\frac{k\bar r}{\varepsilon}$. By choosing $k=\frac{C^2}{2}$, we get

$$\alpha+F(x,t,p)\le\varepsilon.$$

This inequality shows that $v^\varepsilon=u^\varepsilon-\varepsilon t$ is a viscosity sub-solution of (2.1) with $v^\varepsilon(x,0)=u^\varepsilon(x,0)$. By the comparison principle, we have

$$\begin{aligned}v^\varepsilon(x,t)-u(x,t)&\le \sup_{x\in\mathbb{R}}(v^\varepsilon(x,0)-u^0(x)),\\ &\le \sup_{x\in\mathbb{R}}(u^\varepsilon(x,0)-u^0(x)),\\ &\le \sup_{x\in\mathbb{R}}\left(\sup_{y\in\mathbb{R}}\left\{u^0(y)-\frac{|x-y|^2}{2\varepsilon}\right\}-u^0(x)\right),\\ &\le \sup_{x,y\in\mathbb{R}}\left(\gamma|x-y|-\frac{|x-y|^2}{2\varepsilon}\right),\\ &\le \sup_{r\ge 0}\left(\gamma r-\frac{r^2}{2\varepsilon}\right)=\frac{\gamma^2\varepsilon}{2},\end{aligned}$$

where γ is the Lipschitz constant of the function u^0, and $r = |x-y|$. This altogether shows the following inequality for $x, y \in \mathbb{R}$:

$$u(y,t) - e^{kt}\frac{|x-y|^2}{2\varepsilon} \leq u^\varepsilon(x,t) \leq u(x,t) + \varepsilon t + \frac{\gamma^2\varepsilon}{2}. \tag{2.14}$$

Remark here that k is a fixed; previously chosen constant. Inequality (2.14) yields:

$$u(y,t) - u(x,t) \leq e^{kt}\frac{|x-y|^2}{2\varepsilon} + \left(t + \frac{\gamma^2}{2}\right)\varepsilon = \zeta/\varepsilon + \beta\varepsilon, \tag{2.15}$$

where $\zeta = e^{kt}\frac{|x-y|^2}{2}$ and $\beta = \left(t + \frac{\gamma^2}{2}\right)$. We minimize inequality (2.15) over ε to obtain,

$$\begin{aligned} u(y,t) - u(x,t) &\leq 2\sqrt{\zeta\beta}, \\ &\leq e^{\frac{kt}{2}}\sqrt{2}\sqrt{t + \frac{\gamma^2}{2}}|x-y|. \end{aligned}$$

Since this inequality holds $\forall x, y \in \mathbb{R}$, exchanging x with y yields,

$$|u(x,t) - u(y,t)| \leq C(F, u_0)|x-y| \quad \forall x, y \in \mathbb{R} \text{ and } t \in [0,T].$$

This shows that the function u is Lipschitz continuous in x, uniformly in time t. To prove the Lipschitz continuity in time, we mainly use the result of [20, Theorem 3]) with the fact that $u_t = -F(x,t,u_x)$, and the boundedness of the Hamiltonian. □

Remark 2.11 *It is worth mentioning that the space Lipschitz constant of the function u depends on C, where C appears in* (**F1**) *for $p = q$, and on the Lipschitz constant γ of the function u_0. While the time Lipschitz constant depends on the bound of the Hamiltonian.*

2.2 Entropy solution: definition and properties

Definition 2.12 (Entropy sub-/super-solution)
Let $F(x,t,v) = g(x,t)f(v)$ with g, $g_x \in L^\infty_{loc}(Q_T;\mathbb{R})$ and $f \in C^1(\mathbb{R};\mathbb{R})$. A function $v \in L^\infty(Q_T;\mathbb{R})$ is an entropy sub-solution of (2.2) with bounded initial data $v^0 \in L^\infty(\mathbb{R})$ if it satisfies:

$$\begin{aligned} \int_{Q_T} \Big[&\eta_i(v(x,t))\phi_t(x,t) + \Phi(v(x,t))g(x,t)\phi_x(x,t) + \\ h(v(x,t))g_x(x,t)\phi(x,t) \Big] &dxdt + \int_{\mathbb{R}} \eta_i(v^0(x))\phi(x,0)dx \geq 0, \end{aligned} \tag{2.16}$$

$\forall \phi \in C^1_0(\mathbb{R}\times[0,T);\mathbb{R}_+)$, for any non-decreasing convex function $\eta_i \in C^1(\mathbb{R};\mathbb{R})$, $\Phi \in C^1(\mathbb{R};\mathbb{R})$ such that:

$$\Phi' = f'\eta_i', \quad \text{and} \quad h = \Phi - f\eta_i'. \tag{2.17}$$

An entropy super-solution of (2.2) is defined by replacing in (2.16) η_i with η_d; a non-increasing convex function. An entropy solution is defined as being both entropy sub- and super-solution. In other words, it verifies (2.16) for any convex function $\eta \in C^1(\mathbb{R};\mathbb{R})$.

A well know characterization of the entropy solution is that:

Proposition 2.13 *A function $v \in L^\infty(Q_T)$ is an entropy sub-solution of (2.2) if and only if $\forall k \in \mathbb{R}$, $\phi \in C_0^1(\mathbb{R} \times [0,T); \mathbb{R}_+)$, one has:*

$$\int_{Q_T} \Big[(v(x,t)-k)^+ \phi_t(x,t) + sgn^+(v(x,t)-k)(f(v(x,t)) - f(k))g(x,t)\phi_x(x,t) -$$

$$sgn^+(v(x,t)-k)f(k)g_x(x,t)\phi(x,t) \Big] dxdt + \int_R (v^0(x)-k)^+ \phi(x,0)dx \geq 0, \tag{2.18}$$

Where $a^\pm = \frac{1}{2}(|a| \pm a)$ and $sgn^\pm(x) = \frac{1}{2}(sgn(x) \pm 1)$. An entropy super-solution of (2.2) is defined replacing in (2.18) $(\cdot)^+$, sgn^+ by $(\cdot)^-$, sgn^-.

This characterization can be deduced from (2.16), by using regularizations of the function $(\cdot - k)^+$. Also (2.16) may be obtained from (2.18) by approximating any non-decreasing convex function $\eta_i \in C^1(\mathbb{R};\mathbb{R})$ by a sequence of functions of the form: $\eta_i^{(n)}(\cdot) = \sum_1^n \beta_i^{(n)}(\cdot - k_i^{(n)})^+$, with $\beta_i^{(n)} \geq 0$.

Entropy solution was first introduced by Kružkov [22] as the only physically admissible solution among all weak (distributional) solutions to scalar conservation laws. These weak solutions lack the fact of being unique for it is easy to construct multiple weak solutions to Cauchy problems (2.2), see [25].

Our next definition concerns classical sub-/super-solution to scalar conservation laws. This kind of solutions are shown to be entropy solutions, for the details see lemma 3.3.

Definition 2.14 (Classical solution to scalar conservation laws)
Let $F(x,t,v) = g(x,t)f(v)$ with g, $g_x \in L^\infty_{loc}(Q_T;\mathbb{R})$ and $f \in C^1(\mathbb{R};\mathbb{R})$. A function $v \in W^{1,\infty}(Q_T)$ is said to be a classical sub-solution of (2.2) with $v^0(x) = v(x,0)$ if it satisfies

$$v_t(x,t) + (F(x,t,v(x,t)))_x \leq 0 \quad a.e. \text{ in } \quad Q_T. \tag{2.19}$$

Classical super-solutions are defined by replacing "$\leq$" with "$\geq$" in (2.19), and classical solutions are defined to be both classical sub- and super-solutions.

We move now to some results on entropy solutions depicted from [22].

Theorem 2.15 (Kružkov's Existence Theorem)
Let F, v^0 be given by Definition 2.12, and the following conditions hold:

(E0) $f \in C_b^1(\mathbb{R})$,

(E1) g, $g_x \in C_b(\bar{Q}_T)$,

(E2) $g_{xx} \in C(\bar{Q}_T)$,

then there exists an entropy solution $v \in L^\infty(Q_T)$ of (2.2).

In fact, Kružkov's conditions for existence were given for a general flux function [22, Section 4]. However, in Subsection 5.4 of the same paper, a weak version of these conditions, that can be easily checked in the case $F(x,t,v) = g(x,t)f(v)$ and (**E0**)-(**E1**)-(**E2**), is presented. Furthermore, uniqueness follows from the following comparison principle.

Theorem 2.16 (Comparison Principle)
Let F be given by Definition 2.12 with f satisfying **(E0)**, *and g satisfies,*

(E3) $g \in W^{1,\infty}(\bar{Q}_T)$.

Let $u(x,t)$, $v(x,t) \in L^\infty(Q_T)$ be two entropy sub-/super-solutions of (2.2) with initial data u^0, $v^0 \in L^\infty(\mathbb{R})$. Suppose that,

$$u^0(x) \leq v^0(x) \quad a.e. \ in \quad \mathbb{R},$$

then

$$u(x,t) \leq v(x,t) \quad a.e. \ in \quad \bar{Q}_T.$$

Proof. See Section 6, Appendix. □

It is worth noticing that in [22], the proof of the existence of entropy solutions of (2.2) is made through a parabolic regularization of (2.2) and passing to the limit, with respect to the L^1 convergence on compacts, in a convenient space.

At this stage, we are ready to present a relation that sometimes hold between scalar conservation laws and Hamilton-Jacobi equations in one-dimensional space.

2.3 Entropy-Viscosity relation

Formally, by differentiating (2.1) with respect to x and defining $v = u_x$, we see that (2.1) is equivalent to the scalar conservation law (2.2) with $v^0 = u^0_x$ and the same F. This equivalence of the two problems has been exploited in order to translate some numerical methods for hyperbolic conservation laws to methods for Hamilton-Jacobi equations. Moreover, several proofs were given in the one dimensional case. The usual proof of this relation depends strongly on the known results about existence and uniqueness of the solutions of the two problems together with the convergence of the viscosity method (see [8, 23, 27]). Another proof of this relation could be found in [5] via the definion of viscosity/entropy inequalities, while a direct proof could also be found in [21] using the front tracking method. The case of a Hamiltonian of the form (2.3) is also treated even when $g(x,t)$ is allowed to be discontinuous in the (x,t) plane along a finite number of (possibly intersected) curves, see [29].

In our work, the above stated relation will be successfully used to get some gradient estimates of κ. Although several approaches were given to establish this connection, we will present for the reader's convenience, a proof similar to that given in [8, Theorem 2.2]. For every Hamiltionian/flux function $F = gf$ and every $u^0 \in Lip(\mathbb{R})$, let

$$\mathcal{EV} = \{(\mathbf{V0}), (\mathbf{V1}), (\mathbf{V2}), (\mathbf{E0}), (\mathbf{E1}), (\mathbf{E2}), (\mathbf{E3})\},$$

in other words,

$$\mathcal{EV} = \left| \begin{array}{l} \text{The set of all conditions on } f \text{ and } g \text{ ensuring the} \\ \text{existence and uniqueness of a Lipschitz continuous viscosity} \\ \text{solution } u \in Lip(\bar{Q}_T) \text{ of (2.1), and of an entropy} \\ \text{solution } v \in L^\infty(Q_T) \text{ of (2.2), with } v^0 = u^0_x \in L^\infty(\mathbb{R}). \end{array} \right.$$

Theorem 2.17 (A link between viscosity and entropy solutions)
Let $F = gf$ with $g \in C^2(\bar{Q}_T)$, $u^0 \in Lip(\mathbb{R})$ and $\mathcal{EV}$ satisfied. Then,

$$v = u_x \quad a.e. \ in \quad Q_T.$$

Sketch of the proof. Let $\varepsilon > 0$ and $\delta > 0$. We start the prove by making a parabolic regularization of equation (2.1) and a smooth regularization of u_0 and we solve the following parabolic equation:

$$\begin{cases} u_t^{\varepsilon,\delta} + F(x,t,u_x^{\varepsilon,\delta}) = \epsilon u_{xx}^{\varepsilon,\delta} & \text{in} \quad \mathbb{R}\times(0,T), \\ u^{\varepsilon,\delta}(x,0) = u^{0,\delta}(x) & \text{in} \quad \mathbb{R}. \end{cases} \tag{2.20}$$

For the sake of simplicity, we will denote $u^{\varepsilon,\delta}$ by w and $u^{0,\delta}$ by w^0. Note that the first equation of (2.20) can be viewed as the heat equation with a source term F. Thus, we have:

$$\begin{cases} w_t - \varepsilon w_{xx} = F[w](x,t) & \text{in} \quad Q_T, \\ w(x,0) = w^0 & \text{in} \quad \mathbb{R}, \end{cases} \tag{2.21}$$

with $F[w](x,t) = F(x,t,w_x(x,t))$. From the classical theory of heat equations, since $F[w] \in L^p_{loc}(Q_T)$ and $w^0 \in W^{1,p}_{loc}(\mathbb{R})$, there exists a unique solution w of (2.21) such that

$$w \in W^{2,1}_p(\Omega) \quad \forall \Omega \subset\subset Q_T \ \text{ and } \ 1 < p < \infty.$$

Here the space $W^{2,1}_p(\Omega)$, $p \geq 1$ is the Banach space consisting of all functions $w \in L^p(\Omega)$ having generalized derivatives of the form w_t and w_{xx} in $L^p(\Omega)$. For more details, see [24, Theorem 9.1]. We also notice that the space $W^{2,1}_p(\Omega)$ is continuously injected in the Hölder space $C^{\alpha,\alpha/2}(\Omega)$ for $\alpha = 2 - \frac{3}{p}$ and $p > \frac{3}{2}$, see [24]. We use now a bootstrap argument to increase the regularity of w, taking in each stage, the new regularity of $F[w]$ and the regularity of w^0. Finally, we get that $w \in C^{3,1}(\mathbb{R}\times[0,T))$ (three times continuously differentiable in space and one time continuously differentiable in time). From the maximum principle and the L^p-estimates of the heat equation, see [24, 3], it follows the uniform bound of $u^{\varepsilon,\delta}$ in $W^{1,p}_{loc}(Q_T)$, for $p > 2$. Therefore, we get as $\delta \to 0$ and $\varepsilon \to 0$ that:

$$u^{\varepsilon,\delta} \to u \quad \text{in} \quad C(\mathbb{R}\times[0,T)),$$

with $u(x,0) = u^0$. We now make use of the stability theorem, [2, Théorème 2.3], twice on the equation (2.20) to get that the limit u is the unique viscosity solution of (2.1). Hence, we have for any $\phi \in C_0^\infty(Q_T)$

$$\begin{aligned} \lim_{\varepsilon\to 0,\,\delta\to 0} \int_0^T \int_{\mathbb{R}} u_x^{\varepsilon,\delta}\phi \, dx\, dt &= -\lim_{\varepsilon\to 0,\,\delta\to 0} \int_0^T \int_{\mathbb{R}} u^{\varepsilon,\delta}\phi_x \, dx\, dt \\ &= -\int_0^T \int_{\mathbb{R}} u\phi_x \, dx\, dt = \int_0^T \int_{\mathbb{R}} u_x\phi \, dx\, dt. \end{aligned}$$

The appearance of u_x follows since $u \in Lip(\bar{Q}_T)$. Moreover, as a regular solution, the function $v^{\varepsilon,\delta} = u_x^{\varepsilon,\delta}$ solves the derived problem

$$\begin{cases} v_t^{\varepsilon,\delta} + (F(x,t,v^{\varepsilon,\delta}))_x = \epsilon v_{xx}^{\varepsilon,\delta} & \text{in} \quad \mathbb{R}\times(0,T), \\ v^{\varepsilon,\delta}(x,0) = u_x^{0,\delta}(x) & \text{in} \quad \mathbb{R}, \end{cases} \tag{2.22}$$

and, according to [22, Theorem 4], the sequence $v^{\varepsilon,\delta}$ converge in $L^1_{loc}(\bar{Q}_T)$, as $\varepsilon \to 0$ and $\delta \to 0$, to the entropy solution v of (2.2). Then, for any $\phi \in C_0^\infty(Q_T)$,

$$\lim_{\varepsilon\to 0\,\delta\to 0} \int_0^T \int_{\mathbb{R}} v^{\varepsilon,\delta}\phi \, dx\, dt = \int_0^T \int_{\mathbb{R}} v\phi \, dx\, dt.$$

Consequently,

$$\int_0^T \int_{\mathbb{R}} u_x\phi \, dx\, dt = \int_0^T \int_{\mathbb{R}} v\phi \, dx\, dt,$$

and $u_x = v$ a.e. in Q_T. □

Remark 2.18 *The converse of the previous theorem holds under certain assumptions (see [21, 7]).*

Remark 2.19 *In the multidimensional case this one-to-one correspondence no longer exists, instead the gradient $v = \nabla u$ satisfies formally a non-strict hyperbolic system of conservation laws (see [27, 23]).*

Throughout Sections 3 and 4, ρ will always be the solution of the heat equation (1.5). The properties of the solution of the heat equation with such a regular initial data will be frequently used, we refer the reader to [3, 13] for details.

3 The approximate problem

In this section, we approximate (1.4) and we pose a more restrictive condition (see condition (1.8)) on the gradient of the initial data than of the physicaly relevent one (1.6). We prove a result of existence and uniqueness of this approximate problem, namely Theorem 1.2, and the reader will notice at the end of this section that this restrictive condition is satisfied for all time, and this what cancels the approximation in the structure of (1.4) and returns it to its original one. Finaly we present the proof of Theorem 1.3.

For every $a > 0$, we build up an approximation function $f_a \in C_b^1(\mathbb{R})$ of the function $\frac{1}{x}$ defined by:

$$f_a(x) = \begin{cases} \dfrac{1}{x} & \text{if} \quad x \geq a, \\ \dfrac{2a - x}{a^2 + a^2(x-a)^2} & \text{otherwise.} \end{cases} \tag{3.1}$$

Proposition 3.1 *For any $a > 0$, let f_a be defined by (3.1) and $H \in C^1(\mathbb{R})$ be a scalar-valued function. If*

$$F_a(x,t,u) = -H(\rho_x(x,t))\rho_{xx}(x,t)f_a(u) \tag{3.2}$$

and $\kappa^0 \in Lip(\mathbb{R})$, then the Hamilton-Jacobi equation

$$\begin{cases} \kappa_t + F_a(x,t,\kappa_x) = 0 & \text{in} \quad Q_T, \\ \kappa(x,0) = \kappa^0(x) & \text{in} \quad \mathbb{R}, \end{cases} \tag{3.3}$$

has a unique viscosity solution $\kappa \in Lip(\bar{Q}_T)$.

Proof. The proof is easily concluded from Theorems 2.6, 2.7 and Proposition 2.10, after checking that the conditions (**V0**)-(**V1**)-(**V2**) are satisfied with

$$g(x,t) = -H(\rho_x(x,t))\rho_{xx}(x,t). \tag{3.4}$$

The condition (**V0**) is trivial, while for (**V1**), we just use the fact that H is bounded on compacts and the fact that $|\rho_x(x,t)| \leq ||\rho_x^0||_{L^\infty(\mathbb{R})}$ in $\bar{Q}_T$. For the condition (**V2**), the regularity of ρ and H permits to compute the spatial derivative of g in $\bar{Q}_T$, thus we have:

$$g_x = -(H'(\rho_x)\rho_{xx}^2 + H(\rho_x)\rho_{xxx}).$$

The uniform bound of the spatial derivatives, up to the third order, of the solution of the heat equation, and the boundedness of H' on compacts gives immediately (**V2**). □

In the following proposition, we show a lower-bound estimate for the gradient of κ obtained in Proposition 3.1. It is worth mentioning that a result of lower-bound gradient estimates for first-order Hamilton-Jacobi equations could be found in [26, Theorem 4.2]. However, this result holds for Hamiltonians $F(x,t,u)$ that are convex in the u-variable, using only the viscosity theory techniques. This is not the case here, and in order to obtain our lower-bound estimates, we need to use the viscosity/entropy theory techniques. In particular, we have the following:

Proposition 3.2 *Let $G \in C^3(\mathbb{R};\mathbb{R})$ satisfying the following conditions:*

(G1) $G(x) \geq G(0) > 0,$

(G2) $G'' \geq 0.$

Moreover, let

$$H = GG' \quad \textit{and} \quad 0 < a \leq G(0).$$

If κ^0 satisfies:

$$\kappa_x^0(x) \geq G(\rho_x^0(x)), \quad \textit{a.e. in} \quad \mathbb{R},$$

then the solution κ obtained from Proposition 3.1 satisfies:

$$\kappa_x(x,t) \geq G(\rho_x(x,t)) \quad \textit{a.e. in} \quad \bar{Q}_T. \tag{3.5}$$

In order to prove Proposition 3.2, we first show that $G(\rho_x)$ is an entropy sub-solution of

$$\begin{cases} \omega_t + (F(x,t,\omega))_x = 0 & \text{in} \quad Q_T, \\ \omega(x,0) = \omega^0(x) & \text{in} \quad \mathbb{R}, \end{cases} \tag{3.6}$$

with $w^0 = G(\rho_x^0)$ and F is the same as in (3.2). Before going further, we will pause to prove a lemma which makes it easier to reach our goal.

Lemma 3.3 (Classical sub-solutions are entropy sub-solutions)
Let $v \in W^{1,\infty}(Q_T)$ be a classical sub-solution of (2.2) with $v^0(x) = v(x,0)$, then v is an entropy sub-solution.

Proof. Let η_i, Φ, h and ϕ be given by Definition 2.12. Multiplying inequality (2.19) by $\eta_i^{'}(v)\phi$ does not change its sign. Hence, after developing, we have:

$$\eta_i^{'}(v)v_t\phi + \eta_i^{'}(v)g_x f(v)\phi + \eta_i^{'}(v)gf^{'}(v)v_x\phi \leq 0, \quad \text{a.e. in } \ Q_T, \tag{3.7}$$

and since v is Lipschitz continuous, we use the chain-rule formula together with (2.17) to rewrite (3.7) as:

$$(\eta_i(v))_t\,\phi + g_x f(v)\eta_i^{'}(v)\phi + g(\Phi(v))_x\,\phi \leq 0, \quad \text{a.e. in } \ Q_T. \tag{3.8}$$

Upon integrating (3.8) over Q_T and transferring derivatives with respect to t and x to the test function, we obtain:

$$\int_{Q_T} \Big[\eta_i(v(x,t))\phi_t(x,t) + \Phi(v(x,t))g(x,t)\phi_x(x,t) + \\ h(v(x,t))g_x(x,t)\phi(x,t) \Big] dxdt + \int_{\mathbb{R}} \eta_i(v^0(x))\phi(x,0)dx \geq 0, \tag{3.9}$$

which ends the proof. □

Following same arguments, classical super-solutions are shown to entropy super-solutions. We return now to the function $G(\rho_x)$ and we are ready to show that it is indeed an entropy sub-solution of (3.6). In particular, we have the following:

Lemma 3.4 *The function $G(\rho_x)$ defined on Q_T is a classical sub-solution of (3.6) with initial data $G(\rho_x^0)$, hence an entropy sub-solution.*

Proof of Lemma 3.4. First, it is easily seen that $G(\rho_x) \in W^{1,\infty}(Q_T)$. Define the scalar valued quantity B on Q_T by:

$$B(x,t) = \partial_t(G(\rho_x(x,t))) + \partial_x(F(x,t,G(\rho_x(x,t)))).$$

Since $0 < a \leq G(0)$, we use (G1) to get $f_a(G(\rho_x)) = 1/G(\rho_x)$ and we observe that,

$$\begin{aligned} B &= G^{'}(\rho_x)\rho_{xt} - \partial_x\left(\frac{H(\rho_x)\rho_{xx}}{G(\rho_x)}\right) \\ &= G^{'}(\rho_x)\rho_{xxx} - \left(\frac{G(\rho_x)[H^{'}(\rho_x)\rho_{xx}^2 + H(\rho_x)\rho_{xxx}] - (G^{'}(\rho_x)\rho_{xx}^2 H(\rho_x))}{G^2(\rho_x)}\right) \\ &= \frac{G(\rho_x)\rho_{xxx}(G(\rho_x)G^{'}(\rho_x) - H(\rho_x)) - \rho_{xx}^2(H^{'}(\rho_x)G(\rho_x) - H(\rho_x)G^{'}(\rho_x))}{G^2(\rho_x)} \\ &= -\rho_{xx}^2 G^{''}(\rho_x). \end{aligned}$$

The condition (G2) gives immediately that $B \leq 0$. This proves that $G(\rho_x)$ is a classical sub-solution of equation (3.6) and hence an entropy sub-solution. □

Proof of Proposition 3.2. From the definition of H and the properties of ρ, it is easy to check that $g \in C^2(\bar{Q}_T)$ and that $\mathcal{EV}$ is fully satisfied. Hence, we are in the framework of Theorem 2.17 with $u^0 = \kappa^0$. This theorem gives that κ_x is the unique entropy solution of (3.6) with $w^0 = \kappa_x^0$. Moreover, by the previous lemma, $G(\rho_x)$ is an entropy sub-solution of (3.6). Since

$$\kappa_x^0 \geq G(\rho_x^0), \quad \text{a.e. in} \quad \mathbb{R},$$

we can apply the Comparison Theorem 2.16 to get the desired result. $\square$

It is worth notable here that we do not know how to obtain the lower-bound on the spatial gradient κ_x using the viscosity framework directly. However, for the case of the upper-bound, we can do so (see Remark 4.1). At this stage, fix some $\epsilon > 0$, and let

$$G^\epsilon(x) = \sqrt{x^2 + \epsilon^2} \quad \text{and} \quad a = G^\epsilon(0) = \epsilon.$$

It is clear that $G^\epsilon(x)$ satisfies the conditions (G1)-(G2) with

$$H^\epsilon(x) = x,$$

and the Hamiltonian F from (3.2) takes now the following shape:

$$F_\epsilon(x,t,u) = -\rho_x(x,t)\rho_{xx}(x,t)f_\epsilon(u). \tag{3.10}$$

Moreover, we have the following corollary which is is an immediate consequence of Propositions 3.1 and 3.2.

Corollary 3.5 *There exists a unique viscosity solution $\kappa \in Lip(\bar{Q}_T)$ of*

$$\begin{cases} \kappa_t + F_\epsilon(x,t,\kappa_x) = 0 & \text{in} \quad Q_T, \\ \kappa(x,0) = \kappa^0 \in Lip(\mathbb{R}) & \text{in} \quad \mathbb{R}, \end{cases} \tag{3.11}$$

with κ_x^0 satisfies:

$$\kappa_x^0 \geq \sqrt{(\rho_x^0)^2 + \epsilon^2} \qquad a.e. \ \ in \quad \mathbb{R}. \tag{3.12}$$

Moreover, this solution κ satisfies:

$$\kappa_x \geq \sqrt{\rho_x^2 + \epsilon^2} \qquad a.e. \ \ in \quad \bar{Q}_T. \tag{3.13}$$

The following lamma will be used in the proof of Theorem 1.2.

Lemma 3.6 *Let $\bar{c}$ be an arbitrary real constant and take $\psi \in Lip(\mathbb{R};\mathbb{R})$ satisfying:*

$$\psi_x \geq \bar{c} \quad a.e. \ \ in \quad \mathbb{R}.$$

If $\zeta \in C^1(\mathbb{R};\mathbb{R})$ is such that $\psi - \zeta$ has a local maximum or local minimum at some point $x_0 \in \mathbb{R}$, then

$$\zeta_x(x_0) \geq \bar{c}.$$

Proof. Suppose that $\psi - \zeta$ has a local minimum at the point x_0; this ensures the existence of a certain $r > 0$ such that

$$(\psi - \zeta)(x) \geq (\psi - \zeta)(x_0) \quad \forall x; \ |x - x_0| < r.$$

We argue by contradiction. Assuming $\zeta_x(x_0) < \bar{c}$ leads, from the continuity of ζ_x, to the existence of $r' \in (0, r)$ such that

$$\zeta_x(x) < \bar{c} \quad \forall x;\ |x - x_0| < r'. \tag{3.14}$$

Let y_0 be a point such that $|y_0 - x_0| < r'$ and $y_0 < x_0$. Reexpressing (3.14), we get

$$(\zeta - \bar{c}x)_x(x) < 0 \quad \forall x \in (y_0, x_0),$$

and hence

$$\int_{y_0}^{x_0} [(\psi - \bar{c}x)_x(x) - (\zeta - \bar{c}x)_x(x)]dx > 0,$$

which implies that

$$(\psi - \zeta)(x_0) > (\psi - \zeta)(y_0),$$

and hence a contradiction. We remark that the case of a local maximum can be treated in a similar way. □

Now, we are ready to present the proofs of the first two theorems announced in Section 1.

Proof of Theorem 1.2. Let $\kappa \in Lip(\bar{Q}_T)$ be the solution of (3.11) obtained in Corollary 3.5. Let us show that it is the unique viscosity solution of (1.4) among those verifying (3.13). To do this, we consider a test function $\phi \in C^1(Q_T)$ such that $\kappa - \phi$ has a local minimum at some point $(x_0, t_0) \in Q_T$. Proposition 2.10, together with inequality (3.13) gives that

$$\kappa(., t_0) \in Lip(\mathbb{R}) \quad \text{and} \quad \kappa_x(., t_0) \geq \epsilon \ \text{ a.e. in } \ \mathbb{R}.$$

We make use of Lemma 3.6 with $\psi(.) = \kappa(., t_0)$ and $\zeta(.) = \phi(., t_0)$ to get

$$\phi_x(x_0, t_0) \geq \epsilon. \tag{3.15}$$

Since κ is a viscosity super-solution of

$$\kappa_t - f_\epsilon(\kappa_x)\rho_x\rho_{xx} = 0 \quad \text{in} \quad Q_T,$$

we have

$$\phi_t(x_0, t_0) - f_\epsilon(\phi_x(x_0, t_0))\rho_x(x_0, t_0)\rho_{xx}(x_0, t_0) \geq 0.$$

However, from (3.15), we get

$$\phi_t(x_0, t_0)\phi_x(x_0, t_0) - \rho_x(x_0, t_0)\rho_{xx}(x_0, t_0) \geq 0,$$

and hence κ is a viscosity super-solution of

$$\kappa_t\kappa_x = \rho_x\rho_{xx} \quad \text{in} \quad Q_T.$$

In the same way, we can show that κ is a viscosity sub-solution of the above equation and hence a viscosity solution. The uniqueness of this solution comes from the uniqueness of the viscosity solution of (3.11) by reversing the above reasoning. □

Remark 3.7 *Notice that the first equation of (1.4) can be viewed as a Hamilton-Jacobi equation of the type*

$$F(X, \nabla\kappa) = 0 \quad in \quad Q_T,$$

where $F : Q_T \times \mathbb{R}^2 \mapsto \mathbb{R}$ *defined by:*

$$F(X, p) = p_1 p_2 - \rho_x(X)\rho_{xx}(X),$$

with $X = (x, t)$ *and* $p = (p_1, p_2)$.

Proof of Theorem 1.3. Let $\theta = \kappa_x$. By Theorem 2.17, θ is the unique entropy solution of

$$\begin{cases} \theta_t = (\rho_x \rho_{xx} f_\epsilon(\theta))_x & \text{in} \quad Q_T, \\ \theta(x, 0) = \theta^0(x) & \text{in} \quad \mathbb{R}, \end{cases}$$

with

$$\theta^0(x) = \kappa_x^0(x) \geq \sqrt{(\rho_x^0)^2 + \epsilon^2}, \quad \text{a.e. in} \quad \mathbb{R}.$$

Moreover, from Corollary 3.5, we have

$$\theta \geq \sqrt{\rho_x^2 + \epsilon^2} \quad \text{a.e. in} \quad \bar{Q}_T,$$

from which we deduce that $f_\epsilon(\theta) = \frac{1}{\theta}$ and hence our theorem holds. □

4 Proof of Theorem 1.6

We turn our attention now to Theorem 1.6. Let $0 < \epsilon < 1$ be a fixed constant and take

$$\kappa^{0,\epsilon}(x) = \kappa^0(x) + \epsilon x. \tag{4.1}$$

It is easy to check that the function $\kappa^{0,\epsilon}$ belongs to $Lip(\mathbb{R})$, and by condition (1.6) we get for a.e. $x \in \mathbb{R}$,

$$\begin{aligned} \kappa_x^{0,\epsilon}(x) &= \kappa_x^0(x) + \epsilon, \\ &\geq \sqrt{(\rho_x^0(x))^2 + \epsilon^2}. \end{aligned}$$

From Theorem 1.2, there exists a family of viscosity solutions $\kappa^\epsilon \in Lip(\bar{Q}_T)$ to the initial value problem (1.4) that satisfy:

$$\kappa_x^\epsilon \geq \sqrt{\rho_x^2 + \epsilon^2} \quad \text{a.e.} \quad \text{in} \quad \bar{Q}_T.$$

We will try to extract a subsequence of κ^ϵ that converges, in a suitable space, to the desired solution

4.1 Gradient estimates.

Uniform bounds for the space-time gradients of κ^ϵ will play an essential role in the determination of our subsequence.

I. ϵ-uniform upper-bound for κ_t^ϵ.
Starting with the time gradient, we have for a.e. $(x,t) \in Q_T$:

$$\kappa_t^\epsilon(x,t)\kappa_x^\epsilon(x,t) = \rho_x(x,t)\rho_{xx}(x,t), \tag{4.2}$$

and

$$\kappa_x^\epsilon(x,t) \geq \sqrt{\rho_x^2(x,t)+\epsilon^2} > 0 \quad \text{a.e. in} \quad \bar{Q}_T. \tag{4.3}$$

If $\rho_x(x,t) = 0$ for some Lebesgue point (x,t) of κ_x^ϵ and κ_t^ϵ, it follows from (4.2) and (4.3) that $\kappa_t^\epsilon(x,t) = 0$. Otherwise, and since by (4.3) $\kappa_x^\epsilon \geq |\rho_x|$, we conclude that:

$$|\kappa_t^\epsilon| \leq ||\rho_{xx}^0||_{L^\infty(\mathbb{R})} \quad \text{a.e. in} \quad Q_T, \tag{4.4}$$

and hence we obtain an ϵ-uniform bound of κ_t^ϵ.

For the space gradient, we argue in a slightly different way. The key point for obtaining the uniform bound of κ_t^ϵ was the minoration of κ_x^ϵ by $|\rho_x|$ so, roughly speaking, if we want to follow the same previous steps using the symmetry of (4.2) in κ_t^ϵ and κ_x^ϵ, one should also have an appropriate minoration of $|\kappa_t^\epsilon|$ by a well controlled function which no longer exists.

II. Formal calculus and best candidate.
We seek to find the best candidate to be an upper-bound of κ_x^ϵ. For this reason, we regard formally what is happening at the maximum of κ_x^ϵ. Dividing both sides of (4.2) by κ_x^ϵ and differentiating with respect to the spatial variable, we get:

$$\kappa_{xt}^\epsilon = \frac{\rho_{xx}^2 + \rho_x\rho_{xxx}}{\kappa_x^\epsilon} - \frac{\kappa_{xx}^\epsilon\rho_x\rho_{xx}}{(\kappa_x^\epsilon)^2}. \tag{4.5}$$

Notice that $\kappa_{xx}^\epsilon = 0$ at the maximum of κ_x^ϵ. Multiplying equality (4.5) by κ_x^ϵ and integrating between 0 and t, we obtain:

$$\int_0^t \frac{d}{d\tau}\left(\frac{1}{2}(\kappa_x^\epsilon)^2\right)d\tau = \int_0^t (\rho_{xx}^2 + \rho_x\rho_{xxx})d\tau,$$

then

$$(\kappa_x^\epsilon(x,t))^2 = (\kappa_x^{0,\epsilon}(x))^2 + 2\int_0^t (\rho_{xx}^2(x,t) + \rho_x(x,t)\rho_{xxx}(x,t))d\tau,$$

and hence,

$$|\kappa_x^\epsilon| \leq \sqrt{2c_1 t + c_2},$$

where

$$c_1 = ||(\rho_{xx}^0)^2||_{L^\infty(\mathbb{R})} + ||\rho_x^0||_{L^\infty(\mathbb{R})}||\rho_{xxx}^0||_{L^\infty(\mathbb{R})},$$

and

$$c_2 = (||\kappa_x^0||_{L^\infty(\mathbb{R})} + 1)^2.$$

The reason of taking c_2 as above easily follows since $\kappa_x^{0,\epsilon} = \kappa_x^0 + \epsilon$, by taking ϵ small enough, namely less than 1.

III. ϵ-uniform upper-bound for κ_x^ϵ.
Define the function S by:

$$S(x,t) = \sqrt{2c_1 t + c_2}.$$

Let us show that S is an entropy super-solution of (3.6) with F given by (3.10) and $w^0(x) = S(x,0)$. Indeed, it remark that $S \in W^{1,\infty}(Q_T)$, and we know that for every $(x,t) \in Q_T$ we have,

$$S(x,t) \geq \sqrt{c_2} = ||\kappa_x^0||_{L^\infty(\mathbb{R})} + 1 \geq \epsilon,$$

then

$$f_\epsilon(S(x,t)) = \frac{1}{S(x,t)} \quad \forall (x,t) \in Q_T. \tag{4.6}$$

The regularity of the function S permits to inject it directly into the first equation of (3.6). Therefore, using (4.6), we have

$$\begin{aligned} S_t - \left(\frac{\rho_x \rho_{xx}}{S}\right)_x &= \frac{c_1}{\sqrt{2c_1 t + c_2}} - \frac{\rho_{xx}^2 + \rho_x \rho_{xxx}}{\sqrt{2c_1 t + c_2}}, \\ &= \frac{c_1 - (\rho_{xx}^2 + \rho_x \rho_{xxx})}{\sqrt{2c_1 t + c_2}}, \\ &\geq 0, \end{aligned}$$

which proves, by Lemma 3.3, that S is an entropy super-solution of (3.6). From the discussion of the proof of Proposition 3.2, we know that κ_x^ϵ is an entropy solution of (3.6) hence an entropy sub-solution. Since for $\epsilon < 1$ and a.e. $x \in \mathbb{R}$, we have,

$$\begin{aligned} \kappa_x^{0,\epsilon}(x) &= \kappa_x^0(x) + \epsilon, \\ &\leq ||\kappa_x^0||_{L^\infty(\mathbb{R})} + 1, \\ &\leq \sqrt{c_2} = S(x,0), \end{aligned}$$

then we can use the Comparison Theorem 2.16 of scalar conservation laws to obtain:

$$\kappa_x^\epsilon(x,t) \leq \sqrt{c_1 t + c_2} \leq \sqrt{c_1 T + c_2} \quad \text{a.e. in} \quad \bar{Q}_T, \tag{4.7}$$

and hence we get an ϵ-uniform bound for κ_x^ϵ.

Remark 4.1 *We were able to obtain this ϵ-uniform upper-bound of κ_x^ϵ by using the viscosity theory techniques. In fact, we claim that $\zeta^{1,\epsilon}(x,y,t) = \kappa^\epsilon(x,t) - \kappa^\epsilon(y,t)$ and $\zeta^2(x,y,t) = (x-y)S(t)$ are two viscosity sub-/super-solutions of the following Hamilton-Jacobi equation:*

$$\frac{\partial w}{\partial t} = F(x,t,w_x) - F(y,t,-w_y) \quad \text{in} \quad \mathcal{D} = \{(x,y,t);\ x > y \ \text{ and } \ t > 0\}$$

with initial data $\zeta^{1,\epsilon}(x,y,0) = \kappa^{0,\epsilon}(x) - \kappa^{0,\epsilon}(y)$ and $\zeta^2(x,y,0) = (x-y)S(0)$ respectively. Here F is given by (3.10). The claim is easy for ζ^2, and we refer to [9] when κ^ϵ is a continuous viscosity solution of (3.11). We also notice that: $\zeta^{1,\epsilon}(x,y,0) \leq \zeta^2(x,y,0)$ $\forall (x,y,0) \in \mathcal{D}$, and $\zeta^{1,\epsilon}(x,y,t) = \zeta^2(x,y,t) = 0$ for $x = y$, $t \geq 0$. Moreover, since $\zeta^{1,\epsilon}$ and ζ^2 are continuous functions, we use the comparison principle of viscosity solutions (see [2]) to obtain:

$$\kappa^\epsilon(x,t) - \kappa^\epsilon(y,t) \leq (x-y)S(t) \quad \forall (x,y,t) \in \bar{\mathcal{D}},$$

hence, the estimate (4.7) holds.

4.2 Local boundedness in $W^{1,\infty}$.

We now show that the family $(\kappa^\epsilon)_{0<\epsilon<1}$ is locally bounded in $W^{1,\infty}(Q_T)$. Let $K \subset\subset Q_T$ be a compactly contained subset of Q_T, and $(x,t) \in K$. Since κ^ϵ is Lipschitz continuous, we can write,

$$|\kappa^\epsilon(x,t) - \kappa^{0,\epsilon}(0)| \leq C^\epsilon_{lip}\, |(x,t)|,$$

where C^ϵ_{lip} is the Lipschitz constant of κ^ϵ which is independent of ϵ from the previous estimates, namely (4.4) and (4.7). Call this constant $\bar{C}$. From the definition of $\kappa^{0,\epsilon}(0)$ given by (4.1), it follows that,

$$\begin{aligned} |\kappa^\epsilon(x,t)| &\leq \bar{C}\, |(x,t)| + |\kappa^0(0)|, \\ &\leq \bar{C} \max_{(y,\tau)\in K} |(y,\tau)| + |\kappa^0(0)|, \end{aligned}$$

which is finite since K is bounded and hence, $(\kappa^\epsilon)_{0<\epsilon<1}$ is uniformly bounded in $C(K)$. This, together with the uniform gradient estimates, gives the local boundedness of κ^ϵ in $W^{1,\infty}(\bar{Q}_T)$.

4.3 Proof of theorem 1.6

At this point, we have the necessary tools to give the proof of Theorem 1.6. We first recall that κ^ϵ is a viscosity solution of an equation of the type (4.2), with a Hamiltonian independent of ϵ (see Remark 3.7) and $\kappa^{0,\epsilon} \to \kappa^0$ locally uniformly in $\mathbb{R}$. By Ascoli's Theorem, there is a subsequence, called again κ^ϵ, that converges to $\kappa \in Lip(\bar{Q}_T)$ locally uniformly, and by the stability theorem (see [2, Theorem 2.3]), κ is a viscosity solution of the initial value problem

$$\begin{cases} \kappa_t \kappa_x = \rho_x \rho_{xx} & \text{in} \quad Q_T, \\ \kappa(x,0) = \kappa^0(x) & \text{in} \quad \mathbb{R}. \end{cases} \tag{4.8}$$

To end the proof, we still have to show the inequality

$$\kappa_x \geq |\rho_x| \quad \text{a.e. in} \quad \bar{Q}_T.$$

Again by Theorem 1.2, our κ^ϵ verifies for a.e. $(x,t) \in \bar{Q}_T$,

$$\begin{aligned} \kappa^\epsilon_x(x,t) &\geq \sqrt{\rho_x^2(x,t) + \epsilon^2} \\ &> |\rho_x(x,t)|, \end{aligned}$$

then for (y,t), $(x,t) \in Q_T$ close enough, with ρ_x a continuous function, the following inequality hold

$$\frac{\kappa^\epsilon(y,t) - \kappa^\epsilon(x,t)}{x-y} > |\rho_x(x,t)|.$$

Using the local uniform convergence of κ^ϵ to κ, we get a similar inequality with κ^ϵ replaced with κ and hence

$$\kappa_x \geq |\rho_x| \quad \text{a.e. in} \quad \bar{Q}_T.$$

$\square$

5 Problem with boundary conditions

In this part of the paper, we deal with the same problem structure but with boundary conditions of the Dirichlet type. This sort of boundary conditions arises naturally in a special model of dislocation dynamics and will be explained in the following subsection. Our notations are kept untouched; the terms θ^+, θ^-, ρ and κ still have the same physical meaning, while the domain is changed into the open and bounded interval

$$I = (0, 1),$$

of the real line. Although this problem seems to be an independent one, we will try to benefit the results of the previous sections by considering a trick of extension and restriction, in order to apply some of the previous results of the whole space problem.

5.1 Brief physical motivation

To illustrate some physical motivations of the boundary value problem, we consider a constrained channel deforming in simple shear (see [18]). A channel of width 1 in the x-direction and infinite extension in the y-direction is bounded by walls that are impenetrable for dislocations (see Figure 1). The motion of the positive and negative dislocations corresponds to the x-direction. This is a simplified version of a system studied by Van der Giessen and coworkers

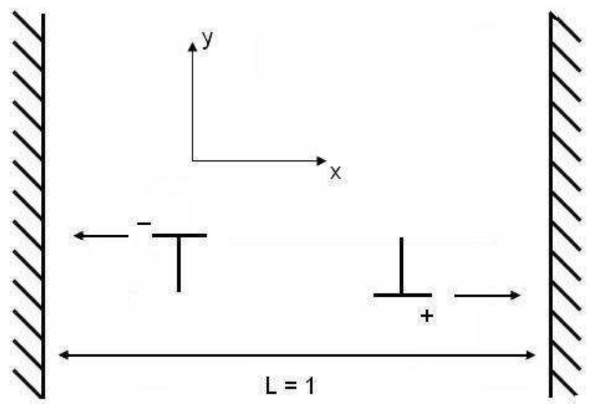

Figure 1: Geometry of a constrained channel

[6], where the simplifications stem from the fact that:

- only a single slip system is assumed to be active, such that reactions between dislocations of different type need not be considered;
- the boundary conditions reduce to "no flux" conditions for the dislocation fluxes at the boundary walls.

The mathematical formulation of this model, as expressed in [18], is the system (1.2) posed on $I \times (0,T)$:

$$\begin{cases} \partial_t \theta^+(x,t) - \partial_x \left(\theta^+(x,t) \left(\dfrac{\theta_x^+(x,t) - \theta_x^-(x,t)}{\theta^+(x,t) + \theta^-(x,t)} \right) \right) = 0, \\ \partial_t \theta^-(x,t) + \partial_x \left(\theta^-(x,t) \left(\dfrac{\theta_x^+(x,t) - \theta_x^-(x,t)}{\theta^+(x,t) + \theta^-(x,t)} \right) \right) = 0. \end{cases} \tag{5.1}$$

To formulate heuristically the boundary conditions at the walls located at $x = 0$ and $x = 1$, we note that the dislocation fluxes at the walls must be zero, which requires that

$$\overbrace{\partial_x(\theta^+ - \theta^-)}^{\Phi} = 0, \qquad \text{at} \qquad x \in \{0,1\}. \tag{5.2}$$

Rewriting system (5.1) in a special integrated form in terms of ρ, κ and Φ, we get

$$\begin{cases} \kappa_t = (\rho_x/\kappa_x)\Phi, \\ \rho_t = \Phi. \end{cases} \tag{5.3}$$

Using (5.2) into the system (5.3), we can formally deduce that ρ and κ are constants along the boundary walls. Therefore, the remaining of this paper focuses attention on the study of the following coupled Dirichlet boundary problems:

$$\begin{cases} \rho_t = \rho_{xx}, & \text{in} \quad I \times (0,\infty), \\ \rho(x,0) = \rho^0(x), & \text{in} \quad I, \\ \rho(0,t) = \rho(1,t) = 0, & \forall t \in [0,\infty), \end{cases} \tag{5.4}$$

and

$$\begin{cases} \kappa_t \kappa_x = \rho_t \rho_x, & \text{in} \quad I \times (0,T), \\ \kappa(x,0) = \kappa^0(x), & \text{in} \quad I, \\ \kappa(0,t) = \kappa(0,0) \quad \text{and} \quad \kappa(1,t) = \kappa(1,0), & \forall t \in [0,T]. \end{cases} \tag{5.5}$$

Denote I_T by:

$$I_T = I \times (0,T).$$

There are two natural assumptions concerning ρ^0 and κ^0, the first one is again the positivity of the dislocation densities θ^+ and θ^- at the initial time, which yields to the following condition:

$$\kappa_x^0 \geq |\rho_x^0|, \tag{5.6}$$

and the second one has to do with the balance of the physical model that starts with the same number of positive and negative dislocations. In other words, if n^+ and n^- are the total number of positive and negative dislocations respectively at $t = 0$ then:

$$\begin{aligned} \rho^0(1) - \rho^0(0) &= \int_0^1 \rho_x^0(x)\,dx, \\ &= \int_0^1 (\theta^+(x,0) - \theta^-(x,0))\,dx, \\ &= n^+ - n^- = 0, \end{aligned}$$

this shows that $\rho^0(1) = \rho^0(0)$ and this is what appears in (5.4). Up to now, formal relations between the initial conditions are only expressed. Whereas, required regularity, together with the announcement of the main results will be stated in the next subsection.

5.2 Statement of the main results on a bounded interval

From now on, the reader should not be confused with the term ρ that will always be the unique solution of the classical heat equation (5.4). The two main theorems that we are going to prove are:

Theorem 5.1 (Existence and uniqueness of a viscosity solution)
Let $T > 0$ and $\epsilon > 0$ be two constants. Take $\kappa^0 \in Lip(I)$ and $\rho^0 \in C_0^\infty(I)$ satisfying:

$$\kappa_x^0 \geq G(\rho_x^0) \quad a.e. \ in \quad I,$$

where

$$G(x) = \sqrt{x^2 + \epsilon^2},$$

then there exists a viscosity solution $\kappa \in Lip(\bar{I}_T)$ of (5.5), unique among those satisfying:

$$\kappa_x \geq G(\rho_x) \quad a.e. \ in \ \ \bar{I}_T. \tag{5.7}$$

Theorem 5.2 (Existence of a viscosity solution)
Let $T > 0$ and $\kappa^0 \in Lip(I)$. Under the condition (5.6) satisfied a.e. in I, there exists a viscosity solution $\kappa \in Lip(\bar{I}_T)$ of (5.5) satisfying:

$$\kappa_x \geq |\rho_x|, \quad a.e. \ in \quad \bar{I}_T.$$

5.3 Preliminary results

Before proceeding with the proof of our theorems, we have to introduce some essential tools that are the core of the "extension and restriction" method that we are going to use.

Extension of ρ over $\mathbb{R} \times [0, T]$.
Consider the function $\hat{\rho}$ defined on $[0, 2] \times [0, T]$ by

$$\hat{\rho}(x,t) = \begin{cases} \rho(x,t) & \text{if} \quad (x,t) \in \bar{I}_T, \\ -\rho(2-x,t) & \text{otherwise}, \end{cases} \tag{5.8}$$

this is just a C^1 antisymmetry of ρ with respect to the line $x = 1$. The continuation of $\hat{\rho}$ to $\mathbb{R} \times [0, T]$ is made by spatial periodicity of period 2. A simple computation yields, for $(x,t) \in (1,2) \times (0,T)$:

$$\hat{\rho}_t(x,t) = -\rho_t(2-x,t) \quad \text{and} \quad \hat{\rho}_{xx}(x,t) = -\rho_{xx}(2-x,t),$$

and hence it is easy to verify that $\hat{\rho}\,|_{[1,2]\times[0,T]}$ solves (5.4) with I replaced with the interval $(1,2)$ and ρ^0 replaced with its symmetry with respect to the point $x = 1$; the boundary conditions are unchanged and the regularity of the initial condition is conserved. To be more precise, we write down some useful properties of $\hat{\rho}$.

Regularity properties of $\hat{\rho}$.

Let r and s are two positive integers such that $s \leq 2$. From the construction of $\hat{\rho}$ and the above discussion, we get the following:

$$\begin{aligned}
&\text{i) } \hat{\rho}_t \text{ and } \hat{\rho}_x \text{ are in } C(\mathbb{R} \times [0,T]),\\
&\text{ii) } \hat{\rho} = 0 \text{ on } \ \mathbb{Z} \times [0,T],\\
&\text{iii) } \hat{\rho}_t = \hat{\rho}_{xx} \text{ on } (\mathbb{R} \setminus \mathbb{Z}) \times (0,T),\\
&\text{iv) } ||\partial_t^r \partial_x^s \hat{\rho}(.,t)||_{L^\infty(\mathbb{R})} \leq C, \ \forall t \in [0,T],
\end{aligned} \tag{5.9}$$

Where C is a certain constant and the limitation $s \leq 2$ comes from the spatial antisymmetry. These conditions are valid thanks to the way of construction of the function $\hat{\rho}$ and to the maximum principle of the solution of the heat equation on bounded domains (see [3, 13]).

Let

$$\hat{g}(x,t) = -\hat{\rho}_t(x,t)\hat{\rho}_x(x,t). \tag{5.10}$$

From the above discussion, it is worth noticing that this function is a Lipschitz continuous function in the x-variable.

The following three lemmas will be used in the proof of Theorem 5.1.

Lemma 5.3 (Entropy sub-solution)
The function $G(\hat{\rho}_x)$ is an entropy sub-solution of

$$\begin{cases} w_t + (\hat{g} f_\epsilon(w))_x = 0, & \textit{in} \quad Q_T,\\ w(x,0) = w^0(x) & \textit{in} \quad \mathbb{R}, \end{cases} \tag{5.11}$$

where f_ϵ is given by (3.1), and $w^0(x) = G(\hat{\rho}_x(x,0))$.

Proof. Similar to Lemma 3.4. □

Lemma 5.4 (Differentiability property)
Let $u(x,t)$ be a differentiable function with respect to (x,t) a.e. in Q_T. Define the set M by:

$$M = \{x \in \mathbb{R}; \ \ u \textit{ is differentiable a.e. in } \{x\} \times (0,T)\},$$

then M is dense in $\mathbb{R}$.

Proof. Define $\mathcal{L}^n$, $n \in \mathbb{N}$ to be the Lebesgue n-dimensional measure. Let $N \subset Q_T$ be the set defined by:

$$N = \{(x,t) \in Q_T; \ \ u \text{ is not differentiable on } (x,t)\},$$

and let $\mathbb{I}_N$ be the characteristic function of the set N. Since $\mathcal{L}^2(N) = 0$, we can write,

$$\int_{Q_T} \mathbb{I}_N(x,t)dxdt = 0.$$

Using Fubini's theorem we get

$$\int_{\mathbb{R}} g(x)dx = 0, \quad \text{with} \quad g(x) = \left(\int_0^T \mathbb{I}_N(x,t)dt\right) \geq 0,$$

then

$$g = 0 \quad \text{a.e. in} \quad \mathbb{R}$$

and consequently

$$J = \{x;\ g(x) \neq 0\} \quad \text{verifies} \quad \mathcal{L}^1(J) = 0.$$

In other words,

$$\forall x \in \mathbb{R} \setminus J,\ \ u(x, \cdot) \text{ is differentiable with respect to } (x,t) \text{ a.e. in } (0,T),$$

hence $\mathbb{R} \setminus J \subset M$ which implies our lemma. □

In the next lemma, we show a lower-bound estimate for the gradient of $\hat{\kappa}$ analogue to (5.7). This was previously done for κ_x in the case where g is a twice continuously differentiable function using mainly Theorems 2.17 and 2.16. Here, the way of extending the function ρ over $\bar{Q}_T$ makes $\hat{g}$ loose some of the regularity stated in Theorem 2.17. However, the following lemma shows that a similar result holds in the case $\hat{g} \in W^{1,\infty}(\bar{Q}_T)$.

Lemma 5.5 *The function $\hat{\kappa}_x \in L^\infty(Q_T)$ is an entropy solution of (5.11) with initial data $w^0 = \hat{\kappa}^0_x \in L^\infty(\mathbb{R})$.*

Proof of Lemma 5.5. Let $\tilde{g}$ be an extension of the function $\hat{g}$ on $\mathbb{R}^2$ defined by:

$$\tilde{g}(x,t) = \begin{cases} \hat{g}(x,t) & \text{if} \quad (x,t) \in \bar{Q}_T, \\ \hat{g}(x,T) & \text{if} \quad t > T, \\ \hat{g}(x,0) & \text{if} \quad t < 0. \end{cases} \tag{5.12}$$

Consider a sequence of mollifiers ξ^n in $\mathbb{R}^2$ and let $\tilde{g}^n = \tilde{g} * \xi^n$. Remark that, from the standard properties of the mollifier sequence, we have $\tilde{g}^n \in C^\infty(\mathbb{R}^2)$ and:

$$\tilde{g}^n \to \hat{g} \ \text{ uniformly on compacts in } \ \bar{Q}_T, \tag{5.13}$$

and

$$\tilde{g}^n_x \to \hat{g}_x \ \text{ in } \ L^p_{loc}(Q_T), \quad 1 \leq p < \infty, \tag{5.14}$$

together with the following estimates:

$$||\partial^r_t \partial^s_x \tilde{g}^n||_{L^\infty(\bar{Q}_T)} \leq ||\partial^r_t \partial^s_x \hat{g}||_{L^\infty(\bar{Q}_T)} \ \text{ for } \ r,s \in \mathbb{N},\ r+s \leq 1. \tag{5.15}$$

Now, take again the Hamilton-Jacobi equation (5.27) with $\hat{g}$ replaced with $\tilde{g}^n$:

$$\begin{cases} u_t + \tilde{g}^n f_\epsilon(u_x) = 0 & \text{in} \quad \mathbb{R} \times (0,T), \\ u(x,0) = \hat{\kappa}^0(x) & \text{in} \quad \mathbb{R}, \end{cases} \tag{5.16}$$

and notice that the above properties of the function $\tilde{g}^n$ enters us into the framework of Theorem 2.17. Thus, we have a unique viscosity solution $\tilde{\kappa}^n \in Lip(\bar{Q}_T)$ of (5.16) with initial condition $\hat{\kappa}^0$ whose spatial derivative $\tilde{\kappa}^n_x \in L^\infty(Q_T)$ is an entropy solution of the corresponding derived equation with initial data $\hat{\kappa}^0_x$. From Remark 2.11 and (5.15), we deduce that the sequence $(\tilde{\kappa}^n)_{n\geq 1}$ is locally uniformly bounded in $W^{1,\infty}(\bar{Q}_T)$ and that:

$$||\tilde{\kappa}^n_x||_{L^\infty(Q_T)} \leq ||\hat{\kappa}^0_x||_{L^\infty(\mathbb{R})} + T||\hat{g}_x||_{L^\infty(Q_T)}||f_\epsilon||_{L^\infty(\mathbb{R})}. \tag{5.17}$$

Moreover, from (5.13), we use again the Stability Theorem of viscosity solutions [2, Theorem 2.3], and we obtain:

$$\tilde{\kappa}^n \to \hat{\kappa} \text{ locally uniformly in } \bar{Q}_T. \tag{5.18}$$

Back to the entropy solution, we write down the entropy inequality (see Definition 2.12) satisfied by $\tilde{\kappa}_x^n$:

$$\int_{Q_T} \Big(\eta(\tilde{\kappa}_x^n)\phi_t + \Phi(\tilde{\kappa}_x^n)\tilde{g}^n\phi_x + h(\tilde{\kappa}_x^n)\tilde{g}_x^n\phi \Big) dxdt + \int_{\mathbb{R}} \eta(\hat{\kappa}_x^0)\phi(x,0)dx \geq 0, \tag{5.19}$$

where η, Φ, h and ϕ are given by Definition 2.12. Taking (5.17) into consideration, we use a property of bounded sequences in $L^\infty(Q_T)$ (see [14, Proposition 3]) that guarantees the existence of a subsequence (call it again $\tilde{\kappa}_x^n$) so that, for any function $\psi \in C(\mathbb{R};\mathbb{R})$,

$$\psi(\tilde{\kappa}_x^n) \to U_\psi \ \text{ weak}-\star \ \text{ in } \ L^\infty(Q_T). \tag{5.20}$$

Furthermore, there exists $\mu \in L^\infty(Q_T \times (0,1))$ such that:

$$\int_0^1 \psi(\mu(x,t,\alpha))d\alpha = U_\psi(x,t), \ \text{ for a.e. } \ (x,t) \in Q_T. \tag{5.21}$$

Applying (5.20) with ψ replaced with η, Φ and h respectively, and using (5.21), we get:

$$\begin{cases} \eta(\tilde{\kappa}_x^n(.)) \to \displaystyle\int_0^1 \eta(\mu(.,\alpha))d\alpha & \text{weak}-\star & \text{in } \ L^\infty(Q_T), \\ \Phi(\tilde{\kappa}_x^n(.)) \to \displaystyle\int_0^1 \Phi(\mu(.,\alpha))d\alpha & \text{weak}-\star & \text{in } \ L^\infty(Q_T), \\ h(\tilde{\kappa}_x^n(.)) \to \displaystyle\int_0^1 h(\mu(.,\alpha))d\alpha & \text{weak}-\star & \text{in } \ L^\infty(Q_T). \end{cases} \tag{5.22}$$

This, together with (5.13), (5.14) permits to pass to the limit in (5.19) in the distributional sense, hence we get:

$$\begin{aligned} \int_{Q_T}\int_0^1 \Big(\eta(\mu(.,\alpha))\phi_t + \Phi(\mu(.,\alpha))\hat{g}\phi_x + h(\mu(.,\alpha))\hat{g}_x\phi \Big) \, dxdtd\alpha + \\ \int_{\mathbb{R}} \eta(\hat{\kappa}_x^0)\phi(x,0)dx \geq 0. \end{aligned} \tag{5.23}$$

In [14, Theorem 3], the function μ satisfying (5.23) is called an entropy process solution. It has been proved to be unique and independent of α. Although this result in [14] was for a divergence-free function $\hat{g} \in C^1(\bar{Q}_T)$, we remark that it can be adapted to the case of any function $\hat{g} \in W^{1,\infty}(\bar{Q}_T)$ (see for instance Remark 6.2 and the proof of [14, Theorem 3]). Using this, we infer the existence of a function $z \in L^\infty(Q_T)$ such that:

$$z(x,t) = \mu(x,t,\alpha), \ \text{ for a.e. } \ (x,t,\alpha) \in Q_T \times (0,1), \tag{5.24}$$

hence, z is an entropy solution of (5.11). We now make use of (5.24) and we apply equality (5.21) for $\psi(x) = x$ to obtain,

$$z = \text{weak}-\star \lim_{n\to\infty} \tilde{\kappa}_x^n \ \text{ in } \ L^\infty(Q_T). \tag{5.25}$$

From (5.25) and (5.18) we deduce that,

$$z(x,t) = \hat{\kappa}_x(x,t) \ \text{ a.e. in } \ Q_T,$$

which completes the proof of Lemma 5.5. □

5.4 Proofs of Theorems 5.1, 5.2

Proof of Theorem 5.1. We extend the function κ^0 to $\hat{\kappa}^0 \in Lip(\mathbb{R})$ in the following way:

$$\hat{\kappa}^0(x) = \begin{cases} \kappa^0(x) & \text{if} \quad x \in [0,1], \\ (||\rho_x^0||_{L^\infty(I)} + \epsilon)(x-1) + \kappa^0(1) & \text{if} \quad x \geq 1, \\ (||\rho_x^0||_{L^\infty(I)} + \epsilon)x + \kappa^0(0) & \text{if} \quad x \leq 0. \end{cases} \tag{5.26}$$

Consider the initial value problem defined by:

$$\begin{cases} u_t + \hat{g} f_\epsilon(u_x) = 0 & \text{in} \quad \mathbb{R} \times (0,T), \\ u(x,0) = \hat{\kappa}^0(x) & \text{in} \quad \mathbb{R}. \end{cases} \tag{5.27}$$

This is a Hamilton-Jacobi equation with a Hamiltonian $F \in C(\bar{Q}_T \times \mathbb{R})$ defined by:

$$F(x,t,u) = \hat{g}(x,t) f_\epsilon(u).$$

From the regularity properties of $\hat{\rho}$, we can directly see that (**V0**)-(**V1**)-(**V2**) are satisfied; this is quite similar to what was done in Proposition 3.1. Since $\hat{\kappa}^0$ is a Lipschitz continuous function, we deduce from Theorems 2.6, 2.7 and Proposition 2.10 the existence and uniqueness of a viscosity solution $\hat{\kappa} \in Lip(\bar{Q}_T)$ of (5.27). Moreover, in order to recover the boundary conditions given by (5.5) on $\partial I \times [0,T]$, we proceed as follows. Let M be the set defined by Lemma 5.4 and let $x \in M$. For every $t \in [0,T]$, we write:

$$|\hat{\kappa}(x,t) - \hat{\kappa}(x,0)| \leq \int_0^t |\hat{\kappa}_s(x,s)| ds \leq \int_0^t |F(x,s,\hat{\kappa}_x(x,s))| ds \leq \int_0^t \left(|F(0,s,\hat{\kappa}_x(x,s))| + C|x|\right) ds.$$

In these inequalities we have used the fact that $\hat{\kappa}$ is a Lipschitz continuous viscosity solution of (5.27) and hence it verifies the equation in Q_T at the points where it is differentiable (see for instance [2]). Also, we have used the condition (**F1**) with $p = q$ and $C_R = C$, a constant independant of R. Now from (5.9)-(ii), we deduce that:

$$|F(0,s,\hat{\kappa}_x(x,s))| = |\hat{\rho}_x(0,s)\hat{\rho}_t(0,s) f_\epsilon(\hat{\kappa}_x(x,s))| = 0, \quad \text{for a.e.} \quad s \in (0,t),$$

and hence we get

$$|\hat{\kappa}(x,t) - \hat{\kappa}(x,0)| \leq C|x|t. \tag{5.28}$$

Since M is a dense subset of $\mathbb{R}$, we pass to the limit in (5.28) as $x \to 0$ and the equality

$$\hat{\kappa}(0,t) = \hat{\kappa}(0,0) = \kappa^0(0) \quad \forall t \in [0,T]$$

holds. Similarly, we can verify that $\hat{\kappa}(1,t) = \hat{\kappa}(1,0) = \kappa^0(1)$ for all $t \in [0,T]$.

Existence. The extension $\hat{\kappa}^0$ of κ^0 outside the interval I is a linear extension of slope $||\rho_x^0||_{L^\infty(I)} + \epsilon$, therefore we have,

$$\hat{\kappa}_x^0(\cdot) \geq \sqrt{(\hat{\rho}_x^0(\cdot))^2 + \epsilon^2} = G(\hat{\rho}_x^0(\cdot)), \quad \text{a.e. in } \mathbb{R}. \tag{5.29}$$

From Lemma 5.5, we know that $\hat{\kappa}_x$ is an entropy solution of equation (5.11) and from Lemma 5.3, we know that $G(\hat{\rho}_x)$ is an entropy sub-solution of (5.11). Since (5.29) holds, we use the Comparison Theorem 2.16 to get,

$$\hat{\kappa}_x(x,t) \geq \sqrt{\hat{\rho}_x^2(x,t) + \epsilon^2} \geq \epsilon > 0, \quad \text{for a.e. } (x,t) \in \bar{Q}_T. \tag{5.30}$$

Take κ to be the restriction of $\hat{\kappa}$ on $\bar{I}_T$ where $\hat{\kappa}^0$ and $\hat{\rho}$ have their automatic replacements κ^0 and ρ respectively on this subdomain. It is clear that $\kappa \in Lip(\bar{I}_T)$ is a viscosity solution of:

$$\begin{cases} \kappa_t + g f_\epsilon(\kappa_x) = 0 & \text{in} \quad I_T, \\ \kappa(x,0) = \kappa^0(x) & \text{in} \quad I, \\ \kappa(0,t) = \kappa^0(0) \quad \text{and} \quad \kappa(1,t) = \kappa^0(1) & \forall\, 0 \le t \le T, \end{cases} \tag{5.31}$$

where $g(x,t) = -\rho_t(x,t)\rho_x(x,t)$ and $\kappa_x(x,t) \ge G(\rho_x(x,t))$ for a.e. $(x,t) \in \bar{I}_T$. We also notice that κ is a viscosity solution of (5.5), for it suffices to follow the same steps of the passage from the viscosity solution of (3.11) to the viscosity solution of (1.4) (see the proof of Theorem 1.2 for details).

Uniqueness. Since the function

$$\bar{H}(x,t,u) = g(x,t) f_\epsilon(u) \in C(\bar{I}_T \times \mathbb{R})$$

satisfies for a fixed t:

$$|\bar{H}(x,t,u) - \bar{H}(y,t,u)| \le C(|x-y|(1+|u|)),$$

for every $x,\ y \in (0,1)$ and $u \in \mathbb{R}$, we use [2, Theorem 2.8] to show that κ is the unique viscosity solution of (5.31). We claim that κ is the unique viscosity solution of (5.5). Indeed, we can also follow the same mechanism as in the proof of Theorem 1.2. □

We now move towards the proof of Theorem 5.2 that has the same flavor of what was done in Section 4. We just need to care about the change in the structure of our problem and the boundary conditions. Our first step will be the following lemma.

Lemma 5.6 *Let c_1 and c_2 be two positive constants defined respectively by:*

$$c_1 = ||(\rho^0_{xx})^2||_{L^\infty(I)} + ||\rho^0_x||_{L^\infty(I)} ||\rho^0_{xxx}||_{L^\infty(I)},$$

and

$$c_2 = (||\kappa^0_x||_{L^\infty(I)} + 1)^2.$$

Then the function $\bar{S}$ defined on Q_T by:

$$\bar{S}(x,t) = \sqrt{2c_1 t + c_2}$$

is an entropy super-solution of (5.11) with

$$w^0(x) = \bar{S}(x,0) = ||\kappa^0_x||_{L^\infty(I)} + 1.$$

Proof. See Subsection 5.1-III. □

Proof of Theorem 5.2. Let $\epsilon > 0$ be a fixed constant. Define $\hat{\kappa}^{0,\epsilon} \in Lip(\mathbb{R})$ by:

$$\hat{\kappa}^{0,\epsilon}(x) = \begin{cases} \kappa^0(x) + \epsilon x & \text{if} \quad x \in [0,1], \\ (||\kappa^0_x||_{L^\infty(I)} + \epsilon)(x-1) + (\kappa^0(1) + \epsilon) & \text{if} \quad x \ge 1, \\ (||\kappa^0_x||_{L^\infty(I)} + \epsilon)x + \kappa^0(0) & \text{if} \quad x \le 0. \end{cases} \tag{5.32}$$

Since $\kappa_x^0 \geq |\rho_x^0|$ a.e. in I, it is clear that for a.e. $x \in \mathbb{R}$ we have

$$\hat{\kappa}_x^{0,\epsilon} \geq G(\hat{\rho}_x^0),$$

and hence, from the discussion of the proof of Theorem 5.1, there exists a unique viscosity solution $\hat{\kappa}^\epsilon \in Lip(\bar{Q}_T)$ of

$$\begin{cases} \hat{\kappa}_t^\epsilon \hat{\kappa}_x^\epsilon = \hat{\rho}_t \hat{\rho}_x & \text{in} \quad Q_T, \\ \hat{\kappa}^\epsilon(x,0) = \hat{\kappa}^{0,\epsilon}(x) \in Lip(\mathbb{R}) & \text{in} \quad \mathbb{R}, \end{cases} \tag{5.33}$$

unique among those satisfying:

$$\hat{\kappa}_x^\epsilon \geq G(\hat{\rho}_x) \quad \text{a.e. in} \quad \bar{Q}_T. \tag{5.34}$$

Assume without loss of generality that $\epsilon < 1$. The ϵ-uniform bound for $\hat{\kappa}_t^\epsilon$ is trivial, it suffices to use directly the equation satisfied by $\hat{\kappa}^\epsilon$ together with (5.34). And the ϵ-uniform bound for $\hat{\kappa}_x^\epsilon$ follows from Lemma 5.6 and Theorem 2.16 since

$$\hat{\kappa}_x^\epsilon(x,0) \leq ||\kappa_x^0||_{L^\infty(I)} + \epsilon \leq ||\kappa_x^0||_{L^\infty(I)} + 1 = \sqrt{c_2} = \bar{S}(x,0).$$

Following exactly the same technic of Section 4, namely the proof of Theorem 1.6, we get that the sequence $\hat{\kappa}^\epsilon$ converges locally uniformly to $\hat{\kappa}$ in $\bar{Q}_T$ with $\hat{\kappa} \in Lip(\bar{Q}_T)$ satisfies,

$$\hat{\kappa}_x \geq |\hat{\rho}_x| \quad \text{a.e. in} \quad \bar{Q}_T \tag{5.35}$$

and

$$\hat{\kappa}(x,0) = \hat{\kappa}_0(x) \quad \text{in} \quad \mathbb{R}, \tag{5.36}$$

where $\hat{\kappa}_0$ is the uniform limit of the sequence $\hat{\kappa}^{0,\epsilon}$ in $\mathbb{R}$. Theorem 5.1 guarantees that

$$\hat{\kappa}^\epsilon(0,t) = \hat{\kappa}^{0,\epsilon}(0) = \kappa^0(0), \tag{5.37}$$

and

$$\hat{\kappa}^\epsilon(1,t) = \hat{\kappa}^{0,\epsilon}(1) = \kappa^0(1) + \epsilon, \tag{5.38}$$

for all $t \in [0,T]$. From (5.37), (5.38) and the pointwise convergence, up to a subsequence, of $\hat{\kappa}^\epsilon$ to $\hat{\kappa}$, we deduce that

$$\hat{\kappa}(0,t) = \lim_{\epsilon \to 0} \hat{\kappa}^\epsilon(0,t) = \kappa^0(0), \quad \forall t \in [0,T], \tag{5.39}$$

and

$$\hat{\kappa}(1,t) = \lim_{\epsilon \to 0} \hat{\kappa}^\epsilon(1,t) = \lim_{\epsilon \to 0}(\kappa^0(1) + \epsilon) = \kappa^0(1) \quad \forall t \in [0,T]. \tag{5.40}$$

Take κ to be the restriction of $\hat{\kappa}$ over $\bar{I}_T$; $\hat{\rho}$ and $\hat{\kappa}_0$ have their automatic replacements ρ and κ^0 respectively on this restricted domain. From (5.35), (5.36), (5.39) and (5.40), we deduce that κ is the required solution. □

6 Appendix: Proof of Theorem 2.16

We will work on the entropy inequality (2.18) satisfied by u and its analogue satisfied by v, using the dedoubling variable technique of Kruzhkov (see [22]) and following the same steps of [14, Theorem 3], taking into consideration the new modifications arising from the fact that we are dealing with sub-/super-entropy solutions and the fact that $g \in W^{1,\infty}(\bar{Q}_T)$ is not a gradient-free function.

The proof can be divided into three steps. Denote B_r by $B_r = \{x \in \mathbb{R};\ \ |x| \leq r\}$ for any $r > 0$, $F^{\pm}(u,v) = sgn^{\pm}(u-v)(f(u) - f(v))$,

$$y^{\infty} = ||y||_{L^{\infty}(Q_T)} \quad \text{for every} \quad y \in L^{\infty}(Q_T) \tag{6.1}$$

and

$$M_f = \max_{|x| \leq \max(u^{\infty}, v^{\infty})} |f^{'}(x)|. \tag{6.2}$$

In step 1, we prove that the initial conditions u^0, v^0 satisfy for any $a > 0$:

$$\lim_{\tau \to 0} \frac{1}{\tau} \int_0^{\tau} \int_{B_a} (u(x,t) - u^0(x))^+ dxdt = 0, \tag{6.3}$$

$$\lim_{\tau \to 0} \frac{1}{\tau} \int_0^{\tau} \int_{B_a} (v(x,t) - v^0(x))^- dxdt = 0, \tag{6.4}$$

respectively.

In step 2, The following relation between u and v is shown:

$$\int_{Q_T} \left[(u(x,t) - v(x,t))^+ \psi_t + F^+(u(x,t), v(x,t)) g(x,t) \psi_x \right] dxdt \geq 0, \tag{6.5}$$

for every $\psi \in C_0^1(\mathbb{R} \times (0,T); \mathbb{R}_+)$.

After that, we define $A(t)$ for $0 < t < \min(T, \frac{a}{\omega})$ and $\omega = g^{\infty} M_f$, by:

$$A(t) = \int_{B_{a-\omega t}} (u(x,t) - v(x,t))^+ \, dx. \tag{6.6}$$

In step 3, we show that A is non-increasing a.e. in $(0, \min(T, \frac{a}{\omega}))$ and we deduce that

$$u(x,t) \leq v(x,t) \quad \text{a.e. in} \quad Q_T.$$

Step 1: Proof of (6.3), (6.4).

Let ξ^n be a sequence of mollifiers in $\mathbb{R}$ with $\xi^1 = \xi$. Recall that the function $\xi \in C_0^{\infty}(\mathbb{R})$ satisfies the following properties:

$$\begin{aligned}
&\operatorname{supp}(\xi) = \{x \in \mathbb{R},\ \xi(x) \neq 0\} \subset B_1; \\
&\xi \geq 0, \quad \xi(-x) = \xi(x); \\
&\int_{B_1} \xi(x) dx = 1; \\
&\xi^n(x) = n\xi(nx).
\end{aligned} \tag{6.7}$$

Let $\tau \in \mathbb{R}$ such that $0 < \tau < T$ and define the function γ by:

$$\gamma(t) = \begin{cases} \dfrac{\tau - t}{\tau} & \text{if} \quad 0 \le t \le \tau, \\ 0 & \text{if} \quad t > \tau. \end{cases} \tag{6.8}$$

Take $a > 0$ and a test function $\psi \in C_0^\infty(\mathbb{R}; \mathbb{R}_+)$ such that,

$$\psi(x) = 1 \quad \text{for} \quad x \in B_a.$$

Let $y \in \mathbb{R}$ be a Lebesgue point of u^0 and we make use of inequality (2.18) with $k = u^0(y)$ and the test function $\phi(x,t) = \psi(x)\gamma(t)\xi^n(x-y)$ (this is possible since ϕ is a permissible test function). Integrating the resulting inequality with respect to y over $\mathbb{R}$ yields:

$$\mathcal{T}_1(n,\tau) + \mathcal{T}_2(n,\tau) + \mathcal{T}_3(n,\tau) + \mathcal{T}_4(n) \ge 0, \tag{6.9}$$

with

$$\mathcal{T}_1(n,\tau) = -\frac{1}{\tau}\int_0^\tau \int_{\mathbb{R}^2} (u(x,t) - u^0(y))^+ \psi(x)\xi^n(x-y)\, dxdydt, \tag{6.10}$$

$$\mathcal{T}_2(n,\tau) = \int_0^\tau \int_{\mathbb{R}^2} F^+(u(x,t), u^0(y)) g(x,t)\gamma(t)(\psi(x)\xi^n(x-y))_x dxdydt, \tag{6.11}$$

$$\mathcal{T}_3(n,\tau) = -\int_0^\tau \int_{\mathbb{R}^2} sgn^+(u(x,t) - u^0(y)) f(u^0(y)) \\ g_x(x,t)\gamma(t)\psi(x)\xi^n(x-y) dxdydt \tag{6.12}$$

and

$$\mathcal{T}_4(n) = \int_{\mathbb{R}^2} (u^0(x) - u^0(y))^+ \psi(x)\xi^n(x-y) dxdy. \tag{6.13}$$

Using the change of variables: $x = x'$, $y = x' - \frac{y'}{n}$ in (6.10), and denoting again by (x,y) the new variables (x', y') yields:

$$\mathcal{T}_1(n,\tau) = -\frac{1}{\tau}\int_0^\tau \int_{B_1}\int_{\mathbb{R}} \left(u(x,t) - u^0\left(x - \frac{y}{n}\right)\right)^+ \psi(x)\xi(y)\, dxdydt, \tag{6.14}$$

Using that,

$$(u-v)^+ - (u-w)^+ \le (w-v)^+ \quad \forall u,v,w \in \mathbb{R}, \tag{6.15}$$

we infer that:

$$\mathcal{T}_1(n,\tau) + \overbrace{\frac{1}{\tau}\int_0^\tau \int_{\mathbb{R}} (u(x,t) - u^0(x))^+ \psi(x) dxdt}^{\mathcal{T}^*(\tau)} \le \\ \psi^\infty \int_{K_\psi}\int_{B_1} \left| u^0\left(x - \frac{y}{n}\right) - u^0(x)\right| \xi(y) dydx, \tag{6.16}$$

where K_ψ is the support of ψ. Same upper-bound, independent of τ, could be obtained for $\mathcal{T}_4(n)$. Furthermore, since $u^0 \in L^\infty(\mathbb{R})$, thus integrable over K_ψ, we use the Lebesgue differentiation Theorem to show that the right side of (6.16) tends to 0 when n becomes large. Now, let $\epsilon > 0$, $\exists n_0$ such that

$$\mathcal{T}_1(n_0,\tau) + \mathcal{T}^*(\tau) < \frac{\epsilon}{4} \quad \text{and} \quad \mathcal{T}_4(n_0) < \frac{\epsilon}{4}, \quad \forall \tau > 0. \tag{6.17}$$

We also remark that the integrands of the right hand sides of (6.11) and (6.12) are bounded and hence, for this particular n_0 we can choose some τ_0 such that $\forall\, 0 < \tau < \tau_0$, we have:

$$\mathcal{T}_2(n_0, \tau) < \frac{\epsilon}{4} \quad \text{and} \quad \mathcal{T}_3(n_0, \tau) < \frac{\epsilon}{4}. \tag{6.18}$$

From (6.17), (6.18) and (6.9), we infer that,

$$0 < \mathcal{T}^*(\tau) < \epsilon, \quad \forall 0 < \tau < \tau_0.$$

Since $\psi(x) = 1$ over B_a, (6.3) is proven. Arguing in the same way, we can prove (6.4). The slight difference is using a similar inequality of (6.15) with $(\cdot)^+$ replaced with $(\cdot)^-$.

Step 2: Proof of (6.5).

It suffices to prove (6.5) for any function $\psi \in C_0^\infty(Q_T; \mathbb{R}_+)$. We may also assume, without loss of generality, that there is some $c > 0$ such that $\psi(x,t) = 0$ for $t \in (0, c) \cup (T - c, T)$. For $n > \frac{1}{c}$, let ξ^n be the usual mollifier sequence in $\mathbb{R}$ and consider the function $\phi(x,t,y,s)$ defined for $(x,t) \in Q_T$ and $(y,s) \in Q_T$ by,

$$\phi(x,t,y,s) = \psi\left(\frac{x+y}{2}, \frac{t+s}{2}\right) \xi^n(x-y)\xi^n(t-s).$$

The function ϕ hence satisfies

$$\phi(.,.,y,s) \in C_0^\infty(Q_T; \mathbb{R}_+) \quad \text{and} \quad \phi(x,t,.,.) \in C_0^\infty(Q_T; \mathbb{R}_+).$$

Fix some $(y,s) \in Q_T$ for which the function v is well defined (this is valid almost everywhere). Since u is an entropy sub-solution of (2.2), we consider the relation (2.18) satisfied by u with $k = v(y,s)$ and the test function $\phi(.,.,y,s)$. Upon integrating this inequality with respect to (y,s) over Q_T, we get:

$$\begin{aligned} \int_{Q_T^2} \big\{ &(u(x,t) - v(y,s))^+ \phi_t(x,t,y,s) + F^+(u(x,t), v(y,s)) g(x,t) \phi_x(x,t,y,s) \\ &- sgn^+(u(x,t) - v(y,s)) f(v(y,s)) g_x(x,t) \phi(x,t,y,s) \big\} \, dxdtdyds \geq 0. \end{aligned} \tag{6.19}$$

Similar inequality could be obtained since v is an entropy super-solution of (2.2). We just swap $+$, u and (x,t) with $-$, v and (y,s) respectively, hence:

$$\begin{aligned} \int_{Q_T^2} \big\{ &(v(y,s) - u(x,t))^- \phi_s(x,t,y,s) + F^-(v(y,s), u(x,t)) g(y,s) \phi_y(x,t,y,s) \\ &- sgn^-(v(y,s) - u(x,t)) f(u(x,t)) g_x(y,s) \phi(x,t,y,s) \big\} \, dxdtdyds \geq 0. \end{aligned} \tag{6.20}$$

Summing (6.19) and (6.20) and using the elementary identities:

$$x^- = (-x)^+ \quad \text{and} \quad sgn^-(x) = -sgn^+(-x), \quad \forall x \in \mathbb{R},$$

we get, for $u = u(x,t)$ and $v = v(y,s)$,

$$\mathcal{Z}_1 + \mathcal{Z}_2 + \mathcal{Z}_3 \geq 0, \tag{6.21}$$

with:

$$\mathcal{Z}_1 = \int_{Q_T^2} (u-v)^+(\phi_t+\phi_s)(x,y,t,s)dxdtdyds, \tag{6.22}$$

$$\mathcal{Z}_2 = \int_{Q_T^2} F^+(u,v)[g(x,t)\phi_x(x,y,t,s)+g(y,s)\phi_y(x,y,t,s)]dxdtdyds, \tag{6.23}$$

$$\mathcal{Z}_3 = \int_{Q_T^2} sgn^+(u-v)[f(u)g_x(y,s)-f(v)g_x(x,t)]\phi(x,y,t,s)dxdtdyds. \tag{6.24}$$

We now compute the first partial derivatives of the function ϕ. For $(x,t,y,s)\in Q_T\times Q_T$, we have:

$$\begin{aligned}\phi_t(x,t,y,s) = \xi^n(x-y)\left(\frac{1}{2}\psi_t\left(\frac{x+y}{2},\frac{t+s}{2}\right)\xi^n(t-s)\right.\\ \left.+\psi\left(\frac{x+y}{2},\frac{t+s}{2}\right)\xi^{n'}(t-s)\right),\end{aligned} \tag{6.25}$$

$$\begin{aligned}\phi_s(x,t,y,s) = \xi^n(x-y)\left(\frac{1}{2}\psi_t\left(\frac{x+y}{2},\frac{t+s}{2}\right)\xi^n(t-s)\right.\\ \left.-\psi\left(\frac{x+y}{2},\frac{t+s}{2}\right)\xi^{n'}(t-s)\right),\end{aligned} \tag{6.26}$$

$$\begin{aligned}\phi_x(x,t,y,s) = \xi^n(t-s)\left(\frac{1}{2}\psi_x\left(\frac{x+y}{2},\frac{t+s}{2}\right)\xi^n(x-y)\right.\\ \left.+\psi\left(\frac{x+y}{2},\frac{t+s}{2}\right)\xi^{n'}(x-y)\right),\end{aligned} \tag{6.27}$$

$$\begin{aligned}\phi_y(x,t,y,s) = \xi^n(t-s)\left(\frac{1}{2}\psi_x\left(\frac{x+y}{2},\frac{t+s}{2}\right)\xi^n(x-y)\right.\\ \left.-\psi\left(\frac{x+y}{2},\frac{t+s}{2}\right)\xi^{n'}(x-y)\right).\end{aligned} \tag{6.28}$$

Using these relations in (6.21) and performing the following change of variables,

$$x^{'} = (x+y)/2,\ y^{'} = n(x-y),\ t^{'} = (t+s)/2,\ s^{'} = n(t-s);$$

denote the new variables $x^{'}$, $t^{'}$, $y^{'}$, $s^{'}$ by x, t, y, s and $\mathcal{Q}_4 = Q_T\times B_1^2$. Also, for the simplicity of expressions, denote

$$x^+ = x+\frac{y}{2n},\quad t^+ = t+\frac{s}{2n},\quad x^- = x-\frac{y}{2n},\quad t^- = t-\frac{s}{2n}.$$

This altogether yields:

$$\mathcal{X}_1+\mathcal{X}_2+\mathcal{X}_3+\mathcal{X}_4 \geq 0, \tag{6.29}$$

with:

$$\mathcal{X}_1 = \int_{\mathcal{Q}_4}(u(x^+,t^+)-v(x^-,t^-))^+\psi_t(x,t)\xi(y)\xi(s)dxdtdyds, \tag{6.30}$$

$$\begin{aligned}\mathcal{X}_2 = \frac{1}{2}\int_{\mathcal{Q}_4} F^+(u(x^+,t^+),v(x^-,t^-))(g(x^+,t^+)+g(x^-,t^-))\times\\ \psi_x(x,t)\xi(y)\xi(s)dxdtdyds,\end{aligned} \tag{6.31}$$

$$\mathcal{X}_3 = \int_{\mathcal{Q}_4} F^+(u(x^+,t^+), v(x^-,t^-))(g(x^+,t^+) - g(x^-,t^-)) \times \\ \psi(x,t) n\xi'(y)\xi(s) dxdtdyds, \tag{6.32}$$

$$\mathcal{X}_4 = \int_{\mathcal{Q}_4} sgn^+(u(x^+,t^+) - v(x^-,t^-)) \left[f(u(x^+,t^+)) g_x(x^-,t^-) - \\ f(v(x^-,t^-)) g_x(x^+,t^+) \right] \psi(x,t)\xi(y)\xi(s) dxdtdyds. \tag{6.33}$$

At this point, it is worth mentioning that we will frequently use the following Lemma from [23].

Lemma 6.1 *If $\Gamma \in Lip(\mathbb{R})$ satisfies $|\Gamma(u) - \Gamma(v)| \leq C_0|u - v|$, then the function*

$$H(u,v) = sgn^+(u-v)(\Gamma(u) - \Gamma(v))$$

satisfies $|H(u,v) - H(u',v')| \leq C_0(|u - u'| + |v - v'|$ (see [22, Lemma 3]).

Consider now (6.30). Since $(u-v)^+ = sgn^+(u-v)(u-v)$, we make use of Lemma 6.1 to obtain:

$$\left| \mathcal{X}_1 - \int_{Q_T} (u(x,t) - v(x,t))^+ \psi_t(x,t) dxdt \right| \leq \\ \left\{ \int_{K_\psi} \int_{B_1^2} |u(x^+,t^+) - u(x,t)| (\psi_t)^\infty \xi(y)\xi(s) dxdtdyds \right. \\ \left. + \int_{K_\psi} \int_{B_1^2} |v(x^-,t^-) - v(x,t)| (\psi_t)^\infty \xi(y)\xi(s) dxdtdyds \right\},$$

where, by the Lebesgue Differentiation/Dominated Theorems, the right hand side of this inequality tends to 0 as $n \to \infty$, and hence:

$$\mathcal{X}_1 \to \int_{Q_T} (u(x,t) - v(x,t))^+ \psi_t(x,t) dxdt \quad \text{as } n \to \infty. \tag{6.34}$$

Let us now turn to (6.31); using the fact that $g \in W^{1,\infty}(Q_T)$ and hence Lipschitz continuous over the compact K_ψ, and the fact that $F^+(u,v)$ is Lipschitz continuous in u and v (see Lemma 6.1), we get:

$$\left| \mathcal{X}_2 - \int_{Q_T} F^+(u(x,t), v(x,t)) g(x,t) \psi_x(x,t) dxdt \right| \leq \\ g^\infty M_f \psi_x^\infty \left\{ \int_{K_\psi} \int_{B_1^2} |u(x^+,t^+) - u(x,t)| \xi(y)\xi(s) dxdtdyds \right. \\ \left. + \int_{K_\psi} \int_{B_1^2} |v(x^-,t^-) - v(x,t)| \xi(y)\xi(s) \right\} dxdtdyds \\ + \frac{1}{n} C((g_x)^\infty, (g_t)^\infty, (\psi_x)^\infty, M_f, u^\infty, v^\infty, T), \tag{6.35}$$

and also, by the Lebesgue Differentiation/Dominated Theorems, the left hand side of this inequality tends to 0 as $n \to \infty$, hence:

$$\mathcal{X}_2 \to \int_{Q_T} F^+(u(x,t), v(x,t)) g(x,t) \psi_x(x,t) dxdt \quad \text{as } n \to \infty. \tag{6.36}$$

We now study the two terms $\mathcal{X}_3^n$ and $\mathcal{X}_4^n$. From the fact that $g \in W^{1,\infty}(\bar{Q}_T)$, we remark that for a.e. $(x,t,y,s) \in Q_T \times Q_T$, we have:

$$g(x^-,t^-) - g(x^+,t^+) = g_x(x^-,t^-)(-y/n) + g_t(x^-,t^-)(-s/n) + o\left(\frac{1}{n}\right).$$

We also remark that the term $g_x(x^+,t^+)$ in $\mathcal{X}_4^n$ could be replaced with $g_x(x^-,t^-)$, since this adds a term that approaches 0 as n becomes large. This term will be omitted throughout what follows and we denote the new $\mathcal{X}_4^n$ by $\tilde{\mathcal{X}}_4^n$. From these two remarks, we rewrite $\mathcal{X}_3^n$ and $\tilde{\mathcal{X}}_4^n$ to get:

$$\begin{aligned}\mathcal{X}_3^n = \int_{\mathcal{Q}_4} & sgn^+(u(x^+,t^+) - v(x^-,t^-))(f(u(x^+,t^+)) - f(v(x^-,t^-))) \\ & (yg_x(x^-,t^-) + sg_t(x^-,t^-))\psi(x,t)\xi'(y)\xi(s)dx\,dt\,dy\,ds + \mathcal{L}(n),\end{aligned} \tag{6.37}$$

where $\mathcal{L}(n) \to 0$ as $n \to \infty$, and

$$\begin{aligned}\tilde{\mathcal{X}}_4^n = \int_{\mathcal{Q}_4} & sgn^+(u(x^+,t^+) - v(x^-,t^-))(f(u(x^+,t^+)) - f(v(x^-,t^-))) \\ & g_x(x^-,t^-)\psi(x,t)\xi(y)\xi(s)dx\,dt\,dy\,ds.\end{aligned} \tag{6.38}$$

The term $\mathcal{L}(n)$ will also be omitted for simplification and we denote the new $\mathcal{X}_3^n$ by $\tilde{\mathcal{X}}_3^n$. Let $\mathcal{X}_{34}^n = \tilde{\mathcal{X}}_3^n + \tilde{\mathcal{X}}_4^n$, hence:

$$\begin{aligned}\mathcal{X}_{34}^n = & \overbrace{\int_{\mathcal{Q}_4} F^+(u(x^+,t^+), v(x^-,t^-))g_x(x^-,t^-)\psi(x,t)(y\xi(y)\xi(s))_y dx\,dt\,dy\,ds}^{\mathcal{X}_{34}^{1n}} \\ & + \overbrace{\int_{\mathcal{Q}_4} F^+(u(x^+,t^+), v(x^-,t^-)))g_t(x^-,t^-)\psi(x,t)(s\xi(y)\xi(s))_y dx\,dt\,dy\,ds}^{\mathcal{X}_{34}^{2n}}.\end{aligned} \tag{6.39}$$

In $\mathcal{X}_{34}^{1n}$ and $\mathcal{X}_{34}^{2n}$, the term $\psi(x,t)$ could be replaced with $\psi(x^-,t^-)$, for this also adds a term getting small when $n \to \infty$. We keep the same notations for $\mathcal{X}_{34}^{1n}$ and $\mathcal{X}_{34}^{2n}$. Since $y\xi(y)\xi(s)$ is a compactly supported smooth function in $\mathcal{Q}_4$, we have:

$$\int_{\mathcal{Q}_4} F^+(u(x^-,t^-), v(x^-,t^-))g_x(x^-,t^-)\psi(x^-,t^-)(y\xi(y)\xi(s))_y dx\,dt\,dy\,ds = 0. \tag{6.40}$$

Moreover, since $F^+(u,v)$ is Lipschitz continuous, we obtain:

$$\begin{aligned}& \left|\mathcal{X}_{34}^{1n} - \int_{\mathcal{Q}_4} F^+(u(x^-,t^-), v(x^-,t^-))g_x(x^-,t^-)\psi(x^-,t^-)(y\xi(y)\xi(s))_y dx\,dt\,dy\,ds\right| \\ & \leq M_f(g_x)^\infty \psi^\infty \int_{K_\psi}\int_{B_1^2} |u(x^+,t^+) - u(x^-,t^-)| dx\,dt\,dy\,ds,\end{aligned} \tag{6.41}$$

where K_ψ is the support of ψ. Therefore, by the Lebesgue Differentiation/Dominated Theorems, we deduce that the right hand side of (6.41) tends to 0 as $n \to \infty$, hence we have:

$$\mathcal{X}_{34}^{1n} \to 0 \quad \text{as} \quad n \to \infty. \tag{6.42}$$

In a similar way we can show that

$$\mathcal{X}_{34}^{2n} \to 0 \quad \text{as} \quad n \to \infty. \tag{6.43}$$

From (6.34), (6.36), (6.42) and (6.43), passing to the limit in (6.29) yields (6.5), which concludes the proof of step 2.

Step 3: $u(x,t) \le v(x,t)$ **a.e. in** Q_T**.**

Let us first show that the function $A(t)$ defined in (6.6) is non-increasing a.e. in $(0, \min(T, \frac{a}{\omega}))$. Take $a > 0$ and recall that $\omega = g^\infty M_f$; let $0 < t_1 < t_2 < \min(T, \frac{a}{\omega})$, $0 < \epsilon < \min(t_1, \min(T, \frac{a}{\omega} - t_2)$, and $\delta > 0$. Consider the function $\phi \in C_0^1(\mathbb{R}_+, [0,1])$ such that $\phi(x) = 1\ \forall x \in [0,a]$, $\phi(x) = 0\ \forall x \in [a+\delta, \infty)$, and $\phi' < 0$. Define r_ϵ by:

$$r_\epsilon(t) = \begin{cases} 0 & \text{if} \quad 0 \le t \le t_1 - \epsilon \\ \dfrac{t-(t_1-\epsilon)}{\epsilon} & \text{if} \quad t_1 - \epsilon \le t \le t_1 \\ 1 & \text{if} \quad t_1 \le t \le t_2 \\ \dfrac{(t_2+\epsilon)-t}{\epsilon} & \text{if} \quad t_2 \le t \le t_2 + \epsilon \\ 0 & \text{if} \quad t_2 + \epsilon \le t \le \infty. \end{cases} \tag{6.44}$$

One can take in (6.5) the permissible test function

$$\psi(x,t) = \phi(|x| + \omega t) r_\epsilon(t).$$

This yields:

$$\begin{aligned} &\overbrace{\frac{1}{\epsilon}\int_{t_1-\epsilon}^{t_1}\int_{\mathbb{R}}(u(x,t)-v(x,t))^+\phi(|x|+\omega t)dxdt}^{E_1(\delta,\epsilon)} - \\ &\overbrace{\frac{1}{\epsilon}\int_{t_2}^{t_2+\epsilon}(u(x,t)-v(x,t))^+\phi(|x|+\omega t)dxdt}^{E_2(\delta,\epsilon)} \ge E(\delta,\epsilon), \end{aligned} \tag{6.45}$$

with

$$\begin{aligned} E(\delta,\epsilon) \quad &= -\int_0^T\int_{\mathbb{R}}[\omega(u(x,t)-v(x,t))^+ + sgn^+((u(x,t)-v(x,t)))\times \\ &(f(u(x,t)) - f(v(x,t)))\frac{x}{|x|}g(x,t)]\phi'(|x|+\omega t)r_\epsilon(t)dxdt. \end{aligned} \tag{6.46}$$

We claim that $E(\delta,\epsilon) \ge 0$. Indeed, since $\phi' \le 0$ and $r_\epsilon \ge 0$, it suffices to show that

$$\begin{aligned} &\omega(u(x,t)-v(x,t))^+ + sgn^+((u(x,t)-v(x,t)))\times \\ &(f(u(x,t)) - f(v(x,t)))\frac{x}{|x|}g(x,t) \ge 0 \quad \text{a.e. in} \quad Q_T. \end{aligned} \tag{6.47}$$

Two cases can be considered, either $u(x,t) \le v(x,t)$; in this case it is easy to verify (6.47), or $u(x,t) > v(x,t)$; in this case we use, from the definition of ω, the fact that

$$(f(u(x,t)) - f(v(x,t)))\frac{x}{|x|}g(x,t) \ge -\omega(u(x,t)-v(x,t)),$$

hence our claim holds. Relation (6.45) now holds with $E(\delta, \epsilon)$ replaced with 0. We regard the integrand term of $E_1(\delta, \epsilon)$ in (6.45) and we notice that for $t_1 - \epsilon < t < t_1$, we have:

$$(u(x,t) - v(x,t))^+ \phi(|x| + \omega t) = (u(x,t) - v(x,t))^+ \phi(|x| + \omega t) \mathbb{I}_{A_\delta},$$

where $\mathbb{I}_{A'_\delta}$ is the characteristic function of the set A_δ defined by:

$$A'_\delta = \{(x,t);\ t_1 - \epsilon < t < t_1,\ 0 < |x| + \omega t < a + \delta\}.$$

Remark that the set A'_δ shrinks, as δ becomes small, to

$$A' = \{(x,t);\ t_1 - \epsilon < t < t_1,\ 0 < |x| + \omega t \leq a\}$$

with $\phi(|x| + \omega t) \equiv 1$ over A. It is easy now to see that as $\delta \to 0$

$$(u(x,t) - v(x,t))^+ \phi(|x| + \omega t) \mathbb{I}_{A'_\delta} \to (u(x,t) - v(x,t))^+ \mathbb{I}_A \text{ a.e. in } Q_T.$$

However, since $(u(x,t) - v(x,t))^+ \in L^\infty(Q_T)$, we use the Lebesgue Dominated Theorem to get:

$$E_1(\delta, \epsilon) \to \frac{1}{\epsilon} \int_{t_1 - \epsilon}^{t_1} \int_{B_{a - \omega t}} (u(x,t) - v(x,t))^+ dxdt \text{ as } \delta \to 0, \tag{6.48}$$

in other words,

$$E_1(\delta, \epsilon) \to \frac{1}{\epsilon} \int_{t_1 - \epsilon}^{t_1} A(t) dt \text{ as } \delta \to 0, \tag{6.49}$$

with $A(t)$ given by (6.6). Similar arguments shows that:

$$E_2(\delta, \epsilon) \to \frac{1}{\epsilon} \int_{t_2}^{t_2 - \epsilon} A(t) dt \text{ as } \delta \to 0. \tag{6.50}$$

Note that $A \in L^1(0,T)$; let t_1 and t_2 be Lebesgue points of the function A such that $0 < t_1 < t_2 < \min(T, \frac{a}{\omega})$, one can easily deduce from (6.49), (6.49) and (6.45) letting ϵ tends to 0 that

$$A(t_1) \geq A(t_2),$$

hence A is a.e. non-increasing. We use this property enjoyed by A to get the comparison principle. In fact, using the elementary identities:

$$\begin{aligned} (u - v)^+ &\leq (u - w)^+ + (v - w)^- \\ (u - v)^- &\leq (u - w)^- + (v - w)^+ \end{aligned}$$

$\forall\, u, v, w \in \mathbb{R}$, we calculate for a.e. $(x,t) \in Q_T$:

$$(u(x,t) - v(x,t))^+ \leq (u(x,t) - u^0(x))^+ + (v(x,t) - v^0(x))^- + (u^0(x) - v^0(x))^+.$$

Since $u^0(x) \leq v^0(x)$ a.e. in $\mathbb{R}$, we get for a.e. $(x,t) \in Q_T$:

$$(u(x,t) - v(x,t))^+ \leq (u(x,t) - u^0(x))^+ + (v(x,t) - v^0(x))^-. \tag{6.51}$$

Using (6.51), for $\tau \in (0,T)$, we calculate:

$$\begin{aligned} &\frac{1}{\tau}\int_0^\tau A(t)dt \le \frac{1}{\tau}\int_0^\tau \int_{B_a} (u(x,t)-v(x,t))^+ dxdt \le \\ &\frac{1}{\tau}\int_0^\tau \int_{B_a} (u(x,t)-u^0(x))^+ dxdt + \frac{1}{\tau}\int_0^\tau \int_{B_a} (v(x,t)-v^0(x))^- dxdt. \end{aligned} \tag{6.52}$$

From (6.3), (6.4) and the passage to the limit as $\tau \to 0$ in (6.52), we deduce that,

$$\frac{1}{\tau}\int_0^\tau A(t)dt \to 0 \quad \text{as} \quad \tau \to 0. \tag{6.53}$$

Thus, since A is a.e. non-increasing on $(0,\tau)$, and $A(t) \ge 0$ for a.e. $t \in (0, \min(T, \frac{a}{\omega}))$, one then has

$$A(t) = 0 \quad \text{for a.e.} \quad t \in \left(0, \min\left(T, \frac{a}{\omega}\right)\right).$$

Since a is arbitrary, we deduce that,

$$u(x,t) \le v(x,t) \quad \text{a.e. in} \quad Q_T.$$

□

Remark 6.2 *In [14], the entropy process solution $\mu(x,t,\alpha)$ was proved to be independent of α for a divergence-free function $g \in C^1(\bar{Q}_T)$. However, for the case of a general non divergence-free function $g \in W^{1,\infty}(\bar{Q}_T)$, same result can be shown by adapting the same proof as in [14, Theorem 3] taking into account the slight modifications that could be deduced from the proof of Theorem (2.16). More precisely, the treatment of the two terms $\mathcal{X}_3^n$ and $\mathcal{X}_4^n$ in Step 2.*

Acknowledgments
The author would like to thank R. Monneau and C. Imbert for fruitful discussions in the preparation of this paper. We also thank A. El-Hajj, R. Eymard, N. Forcadel, M. Jazar and J. Vovelle for their remarks. Finally, this work was partially supported by The Mathematical Analysis and Applications Arab Network (MA3N) and by The contract JC called "ACI jeunes chercheuses et jeunes chercheurs" of the French Ministry of Research (2003-2007).

References

[1] F. Bachmann, J. Vovelle, Existence and uniqueness of entropy solution of scalar conservation law with a flux function involving discontinuous coefficients, Comm. Partial Differential Equations, vol. 31, p. 371-395, 2006.

[2] G. Barles, Solutions de viscosité des équations de Hamilton-Jacobi, Springer-Verlag, Paris, 1994.

[3] H. Brézis Analyse fonctionelle. Théorie et applications. Collection Mathématiques Appliquées pour la Maîtrise. Masson, Paris, 1983. xiv+234 pp. ISBN 2-225-77198-7.

[4] M. Cannone, A. El-Hajj, R. Monneau, F. Ribaud, Global existence of a system of non-linear transport equations describing the dynamics of dislocation densities, work in progress.

[5] V. Caselles, Scalar conservation laws and Hamilton-Jacobi equations in one space variables, Nonlinear Anal. 18 (1992), no.5, 461-469.

[6] H. H. M. Cleveringa, E. Van der Giessen, A. Needleman, Acta Mat 54 (1997), pp. 3164.

[7] G. M. Coclite, N. H. Risebro, Viscosity solutions of Hamilton-Jacobi equations with discontinuous coefficients, preprint 2005.

[8] L. Corrias, M. Falcone, R. Natalini, Numerical schemes for conservation laws via Hamilton-Jacobi equations, Math. Comp. 64 (210) (1995) 555-580, S13-S18.

[9] M. G. Crandall, P. L. Lions, On existence and uniqueness of solutions of Hamilton-Jacobi equations, Nonlinear Anal. Methods and Applications. Vol. 10. No.4, pp. 353-370, 1986.

[10] M. G. Crandall, P. L. Lions, Viscosity solutions of Hamilton-Jacobi equations, Trans. American Mathematical Society, Volume 277, Number 1, May 1983.

[11] A. El-Hajj, Global existence and uniqueness for a non-conservative Burgers type system describing the dynamics of dislocations densities, submitted to SIAM Journal on Mathematical Analysis.

[12] A. El-Hajj, N. Forcadel, A convergent scheme for a non-local coupled system modelling dislocations densities dynamics, to appear in Mathematics of conputation (2006)

[13] L. C. Evans, Partial Defferential Equations, American Mathematical Society, Providence, RI, 1998.

[14] R. Eymard, T. Gallouët, R. Herbin, Existence and uniqueness of the entropy solution to a nonlinear hyperbolic equation, Chin. Ann. of Math. **16B: 1** (1995), 1-14.

[15] J. Friedel, Dislocations, Oxford/ Pergamon Press, 1964.

[16] T. Gimse, N. H. Risebro, A note on reservoir simulation for heterogeneous porous media, Transport Porous Media 10 (1993), 257-270.

[17] I. Groma, P. Balogh, Investigation of dislocation pattern formation in a two-dimentional self-consistent field approximation, Acta Materialia, 47 (1999), pp. 3674-3654.

[18] I. Groma, F. F. Czikor, M. Zaiser, Spatial correlations and higher-order gradient terms in a continuum description of dislocation dynamics, Acta Materialia 51 (2003) 1271-1281.

[19] J. R. Hirth, L. Lothe, Theory of dislocations, second Edition. Malabar, Florida : Krieger, (1992).

[20] H. Ishii, Existence and uniqueness of solutions of Hamilton-Jacobi equations, Funkcialaj Ekvacioj, 29 (1986) 167-188.

[21] K. H. Karlsen, N. H. Risebro, A note on front tracking and the equivalence between viscosity solutions of Hamilton-Jacobi equations and entropy solutions of scalar conservation laws, Nonlinear Anal. 50 (2002) 455-469.

[22] S. N. Kruskov, First order quasilinear equations with several space variables, Math. USSR. Sb. 10 (1970), 217-243.

[23] S. N. Kruskov, The Cauchy problem in the large for non-linear equations and for certain first-order quasilinear systems with several variables, Dokl. Akad. Nauk SSSR 155 (1964) 743-746.

[24] O. A. Ladyzhenskaja, V. A. Solonmikov, N. N. Uralceva, Linear and Quasilinear equations of parabolic type. Translations of Mathematical Monographs. Volume 23. 1968.

[25] P. D. Lax, Hyperbolic systems of conservation laws and the mathematical theory of shock waves. SIAM, Philadelphia, Pa., 1973. v+48 pp.

[26] O. Ley, Lower-bound gradient estimates for first-order Hamilton-Jacobi equations and applications to the regularity of propagating fronts, Adv. in Differential Equations, Volume 6, Number 5, (2001), 547-576.

[27] P. L. Lions, Generalized solutions of Hamilton-Jacobi equations, Pitman (Advanced Publishing Program), Boston, MA, 1982.

[28] F. R. N. Nabarro, Theory of crystal dislocations, Oxford, Clarendon Press, (1969).

[29] D. Ostrov, Solutions of Hamilton-Jacobi equations and scalar conservation laws with discontinuous space-time dependence, J. Differential Equations 182, 51-77 (2002).

[30] N. Seguin, J. Vovelle, Analysis and approximation of a scalar conservation law with a flux function with discontinuous coefficients, Math. Models Methods Appl. Sci., Vol. 13, No. 2 (2003) 221-257.

[31] G. B. Whitham, Linear and Nonlinear Waves, Wiley, New York, 1974.