Global existence for a system of non-linear and non-local transport equations describing the dynamics of dislocation densities

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Abstract

In this paper, we study the global in time existence problem for the Groma-Balogh model describing the dynamics of dislocation densities. This model is a bi-dimensional model where the dislocation densities satisfy a system of transport equations where the velocity vector field is the shear stress in the material, solving the equations of elasticity. This shear stress can be expressed as some Riesz transform of the dislocation densities. The main tool in the proof of this result is the existence of an entropy for this system.

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Key words: Cauchy's problem, system of non-linear transport equations, system of non-local transport equations, system of hyperbolic equations, entropy, Riesz transform, Zygmund space, dynamics of dislocation densities.

1 Introduction

1.1 Physical motivation and presentation of the model

Real crystals show certain defects in the organization of their crystalline structure, called dislocations. These defects were introduced in the Thirties by Taylor, Orowan and Polanyi as the principal explanation of plastic deformation at the microscopic scale of materials.

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In a particular case where these defects are parallel lines in the three-dimensional space, their cross-section can be viewed as points in a plane. Under the effect of an exterior stress, dislocations can be moved. In the special case of what is called "edge dislocations", these dislocations move in the direction of their "Burgers vector" which has a fixed direction. (cf J. Hith and J. Lothe [34] for more physical description).

In this work, we are interested in the mathematical study of a model introduced by I. Groma, P. Balogh in [31] and [32]. In this model we consider two types of dislocations in the plane (x_1, x_2). Typically for a given velocity field, those dislocations of type $(+)$ propagate in the direction $+\vec{b}$ where $\vec{b} = (1, 0)$ is the Burgers vector, while those of type $(-)$ propagate in the direction $-\vec{b}$ (see Figure 1.1).

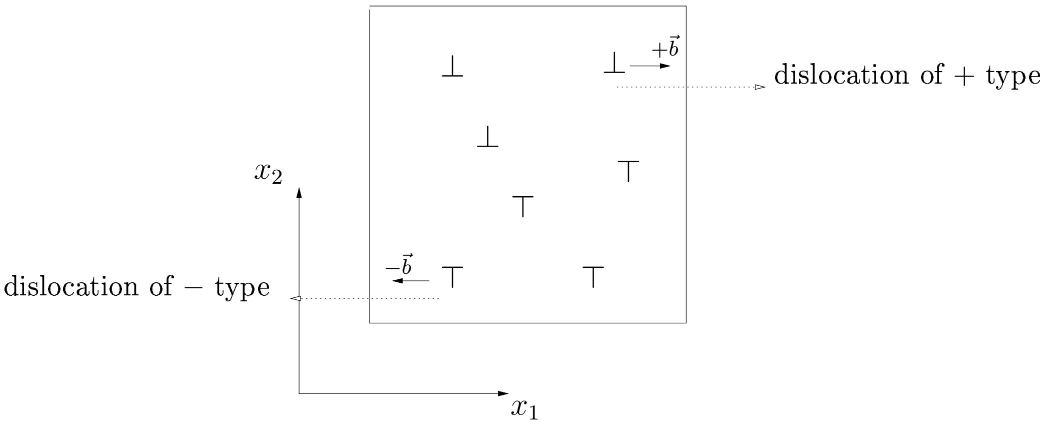

Figure 1: Groma-Balogh 2D model.

Here the velocity vector field is the shear stress in the material, solving the equations of elasticity. It turns out that this shear stress can be expressed as some Riesz transform of the solution (see Section 2). More precisely our non-linear and non-local system of transport equations is the following:

$$\left\{\begin{aligned} \frac{\partial \rho^+}{\partial t}(x,t) &= - \quad \left(R_1^2 R_2^2 \left(\rho^+(\cdot,t) - \rho^-(\cdot,t)\right)(x)\right) \frac{\partial \rho^+}{\partial x_1}(x,t) \quad \text{in } \mathcal{D}'(\mathbb{R}^2 \times (0,T)), \\ \frac{\partial \rho^-}{\partial t}(x,t) &= \qquad \left(R_1^2 R_2^2 \left(\rho^+(\cdot,t) - \rho^-(\cdot,t)\right)(x)\right) \frac{\partial \rho^-}{\partial x_1}(x,t) \quad \text{in } \mathcal{D}'(\mathbb{R}^2 \times (0,T)). \end{aligned}\right. \tag{P}$$

The unknowns of the system (P) are the scalar functions ρ^+ and ρ^- at the time t and the position $x = (x_1, x_2)$, that we denote for simplification by $\rho^\pm$. These terms correspond to the plastic deformations in a crystal. Their derivative in the x_1 direction (i.e. the direction of Burgers vector $\vec{b}$), $\dfrac{\partial \rho^\pm}{\partial x_1}$ represents the dislocation densities of $\pm$ type. The operators R_1 (resp. R_2) are the Riesz transformations associated to x_1 (resp. x_2) (for a precise definition of R_i, $i = 1, 2$, see Definition 1.1).

In fact, this 2D model has been generalized later in 2003 by I. Groma, F. Csikor and M. Zaiser in a model taking into account the back stress describing more carefully boundary layers (see [33] for further details). The Groma-Balogh model neglects in particular the short range dislocation-dislocation correlations in one slip direction. For an extension to multiple slip see S. Yefimov and E. Van der Giessen [53, ch. 5.] and [54]. This multiple slip version of the Groma-Balogh model presents some analogies with some traffic flow models (see O. Biham et al. [8] and J. Török, J. Kertész [51]). See also V. S. Deshpande et al. [19] for a similar model with boundary conditions and exterior forces. Recently, A. EL-Azab [21], M. Zaiser, T. Hochrainer [35], [55], [56] and R. Monneau [42] were interested in modeling the dynamics of dislocation densities in the three-dimensional space, but many more open questions have to be solved for establishing a satisfactory three-dimensional theory of dislocations dynamics and for getting rigorous results.

From a technical point of view, (P) is related to other well known models, such as the transport equation with a low regularity vector field. This equation was studied in the work of R. J. Diperna, P. L. Lions [20] and L. Ambrosio [4], where the authors showed the existence and uniqueness of renormalized solutions by considering vector fields in $L^1((0,T); W^{1,1}_{loc}(\mathbb{R}^N))$ and $L^1((0,T); BV_{loc}(\mathbb{R}^N))$ respectively in both cases with bounded divergence. On the contrary in system (P), we are only able to prove that for the constructed solution, the vector field is in $L^2((0,T); W^{1,2}_{loc}(\mathbb{R}^2))$ without any better estimate on the divergence of the vector field.

We stress out the attention of the reader that there was no any existence and uniqueness result for (P) . In this paper we prove that (P) admits a “global in time” solution.

More generally in the frame of symmetric hyperbolic system, we refer to the book of D. Serre [47, Vol I, Th 3.6.1], for a typical result of local existence and uniqueness in $C([0,T); H^s(\mathbb{R}^N)) \cap C^1([0,T); H^{s-1}(\mathbb{R}^N))$, with $s > \frac{N}{2} + 1$, by considering initial data in $H^s(\mathbb{R}^N)$. This result remains local in time, even in dimension $N = 2$.

We can also remark that in the case where we multiply the right side of the two equations in system (P) by -1, we get a quasi-geostrophic-like system. For those who are concerned in quasi-geostrophic systems, we refer to P. Constantin et al. [16], to [17] for certain 2D numerical results. We also refer to J. Wu [52, Th 4.1] for 2D local existence and uniqueness results in Hölder spaces and to A. Córdoba, D. Córdoba [18], D. Chae, A. Córdoba [12] for blow-up results in finite time, in dimension one.

Let us also mention some related Vlasov-Poisson models (see J. Nieto et al. [43] for instance) and a related model in superconductivity studied by N. Masmoudi et al. [41] and by L. Ambrosio et al. [5]. These models were derived from some Vlasov-Poison-

Fokker-Planck models (see for instance T. Goudon et al. [30], and P. Chavanis et al. [13] for an overview of similar models). It is also worth mentioning that this model is related to Vlasov-Navier-Stokes equation see T. Goudon et al. [28], [29].

1.2 Main result

In the present paper, we prove a "global in time" existence result for the system (P) describing the dynamics of dislocation densities.
In this work we consider the following initial conditions:

$$\rho^{\pm}(x_1, x_2, t = 0) = \rho_0^{\pm}(x_1, x_2) = \rho_0^{\pm,per}(x_1, x_2) + Lx_1, \tag{IC}$$

where $\rho^{\pm,per}$ is a 1-periodic function in x_1 and x_2. The periodicity is a way of studying the bulk behavior of the material away from its boundary. Here L is a given positive constant that represents the initial total dislocation densities of $\pm$ type on the periodic cell.

First of all we give some results which prove that the bilinear term in (P) is well defined.

Definition 1.1 (Riesz transform)
Let $p > 1$ and $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, the periodic square $[0, 1) \times [0, 1)$. If $f \in L^p(\mathbb{T}^2)$, we define R_i for $i \in \{1, 2\}$ as the Riesz transforms over $\mathbb{T}^2$ such that the Fourier series coefficients are given by:
i) $c_{(0,0)}(R_i f) = 0$,
ii) $c_k(R_i f) = \dfrac{k_i}{|k|} c_k(f)$ *for* $k = (k_1, k_2) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$,
where $c_k(f) = \displaystyle\int_{\mathbb{T}^2} f(x) e^{-2\pi i k\cdot x} d^2x$.

Definition 1.2 (The space $L \log L$)
We define $L \log L(\mathbb{T}^2)$ as the following special case of Zygmund spaces (see C. Bennett and R. Sharpley [7, Page 243]):

$$L \log L(\mathbb{T}^2) = \left\{ f \in L^1(\mathbb{T}^2) \text{ such that } \int_{\mathbb{T}^2} |f| \log(e + |f|) < +\infty \right\}.$$

This space is endowed with the norm

$$\|f\|_{L \log L(\mathbb{T}^2)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{T}^2} \frac{|f|}{\lambda} \log\left(e + \frac{|f|}{\lambda}\right) \leq 1 \right\},$$

which is due to Luxemburg (see R. A. Adams [1, (13), Page 234]).

For other equivalent definitions of Zygmund spaces (see P. Koosis [38, Page 96], E. M. Stein [49, Page 43] and A. Zygmund [57]). We now present the following proposition.

Proposition 1.3 (Meaning of the bilinear term)
Let $T > 0$, f and g be two functions defined on $\mathbb{T}^2 \times (0,T)$, such that $f \in L^1((0,T); W^{1,2}(\mathbb{T}^2))$ and $g \in L^\infty((0,T); L\log L(\mathbb{T}^2))$ then,

$$fg \in L^1(\mathbb{T}^2 \times (0,T)).$$

The proof of this proposition is given in Subsection 4.2. We can now state our main result.

Theorem 1.4 (Global existence)
For all $T, L > 0$, and for every initial data $\rho_0^\pm \in L^2_{loc}(\mathbb{R}^2)$ with

(H1) $\rho_0^\pm(x_1+1, x_2) = \rho_0^\pm(x_1, x_2) + L$, *a.e. in* $\mathbb{R}^2$,

(H2) $\rho_0^\pm(x_1, x_2+1) = \rho_0^\pm(x_1, x_2)$, *a.e. in* $\mathbb{R}^2$,

(H3) $\dfrac{\partial {\rho_0}^\pm}{\partial x_1} \geq 0$, *a.e. in* $\mathbb{R}^2$,

(H4) $\left\| \dfrac{\partial {\rho_0}^\pm}{\partial x_1} \right\|_{L\log L(\mathbb{T}^2)} \leq C$, *with* $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$,

the system (P)-(IC) admits solutions $\rho^\pm \in L^\infty((0,T); L^2_{loc}(\mathbb{R}^2)) \cap C([0,T); L^1_{loc}(\mathbb{R}^2))$ in the distributional sense. These solutions satisfy $(H1)$, $(H2)$, $(H3)$ and $(H4)$ for a.e. $t \in (0,T)$. Moreover, we have:

(P1) $R_1^2 R_2^2 (\rho^+ - \rho^-) \in L^2((0,T); W^{1,2}_{loc}(\mathbb{R}^2))$.

Remark 1.5 (Bilinear term)
It is clear here that the bilinear term is always defined via $(P1)$ and Proposition 1.3.

Remark 1.6 (Entropy and energy inequalities)
It turns out that the constructed solution also satisfies the following fundamental entropy inequality (as a consequence of Lemma 7.7), for a.e. $t \in (0,T)$,

$$\int_{\mathbb{T}^2} \sum_{\pm} \frac{\partial \rho^\pm}{\partial x_1}(\cdot,t) \log\left(\frac{\partial \rho^\pm}{\partial x_1}(\cdot,t)\right) + \int_0^t \int_{\mathbb{T}^2} \left(R_1 R_2 \left(\frac{\partial \rho^+}{\partial x_1} - \frac{\partial \rho^-}{\partial x_1}\right)\right)^2 \leq C_1, \qquad (1.1)$$

with $C_1 = C_1\left(\left\|\dfrac{\partial {\rho_0}^\pm}{\partial x_1}\right\|_{L\log L(\mathbb{T}^2)}\right)$.

Moreover, (at least formally for enough regular functions) the following energy inequality holds:

$$\frac{1}{2}\int_{\mathbb{T}^2} \left(R_1 R_2 (\rho^+ - \rho^-)(\cdot,t)\right)^2 + \int_0^t \int_{\mathbb{T}^2} \left(R_1^2 R_2^2 (\rho^+ - \rho^-)\right)^2 \left(\frac{\partial \rho^+}{\partial x_1} + \frac{\partial \rho^-}{\partial x_1}\right) \leq C_2,$$

with $C_2 = C_2\left(\left\|\rho_0^+ - \rho_0^-\right\|_{L^2(\mathbb{T}^2)}\right)$.

Remark 1.7 (Bounds on the solution)
If we denote $\rho = \rho^+ - \rho-$, then there exists a constant C independent on T, and a constant C_T depending on T such that,

$$(E1)\ \|\rho^\pm - Lx_1\|_{L^\infty((0,T);L^2(\mathbb{T}^2))} \le C_T, \qquad (E2)\ \|R_1^2R_2^2\rho\|_{L^\infty((0,T);BMO(\mathbb{T}^2))} \le C,$$

$$(E3)\ \left\|\frac{\partial \rho^\pm}{\partial x_1}\right\|_{L^\infty((0,T);L\log L(\mathbb{T}^2))} \le C, \qquad (E4)\ \|R_1^2R_2^2\rho\|_{L^2((0,T);W^{1,2}(\mathbb{T}^2))} \le C,$$

$$(E5)\ \left\|\frac{\partial \rho^\pm}{\partial t}\right\|_{L^2((0,T);W^{-2,2}(\mathbb{T}^2))} \le C_T, \qquad (E6)\ \left\|R_1^2R_2^2\frac{\partial \rho}{\partial t}\right\|_{L^2((0,T);W^{-1,2}(\mathbb{T}^2))} \le C_T,$$

where $W^{-1,2}(\mathbb{T}^2)$ and $W^{-2,2}(\mathbb{T}^2)$ are respectively the dual spaces of $W^{1,2}(\mathbb{T}^2)$ and $W^{2,2}(\mathbb{T}^2)$. The space BMO is the set of bounded mean oscillation functions that will be precised in the sequel (see Definition 7.1).

In order to prove our main theorem we regularize the system (P) by the mean of the viscosity term ($\varepsilon\Delta\rho^\pm$) and the initial data (IC) by classical convolution. Then, using a fixed point Theorem, we prove that our regularized system admits local in time solutions. Moreover, as we get some ε-independent *a priori* estimates we will be able to extend our local in time solution into a global one. This turns out to be possible thanks to the entropy inequality (1.1). Then, joined with other *a priori* estimates, it will be possible to prove some compactness properties and pass to the limit as ε goes to 0 is the ε-problem.

In a particular sub-case of this model where the dislocation densities depend on a single variable $x = x_1 + x_2$, the existence and uniqueness of a Lipschitz viscosity solution was proved in A. El Hajj, N. Forcadel [23]. Also the existence and uniqueness of a strong solution in $W^{1,2}_{loc}(\mathbb{R} \times [0,T))$ was proved in A. El Hajj [22]. Concerning the model of I. Groma, F. Csikor, M. Zaiser [33] which takes into consideration the short range dislocation-dislocation correlations giving a parabolic-hyperbolic system, let us mention the work of H. Ibrahim [37] where a result of existence and uniqueness of a viscosity solution is given but only for a one-dimensional model.

Our study of the dynamics of dislocation densities in a special geometry is related to the more general dynamics of dislocation lines. We refer the interested reader to the work of O. Alvarez et al. [3], for a local existence and uniqueness of some non-local Hamilton-Jacobi equation. We also refer to O. Alvarez et al. [2] and G. Barles, O. Ley [6] for some long time existence results.

1.3 Organization of the paper

First, in Section 2, we recall the physical derivation of system (P). In Section 3, we give our notation for the sequel of the paper. In Section 4, we give the proof of Proposition 1.3. We also prove that the bilinear term of our system has a better mathematical meaning (see Proposition 4.7). Next, in Section 5, we regularize the initial conditions and we prove that the system (P), modified by a term $(\varepsilon\Delta\rho^{\pm})$, admits local in time solutions (in the "Mild" sense). This will be achieved by using an application of a fixed point Theorem. In Section 6, we prove that the obtained solutions are regular and increasing for all $t \in (0,T)$, for increasing initial data. In Section 7, we prove some ε-uniform *a priori* estimates for the regularized solution obtained in Section 6. Then thanks to these *a priori* estimates, we prove in Section 8 that the local in time solutions constructed in Section 6 are in fact global in time for the ε-problem. Finally, in Section 9, we achieve the proof of our main Theorem, passing to the limit in the equation as ε goes to 0, and using some compactness properties inherited from our a priori estimates.

2 Physical derivation of the model

In this section we explain how to get physically the system (P). We consider a three-dimensional crystal, with displacement

$$u = (u_1, u_2, u_3) : \mathbb{R}^3 \to \mathbb{R}^3.$$

For $x = (x_1, x_2, x_3)$, and an orthogonal basis (e_1, e_2, e_3), we define the total strain by:

$$\varepsilon(u) = \frac{1}{2}(\nabla u + {}^t\nabla u), \quad \text{i.e.} \quad \varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i, j = 1, 2, 3.$$

This total strain is decomposed as

$$\varepsilon(u) = \varepsilon^e(u) + \varepsilon^p,$$

with $\varepsilon^e(u)$ is the elastic strain and ε^p the plastic strain which is defined by:

$$\varepsilon^p = \varepsilon^0\gamma, \tag{2.2}$$

with $\varepsilon^0 = \frac{1}{2}(\vec{e_1} \otimes \vec{e_2} + \vec{e_2} \otimes \vec{e_1})$ in the special case of a single slip system where dislocations move in the plane $\{x_2 = 0\}$ with Burgers vector $\vec{b} = e_1$. Here γ is the resolved plastic strain, and will be precised later. The stress in the crystal satisfies the equation of elasticity div $\sigma = 0$ and is given by,

$$\sigma = \Lambda : \varepsilon^e(u), \tag{2.3}$$

where for $i, j = 1, 2, 3$,

$$(\Lambda : \varepsilon^e(u))_{ij} = 2\mu\varepsilon^e_{ij}(u) + \lambda\delta_{ij}tr(\varepsilon^e(u)), \tag{2.4}$$

and $\lambda, \mu > 0$ are the constants of Lamé coefficients of the crystal (here, for simplification, assumed isotropic).
We now assume that we are in a particular geometry where the dislocations are straight lines parallel to the direction e_3 and that the problem is invariant by translation in the x_3 direction. Moreover we assume that $u_3 = 0$. Then, this problem reduces to a bi-dimensional problem with u_1, u_2 only depending on (x_1, x_2) and so we can express the resolved plastic strain γ as

$$\gamma = \rho^+ - \rho^-,$$

where $\dfrac{\partial \rho^+}{\partial x_1}$ and $\dfrac{\partial \rho^-}{\partial x_1}$ are respectively the densities of dislocations of Burgers vectors given by $\vec{b} = e_1$ and $\vec{b} = -e_1$.
Furthermore, these dislocation densities are transported in the direction of the Burgers vectors by a velocity. This velocity is given by the resolved shear stress $(\sigma : \varepsilon^0)$ up to sign of the Burgers vectors. More precisely, we have:

$$\frac{\partial \rho^\pm}{\partial t} = \pm(\sigma : \varepsilon^0) e_1 . \nabla \rho^\pm.$$

Finally, the functions $\rho^\pm$ and u are solutions of the coupled system (see I. Groma, P. Balogh [32], [31]),

$$\left\{ \begin{array}{lll} \operatorname{div} \sigma & = 0 & \text{in } \mathbb{R}^2 \times (0,T), \\ \sigma & = \Lambda : (\varepsilon(u) - \varepsilon^p) & \text{in } \mathbb{R}^2 \times (0,T), \\ \varepsilon(u) & = \frac{1}{2}(\nabla u + {}^t\nabla u) & \text{in } \mathbb{R}^2 \times (0,T), \\ \varepsilon^p & = \varepsilon^0 (\rho^+ - \rho^-) & \text{in } \mathbb{R}^2 \times (0,T), \\ & & \\ \dfrac{\partial \rho^\pm}{\partial t} & = \pm(\sigma : \varepsilon^0) e_1 . \nabla \rho^\pm & \text{in } \mathbb{R}^2 \times (0,T), \end{array} \right. \tag{2.5}$$

$i.e$ in coordinates,

$$\left\{ \begin{array}{lll} \displaystyle\sum_{j=1,2} \frac{\partial \sigma_{ij}}{\partial x_j} & = 0 & \text{in } \mathbb{R}^2 \times (0,T), \\ \sigma_{ij} & = 2\mu \varepsilon^e_{ij}(u) + \lambda \delta_{ij} tr(\varepsilon^e(u)) & \text{in } \mathbb{R}^2 \times (0,T), \\ \varepsilon_{ij}(u) & = \dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\right) & \text{in } \mathbb{R}^2 \times (0,T), \\ \varepsilon^p_{ij} & = \dfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(\rho^+ - \rho^-) & \text{in } \mathbb{R}^2 \times (0,T), \\ & & \\ \dfrac{\partial \rho^\pm}{\partial t} & = \pm \sigma_{12} e_1 . \nabla \rho^\pm & \text{in } \mathbb{R}^2 \times (0,T), \end{array} \right. \tag{2.6}$$

where the unknowns of the system are $\rho^{\pm}$ and the displacement (u_1, u_2).

Then the following lemma holds.

***Lemma* 2.1 (Equivalence between 2-D systems)**
Assume that (u_1, u_2) and $\rho^+ - \rho^-$ are $\mathbb{Z}^2$-periodic functions. Then the 2-D problem (2.6), is equivalent to the following 2-D problem

$$\begin{cases} \dfrac{\partial \rho^+}{\partial t} = - & C_1 \left(R_1^2 R_2^2 \left(\rho^+ - \rho^-\right)\right) \dfrac{\partial \rho^+}{\partial x_1} \quad \text{in } \mathbb{R}^2 \times (0, T), \\ \\ \dfrac{\partial \rho^-}{\partial t} = & C_1 \left(R_1^2 R_2^2 \left(\rho^+ - \rho^-\right)\right) \dfrac{\partial \rho^-}{\partial x_1} \quad \text{in } \mathbb{R}^2 \times (0, T), \end{cases} \tag{2.7}$$

where $C_1 = 4\dfrac{(\lambda + \mu)\mu}{\lambda + 2\mu}$.

As the constant C_1 is non-negative, rescaling in time in system (2.7), we can replace this constant by 1.

Proof of Lemma 2.1:
We can rewrite the first equation of (2.5) as

$$\text{div}\left(2\mu\varepsilon(u) + \lambda tr(\varepsilon(u)) I_d\right) = \text{div}\left(2\mu\varepsilon^p + \lambda tr(\varepsilon^p) I_d\right).$$

This implies that:

$$\mu \Delta u + (\lambda + \mu)\nabla(\text{div } u) = \mu \begin{pmatrix} \dfrac{\partial}{\partial x_2}(\rho^+ - \rho^-) \\ \\ \dfrac{\partial}{\partial x_1}(\rho^+ - \rho^-) \end{pmatrix}. \tag{2.8}$$

We now derive the first equation and the second equation of the previous system with respect to x_1 and x_2 respectively. We obtain

$$\mu \Delta \begin{pmatrix} \dfrac{\partial u_1}{\partial x_1} \\ \\ \dfrac{\partial u_2}{\partial x_2} \end{pmatrix} + (\lambda + \mu) \begin{pmatrix} \dfrac{\partial^2}{\partial x_1^2}(\text{div } u) \\ \\ \dfrac{\partial^2}{\partial x_2^2}(\text{div } u) \end{pmatrix} = \mu \begin{pmatrix} \dfrac{\partial^2}{\partial x_1 \partial x_2}(\rho^+ - \rho^-) \\ \\ \dfrac{\partial^2}{\partial x_2 \partial x_1}(\rho^+ - \rho^-) \end{pmatrix}.$$

Now, by adding the two above equations, we get

$$(\lambda + 2\mu)\Delta(\text{div } u) = 2\mu \frac{\partial^2}{\partial x_1 \partial x_2}(\rho^+ - \rho^-).$$

Applying Δ^{-1} to this expression we get (div u) that we plug it into (2.8). Which leads to

$$\Delta u = \begin{pmatrix} \dfrac{\partial}{\partial x_2}(\rho^+ - \rho^-) \\ \\ \dfrac{\partial}{\partial x_1}(\rho^+ - \rho^-) \end{pmatrix} - 2\frac{(\lambda+\mu)}{(\lambda+2\mu)}\nabla\Delta^{-1}\frac{\partial^2}{\partial x_1 \partial x_2}(\rho^+ - \rho^-). \tag{2.9}$$

As, previously, we derive the first equation and the second equation of system (2.10) with respect to x_2 and x_1 respectively, and obtain

$$\Delta \begin{pmatrix} \dfrac{\partial u_1}{\partial x_2} \\ \\ \dfrac{\partial u_2}{\partial x_1} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial^2}{\partial x_2^2}(\rho^+ - \rho^-) \\ \\ \dfrac{\partial^2}{\partial x_1^2}(\rho^+ - \rho^-) \end{pmatrix} - 2\frac{(\lambda+\mu)}{(\lambda+2\mu)}\Delta^{-1} \begin{pmatrix} \dfrac{\partial^4}{\partial x_1^2 \partial x_2^2}(\rho^+ - \rho^-) \\ \\ \dfrac{\partial^4}{\partial x_1^2 \partial x_2^2}(\rho^+ - \rho^-) \end{pmatrix}. \tag{2.10}$$

Now, adding the two above equations, we infer that

$$\Delta\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \Delta(\rho^+ - \rho^-) - 4\frac{(\lambda+\mu)}{(\lambda+2\mu)}\Delta^{-1}\frac{\partial^4}{\partial x_1^2 \partial x_2^2}(\rho^+ - \rho^-). \tag{2.11}$$

Using that

$$(\sigma : \varepsilon^0) = \sigma_{12} = 2\mu(\varepsilon^e(u))_{12} = \mu\left(\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) - (\rho^+ - \rho^-)\right),$$

together with equation (2.11), this yields

$$(\sigma : \varepsilon^0) = -4\frac{(\lambda+\mu)\mu}{(\lambda+2\mu)}\Delta^{-2}\frac{\partial^4}{\partial x_1^2 \partial x_2^2}(\rho^+ - \rho^-) = -C_1\left(R_1^2 R_2^2(\rho^+ - \rho^-)\right).$$

Hence we see that the system (2.5) can be rewritten as (2.7). □

Remark 2.2 (Property of the elastic energy)
If we define the elastic energy by

$$E = \frac{1}{2}\int_{\mathbb{R}^2/\mathbb{Z}^2} \Lambda : (\varepsilon^e(u) : \varepsilon^e(u)),$$

with $\varepsilon^e(u) = \varepsilon(u) - \varepsilon^0(\rho^+ - \rho^-)$. *Then, since by the equation of elasticity* $\dfrac{\partial E}{\partial u} = 0$, *we can notice if* $\dfrac{\partial \rho^+}{\partial x_1}, \dfrac{\partial \rho^-}{\partial x_1} \geq 0$ *that,*

$$\frac{dE}{dt} = -\int_{\mathbb{R}^2/\mathbb{Z}^2} (\Lambda : \varepsilon^e(u)) : \varepsilon^0 \frac{\partial(\rho^+ - \rho^-)}{\partial t} = -\int_{\mathbb{R}^2/\mathbb{Z}^2} \sigma_{12}^2 \left(\frac{\partial \rho^+}{\partial x_1} + \frac{\partial \rho^-}{\partial x_1}\right) \leq 0.$$

This formal result indicates that the elastic energy is a non-increasing quantity in this model. Hence, the elastic energy E is a Lyapunov functional for our dissipative model.

3 Notation

In what follows, we are going to use the following notation:

1. $\rho = \rho^{+} - \rho^{-}$,
2. $\rho^{\pm,per}(x_1, x_2, t) = \rho^{\pm}(x_1, x_2, t) - Lx_1$,
3. $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ the periodic interval $[0,1)$, and $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ the periodic square $[0,1) \times [0,1)$.
4. Let f be a function defined on $\mathbb{R}^2 \times (0,T)$ having values in $\mathbb{R}^2$, we denote by $f(t) = f(.,t) : x \longmapsto f(x,t)$.
5. We write $\int_{\mathbb{T}}$ in place of $\int_0^1$
6. Let E be a Banach space and $f = (f_1, f_2)$ a vector such that $f_i \in E$ for $i \in \{1,2\}$. The norm of f in E^2 will be defined as $\|f\|_{E^2} = \max(\|f_1\|_E, \|f_2\|_E)$.
7. Throughout the paper, C is an arbitrary positive constant.

4 Concerning the meaning of the solution of (P)

In this Section we prove Proposition 1.3. This shows that if (P) admits solutions verifying the conditions of Theorem 1.4, then we can give a mathematical meaning to the bilinear term. In order to do this, we need to define some functional spaces and recall some of their properties, that will be used later in our work.

4.1 Properties of some useful Orlicz spaces

We recall the definition of Orlicz spaces and some of their properties. For details, we refer to R. A. Adams [1, Ch. 8] and M. M. Rao, Z. D. Ren [46].
A real valued function $A : [0,+\infty) \to \mathbb{R}$ is called a Young function if it has the following properties (see R. O'Neil [44, Def 1.1]):

- A is a continuous, non-negative, non-decreasing and convex function.
- $A(0) = 0$ and $\lim\limits_{t \to +\infty} A(t) = +\infty$.

Let $A(\cdot)$ be a Young function. The Orlicz class $K_A(\mathbb{T}^2)$ is the set of (equivalence classes of) real-valued measurable function h on $\mathbb{T}^2$ satisfying

$$\int_{\mathbb{T}^2} A(|h(x)|) < +\infty.$$

The Orlicz space $L_A(\mathbb{T}^2)$ is the linear hull of $K_A(\mathbb{T}^2)$ supplemented with the Luxemburg norm,

$$\|f\|_{L_A(\mathbb{T}^2)} = \inf\left\{\lambda > 0 : \int_{\mathbb{T}^2} A\left(\frac{|h(x)|}{\lambda}\right) \leq 1\right\}.$$

Endowed with this norm, the Orlicz space $L_A(\mathbb{T}^2)$ is a Banach space. For example if $A(t) = t^p$ for $p \geq 1$, the Orlicz space is the usual Lebesgue space $L^p(\mathbb{T}^2)$.

Remark 4.1 (Separability)
If A is Δ_2-regular (i.e. there exists a positive constant δ such that for all $t \geq 0$, $A(2t) \leq \delta A(t)$), then the Orlicz space $L_A(\mathbb{T}^2)$ is separable (see M. M. Rao and Z. D. Ren [46, Th 1, Page 87]). In particular this holds for $L\log L(\mathbb{T}^2)$.

Definition 4.2 (Some Orlicz spaces)
For $\alpha \geq 1$, we denote by

$$EXP_\alpha(\mathbb{T}^2), \textit{ the Orlicz space defined by the function } A(t) = e^{t^\alpha} - 1.$$

Another space of interest will be the Zygmund space

$$L\log^\beta L(\mathbb{T}^2), \textit{ the Orlicz space defined by the function } A(t) = t(\log(e+t))^\beta, \textit{ for } \beta \geq 0.$$

Observe that those spaces are Banach spaces and that $EXP_{\frac{1}{\beta}}(\mathbb{T}^2)$ is the dual of $L\log^\beta L(\mathbb{T}^2)$, for $0 < \beta \leq 1$ (see C. Bennett and R. Sharpley [7, Def 6.11]). It is worth noticing that $L\log^1 L(\mathbb{T}^2) = L\log L(\mathbb{T}^2)$.

Let us recall some useful properties of these spaces. The first one is the generalized Hölder inequality.

***Lemma* 4.3 (Generalized Hölder inequality)**
i) Let $f \in EXP_2(\mathbb{T}^2)$ and $g \in L\log^{\frac{1}{2}} L(\mathbb{T}^2)$, Then there exists a constant C such that (see R. O'Neil [44, Th 2.3]),

$$\|fg\|_{L^1(\mathbb{T}^2)} \leq C\|f\|_{EXP_2(\mathbb{T}^2)}\|g\|_{L\log^{\frac{1}{2}} L(\mathbb{T}^2)}.$$

ii) Let $f \in EXP_2(\mathbb{T}^2)$ and $g \in L\log L(\mathbb{T}^2)$. Then there exists a constant C such that (see R. O'Neil [44, Th 2.3]),

$$\|fg\|_{L\log^{\frac{1}{2}} L(\mathbb{T}^2)} \leq C\|f\|_{EXP_2(\mathbb{T}^2)}\|g\|_{L\log L(\mathbb{T}^2)}.$$

Remark 4.4
For the proof of this lemma see also M. M. Rao and Z.D. Ren [46, Th 7, Page 64], J. Hogan et al. [36, Th A.1].

The second property is the Trudinger embedding,

Lemma **4.5 (Continuous Trudinger embedding)**
We have the following continuous injection (see N. S. Trudinger [50] and R. A. Adams [1, Th 8. 25]):

$$W^{1,2}(\mathbb{T}^2) \hookrightarrow EXP_2(\mathbb{T}^2).$$

Finally, we have the following embedding.

Lemma **4.6 (Properties of the Zygmund space)**
For $1 < p < +\infty$, $\alpha \geq 1$ and $\beta \geq 0$ we have the following continuous embedding:

$$L^\infty(\mathbb{T}^2) \hookrightarrow EXP_\alpha(\mathbb{T}^2) \hookrightarrow L^p(\mathbb{T}^2) \hookrightarrow L\log^\beta L(\mathbb{T}^2) \hookrightarrow L^1(\mathbb{T}^2).$$

For the proof, see for instance R. A. Adams [1, Th 8.12].

4.2 Sharp estimate of the bilinear term

Now, we propose to verify with the help of the following proposition that the system (P) has indeed a sense, and first prove a better estimate than those mentioned in Proposition 1.3. Namely, we have the following.

Proposition 4.7 (Estimate of the bilinear term)
Let $T > 0$, f and g be two functions defined on $\mathbb{T}^2 \times (0, T)$, such that,

(1) $f \in L^2((0,T); W^{1,2}(\mathbb{T}^2))$,

(2) $g \in L^\infty((0,T); L\log L(\mathbb{T}^2))$. *Then,*

$$fg \in L^2((0,T); L\log^{\frac{1}{2}} L(\mathbb{T}^2)),$$

and for a positive constant C, we have:

$$\|fg\|_{L^2((0,T);L\log^{\frac{1}{2}} L(\mathbb{T}^2))} \leq C\|f\|_{L^2((0,T);W^{1,2}(\mathbb{T}^2))}\|g\|_{L^\infty((0,T);L\log L(\mathbb{T}^2))}.$$

Proof of Proposition 4.7:
First of all, according to the generalized Hölder inequality Lemma 4.3 (ii), we know that

$$\|f(t)g(t)\|^2_{L\log^{\frac{1}{2}} L(\mathbb{T}^2)} \leq C\|f(t)\|^2_{EXP_2(\mathbb{T}^2)}\|g(t)\|^2_{L\log L(\mathbb{T}^2)}.$$

Integrating on $(0,T)$, we infer that,

$$\int_0^T \|f(t)g(t)\|^2_{L\log^{\frac{1}{2}} L(\mathbb{T}^2)} \leq C\int_0^T \|f(t)\|^2_{EXP_2(\mathbb{T}^2)}\|g(t)\|^2_{L\log L(\mathbb{T}^2)}.$$

Knowing that $g \in L^\infty((0,T); L\log L(\mathbb{T}^2))$, we have,

$$\|fg\|^2_{L^2((0,T);L\log^{\frac{1}{2}} L(\mathbb{T}^2))} \leq C\|g\|^2_{L^\infty((0,T);L\log L(\mathbb{T}^2))}\|f\|^2_{L^2((0,T);EXP_2(\mathbb{T}^2))}.$$

Now, by the Trudinger inequality Lemma 4.5, we get,

$$\|fg\|^2_{L^2((0,T);L\log^{\frac{1}{2}} L(\mathbb{T}^2))} \leq C\|g\|^2_{L^\infty((0,T);L\log L(\mathbb{T}^2))}\|f\|^2_{L^2((0,T);W^{1,2}(\mathbb{T}^2))}.$$

□

Proof of Proposition 1.3:

We proceed as in the proof of Proposition 4.7. We use Lemma 4.3 (i), and integrate in time, thanks to the Trudinger inequality (Lemma 4.5) and the continuous injection $L\log L(\mathbb{T}^2) \hookrightarrow L\log^{\frac{1}{2}} L(\mathbb{T}^2)$. □

5 Local existence of solutions of a regularized system

In this Section, we prove a local in time existence for the system (P), modified by the term $\varepsilon\Delta\rho^\pm$, and for smoothed data. This modification brings us to study, for all $0 < \varepsilon \leq 1$, the following system:

$$\left\{\begin{array}{lll} \dfrac{\partial\rho^{+,\varepsilon}}{\partial t} - \varepsilon\Delta\rho^{+,\varepsilon} & = -(R_1^2R_2^2\rho^\varepsilon)\dfrac{\partial\rho^{+,\varepsilon}}{\partial x_1} & \text{in } \mathcal{D}'(\mathbb{R}^2\times(0,T)), \\ \dfrac{\partial\rho^{-,\varepsilon}}{\partial t} - \varepsilon\Delta\rho^{-,\varepsilon} & = \quad (R_1^2R_2^2\rho^\varepsilon)\dfrac{\partial\rho^{-,\varepsilon}}{\partial x_1} & \text{in } \mathcal{D}'(\mathbb{R}^2\times(0,T)), \end{array}\right. \qquad (P_\varepsilon)$$

where $\rho^\varepsilon = \rho^{+,\varepsilon} - \rho^{-,\varepsilon}$, with the following regular initial data:

$$\rho^{\pm,\varepsilon}(x,0) = \rho_0^{\pm,\varepsilon}(x) = \rho_0^{\pm,per} * \eta_\varepsilon(x) + (L+\varepsilon)x_1 = \rho_0^{\pm,\varepsilon,per}(x) + L_\varepsilon x_1, \qquad (IC_\varepsilon)$$

where $\eta_\varepsilon(\cdot) = \frac{1}{\varepsilon^2}\eta(\frac{\cdot}{\varepsilon})$, such that $\eta \in C_c^\infty(\mathbb{R}^2)$ is a non-negative function and $\int_{\mathbb{R}^2}\eta = 1$.

Remark 5.1
We consider L_ε to obtain strongly monotonous initial data $\rho_0^{\pm,\varepsilon}$. This condition will be useful in the proof of Lemma 7.7.

If we let $\rho^{\pm,\varepsilon,per} = \rho^{\pm,\varepsilon} - L_\varepsilon x_1$, we know that the system (P_ε) is equivalent to,

$$\frac{\partial\rho^{\pm,\varepsilon}}{\partial t} - \varepsilon\Delta\rho^{\pm,\varepsilon,per} = \mp(R_1^2R_2^2\rho^\varepsilon)\frac{\partial\rho}{\partial x_1}^{\pm,\varepsilon,per} \mp L_\varepsilon(R_1^2R_2^2\rho^\varepsilon) \text{ in } \mathcal{D}'(\mathbb{T}^2\times(0,T)), \quad (P_\varepsilon^{per})$$

with initial conditions,

$$\rho^{\pm,\varepsilon,per}(x,0) = \rho_0^{\pm,\varepsilon}(x) - L_\varepsilon x_1 = \rho_0^{\pm,\varepsilon,per}(x). \qquad (IC_\varepsilon^{per})$$

Remark 5.2
The properties of the mollifier $(\eta_\varepsilon)_\varepsilon$ *and the fact that* $\rho_0^{\pm,per} \in L^2(\mathbb{T}^2)$ *implies that* $\rho_0^{\pm,\varepsilon,per} \in C^\infty(\mathbb{T}^2)$. *In particular,* $\rho_0^{\pm,\varepsilon,per} \in W^{1,p}(\mathbb{T}^2)$ *for all* $1 \le p \le +\infty$.

The following theorem is a local existence result (in the "Mild" sense) of the regularized system (P_ε)-(IC_ε). This result is achieved in a super-critical space. Here particularly we chose the space of functions $C([0,T); W^{1,\frac{3}{2}}_{loc}(\mathbb{R}^2))$. Later, in Section 6, we will improve the regularity of the solution.

Theorem 5.3 (Local existence result)
For all initial data $\rho_0^\pm \in L^2_{loc}(\mathbb{R}^2)$ *verifying* $(H1)$ *and* $(H2)$*, there exists*

$$T^\star(\|\rho_0^{\pm,\varepsilon,per}\|_{W^{1,\frac{3}{2}}(\mathbb{T}^2)}, L, \varepsilon) > 0,$$

such that the system (P_ε)-(IC_ε) *admits solutions* $\rho^{\pm,\varepsilon} \in C([0,T^\star); W^{1,\frac{3}{2}}_{loc}(\mathbb{R}^2))$, *satisfying* $(H1)$ *and* $(H2)$ *for a.e.* $t \in (0,T^\star)$.

Before proving Theorem 5.3, let us recall some well known results.

5.1 Useful results

We first start with reformulation of system (P_ε^{per})-(IC_ε^{per}) as an integral system.

***Lemma* 5.4 (Mild solutions are solutions in the distributional sense)**
If $\rho^{\pm,\varepsilon,per} \in C([0,T); W^{1,\frac{3}{2}}(\mathbb{T}^2))$ *are solutions of the following integral problem:*

$$\begin{aligned}\rho^{\pm,\varepsilon,per}(\cdot,t) = S_\varepsilon(t)\rho_0^{\pm,\varepsilon,per} \quad & \mp L_\varepsilon \int_0^t S_\varepsilon(t-s)\left(R_1^2R_2^2\rho^\varepsilon(s)\right)ds \\ & \mp \int_0^t S_\varepsilon(t-s)\left(\left(R_1^2R_2^2\rho^\varepsilon(s)\right)\frac{\partial \rho}{\partial x_1}^{\pm,\varepsilon,per}(s)\right)ds,\end{aligned} \qquad (In_\varepsilon)$$

where $S_\varepsilon(t) = S_1(\varepsilon t)$, *and* $S_1(t) = e^{t\Delta}$ *is a the heat semi-group, then* $\rho^{\pm,\varepsilon,per}$ *are solutions of the system* (P_ε^{per})-(IC_ε^{per}) *in the distributional sense.*

For the proof of Lemma 5.4, see A. Pazy [45, Th 5.2, Page 146].

Remark 5.5
We notice that the product $(R_1^2R_2^2\rho^\varepsilon)\dfrac{\partial \rho}{\partial x_1}^{\pm,\varepsilon,per}$ *is well defined in* $C([0,T); L^{\frac{6}{5}}(\mathbb{T}^2))$ *since* $C([0,T); W^{1,\frac{3}{2}}(\mathbb{T}^2)) \hookrightarrow C([0,T); L^6(\mathbb{T}^2))$.

***Lemma* 5.6 (Time continuity)**
Let $T > 0$. *If* $\rho^{\pm,\varepsilon,per} \in L^\infty((0,T); W^{1,\frac{3}{2}}(\mathbb{T}^2))$ *are solutions of integral problem* (In_ε), *then* $\rho^{\pm,\varepsilon,per} \in C([0,T); W^{1,\frac{3}{2}}(\mathbb{T}^2))$.

For the proof of Lemma 5.4, see A. Pazy [45, 7.3, Page 212].

We now recall the Picard fixed point result which will be applied in Subsection 5.2 to the space $E = \Big(L^\infty((0,T); W^{1,\frac{3}{2}}(\mathbb{T}^2))\Big)^2$ in order to prove, the existence of solutions.

***Lemma* 5.7 (Picard Fixed point Theorem)**
Let E be a Banach space, B is a continuous bilinear application over $E \times E$ having values in E, and A a continuous linear application over E having values in E such that:

$$\|B(x,y)\|_E \leq \eta \|x\|_E \|y\|_E \quad \textit{for all} \quad x, y \in E,$$
$$\|A(x)\|_E \leq \mu \|x\|_E \quad \textit{for all} \quad x \in E,$$

where $\eta > 0$ and $\mu \in (0,1)$ are two given constants. Then, for every $x_0 \in E$ verifying

$$\|x_0\|_E < \frac{1}{4\eta}(1-\mu)^2,$$

the equation $x = x_0 + B(x,x) + A(x)$ admits a solution in E.

For the proof of Lemma 5.7, see M. Cannone [10, Lemma 4.2.14].

***Lemma* 5.8 (Decay estimate)**
Let $r, p, q \geq 1$. Then, for all functions $f \in L^q(\mathbb{T}^2)$ and $g \in L^p(\mathbb{T}^2)$, where $\dfrac{1}{r} \leq \dfrac{1}{q} + \dfrac{1}{p}$, we have, for $S_1(t) = e^{t\Delta}$, the following estimates:

i) $\|S_1(t)(fg)\|_{L^r(\mathbb{T}^2)} \leq C t^{-\left(\frac{1}{p}+\frac{1}{q}-\frac{1}{r}\right)} \|f\|_{L^q(\mathbb{T}^2)} \|g\|_{L^p(\mathbb{T}^2)}$ *for all $t > 0$,*

ii) $\|\nabla S_1(t)(fg)\|_{L^r(\mathbb{T}^2)} \leq C t^{-\left(\frac{1}{2}+\frac{1}{p}+\frac{1}{q}-\frac{1}{r}\right)} \|f\|_{L^q(\mathbb{T}^2)} \|g\|_{L^p(\mathbb{T}^2)}$ *for all $t > 0$,*

with $C = C(r,p,q)$ is a positive constant.

Proof of Lemma 5.8:

In the special case where $q = +\infty$ and $f = 1$, these estimates are the classical version of the L^r-L^p estimates for the heat semi-group (see A. Pazy [45, Lemma 1.1.8, Th 6.4.5]). The statement of Lemma 5.8 then follows by Hölder inequality:

$$\|fg\|_{L^s(\mathbb{T}^2)} \leq \|f\|_{L^q(\mathbb{T}^2)} \|g\|_{L^p(\mathbb{T}^2)}$$

with $\dfrac{1}{s} = \dfrac{1}{q} + \dfrac{1}{p}$. □

Here is now, the demonstration of Theorem 5.3.

5.2 Proof of Theorem 5.3

We rewrite the system (In_ε) in the following vectorial form:

$$\rho_v^\varepsilon(x,t) = S_\varepsilon(t)\rho_{0,v}^\varepsilon + L_\varepsilon \bar{J}_1 \int_0^t S_\varepsilon(t-s)\left(R_1^2R_2^2\rho^\varepsilon(s)\right)ds + \bar{I}_1\int_0^t \left(R_1^2R_2^2\rho^\varepsilon(s)\right)\frac{\partial \rho_v^\varepsilon}{\partial x_1}(s)ds,$$

where $S_\varepsilon(t) = S_1(\varepsilon t)$, $\rho_v^\varepsilon = (\rho^{+,\varepsilon,per}, \rho^{-,\varepsilon,per})$, $\rho_{0,v}^\varepsilon = (\rho_0^{+,\varepsilon,per}, \rho_0^{-,\varepsilon,per})$,
$\bar{I}_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\bar{J}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

which is equivalent to,

$$\rho_v^\varepsilon(x,t) = S_\varepsilon(t)\rho_{0,v}^\varepsilon + B(\rho_v^\varepsilon,\rho_v^\varepsilon)(t) + A(\rho_v^\varepsilon)(t), \tag{5.12}$$

where B is a bilinear map and A is a linear one defined respectively, for every vector $u=(u_1,u_2)$ and $v=(v_1,v_2)$, as follows:

$$B(u,v)(t) = \bar{I}_1\int_0^t S_\varepsilon(t-s)\left(\left(R_1^2R_2^2(u_1-u_2)\right)\frac{\partial v}{\partial x_1}(s)\right)ds, \tag{5.13}$$

$$A(u)(t) = L_\varepsilon \bar{J}_1\int_0^t S_\varepsilon(t-s)\left(R_1^2R_2^2(u_1-u_2)(s)\right)ds. \tag{5.14}$$

Now, we apply Lemma 5.7 to equation (5.12). First of all, we estimate the bilinear term,

$$\begin{aligned}\|B(u,v)(t)\|_{(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2} &\le \left\|\bar{I}_1\int_0^t S_\varepsilon(t-s)\left(\left(R_1^2R_2^2(u_1-u_2)\right)\frac{\partial v}{\partial x_1}(s)\right)ds\right\|_{(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2} \\ &\le \int_0^t\left\|S_\varepsilon(t-s)\left(\left(R_1^2R_2^2(u_1-u_2)\right)\frac{\partial v}{\partial x_1}(s)\right)ds\right\|_{(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2}.\end{aligned}$$

Then, since $W^{1,\frac{3}{2}}(\mathbb{T}^2)\hookrightarrow L^4(\mathbb{T}^2)$, we have,

$$\begin{aligned}\|B(u,v)(t)\|_{(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2} \le &\int_0^t\left\|S_\varepsilon(t-s)\left(\left(R_1^2R_2^2(u_1-u_2)\right)\frac{\partial v}{\partial x_1}(s)\right)ds\right\|_{(L^4(\mathbb{T}^2))^2} \\ &+\int_0^t\left\|\nabla S_\varepsilon(t-s)\left(\left(R_1^2R_2^2(u_1-u_2)\right)\frac{\partial v}{\partial x_1}(s)\right)ds\right\|_{(L^{\frac{3}{2}}(\mathbb{T}^2))^2}.\end{aligned} \tag{5.15}$$

We use Lemma 5.8 (i) with $r=4, q=3, p=\frac{3}{2}$ to estimate the first term and Lemma 5.8 (ii) with $r=\frac{3}{2}, q=4, p=\frac{3}{2}$ to estimate the second term. We get for $0\le t\le T$, and with constants C depending on ε,

$$\|B(u,v)(t)\|_{(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2} \leq C\int_0^t \frac{1}{(t-s)^{\frac{3}{4}}} \left\|R_1^2R_2^2u(s)\right\|_{(L^4(\mathbb{T}^2))^2} \left\|\frac{\partial v}{\partial x_1}(s)\right\|_{(L^{\frac{3}{2}}(\mathbb{T}^2))^2} ds$$

$$\leq C \sup_{0\leq s<T}\left(\|u(s)\|_{(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2}\right) \sup_{0\leq s<T}\left(\|v(s)\|_{(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2}\right)\int_0^t \frac{1}{(t-s)^{\frac{3}{4}}}ds.$$

Here we have used in the second line the property that Riesz transformations are continuous from $L^{\frac{3}{2}}$ onto itself (see A. Zygmund [57, Vol I, Page 254, (2.6)]) and the Sobolev injection. Hence we have,

$$\|B(u,v)\|_{L^\infty((0,T);(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2)} \leq \eta(T)\|u\|_{L^\infty((0,T);(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2)}\|v\|_{L^\infty((0,T);(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2)}, \tag{5.16}$$

with $\eta(T) = C_0T^{\frac{1}{4}}$ for some constant $C_0 > 0$. We estimate the linear term in the same way to get,

$$\|A(u)\|_{L^\infty((0,T);(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2)} \leq L_\varepsilon\eta(T)\|u\|_{L^\infty((0,T);(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2)}. \tag{5.17}$$

Moreover, we know by classical properties of heat semi-group (see A. Pazy [45]) that,

$$\|S_\varepsilon(t)\rho_{0,v}^\varepsilon\|_{L^\infty((0,T);(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2)} \leq \|\rho_{0,v}^\varepsilon\|_{(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2}. \tag{5.18}$$

Now, if we take

$$(T^\star)^{\frac{1}{4}} = \min\left(\frac{1}{2C_0L_\varepsilon}, \frac{1}{16C_0\|\rho_{0,v}^\varepsilon\|_{(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2}}\right), \tag{5.19}$$

we can easily verify that we have the following inequalities:

$$\|\rho_{0,v}^\varepsilon\|_{(W^{1,\frac{3}{2}})^2(\mathbb{T}^2)} < \frac{1}{4\eta(T^\star)}(1-L_\varepsilon\eta(T^\star))^2, \quad \text{and } L_\varepsilon\eta(T^\star) < 1, \tag{5.20}$$

Using inequalities (5.16), (5.17), (5.18), (5.20) and Lemma 5.7 with the space $E = \left(L^\infty((0,T^\star);W^{1,\frac{3}{2}}(\mathbb{T}^2))\right)^2$, we obtain the existence of a solutions $\rho_v^\varepsilon \in \left(L^\infty((0,T^\star);W^{1,\frac{3}{2}}(\mathbb{T}^2))\right)^2$ for the system (5.12). Next, from Lemma 5.6, we deduce that $\rho_v^\varepsilon \in \left(C([0,T^\star);W_{loc}^{1,\frac{3}{2}}(\mathbb{T}^2))\right)^2$.

As a consequence, by Lemma 5.4 we prove that the system (P_ε)-(IC_ε) admits some solutions $\rho^{\pm,\varepsilon} \in C([0,T^\star);W_{loc}^{1,\frac{3}{2}}(\mathbb{R}^2))$, satisfying $(H1)$ and $(H2)$ a.e. $t \in [0,T^\star)$. $\square$

6 Properties of the solutions of (P_ε)-(IC_ε)

In this section, we are going to prove that the solutions of (P_ε)-(IC_ε) obtained by Theorem 5.3 are smooth. Moreover if we assume that the initial data (IC) satisfies $(H3)$, then the solutions are increasing in x_1 for all $t \in (0,T^\star)$.

***Lemma* 6.1 (Smoothness of the solution)**
Let $T > 0$. For all initial data $\rho_0^\pm \in L^2_{loc}(\mathbb{R}^2)$ satisfying $(H1)$ and $(H2)$, if $\rho^{\pm,\varepsilon} \in C([0,T); W^{1,\frac{3}{2}}_{loc}(\mathbb{R}^2))$ are solutions of the system (P_ε)-(IC_ε), then $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}^2 \times [0,T))$.

Proof of Lemma 6.1:
We denote the second term of the system (P_ε^{per}) by,

$$f^{\pm,\varepsilon} = \mp(R_1^2R_2^2\rho^\varepsilon)\frac{\partial\rho}{\partial x_1}^{\pm,\varepsilon,per} \mp L_\varepsilon(R_1^2R_2^2\rho^\varepsilon).$$

Since $W^{1,\frac{3}{2}}(\mathbb{T}^2) \hookrightarrow L^6(\mathbb{T}^2)$, $f^{\pm,\varepsilon} \in L^{\frac{6}{5}}(\mathbb{T}^2 \times (0,T))$. Moreover, we know that $\rho_0^{\pm,\varepsilon,per} \in C^\infty(\mathbb{T}^2)$. We apply the L^p regularity for the heat equation to the system (P_ε^{per})-(IC_ε^{per}), see J. L. Lions, E. Magenes [40, Th 8.2], and deduce that

$$\frac{\partial\rho}{\partial t}^{\pm,\varepsilon,per},\quad \frac{\partial\rho}{\partial x_i}^{\pm,\varepsilon,per},\quad \frac{\partial^2\rho}{\partial x_i\partial x_j}^{\pm,\varepsilon,per} \in L^{\frac{6}{5}}(\mathbb{T}^2 \times (0,T)) \text{ for } \{i,j = 1,2\}.$$

We infer now by Sobolev embedding that $f^{\pm,\varepsilon} \in L^{\frac{3}{2}}(\mathbb{T}^2 \times (0,T))$. We can then iterate the previous argument with a better integrability of $f^{\pm,\varepsilon}$. By bootstrap it follows that $\rho^{\pm,\varepsilon,per} \in C^\infty(\mathbb{T}^2 \times [0,T))$. □

***Lemma* 6.2 (Strong monotonicity of the solution in x_1)**
Let $T > 0$. For all initial data $\rho_0^\pm \in L^2_{loc}(\mathbb{R}^2)$ satisfying $(H1)$, $(H2)$ and $(H3)$, if $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}^2 \times [0,T))$ are solutions of system (P_ε)-(IC_ε), then $\dfrac{\partial\rho^{\pm,\varepsilon}}{\partial x_1} > 0$ for all $t \in (0,T)$.

Proof of Lemma 6.2:
First of all, remark that if $\dfrac{\partial\rho_0^\pm}{\partial x_1} \geq 0$, then $\dfrac{\partial\rho_0^{\pm,\varepsilon}}{\partial x_1} \geq \varepsilon$. Indeed, we have

$$\begin{aligned}\frac{\partial\rho_0^{\pm,\varepsilon}}{\partial x_1} = \frac{\partial\rho_0^{\pm,per}}{\partial x_1} * \eta_\varepsilon + L_\varepsilon &= \left(\frac{\partial\rho_0^{\pm,per}}{\partial x_1} + L\right) * \eta_\varepsilon + \varepsilon \\ &= \left(\frac{\partial\rho_0^{\pm}}{\partial x_1}\right) * \eta_\varepsilon + \varepsilon > 0,\end{aligned}$$

since η_ε is non-negative. Let us write the system obtained by derivation of (P_ε)-(IC_ε) with respect to x_1, that reads for $\theta^{\pm,\varepsilon} = \dfrac{\partial\rho^{\pm,\varepsilon}}{\partial x_1}$,

$$\begin{cases} \dfrac{\partial\theta^{\pm,\varepsilon}}{\partial t} - \varepsilon\Delta\theta^{\pm,\varepsilon} \pm (R_1^2R_2^2\rho^\varepsilon)\dfrac{\partial\theta^{\pm,\varepsilon}}{\partial x_1} \pm \left(R_1^2R_2^2(\theta^{+,\varepsilon} - \theta^{-,\varepsilon})\right)\theta^{\pm,\varepsilon} = 0 \quad \text{in} \quad \mathbb{T}^2 \times (0,T), \\ \theta^{\pm,\varepsilon}(x,0) = \dfrac{\partial\rho_0^{\pm,\varepsilon}}{\partial x_1} \quad \text{in} \quad \mathbb{T}^2. \end{cases}$$

Since $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}^2 \times [0,T))$ and $\theta^{\pm,\varepsilon}(\cdot,0) > \varepsilon$, we deduce from the maximum principle for scalar parabolic equations (see G. Lieberman [39, Th 2.10]), that $\theta^{\pm,\varepsilon} > 0$ on $\mathbb{T}^2 \times (0,T)$. $\square$

Corollary 6.3 (Short time existence of monotone smooth solutions) *For all initial data $\rho_0^\pm \in L^2_{loc}(\mathbb{R}^2)$ satisfying $(H1)$, $(H2)$ and $(H3)$, and all $\varepsilon > 0$, there exists*

$$T^\star(\|\rho_0^{\pm,\varepsilon,per}\|_{W^{1,\frac{3}{2}}(\mathbb{T}^2)}, L, \varepsilon) > 0,$$

such that the system (P_ε)-(IC_ε) admits solutions $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}^2 \times [0,T^\star))$ verifying $(H1)$, $(H2)$ for all $t \in [0,T^\star)$. Moreover $\dfrac{\partial \rho^{\pm,\varepsilon}}{\partial x_1} > 0$ for all $t \in [0,T^\star)$.

Corollary 6.3 is a consequence of Theorem 5.3 and of Lemmata 6.1 and 6.2.

7 ε-Uniform estimates on the solution of the regularized system

In this Section, we prove some fundamental ε-uniform estimates. In the Subsection 7.2 we give some general estimates independent on the system of equations. In the second Subsection 7.3 we establish a priori estimates on the solutions of system (P_ε).

7.1 Properties of Hardy space

Definition 7.1

i) **Hardy space,** *(C. Fefferman, E. M. Stein [26]):*

The Hardy space $\mathcal{H}^1(\mathbb{T}^2)$ is the set of functions $f \in L^1(\mathbb{T}^2)$ such that $R_i f \in L^1(\mathbb{T}^2)$ for $i = 1,2$. This space is endowed with the norm

$$\|f\|_{\mathcal{H}^1(\mathbb{T}^2)} = \|f\|_{L^1(\mathbb{T}^2)} + \sum_{i=1,2} \|R_i f\|_{L^1(\mathbb{T}^2)}.$$

ii) BMO **space,** *(John and Nirenberg, see C. Fefferman [25]):*

We say that $f \in L^1(\mathbb{T}^2)$ belongs to $BMO(\mathbb{T}^2)$ if an only if

$$\|f\|_{BMO} = \sup_B \left(\frac{1}{|B|} \int_B |f(x) - m_B(f)| dx \right) < +\infty \tag{7.21}$$

for every ball $B \subset \mathbb{T}^2$ where $m_B(f) = \dfrac{1}{|B|} \displaystyle\int_B f$.

Here $\|f\|_{BMO}$ defines a norm over $BMO(\mathbb{T}^2)$ quotiented by the constant functions. Moreover, the space $BMO(\mathbb{T}^2)$ is the dual of $\mathcal{H}^1(\mathbb{T}^2)$.

We refer to P. Koosis [38], R. Coifman, Y. Meyer [15], J. B. Garnett [27] and E. M. Stein [49] for other definitions of $\mathcal{H}^1(\mathbb{T}^2)$ and $BMO(\mathbb{T}^2)$. Here, this definition makes a sense thanks to the definition of the Riesz transform for L^p function, and the density in L^1 of the spaces L^p for $p > 1$.
The spaces $\mathcal{H}^1$ and BMO satisfy the following properties:

***Lemma* 7.2 (Stability of Riesz transform)**

(I1) The Riesz transforms R_i, for $i = 1, 2$, are linear continuous operators on $\mathcal{H}^1(\mathbb{T}^2)$ onto itself.

(I2) The Riesz transforms R_i, for $i = 1, 2$, are linear continuous operators on $BMO(\mathbb{T}^2)$ onto itself.

(I3) The Riesz transforms R_i, for $i = 1, 2$, are linear continuous operators on $L^p(\mathbb{T}^2)$, for all $1 < p < +\infty$ onto itself.

For the proof, see R. Coifman, Y. Meyer [15, Chap 5] and A. Zygmund [57, Vol I, Page 254, (2.6)].

***Lemma* 7.3 (Embeddings)**
For $1 < p < +\infty$, we have the following property:

$$L^\infty(\mathbb{T}^2) \hookrightarrow BMO(\mathbb{T}^2) \hookrightarrow EXP(\mathbb{T}^2) \hookrightarrow L^p(\mathbb{T}^2) \hookrightarrow L\log L(\mathbb{T}^2) \hookrightarrow \mathcal{H}^1(\mathbb{T}^2) \hookrightarrow L^1(\mathbb{T}^2).$$

For the proof, see C. Bennett and R. Sharpley [7, (7.22) Page 382 , (6.11) Page 247].

***Lemma* 7.4 (Zygmund's Lemma)**
If $f \geq 0$, then $f \in L\log L(\mathbb{T}^2)$ if and only if $f \in \mathcal{H}^1(\mathbb{T}^2)$. Moreover, there exists a constant C such that,

$$\|f\|_{\mathcal{H}^1(\mathbb{T}^2)} \leq C\left(\int_{\mathbb{T}^2} |f|\log(e + |f|)dx_1dx_2 + 1\right).$$

For the proof, see A. Zygmund [57, Vol. I, Chap 7, (2.8), (2.10)] and P. Koosis [38, Page 96-97]. See also, E. M. Stein [49, 5.3, Page 128] for a proof on $\mathbb{R}^N$. Under the assumptions $(H1)$, $(H2)$, $(H3)$, and $(H4)$, we deduce that $\dfrac{\partial \rho_0{}^{\pm}}{\partial x_1} \in \mathcal{H}^1(\mathbb{T}^2)$.

7.2 Useful estimates

***Lemma* 7.5 (BMO estimate)**
If f is a function defined on $\mathbb{R}^2 \times (0, T)$ and verifies $(H1)$, $(H2)$ and $(H3)$ for a.e. $t \in (0, T)$, then there exists a constant $C = C(L)$ such that,

$$\|R_1R_2f^{per}\|_{L^\infty((0,T);BMO(\mathbb{T}^2))} \leq C,$$

where $f^{per} = f - Lx_1$.

Proof of Lemma 7.5:
According to $(H1)$ and $(H3)$, we know that for a.e (x_2, t)

$$\int_0^1 \left|\frac{\partial f^{per}}{\partial, x_1}\right| dx_1 \le \int_0^1 \left|\frac{\partial f}{\partial x_1} - L\right| dx_1 \le \int_0^1 \left|\frac{\partial f}{\partial x_1}\right| dx_1 + L \le 2L.$$

We apply a "Poincaré-Wirtinger inequality" in x_1 and we deduce that there exists a constant $C = C(L)$ such that,

$$\left\| f^{per} - \int_0^1 f^{per} dx_1 \right\|_{L^\infty(\mathbb{T}^2\times(0,T))} \le C. \tag{7.22}$$

Moreover, $R_1R_2(f^{per} - \int_0^1 f^{per} dx_1) = R_1R_2(f^{per})$ since, we can check that $R_1\left(\int_0^1 f^{per} dx_1\right) = 0$.
We use Lemmata 7.3 and 7.2 $(I2)$ to obtain that $R_1R_2f^{per} \in L^\infty((0,T); BMO(\mathbb{T}^2))$. $\square$

***Lemma* 7.6 ($L\log L$ Estimate)**
Let $(\eta_\varepsilon)_\varepsilon$ be a non-negative mollifier, then for all $f \in L\log L(\mathbb{T}^2)$, the function $f_\varepsilon = f * \eta_\varepsilon$ satisfies

$$\|f - f_\varepsilon\|_{L\log L(\mathbb{T}^2)} \to 0 \qquad as \qquad \varepsilon \to 0.$$

For the proof see R. A. Adams [1, Th 8.20].

7.3 *A priori* Estimates

In this Subsection, we show some ε-uniform estimates on the solutions of the system (P_ε)-(IC_ε) obtained in Corollary 6.3. These estimates will be used, on one hand in Section 8 for the proof of long time existence, and on the other hand, in Subsection 9.2 for ensuring, by compactness, the passage to the limit as ε tends to zero.
The first estimate concerns the physical entropy of the system, and is a key result. It shows that in our model, the dislocations cannot be so concentrated. In other words, the dislocation densities can always be controlled.

***Lemma* 7.7 (Entropy estimate)**
Let $\rho_0^\pm \in L^2_{loc}(\mathbb{R}^2)$. Under the assumptions $(H1)$, $(H2)$, $(H3)$ and $(H4)$, if $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}^2 \times [0,T))$ are solutions of the system (P_ε)-(IC_ε), then there exists a constant C independent of ε such that,

$$\left\|\frac{\partial \rho^{\pm,\varepsilon}}{\partial x_1}\right\|_{L^\infty((0,T);L\log L(\mathbb{T}^2))} + \left\|\frac{\partial}{\partial x_1}(R_1R_2\rho^\varepsilon)\right\|_{L^2(\mathbb{T}^2\times(0,T))} \le C, \tag{7.23}$$

with $C = C\left(\left\|\frac{\partial \rho_0{}^\pm}{\partial x_1}\right\|_{L\log L(\mathbb{T}^2)}\right)$.

Proof of Lemma 7.7:
First of all, we denote $\theta^{\pm,\varepsilon} = \dfrac{\partial \rho^{\pm,\varepsilon}}{\partial x_1}$, $\theta^{\varepsilon} = \theta^{+,\varepsilon} - \theta^{-,\varepsilon}$ and

$$N^{\pm}(t) = \int_{\mathbb{T}^2} \theta^{\pm,\varepsilon}(t) \log(\theta^{\pm,\varepsilon}(t)).$$

Using the fact that $\rho^{\pm,\varepsilon} \in C^{\infty}(\mathbb{R}^2 \times [0,T))$, we can derive $N(t) = N^+(t) + N^-(t)$ with respect to t, since $\theta^{\pm,\varepsilon} > 0$ (see Lemma 6.2), and we obtain:

$$\begin{aligned}
\frac{d}{dt}N(t) &= \int_{\mathbb{T}^2} \sum_{+,-} (\theta^{\pm,\varepsilon})_t \log(\theta^{\pm,\varepsilon}) + \int_{\mathbb{T}^2} \sum_{+,-} (\theta^{\pm,\varepsilon})_t \\
&= \int_{\mathbb{T}^2} \sum_{+,-} \left(\mp (R_1^2 R_2^2 \rho^{\varepsilon})\theta^{\pm,\varepsilon} + \varepsilon \Delta \rho^{\pm,\varepsilon}\right)_{x_1} \log(\theta^{\pm,\varepsilon}) \\
&= \int_{\mathbb{T}^2} \sum_{+,-} \left(\left(\pm (R_1^2 R_2^2 \rho^{\varepsilon})\theta^{\pm,\varepsilon}\right) \frac{\theta^{\pm,\varepsilon}_{x_1}}{\theta^{\pm,\varepsilon}} + \varepsilon \Delta \theta^{\pm,\varepsilon} \log(\theta^{\pm,\varepsilon}) \right) \\
&= -\int_{\mathbb{T}^2} \sum_{+,-} \left(\pm (R_1^2 R_2^2 \theta^{\varepsilon})\theta^{\pm,\varepsilon}\right) - \varepsilon \sum_{+,-} \int_{\mathbb{T}^2} \frac{|\nabla \theta^{\pm,\varepsilon}|^2}{\theta^{\pm,\varepsilon}} \\
&= -\int_{\mathbb{T}^2} \left(R_1^2 R_2^2 \theta^{\varepsilon}\right) \theta^{\varepsilon} - \varepsilon \sum_{+,-} \int_{\mathbb{T}^2} \frac{|\nabla \theta^{\pm,\varepsilon}|^2}{\theta^{\pm,\varepsilon}} \\
&= -\int_{\mathbb{T}^2} (R_1 R_2 \theta^{\varepsilon})^2 - \varepsilon \sum_{+,-} \int_{\mathbb{T}^2} \frac{|\nabla \theta^{\pm,\varepsilon}|^2}{\theta^{\pm,\varepsilon}} \leq 0,
\end{aligned}$$

Integrating in time we get,

$$N(t) + \int_0^t \int_{\mathbb{T}^2} (R_1 R_2 \theta^{\varepsilon})^2 \leq N(0) \leq \int_{\mathbb{T}^2} \sum_{+,-} \theta^{\pm,\varepsilon}(0) \log(e + \theta^{\pm,\varepsilon}(0))$$

Since the initial data (IC) satisfies $(H4)$, we deduce by Lemma 7.6 that there exists a positive constant C independent of ε such that,

$$N(t) + \int_0^t \int_{\mathbb{T}^2} (R_1 R_2 \theta^{\varepsilon})^2 \leq C.$$

Let us now consider,

$$\begin{aligned}
N_1^{\pm}(t) &= \int_{\mathbb{T}^2} \theta^{\pm,\varepsilon}(t) \log(e + \theta^{\pm,\varepsilon}(t)) \\
&= \int_{\mathbb{T}^2 \cap \{0 < \theta^{\pm,\varepsilon} < e\}} \theta^{\pm,\varepsilon}(t) \log(e + \theta^{\pm,\varepsilon}(t)) + \int_{\mathbb{T}^2 \cap \{\theta^{\pm,\varepsilon} \geq e\}} \theta^{\pm,\varepsilon}(t) \log(e + \theta^{\pm,\varepsilon}(t)).
\end{aligned}$$

Using that $x \log(e+x) \leq e \log(2e)$ for all $0 < x \leq e$, we deduce that

$$
\begin{aligned}
N_1^{\pm}(t) &\leq e\log(2e) + \int_{\mathbb{T}^2\cap\{\theta^{\pm,\varepsilon}\geq e\}} \theta^{\pm,\varepsilon}(t)\log(2\theta^{\pm,\varepsilon}(t)) \\
&\leq e\log(2e) + \int_{\mathbb{T}^2} \theta^{\pm,\varepsilon}(t)\log(2) + \int_{\mathbb{T}^2\cap\{\theta^{\pm,\varepsilon}\geq e\}} \theta^{\pm,\varepsilon}(t)\log(\theta^{\pm,\varepsilon}(t)) \\
&\leq e\log(2e) + \log(2)(L+1) + N^{\pm}(t) - \int_{\mathbb{T}^2\cap\{0<\theta^{\pm,\varepsilon}<e\}} \theta^{\pm,\varepsilon}(t)\log(\theta^{\pm,\varepsilon}(t)) \\
&\leq C + N^{\pm}(t),
\end{aligned}
$$

where, in the last line we have used that $-x\log(x) \leq \frac{1}{e}$ for all $0 < x \leq e$. This finally lead to the following estimate:

$$
N_1^{+}(t) + N_1^{-}(t) + \int_0^t \int_{\mathbb{T}^2} (R_1 R_2 \theta^{\varepsilon})^2 \leq C,
$$

which implies (7.23). $\square$

Remark 7.8 **($W^{1,2}$ estimate)**
Since we have

$$
\frac{\partial}{\partial x_2} R_1^2 R_2^2 = R_1 R_2 \left(\frac{\partial}{\partial x_1} R_1 R_2\right) \quad \textit{and} \quad \frac{\partial}{\partial x_2} R_1^2 R_2^2 = R_2^2 \left(\frac{\partial}{\partial x_1} R_1 R_2\right),
$$

we deduce by Lemma 7.2 (I3), that $\nabla\left(R_1^2 R_2^2 \rho^{\varepsilon}\right) \in L^2\left(\mathbb{T}^2\times(0,T)\right)$ uniformly in ε.

Remark 7.9 **($\mathcal{H}^1$ estimate)**
Given $\theta^{\pm,\varepsilon} \geq 0$, we deduce from Lemma 7.4 that $\theta^{\pm,\varepsilon} \in L^{\infty}((0,T);\mathcal{H}^1(\mathbb{T}^2))$, uniformly in ε.

We now present a second a priori estimate.

***Lemma* 7.10** **(L^2 bound on the solutions)**
Let $T>0$. Under the condition $\rho_0^{\pm} \in L^2_{loc}(\mathbb{R}^2)$, and the assumptions $(H1)$, $(H2)$, $(H3)$ and $(H4)$, if $\rho^{\pm,\varepsilon} \in C^{\infty}(\mathbb{R}^2\times[0,T))$ are solutions of system (P_ε)-(IC_ε), then there exists a constant C independent of ε, but depending on T, such that:

$$
\left\|\rho^{\pm,\varepsilon,per}\right\|_{L^{\infty}((0,T);L^2(\mathbb{T}^2))} \leq C,
$$

with $\rho^{\pm,\varepsilon,per} = \rho^{\pm,\varepsilon} - Lx_1$.

Proof of Lemma 7.10:
We want to bound $m^{\pm,\varepsilon}(x_2,t) = \displaystyle\int_{\mathbb{T}} \rho^{\pm,\varepsilon,per}(x_1,x_2,t)dx_1$. There is no problem of regularity since $\rho^{\pm,\varepsilon} \in C^{\infty}(\mathbb{R}^2\times[0,T))$. We integrate equation (P_ε^{per}) with respect to x_1, and then integrate by parts the first term of the right hand side. This leads to,

$$\frac{\partial}{\partial t} m^{\pm,\varepsilon} - \varepsilon \frac{\partial^2 m^{\pm,\varepsilon}}{\partial x_2^2} = \pm \int_{\mathbb{T}} (R_1^2 R_2^2 \frac{\partial \rho^\varepsilon}{\partial x_1})(\rho^{\pm,\varepsilon,per} - m^{\pm,\varepsilon}) dx_1$$
$$\mp L_\varepsilon \int_{\mathbb{T}} (R_1^2 R_2^2 \rho^\varepsilon) dx_1 \pm m^{\pm,\varepsilon} \int_{\mathbb{T}} (R_1^2 R_2^2 \frac{\partial \rho^\varepsilon}{\partial x_1}) dx_1.$$

Using that ρ^ε is a 1-periodic function in x_1, the previous equation is equivalent to,

$$\frac{\partial}{\partial t} m^{\pm,\varepsilon} - \varepsilon \frac{\partial^2 m^{\pm,\varepsilon}}{\partial x_2^2} = \overbrace{\pm \int_{\mathbb{T}} (R_1^2 R_2^2 \frac{\partial \rho^\varepsilon}{\partial x_1})(\rho^{\pm,\varepsilon,per} - m^{\pm,\varepsilon}) dx_1}^{I_1^\pm} \quad \overbrace{\mp L_\varepsilon \int_{\mathbb{T}} (R_1^2 R_2^2 \rho^\varepsilon) dx_1}^{I_2^\mp}. \tag{7.24}$$

Let us denote the right hand side by $g^\pm = I_1^\pm + I_2^\mp$. We now show that $g^\pm \in L^2(\mathbb{T}\times(0,T))$. Indeed, we have,

$$\begin{aligned}
\|I_1^\pm\|_{L^2(\mathbb{T}\times(0,T))} &\le \left\| \int_{\mathbb{T}} (R_1^2 R_2^2 \frac{\partial \rho^\varepsilon}{\partial x_1})(\rho^{\pm,\varepsilon,per} - m^{\pm,\varepsilon}) dx_1 \right\|_{L^2(\mathbb{T}\times(0,T))} \\
&\le \|\rho^{\pm,\varepsilon,per} - m^{\pm,\varepsilon}\|_{L^\infty(\mathbb{T}^2\times(0,T))} \left\| R_1^2 R_2^2 \frac{\partial \rho^\varepsilon}{\partial x_1} \right\|_{L^2(\mathbb{T}^2\times(0,T))} \\
&\le C,
\end{aligned}$$

where for the last line we used Lemma 7.7 to bound $\left\| R_1^2 R_2^2 \frac{\partial \rho^\varepsilon}{\partial x_1} \right\|_{L^2(\mathbb{T}^2\times(0,T))}$ and the fact that the Riesz transforms are continuous from L^2 onto itself. Furthermore, the bound on $\|\rho^{\pm,\varepsilon,per} - m^{\pm,\varepsilon}\|_{L^\infty(\mathbb{T}^2\times(0,T))}$ follows from (7.22).

For the term $I_2^\mp$, recall that $0 < \varepsilon \le 1$, hence

$$\|I_2^\mp\|_{L^2(\mathbb{T}\times(0,T))} \le \left\| (L+1) \int_{\mathbb{T}} (R_1^2 R_2^2 \rho^\varepsilon) dx_1 \right\|_{L^2(\mathbb{T}\times(0,T))} \le C T^{\frac{1}{2}},$$

where for the last inequality we have used that $R_1^2 R_2^2 \rho^\varepsilon \in L^\infty((0,T); BMO(\mathbb{T}^2))$ (see Lemma 7.5) and the embeddings of Lemma 7.3. Therefore, we get,

$$\|g^\pm\|_{L^2(\mathbb{T}\times(0,T))} \le C(1 + T^{\frac{1}{2}}).$$

To end the proof, we multiply equation (7.24) by $m^{\pm,\varepsilon}$, and we integrate in space. This gives,

$$\frac{1}{2} \frac{d}{dt} \|m^{\pm,\varepsilon}(t)\|^2_{L^2(\mathbb{T})} + \varepsilon \left\| \frac{\partial}{\partial x_2} m^{\pm,\varepsilon}(t) \right\|^2_{L^2(\mathbb{T})} = \int_{\mathbb{T}} g^\pm m^{\pm,\varepsilon}.$$

We integrate in time, to obtain,

$$
\begin{aligned}
\frac{1}{2}\|m^{\pm,\varepsilon}\|^2_{L^\infty((0,T);L^2(\mathbb{T}))} &\leq \|g^\pm\|_{L^2(\mathbb{T}\times(0,T))}\|m^{\pm,\varepsilon}\|_{L^2(\mathbb{T}\times(0,T))} + \frac{1}{2}\|m^{\pm,\varepsilon}(0)\|^2_{L^2(\mathbb{T})} \\
&\leq T^{\frac{1}{2}}\|g^\pm\|_{L^2(\mathbb{T}\times(0,T))}\|m^{\pm,\varepsilon}\|_{L^\infty((0,T);L^2(\mathbb{T}))} + \frac{1}{2}\|m^{\pm,\varepsilon}(0)\|^2_{L^2(\mathbb{T})} \\
&\leq T\|g^\pm\|^2_{L^2(\mathbb{T}\times(0,T))} + \frac{1}{4}\|m^{\pm,\varepsilon}\|^2_{L^\infty((0,T);L^2(\mathbb{T}))} + \frac{1}{2}\|m^{\pm,\varepsilon}(0)\|^2_{L^2(\mathbb{T})}.
\end{aligned}
$$

Therefore

$$
\|m^{\pm,\varepsilon}\|^2_{L^\infty((0,T);L^2(\mathbb{T}))} \leq 4T\|g^\pm\|^2_{L^2(\mathbb{T}\times(0,T))} + 2\|m^{\pm,\varepsilon}(0)\|^2_{L^2(\mathbb{T})}.
$$

We now bound the term $\|m^{\pm,\varepsilon}(0)\|^2_{L^2(\mathbb{T})}$. We have,

$$
\begin{aligned}
\|m^{\pm,\varepsilon}(0)\|^2_{L^2(\mathbb{T})} &= \int_{\mathbb{T}}\left|\int_{\mathbb{T}}\rho_0^{\pm,\varepsilon,per}(x_1,x_2)dx_1\right|^2 dx_2 \\
&\leq \|\eta_\varepsilon * \rho_0^{\pm,per}\|^2_{L^2(\mathbb{T}^2)} \\
&\leq \|\rho_0^{\pm,per}\|^2_{L^2(\mathbb{T}^2)},
\end{aligned}
$$

where we have used Hölder's inequality for the second line, and that $\|\eta_\varepsilon\|_{L^1(\mathbb{T}^2)} = 1$. This indicates that for a constant C independent of ε, $\|m^{\pm,\varepsilon}\|_{L^\infty((0,T);L^2(\mathbb{T}))} \leq C$.

Finally, we use estimate (7.22) to deduce that $\rho^{\pm,\varepsilon,per}$ is bounded in $L^\infty((0,T);L^2(\mathbb{T}^2))$ uniformly in ε. $\square$

The following estimate will provide compactness in time of the solution, uniform with respect to ε .

***Lemma* 7.11 (Duality estimate of Riesz transform for the time derivative of the solution)**
Let $T>0$. Under the assumptions $\rho_0^\pm \in L^2_{loc}(\mathbb{R}^2)$, $(H1)$, $(H2)$, $(H3)$ and $(H4)$, if $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}^2 \times [0,T))$ are solutions of the system (P_ε)-(IC_ε), then for all $\psi \in L^2((0,T);W^{1,2}(\mathbb{T}^2))$, there exists a constant C independent of ε such that:

$$
\left|\int_{\mathbb{T}^2\times(0,T)}\psi R_1^2R_2^2\left(\frac{\partial\rho^\varepsilon}{\partial t}\right)\right| \leq C\|\psi\|_{L^2((0,T);W^{1,2}(\mathbb{T}^2))},
$$

where $\rho^\varepsilon = \rho^{+,\varepsilon} - \rho^{-,\varepsilon}$.

Proof of Lemma 7.11:
The idea is somehow to bound $R_1^2R_2^2\left(\frac{\partial\rho^\varepsilon}{\partial t}\right)$ using the available bounds on the right hand side of the equation (P_ε).

We will give a proof by duality. First of all, we subtract the two equations of system (P_ε) to obtain that,

$$\frac{\partial \rho^\varepsilon}{\partial t} = -(R_1^2 R_2^2 \rho^\varepsilon)\left(\frac{\partial \rho^{+\varepsilon}}{\partial x_1} + \frac{\partial \rho^{-\varepsilon}}{\partial x_1}\right) + \varepsilon \Delta \rho^{\pm,\varepsilon}.$$

We apply the Riesz transform $R_1^2 R_2^2$, which gives,

$$R_1^2 R_2^2 \left(\frac{\partial \rho^\varepsilon}{\partial t}\right) = -\overbrace{R_1^2 R_2^2\left((R_1^2 R_2^2 \rho^\varepsilon)\frac{\partial k^\varepsilon}{\partial x_1}\right)}^{I_1} + \overbrace{\varepsilon R_1^2 R_2^2 \left(\Delta \rho^\varepsilon\right)}^{I_2}, \tag{7.25}$$

with $k^\varepsilon = \rho^{+,\varepsilon} + \rho^{-,\varepsilon}$. In what follows, we will prove that for a function $\psi \in L^2((0,T); W^{1,2}(\mathbb{T}^2))$, we can bound $J_i = \displaystyle\int_{\mathbb{T}^2\times(0,T)} \psi I_i$ for $i = 1, 2$.

<u>**Estimate of J_1**</u>: to control J_1, we rewrite it under the following form:

$$\int_{\mathbb{T}^2\times(0,T)} R_1^2 R_2^2 \left((R_1^2 R_2^2 \rho^\varepsilon)\frac{\partial k^\varepsilon}{\partial x_1}\right)\psi = \int_{\mathbb{T}^2\times(0,T)} \left((R_1^2 R_2^2 \rho^\varepsilon)\frac{\partial k^\varepsilon}{\partial x_1}\right) R_1^2 R_2^2(\psi).$$

We use the fact that,

(i) $(R_1^2 R_2^2 \rho^\varepsilon)$ is bounded in $L^\infty((0,T); W^{1,2}(\mathbb{T}^2))$ uniformly in ε (by Lemma 7.7),

(ii) $\dfrac{\partial k^\varepsilon}{\partial x_1}$ is bounded in $L^\infty((0,T); L\log L(\mathbb{T}^2))$, uniformly in ε (by Lemma 7.7).

We deduce from this and from Proposition 4.7, (with $f = R_1^2 R_2^2 \rho^\varepsilon$ and $g = \dfrac{\partial k^\varepsilon}{\partial x_1}$) the following estimate:

$$\begin{aligned}
\left\|(R_1^2 R_2^2 \rho^\varepsilon)\frac{\partial k^\varepsilon}{\partial x_1}\right\|_{L^2((0,T);L\log^{\frac{1}{2}} L(\mathbb{T}^2))} &\le C\|R_1^2 R_2^2 \rho^\varepsilon\|_{L^2((0,T);W^{1,2}(\mathbb{T}^2))}\left\|\frac{\partial k^\varepsilon}{\partial x_1}\right\|_{L^2((0,T);L\log L(\mathbb{T}^2))}\\
&\le C\left\|\frac{\partial k^\varepsilon}{\partial x_1}\right\|_{L^\infty((0,T);L\log L(\mathbb{T}^2))} \le C.
\end{aligned}$$

We use Lemma 4.3 (i), to deduce that

$$\begin{aligned}
|J_1| &\le \left|\int_{\mathbb{T}^2\times(0,T)} \left((R_1^2 R_2^2 \rho^\varepsilon)\frac{\partial k^{\ \varepsilon}}{\partial x_1}\right) R_1^2 R_2^2(\psi)\right|\\
&\le \left\|(R_1^2 R_2^2 \rho^\varepsilon)\frac{\partial k^{\ \varepsilon}}{\partial x_1}\right\|_{L^2((0,T);L\log^{\frac{1}{2}} L(\mathbb{T}^2))}\|R_1^2 R_2^2 \psi\|_{L^2((0,T);EXP_2(\mathbb{T}^2))}\\
&\le C\,\|R_1^2 R_2^2 \psi\|_{L^2((0,T);W^{1,2}(\mathbb{T}^2))}\\
&\le C\,\|\psi\|_{L^2((0,T);W^{1,2}(\mathbb{T}^2))},
\end{aligned} \tag{7.26}$$

where we have used the Trudinger embedding (see Lemma 4.5) in the third line and the fact that Riesz transforms are continuous from $W^{1,2}$ onto itself in the last line.
Estimate of J_2: to estimate J_2, we integrate by parts, to get:

$$J_2 = -\varepsilon \int_{\mathbb{T}^2 \times (0,T)} \nabla (R_1^2 R_2^2 \rho^\varepsilon) \cdot \nabla \psi.$$

Since $R_1^2 R_2^2 \rho^\varepsilon$ is bounded in $L^2((0,T); W^{1,2}(\mathbb{T}^2))$, we deduce that for all $0 < \varepsilon \leq 1$:

$$\begin{aligned} |J_2| &\leq \left| \int_{\mathbb{T}^2 \times (0,T)} \nabla (R_1^2 R_2^2 \rho^\varepsilon) \cdot \nabla \psi \right| \\ &\leq C \, \|R_1^2 R_2^2 \rho^\varepsilon\|_{L^2((0,T);W^{1,2}(\mathbb{T}^2))} \, \|\psi\|_{L^2((0,T);W^{1,2}(\mathbb{T}^2))}. \end{aligned} \tag{7.27}$$

Finally, collecting (7.26) and (7.27) together with (7.25) and the definitions of J_i, for $i = 1, 2$, we get that there exists a constant C independent of ε such that,

$$\left| \int_{\mathbb{T}^2 \times (0,T)} \psi R_1^2 R_2^2 (\frac{\partial \rho^\varepsilon}{\partial t}) \right| \leq C \|\psi\|_{L^2((0,T);W^{1,2}(\mathbb{T}^2))}.$$

□

Remark 7.12 **($W^{-1,2}$ estimate)**
Let $W^{-1,2}(\mathbb{T}^2)$ be the dual space of $W^{1,2}(\mathbb{T}^2)$. Thanks to the previous lemma we deduce that there exists a constant C independent of ε such that,

$$\left\| R_1^2 R_2^2 \left(\frac{\partial \rho^\varepsilon}{\partial t} \right) \right\|_{L^2((0,T);W^{-1,2}(\mathbb{T}^2))} \leq C.$$

These three estimates made in Lemmata 7.7, 7.10 and 7.11 are sufficient to obtain the required compactness. This compactness ensures in Subsection 9.2 the passage to the limit which allows us to show the existence of solutions.

***Lemma* 7.13** **(Duality estimate for the time derivative of the solution)**
Let $T > 0$. Under the assumptions $\rho_0^\pm \in L^2_{loc}(\mathbb{R}^2)$, $(H1)$, $(H2)$, $(H3)$ and $(H4)$, if $\rho^{\pm,\varepsilon} \in C^\infty(\mathbb{R}^2 \times [0,T))$ are solutions of the system (P_ε)-(IC_ε), then for all $\psi \in L^2((0,T); W^{2,2}(\mathbb{T}^2))$, there exists a constant C independent of ε such that,

$$\left| \int_{\mathbb{T}^2 \times (0,T)} \psi \left(\frac{\partial \rho^{\pm,\varepsilon}}{\partial t} \right) \right| \leq C \|\psi\|_{L^2((0,T);W^{2,2}(\mathbb{T}^2))}.$$

The proof of this lemma is similar to that of Lemma 7.11. The only difference is that we integrate by parts the viscosity term twice and use the estimate of Lemma 7.10.

Remark 7.14 **(The sense of the initial condition)**
According to this lemma and Lemma 7.10 , we have $\rho^{\pm,\varepsilon,per} \in C([0,T), W^{-2,2}(\mathbb{T}^2))$ uniformly in ε with $W^{-2,2}(\mathbb{T}^2)$ is the dual space of $W^{2,2}(\mathbb{T}^2)$. This will give later a sense to the limit of the initial conditions.

8 Global existence for the regularized system

In this Section, we will prove the global existence of solutions for the system (P_ε)-(IC_ε) using the previous a priori estimates (proven in Lemmata 7.5 and 7.7).

Before going into the proof, we need the following lemma.

***Lemma* 8.1 ($W^{1,\frac{3}{2}}$ estimate)**
For all initial data $\rho_0^\pm \in L^2_{loc}(\mathbb{R}^2)$ satisfying $(H1)$ and $(H2)$, if $\rho^{\pm,\varepsilon,per} \in C^\infty(\mathbb{T}^2 \times [0,T))$, are solutions of the Mild integral problem (In_ε), then there exists a constant $C = C(\varepsilon, L)$ such that,

$$\|\rho^{\pm,\varepsilon,per}\|_{L^\infty((0,T);W^{1,\frac{3}{2}}(\mathbb{T}^2))} \leq B_0^\pm + CT^{\frac{1}{24}} \|R_1^2R_2^2\rho^\varepsilon\|_{L^\infty((0,T);L^8(\mathbb{T}^2))} \left(\left\| \frac{\partial \rho^{\pm,\varepsilon}}{\partial x_1} \right\|_{L^\infty((0,T);L^1(\mathbb{T}^2))} + 1 \right),$$

where $B_0^\pm = \|\rho_0^{\pm,\varepsilon,per}\|_{W^{1,\frac{3}{2}}(\mathbb{T}^2)}$.

Proof of Lemma 8.1:
If we denote $\rho_v^\varepsilon = (\rho^{+,\varepsilon,per}, \rho^{-,\varepsilon,per})$ and $\rho_{0,v}^\varepsilon = (\rho_0^{+,\varepsilon,per}, \rho_0^{-,\varepsilon,per})$, then we have shown that ρ_v^ε satisfies (5.12), namely,

$$\rho_v^\varepsilon(x,t) = S(t)\rho_{0,v}^\varepsilon + B(\rho_v^\varepsilon, \rho_v^\varepsilon)(t) + A(\rho_v^\varepsilon)(t), \tag{8.28}$$

where B and A are defined in (5.13) and (5.14) respectively and where $S_\varepsilon(t) = S_1(\varepsilon t)$. Moreover, using (5.15) with $u = v = \rho_v^\varepsilon$, we get,

$$\begin{aligned} \|B(\rho_v^\varepsilon, \rho_v^\varepsilon)(t)\|_{(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2} \leq & \int_0^t \left\| S_\varepsilon(t-s) \left(\left(R_1^2R_2^2\rho^\varepsilon(s)\right) \frac{\partial \rho_v^\varepsilon}{\partial x_1}(s) \right) ds \right\|_{(L^4(\mathbb{T}^2))^2} \\ & + \int_0^t \left\| \nabla S_\varepsilon(t-s) \left(\left(R_1^2R_2^2\rho^\varepsilon(s)\right) \frac{\partial \rho_v^\varepsilon}{\partial x_1}(s) \right) ds \right\|_{(L^{\frac{3}{2}}(\mathbb{T}^2))^2}. \end{aligned}$$

We use now Lemma 5.8 (i) with $r = 4, q = \frac{24}{5}, p = 1$ to estimate the first term, and Lemma 5.8 (ii) with $r = \frac{3}{2}, q = 8, p = 1$ to estimate the second term. It gives for $t \in (0,T)$, that,

$$\begin{aligned} \|B(\rho_v^\varepsilon, \rho_v^\varepsilon)(t)\|_{(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2} &\leq C \int_0^t \frac{1}{(t-s)^{\frac{23}{24}}} \left\|R_1^2R_2^2\rho^\varepsilon(s)\right\|_{L^8(\mathbb{T}^2)} \left\| \frac{\partial \rho_v^\varepsilon}{\partial x_1}(s) \right\|_{(L^1(\mathbb{T}^2))^2} ds \\ &\leq C \sup_{0\leq s<T} \left(\left\|R_1^2R_2^2\rho^\varepsilon(s)\right\|_{L^8(\mathbb{T}^2)} \right) \sup_{0\leq s<T} \left(\left\| \frac{\partial \rho_v^\varepsilon}{\partial x_1}(s) \right\|_{(L^1(\mathbb{T}^2))^2} \right) \int_0^t \frac{1}{(t-s)^{\frac{23}{24}}}. \end{aligned}$$

That leads,

$$\|B(\rho_v^\varepsilon,\rho_v^\varepsilon)\|_{L^\infty((0,T);(W^{1,\frac{3}{2}}(\mathbb{T}^2))^2)} \leq CT^{\frac{1}{24}}\|R_1^2R_2^2\rho^\varepsilon\|_{L^\infty((0,T);L^8(\mathbb{T}^2))}\left\|\frac{\partial\rho_v^\varepsilon}{\partial x_1}\right\|_{L^\infty((0,T);(L^1(\mathbb{T}^2))^2)}. \tag{8.29}$$

Similarly, we show that,

$$\|A(\rho_v^\varepsilon)\|_{L^\infty((0,T);W^{1,\frac{3}{2}}(\mathbb{T}^2))} \leq CT^{\frac{1}{24}}\|R_1^2R_2^2\rho^\varepsilon\|_{L^\infty((0,T);L^8(\mathbb{T}^2))}. \tag{8.30}$$

By using (8.29), (8.30) and (5.18), and the equation (8.28) we get the proof. □

Theorem 8.2 (Global existence)
For all $T,\varepsilon>0$ and for all initial data $\rho_0^\pm\in L^2_{loc}(\mathbb{R}^2)$ satisfies $(H1)$, $(H2)$, $(H3)$ and $(H4)$ the system (P_ε)-(IC_ε) admits solutions $\rho^{\pm,\varepsilon}\in C^\infty(\mathbb{R}^2\times[0,T))$. Moreover, this solution satisfies $(H1)$, $(H2)$ and $(H3)$ for all $t\in(0,T)$ and the estimates given in Lemmata 7.7, 7.10 and 7.11.

Proof of Theorem 8.2:
In Theorem 8.2, we prove that the local solutions given by Corollary 6.3 can be extended to some global ones. We argue by contradiction. Suppose that there exists a maximum time T_{max} such that we have the existence of solutions of (P_ε)-(IC_ε) in $C^\infty(\mathbb{R}^2\times[0,T_{max}))$.

For $\delta>0$, we reconsider the system (P_ε) with the initial data

$$\rho_{\delta,max}^{\pm,\varepsilon}=\rho^{\pm,\varepsilon}(x,T_{max}-\delta).$$

we reapply for the second time, the proof of Corollary 6.3, we deduce that there exists a time

$$T^\star_{\delta,max}(\|\rho_{\delta,max}^{\pm,\varepsilon,per}\|_{W^{1,\frac{3}{2}}(\mathbb{T}^2)},L,\varepsilon)>0, \quad \text{where} \quad \rho_{\delta,max}^{\pm,\varepsilon,per}=\rho_{\delta,max}^{\pm,\varepsilon}-Lx_1,$$

such that the system (P_ε)-(IC_ε) admits solutions defined until,

$$T_0=(T_{max}-\delta)+T^\star_{\delta,max}.$$

Moreover, by Lemmata 8.1, 7.7 and 7.5, we know that $\rho_{\delta,max}^{\pm,\varepsilon,per}$ are δ-uniformly bounded in $W^{1,\frac{3}{2}}(\mathbb{T}^2)$. By using (5.19), we deduce that there exists a constant $C(\varepsilon,T_{max},L)>0$ independent of δ such that $T^\star_{\delta,max}\geq C>0$. Then $\liminf_{\delta\to0}T^\star_{\delta,max}\geq C>0$. Hence $T_0>T_{max}$ which gives the contradiction. □

9 Existence of solutions for the system (P)-(IC)

In this section, we will prove that the system (P)-(IC) admits solutions $\rho^\pm$ in the distributional sense. They are the limits of $\rho^{\pm,\varepsilon}$ given by Theorem 8.2 when $\varepsilon\to0$. To do this, we will justify the passage to the limit as ε tends to 0 in the system (P_ε^{per})-(IC_ε^{per}) by using some compactness arguments.

9.1 Preliminary results

***Lemma* 9.1 (Trudinger compact embedding)**
The following injection (see N. S. Trudinger [50]):

$$W^{1,2}(\mathbb{T}^2) \hookrightarrow EXP_\beta(\mathbb{T}^2),$$

is compact, for all $1 \le \beta < 2$.

For the proof of this lemma see also R. A. Adams [1, Th 8.32].

***Lemma* 9.2 (Simon's Lemma)**
Let X, B, Y *three Banach spaces, where* $X \hookrightarrow B$ *with compact embedding and* $B \hookrightarrow Y$ *with continuous embedding. If* $(\rho^n)_n$ *is a sequence such that,*

$$\|\rho^n\|_{L^q((0,T);B)} + \|\rho^n\|_{L^1((0,T);X)} + \left\|\frac{\partial \rho^n}{\partial t}\right\|_{L^1((0,T);Y)} \le C,$$

where $q > 1$ *and* C *is a constant independent of* n, *then* $(\rho^n)_n$ *is relatively compact in* $L^p((0,T);B)$ *for all* $p < q$.

For the proof, see J. Simon [48, Th 6, Page 86].
In order to show the existence of system (P) in Subsection 9.2, we apply this lemma in the particular cases where $B = EXP_\beta(\mathbb{T}^2)$, $X = W^{1,2}(\mathbb{T}^2)$ and $Y = W^{-1,2}(\mathbb{T}^2)$, for $1 < \beta < 2$.

***Lemma* 9.3 (Weak star topology in $L\log L$)**
Let $E_{exp}(\mathbb{T}^2)$ *be the closure in* $EXP(\mathbb{T}^2)$ *of the space of functions bounded on* $\mathbb{T}^2$. *Then* $E_{exp}(\mathbb{T}^2)$ *is a separable Banach space which verifies,*

i) $L\log L(\mathbb{T}^2)$ *is the dual space of* $E_{exp}(\mathbb{T}^2)$.

ii) $EXP_\beta(\mathbb{T}^2) \hookrightarrow E_{exp}(\mathbb{T}^2) \hookrightarrow EXP(\mathbb{T}^2)$ *for all* $\beta > 1$.

For the proof, see R. A. Adams [1, Th 8.16, 8.18, 8.20].

9.2 Proof of Theorem 1.4

Step 1 (Passage to the limit):

First, by Lemma 7.10 we know that for any $T > 0$, the solutions $\rho^{\pm,\varepsilon,per}$ of the system (P_ε^{per})-(IC_ε^{per}) obtained with the help of Theorem 8.2, are ε-uniformly bounded in $L^2(\mathbb{T}^2 \times (0,T))$. Hence, as ε goes to zero, we can extract a subsequence still denoted by $\rho^{\pm,\varepsilon,per}$, that converges weakly in $L^2(\mathbb{T}^2 \times (0,T))$ to some limit $\rho^{\pm,per}$. Then we want to prove that $\rho^\pm = \rho^{\pm,per} + Lx_1$ are solutions of the system (P)-(IC). Indeed, since the passage to the limit in the linear term is trivial in $\mathcal{D}'(\mathbb{T}^2 \times (0,T))$, it suffices to pass to the limit in the non-linear term,

$$(R_1^2 R_2^2 \rho^\varepsilon)\frac{\partial}{\partial x_1}\rho^{\pm,\varepsilon,per}.$$

• From Lemmata 7.7 and 7.5 we know that the term $(R_1^2R_2^2\rho^\varepsilon)$ is ε-uniformly bounded in $L^2((0,T);W^{1,2}(\mathbb{T}^2))$. Then it is in particular ε-uniformly bounded in $L^1((0,T);W^{1,2}(\mathbb{T}^2))$.
• From thes previous point and Lemma 9.1, we know that $(R_1^2R_2^2\rho^\varepsilon)$ is also ε-uniformly bounded in $L^2((0,T);EXP_\beta(\mathbb{T}^2))$ for all $1 \leq \beta < 2$.
• From Lemma 7.11, the term $R_1^2R_2^2(\dfrac{\partial \rho^\varepsilon}{\partial t})$ is ε-uniformly bounded in $L^2((0,T);W^{-1,2}(\mathbb{T}^2))$ and then in $L^1((0,T);W^{-1,2}(\mathbb{T}^2))$.

Collecting this, we get that there exists a constant C independent on ε such that $\bar{\rho}^\varepsilon = R_1^2R_2^2\rho^\varepsilon$ satisfies for some $1<\beta<2$

$$\|\bar{\rho}^\varepsilon\|_{L^2((0,T);EXP_\beta(\mathbb{T}^2))} + \|\bar{\rho}^\varepsilon\|_{L^1((0,T);W^{1,2}(\mathbb{T}^2))} + \left\|\frac{\partial \bar{\rho}^\varepsilon}{\partial t}\right\|_{L^1((0,T);W^{-1,2}(\mathbb{T}^2))} \leq C.$$

Then Lemma 9.2, with $B = EXP_\beta(\mathbb{T}^2)$, $X = W^{1,2}(\mathbb{T}^2)$ and $Y = W^{-1,2}(\mathbb{T}^2)$, shows the relative compactness of $(R_1^2R_2^2\rho^\varepsilon)$ in $L^1((0,T);EXP_\beta(\mathbb{T}^2))$, and then using Lemma 9.3, we have the compactness in $L^1((0,T);E_{exp}(\mathbb{T}^2))$.

Moreover, by Lemma 7.7, we have that $\dfrac{\partial \rho}{\partial x_1}^{\pm,\varepsilon,per}$ is ε-uniformly bounded in $L^\infty((0,T);L\log L(\mathbb{T}^2))$ which is the dual of $L^1((0,T);E_{exp}(\mathbb{T}^2))$ by Lemma 9.3 (see T. Cazenave and A. Haraux [11, Th 1.4.19, Page 17]). Then, this final term converges weakly $\star$ in $L^\infty((0,T);L\log L(\mathbb{T}^2))$ toward $\dfrac{\partial \rho}{\partial x_1}^{\pm,per}$. That enables us to pass to the limit in the bilinear term in the sense

$$L^1((0,T);E_{exp}(\mathbb{T}^2)) - strong \ \times \ L^\infty((0,T);L\log L(\mathbb{T}^2)) - weak\ \star.$$

In what precedes, we have shown that $\rho^\pm = \rho^{\pm,per} + Lx_1$ are solutions of the following equation:

$$\begin{aligned}\frac{\partial \rho^\pm}{\partial t} &= \mp(R_1^2R_2^2\rho)\,\frac{\partial \rho}{\partial x_1}^{\pm,per} \mp L(R_1^2R_2^2\rho)\\ &= \mp(R_1^2R_2^2\rho)\,\frac{\partial \rho}{\partial x_1}^{\pm}.\end{aligned}$$

Therefore $\rho^\pm$ is solutions of system (P) which has the same bounds as $\rho^{\pm,\varepsilon}$. At this stage we remark that, by Proposition 4.7, the second term of the previous system is in $L^2((0,T);L\log^{\frac{1}{2}}L(\mathbb{T}^2))$, which gives that $\dfrac{\partial \rho^\pm}{\partial t} \in L^2((0,T);L\log^{\frac{1}{2}}L(\mathbb{T}^2))$, and then

$\rho^{\pm,per} \in C([0,T); L\log^{\frac{1}{2}} L(\mathbb{T}^2))$.

Step 2 (The initial conditions):

It remains to prove the the initial conditions (IC) coincides with $\rho^{\pm}(\cdot,0)$. Indeed, from the estimates of $\rho^{\pm,\varepsilon,per}$ and $\dfrac{\partial\rho^{\pm,\varepsilon,per}}{\partial t}$ done in Lemmata 7.10 and 7.13, we see that $\rho^{\pm,\varepsilon}$ is ε-uniformly bounded in

$$W^{1,2}((0,T); W^{-2,2}(\mathbb{T}^2)) \hookrightarrow C^{\frac{1}{2}}([0,T); W^{-2,2}(\mathbb{T}^2)),$$

where $W^{-2,2}(\mathbb{T}^2)$ is the dual of $W^{2,2}(\mathbb{T}^2)$. It follows that, there exists a constant C independent on ε, such that, for all $t,s \in [0,T)$:

$$\|\rho^{\pm,\varepsilon,per}(t) - \rho^{\pm,\varepsilon,per}(s)\|_{W^{-2,2}(\mathbb{T}^2)} \leq C|t-s|^{\frac{1}{2}}.$$

In particular if we set $s=0$, we have

$$\|\rho^{\pm,\varepsilon,per}(t) - \rho_0^{\pm,\varepsilon,per}\|_{W^{-2,2}(\mathbb{T}^2)} \leq Ct^{\frac{1}{2}}. \tag{9.31}$$

Now we pass to the limit in (9.31). Indeed, the functions $\rho^{\pm,\varepsilon,per}$ and $\rho_0^{\pm,\varepsilon,per}$ are ε-uniformly bounded in $W^{1,2}((0,T); W^{-2,2}(\mathbb{T}^2))$ and $W^{-2,2}(\mathbb{T}^2)$ respectively. Moreover we know that $\rho^{\pm,\varepsilon,per} - \rho_0^{\pm,\varepsilon,per}$ converges weakly in $L^2(\mathbb{T}^2 \times (0,T))$ to $(\rho^{\pm,\varepsilon,per} - \rho_0^{\pm,\varepsilon,per})$.

Therefore, we can extract a subsequence still denoted by $(\rho^{\pm,\varepsilon,per} - \rho_0^{\pm,\varepsilon,per})$, that weakly converges in $W^{1,2}((0,T); W^{-2,2}(\mathbb{T}^2))$ to $(\rho^{\pm,per} - \rho_0^{\pm,per})$. This is possible because $W^{-2,2}(\mathbb{T}^2) = (W^{2,2}(\mathbb{T}^2))'$ and $W^{1,2}(\mathbb{T}^2) = (W^{-1,2}(\mathbb{T}^2))'$. In particular this subsequence converges, for all $t \in (0,T)$, weakly $\star$ in $L^\infty((0,t); W^{-2,2}(\mathbb{T}^2))$, and consequently it verifies (see for instance H. Brezis [9, Prop. 3.12]),

$$\|\rho^{\pm,per} - \rho_0^{\pm,per}\|_{L^\infty((0,t);W^{-2,2}(\mathbb{T}^2))} \leq \liminf \|\rho^{\pm,\varepsilon,per} - \rho_0^{\pm,\varepsilon,per}\|_{L^\infty((0,t);W^{-2,2}(\mathbb{T}^2))} \leq Ct^{\frac{1}{2}}.$$

From (9.31) we deduce that

$$\|\rho^{\pm,per}(t) - \rho_0^{\pm,per}\|_{W^{-2,2}(\mathbb{T}^2)} \leq Ct^{\frac{1}{2}}.$$

Which proves that $\rho^{\pm}(\cdot,0) = \rho_0^{\pm}$ in $\mathcal{D}'(\mathbb{R}^2)$. $\square$

Remark 9.4
In step **1.** *of the proof, we indirectly used the fact that $\bar{\rho}^\varepsilon$ is bounded in $L^2((0,T); W^{1,2}(\mathbb{T}^2))$ and $\dfrac{\partial\bar{\rho}^\varepsilon}{\partial t}$ is bounded in $L^2((0,T); W^{-1,2}(\mathbb{T}^2))$. The usual compactness result (see L. C. Evans [24, P. 5.9.2]) asserts that we have compactness of the sequance in $C((0,T); L^2(\mathbb{T}^2))$. Here we work in dimension 2, and we use another result which asserts that we have, in particular, compactness in $L^1((0,T); EXP_\beta(\mathbb{T}^2))$ for every $1<\beta<2$.*

Remark 9.5 (BMO **times** $\mathcal{H}^1$)
We notice, using Lemma 7.5 and Remark 7.9, that we can also define the bilinear term of our system as the product duality between $L^2((0,T);\mathcal{H}^1(\mathbb{T}^2))$ *and* $L^2((0,T);BMO(\mathbb{T}^2))$.

Remark 9.6
In our proof, we have indirectly used a kind of compensated compactness technic for Hardy spaces. This technic allows to pass to the limit in a scalar product B.E "weak times weak", if we have some regularity conditions on "div E" and on "curl B" (see R. Coifman et al. [14]). In our case, we do not have enough regularity to do so.

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