

Global continuous solutions to diagonalizable hyperbolic systems with large and monotone data

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Global continuous solutions to diagonalizable hyperbolic systems with large and monotone data

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Abstract

In this paper, we study diagonalizable hyperbolic systems in one space dimension. Based on a new gradient entropy estimate, we prove the global existence of a continuous solution, for large and nondecreasing initial data. Moreover, we show in particular cases some uniqueness results. We also remark that these results cover the case of systems which are hyperbolic but not strictly hyperbolic. Physically, this kind of diagonalizable hyperbolic systems appears naturally in the modelling of the dynamics of dislocation densities.

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1 Introduction and main result

1.1 Setting of the problem

In this paper we are interested in continuous solutions to hyperbolic systems in dimension one. Our work will focus on solution $u(t, x) = (u^i(t, x))_{i=1, \dots, M}$, where M is an integer, of hyperbolic systems which are diagonal, i.e.

$$\partial_t u^i + a^i(u) \partial_x u^i = 0 \quad \text{on} \quad (0, T) \times \mathbb{R} \quad \text{and for} \quad i = 1, \dots, M, \quad (\text{P})$$

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with the initial data:

$$u^i(0, x) = u_0^i(x), \quad x \in \mathbb{R}, \text{ for } i = 1, \dots, M. \quad (\text{ID})$$

For real numbers $\alpha^i \leq \beta^i$, let us consider the box

$$U = \prod_{i=1}^M [\alpha^i, \beta^i]. \quad (1.1)$$

We consider a given function $a = (a^i)_{i=1, \dots, M} : U \rightarrow \mathbb{R}^M$, which satisfies the following regularity assumption:

$$(H1) \quad \left\{ \begin{array}{l} \text{the function } a \in C^\infty(U), \\ \text{there exists } M_0 > 0 \text{ such that for } i = 1, \dots, M, \\ |a^i(u)| \leq M_0 \text{ for all } u \in U, \\ \text{there exists } M_1 > 0 \text{ such that for } i = 1, \dots, M, \\ |a^i(v) - a^i(u)| \leq M_1 |v - u| \text{ for all } v, u \in U. \end{array} \right.$$

We assume, for all $u \in \mathbb{R}^M$, that the matrix

$$(a_{,j}^i(u))_{i,j=1, \dots, M}, \text{ where } a_{,j}^i = \frac{\partial}{\partial u^j} a^i,$$

is non-negative in the positive cone, namely

$$(H2) \quad \left| \begin{array}{l} \text{for all } u \in U, \text{ we have} \\ \sum_{i,j=1, \dots, M} \xi_i \xi_j a_{,j}^i(u) \geq 0 \text{ for every } \xi = (\xi_1, \dots, \xi_M) \in [0, +\infty)^M. \end{array} \right.$$

In (ID), each component u_0^i of the initial data $u_0 = (u_0^1, \dots, u_0^M)$ is assumed satisfy the following property:

$$(H3) \quad \left\{ \begin{array}{l} u_0^i \in L^\infty(\mathbb{R}), \\ u_0^i \text{ is nondecreasing,} \\ \partial_x u_0^i \in L \log L(\mathbb{R}), \end{array} \right| \text{ for } i = 1, \dots, M,$$

where $L \log L(\mathbb{R})$ is the following Zygmund space:

$$L \log L(\mathbb{R}) = \left\{ f \in L^1(\mathbb{R}) \text{ such that } \int_{\mathbb{R}} |f| \ln(1 + |f|) < +\infty \right\}.$$

This space is equipped by the following norm:

$$\|f\|_{L \log L(\mathbb{R})} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}} \frac{|f|}{\lambda} \ln \left(1 + \frac{|f|}{\lambda} \right) \leq 1 \right\},$$

This norm is due to Luxemburg (see Adams [1, (13), Page 234]).

Our purpose is to show the existence of a continuous solution, such that $u^i(t, \cdot)$ satisfies (H3) for all time.

1.2 Main result

It is well-known that for the classical Burgers equation, the solution stays continuous when the initial data is Lipschitz-continuous and non-decreasing. We want somehow to generalize this result to the case of diagonal hyperbolic systems.

Theorem 1.1 (*Global existence of a nondecreasing solution*)

Assume (H1), (H2) and (H3). Then, for all $T > 0$, we have:

i) Existence of a weak solution:

There exists a function u solution of (P)-(ID) (in the distributional sense), where

$u \in [L^\infty((0, T) \times \mathbb{R})]^M \cap [C([0, T]; L \log L(\mathbb{R}))]^M$ and $\partial_x u \in [L^\infty((0, T); L \log L(\mathbb{R}))]^M$, such that for a.e $t \in [0, T]$ the function $u(t, \cdot)$ is nondecreasing in x and satisfies the following L^∞ estimate:

$$\|u^i(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \|u_0^i\|_{L^\infty(\mathbb{R})}, \quad \text{for } i = 1, \dots, M, \quad (1.2)$$

and the gradient entropy estimate:

$$\int_{\mathbb{R}} \sum_{i=1, \dots, M} f(\partial_x u^i(t, x)) dx + \int_0^t \int_{\mathbb{R}} \sum_{i, j=1, \dots, M} a_{ij}^i(u) \partial_x u^i(s, x) \partial_x u^j(s, x) dx ds \leq C_1, \quad (1.3)$$

where

$$f(x) = \begin{cases} x \ln(x) + \frac{1}{e} & \text{if } x \geq 1/e, \\ 0 & \text{if } 0 \leq x \leq 1/e, \end{cases} \quad (1.4)$$

and $C_1(T, M, M_1, \|u_0\|_{[L^\infty(\mathbb{R})]^M}, \|\partial_x u_0\|_{[L \log L(\mathbb{R})]^M})$.

ii) Continuity of the solution:

The solution u constructed in (i) belongs to $C([0, T] \times \mathbb{R})$ and there exists a modulus of continuity $\omega(\delta, h)$, such that for all $(t, x) \in (0, T) \times \mathbb{R}$ and all $\delta, h \geq 0$, we have:

$$|u(t + \delta, x + h) - u(t, x)| \leq C_2 \omega(\delta, h) \quad \text{with } \omega(\delta, h) = \frac{1}{\ln(\frac{1}{\delta} + 1)} + \frac{1}{\ln(\frac{1}{h} + 1)}. \quad (1.5)$$

where $C_2(T, M_1, M_0, \|u_0\|_{[L^\infty(\mathbb{R})]^M}, \|\partial_x u_0\|_{[L \log L(\mathbb{R})]^M})$.

Remark 1.2

Here, we can easily extend the solution u of (P)-(ID), given by Theorem 1.1, on the time interval $[0, +\infty)$.

Our method is based on the following simple remark: if the initial data satisfies (H3) then the solution satisfies (H3) for all t . What seems new is the gradient entropy inequality. The prove of Theorem 1.1 is rather standard. First we regularize the initial data and the system with the addition of a viscosity term, then we show that this regularized system admits a classical solution for short time. We prove the bounds (1.2) and the fundamental gradient entropy inequality (1.3) which allow to get a solution for all time. Finally, these *a priori* estimates ensure enough compactness to pass to the limit when the regularization vanishes and to get the existence of a solution.

Remark 1.3

To guarantee the $L \log L$ bound on the gradient of the solutions. We assumed in (H2) a sign on the left hand side of gradient entropy inequality (1.3).

In the case of 2×2 strictly hyperbolic systems, which corresponds in (P) to the case of $a^1(u^1, u^2) < a^2(u^1, u^2)$. Lax [30] proved the existence of smooth solution of (P)-(ID). This result was also proven by Serre [36, Vol II] in the case of $M \times M$ rich hyperbolic systems (see also Subsection 1.4 for more related references). Their result is limited to the case of strictly hyperbolic systems, here in Theorem 1.1, we treated the case of systems which are hyperbolic but not strictly hyperbolic. See the following Remark for a quite detailed example.

Remark 1.4 (Crossing eigenvalues)

Condition (1.9) on the eigenvalues is required in our framework (Theorem 1.1). Here is a simple example of a 2×2 hyperbolic but not strictly hyperbolic system. We consider solution $u = (u^1, u^2)$ of

$$\left\{ \begin{array}{l} \partial_t u^1 + \cos(u^2) \partial_x u^1 = 0, \\ \partial_t u^2 + u^1 \sin(u^2) \partial_x u^2 = 0, \end{array} \right. \quad \text{on } (0, T) \times \mathbb{R}. \quad (1.6)$$

Assume:

i) $u^1(-\infty) = 0$, $u^1(+\infty) = 1$ and $\partial_x u^1 \geq 0$,

ii) $u^2(-\infty) = -\frac{\pi}{2}$, $u^2(+\infty) = \frac{\pi}{2}$ and $\partial_x u^2 \geq 0$.

Here the eigenvalues $\lambda_1(u^1, u^2) = \cos(u^2)$ and $\lambda_2(u^1, u^2) = u^1 \sin(u^2)$ cross each other at the initial time (and indeed for any time). Nevertheless for $a^1(u^1, u^2) = \cos(u^2)$ and $a^2(u^1, u^2) = u^1 \sin(u^2)$, we can compute

$$(a_{;j}^i(u^1, u^2))_{i,j=1,2} = \begin{pmatrix} 0 & -\sin(u^2) \\ \sin(u^2) & u^1 \cos(u^2) \end{pmatrix},$$

which satisfies (H2) (under assumptions (i) and (ii)). Therefor Theorem 1.1 gives the existence of a solution to (1.6) with (i) and (ii).

Based on the same type of gradient entropy inequality (1.3), it was proved in Cannone et al. [8] the existence of a solution in the distributional sense for a two-dimensional system of two transport equations, where the velocity vector field is non-local.

The uniqueness of the solution is strongly related to the existence of regular (Lipschitz) solutions (see Theorem 7.7). Let us remark that equation (P)-(ID) does not create shocks because the solution (given in Theorem 1.1) is continuous. In this situation, it seems very natural to expect the uniqueness of the solution. Indeed the notion of entropy solution (in particular designed to deal with the discontinuities of weak solutions) does not seem so helpful in this context. Nevertheless the uniqueness of the solution is an open problem in general (even for such a simple system).

We ask the following **Open question**:

Is there uniqueness of the solution given in Theorem 1.1 ?

Now we give the following existence and uniqueness result in $[W^{1,\infty}([0, T] \times \mathbb{R})]^M$, in a special case to simplify the presentation. More precisely we assume

$$(H1') \quad a^i(u) = \sum_{j=1, \dots, M} A_{ij} u^j \text{ for } i = 1, \dots, M \text{ and for all } u \in U,$$

$$(H2') \quad \sum_{i,j=1, \dots, M} A_{ij} \xi_i \xi_j \geq 0 \text{ for every } \xi = (\xi_1, \dots, \xi_M) \in [0, +\infty)^M.$$

Theorem 1.5 (Existence and uniqueness of $W^{1,\infty}$ solution for a particular $A = (A_{ij})_{i,j=1, \dots, M}$)

Assume (H1'). For $T > 0$ and all nondecreasing initial data $u_0 \in [W^{1,\infty}(\mathbb{R})]^M$, the system (P)-(ID) admits a unique solution $u \in [W^{1,\infty}([0, T] \times \mathbb{R})]^M$, in the following cases:

i) $M \geq 2$ and $A_{ij} \geq 0$, for all $j \geq i$.

ii) $M \geq 2$ and $A_{ij} \leq 0$, for all $i \neq j$ and (H2'). And then for all $(t, x) \in [0, T] \times \mathbb{R}$ we have

$$\sum_{i=1, \dots, M} \partial_x u^i(t, x) \leq \sup_{y \in \mathbb{R}} \sum_{i=1, \dots, M} \partial_x u_0^i(y). \quad (1.7)$$

Remark 1.6 (Case of $M = 2$)

In particular for $M = 2$, if (H1'), (H2') and (H3) satisfied then we have, by Theorem 1.5 the existence and uniqueness of a solution in $[W^{1,\infty}([0, T) \times \mathbb{R})]^2$ of (P)-(ID).

In these particular cases of the matrix A , we can prove that $\partial_x u^i$ for $i = 1, \dots, M$, are bounded on $[0, T) \times \mathbb{R}$. Thanks to this better estimates on $\partial_x u^i$, and then on the velocity vector field Au , we prove here the uniqueness of the solution.

In the case of the matrix $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, it was proved in El Hajj, Forcadel [16], the existence and uniqueness of a Lipschitz viscosity solution, and in A. El Hajj [15], the existence and uniqueness of a strong solution in $W_{loc}^{1,2}([0, T) \times \mathbb{R})$.

1.3 Application to diagonalizable systems

Let us first consider a smooth function $u = (u^1, \dots, u^M)$, solution of the following non-conservative hyperbolic system:

$$\begin{cases} \partial_t u(t, x) + F(u) \partial_x u(t, x) = 0, & u(t, x) \in U, x \in \mathbb{R}, t \in (0, T), \\ u(x, 0) = u_0(x) & x \in \mathbb{R}, \end{cases} \quad (1.8)$$

where the space of states U is now an open subset of \mathbb{R}^M , and for each u , $F(u)$ is a $M \times M$ -matrix and the map F is of class $C^1(U)$. We assume that $F(u)$ has M real eigenvalues $\lambda_1(u), \dots, \lambda_M(u)$, and we suppose that we can select bases of right and left eigenvectors $r_i(u), l_i(u)$ normalized so that

$$|r_i| \equiv 1 \quad \text{and} \quad l_i \cdot r_j = \delta_{ij}$$

Remark 1.7 (Riemann invariant)

Recall that locally a necessary and sufficient condition to write

$$l_i(u) = \nabla_u \varphi_i(u),$$

is the Frobenius condition $l_i \wedge dl_i = 0$. In that case the function $\varphi_i(u)$ is solution of the following equation

$$(\varphi_i(u))_t + \lambda_i(u)(\varphi_i(u))_x = 0.$$

We recall that then $\varphi_i(u)$ is called a i -Riemann invariant (see Sevennec [37] and Serre [36, Vol II]). If this is true for any i , we say that the system (1.8) is diagonalizable.

Our theory is naturally applicable to this more general class of systems.

1.4 A brief review of some related literature

Now we recall some well known results for system (1.8).

For a scalar conservation law, this corresponds in (1.8) to the case $M = 1$ and $F(u) = h'(u)$ is the derivative of some flux function h , the global existence and uniqueness of BV solution established by Oleinik [34] in one space dimension. The famous paper of Kruzhkov [28] covers the more general class of L^∞ solutions, in several space dimension. For another alternative approach based on the notion of entropy process solutions, see Eymard et al. [17], see also the kinetic formulation P. L. Lions et al. [33].

We now recall some well-known results for a class of 2×2 strictly hyperbolic systems in one space dimension. Here i.e $F(u)$ has two real, distinct eigenvalues

$$\lambda_1(u) < \lambda_2(u).$$

Lax [30] proved the existence and uniqueness of nondecreasing and smooth solutions of the 2×2 strictly hyperbolic systems. Also in case of 2×2 strictly hyperbolic systems DiPerna [12, 13] showed the global existence of a L^∞ solution. The proof of DiPerna relies on a compensated compactness argument, based on the representation of the weak limit in terms of Young measures, which must reduce to a Dirac mass due to the presence of a large family of entropies. This result is based on the idea of Tartar [39].

For general $M \times M$ strictly hyperbolic systems; i. e. where $F(u)$ has M real, distinct eigenvalues

$$\lambda_1(u) < \dots < \lambda_M(u), \tag{1.9}$$

Bianchini and Bressan proved in [6] a striking global existence and uniqueness result of BV solutions to system (1.8), assuming that the initial data has small total variation. Their existence result is a generalization of Glimm result [20], proved in the conservative case; i.e. $F(u) = Dh(u)$ is the Jacobian of some flux function h and generalized by LeFloch and Liu [31, 32] in the non-conservative case.

We can also mention that, our system (P) is related to other similar models, such as scalar transport equations based on vector fields with low regularity. Such equations were for instance studied by Diperna and Lions in [14]. They have proved the existence (and uniqueness) of a solution (in the renormalized sense), for given vector fields in $L^1((0, +\infty); W_{loc}^{1,1}(\mathbb{R}^N))$ whose divergence is in $L^1((0, +\infty); L^\infty(\mathbb{R}^N))$. This study was generalized by Ambrosio [2], who considered vector fields in $L^1((0, +\infty); BV_{loc}(\mathbb{R}^N))$ with bounded divergence. In the present paper, we work in dimension $N = 1$ and prove the existence (and some uniqueness results) of solutions of the system (P)-(ID) with a velocity vector field $a^i(u)$, $i = 1, \dots, M$. Here, in Theorem 1.1, the divergence of our vector field is only in $L^\infty((0, +\infty), L \log L(\mathbb{R}))$. In this case we proved the existence result thanks to the gradient entropy estimate (1.3), which gives a better estimate on

the solution. However, in Theorem 1.5, the divergence of our vector field is bounded, which allows us to get a uniqueness result for the non-linear system (P).

We also refer to Ishii, Koike [25] and Ishii [24], who showed existence and uniqueness of viscosity solutions for Hamilton-Jacobi systems of the form:

$$\begin{cases} \partial_t u^i + H_i(u, Du^i) = 0 & \text{with } u = (u^i)_i \in \mathbb{R}^M, \text{ for } x \in \mathbb{R}^N, t \in (0, T), \\ u^i(x, 0) = u_0^i(x) & x \in \mathbb{R}, \end{cases} \quad (1.10)$$

where the Hamiltonian H_i is quasi-monotone in u (see Ishii, Koike [25, Th.4.7]). This does not cover our study since our Hamiltonian is not necessarily quasi-monotone.

For hyperbolic and symmetric systems, Gårding has proved in [18] a local existence and uniqueness result in $C([0, T]; H^s(\mathbb{R}^N)) \cap C^1([0, T]; H^{s-1}(\mathbb{R}^N))$, with $s > \frac{N}{2} + 1$ (see also Serre [36, Vol I, Th 3.6.1]), this result being only local in time, even in dimension $N = 1$.

1.5 Miscellaneous extensions to explore in a futur work

1. In Theorem 1.1 we have considered the study of a particular system only to simplify the presentation. This result could be generalized to the following system

$$\partial_t u^i + a^i(u, x, t) \partial_x u^i = h^i(u, x, t) \quad \text{on } (0, T) \times \mathbb{R} \quad \text{and for } i = 1, \dots, M, \quad (P')$$

with suitable conditions on a^i and h^i .

2. If we consider the case where the system (P) is strictly hyperbolic. Based in the result of Bianchini, Bressan [6], we could also prove the uniqueness of the solution, whose existence is given by Theorem 1.1.

3. We could also extend Theorem 1.5 to system (P'), where we replace (i) and (ii) by the following condition

i') For $M \geq 2$, $a_{,j}^i(u, x, t) \geq 0$ for $j \geq i$ and for all $(u, x, t) \in U \times \mathbb{R} \times [0, T)$.

ii') For $M \geq 2$,

$$a_{,j}^i(u, x, t) \leq 0 \quad \text{for all } (u, x, t) \in U \times \mathbb{R} \times [0, +\infty), \quad \text{for all } i \neq j,$$

and we assume that for any $v_i \in \mathbb{R}^M, x_i \in \mathbb{R}$, the matrix

$$b_{ij}(t) = a_{,j}^i(v_i, x_i, t)$$

satisfies for all $t \geq 0$

$$(H2'') \quad \sum_{i,j=1,\dots,M} b_{ij}(t) \xi_i \xi_j \geq 0 \quad \text{for all } \xi = (\xi_1, \dots, \xi_M) \in [0, +\infty)^M.$$

4. We could also prove the uniqueness result in case of $W^{1,\infty}$ solution among weak solution. (and in particular any weak solution is a viscosity solution in the sense of Crandall-Lions [10, 11]).

5. We could propose a numerical scheme and try to prove its convergence.

6. Applications to other equations: Euler, p -systems.

1.6 Organization of the paper

This paper is organized as follows: in the Section 2, we approximate the system (P) and the initial conditions. Then we prove a local in time existence for this approximated system. In Section 3, we prove the global in time existence for the approximated system. In the Section 4, we prove that the obtained solutions are regular and non-decreasing in x for all $t \in (0, T)$. In Section 5, we prove the gradient entropy inequality and some other ε -uniform *a priori* estimates. In Section 6, we prove the main result (Theorem 1.1) passing to the limit as ε goes to 0 and using some compactness properties inherited from our entropy gradient inequality and the *a priori* estimates. In Section 7 we prove some uniqueness results in particular cases (Theorem 1.5). An application to the dynamics of dislocation densities given in Section 8. Finally, in the Appendix, we recall the proof of uniqueness of Lipschitz solution to system (P).

2 Local existence of an approximated system

The system (P) can be written as:

$$\partial_t u + a(u) \diamond \partial_x u = 0, \quad (2.11)$$

where $u := (u^i)_{1,\dots,M}$, $a(u) = (a^i(u))_{1,\dots,M}$ and $U \diamond V$ is the “component by component product” of the two vectors $U, V \in \mathbb{R}^M$. This is the vector in \mathbb{R}^M whose coordinates are given by $(U \diamond V)_i := U_i V_i$:

$$\begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_M \end{bmatrix} \diamond \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_M \end{bmatrix} = \begin{bmatrix} U_1 V_1 \\ U_2 V_2 \\ \vdots \\ U_M V_M \end{bmatrix}.$$

Now, we consider the system (2.11), modified by the term $\varepsilon \partial_{xx} u$, where $\partial_{xx} = \frac{\partial^2}{\partial x^2}$, and for smoothed data. This modification brings us to study, for all $0 < \varepsilon \leq 1$, the following system:

$$\partial_t u^\varepsilon - \varepsilon \partial_{xx} u^\varepsilon = -a(u^\varepsilon) \diamond \partial_x u^\varepsilon, \quad (P_\varepsilon)$$

with the smooth initial data:

$$u^\varepsilon(x, 0) = u_0^\varepsilon(x), \quad \text{with } u_0^\varepsilon(x) := u_0 * \eta_\varepsilon(x), \quad (ID_\varepsilon)$$

where η_ε is a mollifier verify, $\eta_\varepsilon(\cdot) = \frac{1}{\varepsilon}\eta(\frac{\cdot}{\varepsilon})$, such that $\eta \in C_c^\infty(\mathbb{R})$ is a non-negative function and $\int_{\mathbb{R}} \eta = 1$.

Remark 2.1

By classical properties of the mollifier $(\eta_\varepsilon)_\varepsilon$ and the fact that $u_0^\varepsilon \in [L^\infty(\mathbb{R})]^M$, then $u_0^\varepsilon \in [C^\infty(\mathbb{R})]^M \cap [W^{m,\infty}(\mathbb{R})]^M$ for all $m \in \mathbb{N}$.

The global existence of smooth solution of the system (P_ε) is standard. Here, we prove this results only to ensure the reader.

The following theorem is a local existence result (in the "Mild" sense) of the regularized system (P_ε) - (ID_ε) . This result is achieved in a super-critical space. Here particularly we chose the space of functions $[C([0, T]; X(\mathbb{R}))]^M$, where

$$X(\mathbb{R}) = \{u \in L^\infty(\mathbb{R}) \text{ such that } \partial_x u \in L^8(\mathbb{R})\}. \quad (2.12)$$

This space is a Banach space supplemented with the following norm

$$\|u\|_{X(\mathbb{R})} = \|u\|_{L^\infty(\mathbb{R})} + \|\partial_x u\|_{L^8(\mathbb{R})}.$$

Here the espace $L^p(\mathbb{R})$ with $p = 8$ will simplify later in Lemma 4.1 the Bootstrap argument to get smooth solution.

In this Section, we will prove the following

Theorem 2.2 (Local existence result)

For all initial data $u_0^\varepsilon \in [X(\mathbb{R})]^M$ there exists

$$T^* = T^*(M_0, \varepsilon) > 0,$$

such that the system (P_ε) - (ID_ε) admits solutions $u^\varepsilon \in [C([0, T^]; X(\mathbb{R}))]^M$.*

In order to do the proof of Theorem 2.2 in Subsection 2.2 we need to recall in the following Subsection some known results.

2.1 Useful results

Lemma 2.3 (Mild solution)

Let $T > 0$, and $u^\varepsilon \in [C([0, T]; X(\mathbb{R}))]^M$ be a solution of the following integral problem with $u^\varepsilon(t) = u^\varepsilon(t, \cdot)$:

$$u^\varepsilon(t) = S_\varepsilon(t)u_0^\varepsilon - \int_0^t S_\varepsilon(t-s) (a(u^\varepsilon(s)) \diamond \partial_x u^\varepsilon(s)) ds, \quad (IN_\varepsilon)$$

where $S_\varepsilon(t) = S_1(\varepsilon t)$ such that $S_1(t) = e^{t\Delta}$ is the heat semi-group. Then u^ε is a solution of the system (P_ε) - (ID_ε) in the sense of distributions.

For the proof of this lemma, we refer to Pazy [35, Th 5.2. Page 146].

Lemma 2.4 (Picard Fixed Point Theorem, see [26])

Let E be a Banach space, let $B : E \times E \longrightarrow E$ be a continuous map such that:

$$\|B(x, y)\|_E \leq \eta \|y\|_E \text{ for all } x, y \in E,$$

where η is a positive given constant. Then, for every $x_0 \in E$, if

$$0 < \eta < 1,$$

the equation $x = x_0 + B(x, x)$ admits a solution in E .

In order to show the local existence of a solution for (IN_ε) , we will apply Lemma 2.4 in the space $E = [L^\infty((0, T); X(\mathbb{R}))]^M$.

Lemma 2.5 (Time continuity)

Let $T > 0$. If $u^\varepsilon \in [L^\infty((0, T); W^{1,p}(\mathbb{R}))]^M$, $1 \leq p \leq +\infty$, are solutions of integral problem (IN_ε) , then $u^\varepsilon \in [C([0, T]; W^{1,p}(\mathbb{R}))]^M$.

For the proof of Lemma 2.3, see A. Pazy [35, 7.3, Page 212].

Lemma 2.6 (Semi-group estimates)

Let $1 \leq p \leq q \leq +\infty$. Then for all $f \in L^p(\mathbb{R})$ and for all $t > 0$, we have the following estimates:

$$i) \|S_\varepsilon(t)f\|_{L^q(\mathbb{R})} \leq Ct^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\mathbb{R})},$$

$$ii) \|\partial_x S_\varepsilon(t)f\|_{L^p(\mathbb{R})} \leq Ct^{-\frac{1}{2}}\|f\|_{L^p(\mathbb{R})},$$

where $C = C(\varepsilon)$ is a positive constant depending on ε .

For the proof of this Lemma, see Pazy [35, Lemma 1.1.8, Th 6.4.5].

2.2 Proof of Theorem 2.2

Our goal is to show local existence of a solution of (P_ε) using the Picard fixed point Theorem. To be done according Lemma 2.3 it is enough to prove the local existence for the following equation:

$$\begin{aligned} u^\varepsilon(t) &= S_\varepsilon(t)u_0^\varepsilon - \int_0^t S_\varepsilon(t-s) (a(u^\varepsilon(s)) \diamond \partial_x u^\varepsilon(s)) ds, \\ &= S_\varepsilon(t)u_0^\varepsilon + B(u^\varepsilon, u^\varepsilon)(t), \end{aligned} \tag{2.13}$$

with $B(u, v)(t) = - \int_0^t S_\varepsilon(t-s) (a(u)(s) \diamond \partial_x v(s)) ds$.

If we estimate $B(u, v)$, we will obtain, for all $u, v \in [L^\infty((0, T); X(\mathbb{R}))]^M$, where $X(\mathbb{R})$ defined in (2.12), the following:

$$\begin{aligned} \|B(u, v)(t)\|_{[X(\mathbb{R})]^M} &= \left\| \int_0^t S_\varepsilon(t-s) (a(u)(s) \diamond \partial_x v(s)) ds, \right\|_{[L^\infty(\mathbb{R})]^M}, \\ &+ \left\| \int_0^t \partial_x S_\varepsilon(t-s) (a(u)(s) \diamond \partial_x v(s)) ds, \right\|_{[L^8(\mathbb{R})]^M}, \end{aligned} \quad (2.14)$$

where for a function $f = (f^1, \dots, f^M) \in [X(\mathbb{R})]^M$, we note here

$$\|f\|_{[X(\mathbb{R})]^M} = \sup_{i=1, \dots, M} \|f^i\|_{L^\infty(\mathbb{R})} + \sup_{i=1, \dots, M} \|\partial_x f^i\|_{L^8(\mathbb{R})}.$$

Using Lemma 2.6 (i) with $p = 8, q = \infty$ for the first term and Lemma 2.6 (ii) with $p = 8$ for the second term, we obtain that :

$$\begin{aligned} \|B(u, v)(t)\|_{[X(\mathbb{R})]^M} &\leq C \int_0^t \frac{1}{(t-s)^{\frac{7}{16}}} \|a(u(s)) \partial_x v(s)\|_{[L^2(\mathbb{R})]^M} ds, \\ &+ C \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \|a(u(s)) \partial_x v(s)\|_{[L^8(\mathbb{R})]^M} ds. \end{aligned}$$

We use the Hölder inequality, and get, for all $0 < T \leq 1$:

$$\begin{aligned} \|B(u, v)(t)\|_{[X(\mathbb{R})]^M} &\leq CT^{\frac{1}{2}} \|\partial_x v\|_{[L^\infty((0, T); L^8(\mathbb{R}))]^M}, \\ &\leq CT^{\frac{1}{2}} \|v\|_{[L^\infty((0, T); X(\mathbb{R}))]^M}, \end{aligned} \quad (2.15)$$

where $C(M_0, \varepsilon)$. Moreover, we know by classical properties of heat semi-group (see A. Pazy [35]):

$$\|S_\varepsilon(t)u_0^\varepsilon\|_{[L^\infty((0, T); X(\mathbb{R}))]^M} \leq \|u_0^\varepsilon\|_{[X(\mathbb{R})]^M}. \quad (2.16)$$

Now, taking

$$(T^*)^{\frac{1}{2}} = \min\left(\frac{1}{2C}, 1\right), \quad (2.17)$$

we can easily verify that

$$C(T^*)^{\frac{1}{2}} < 1.$$

By applying the Picard Fixed Point Theorem (Lemma 2.4) with $E = [L^\infty((0, T^*); X(\mathbb{R}))]^M$, this proves the existence of a solution $u^\varepsilon \in [L^\infty((0, T^*); X(\mathbb{R}))]^M$ for (2.13).

Then, according to Lemma 2.5, we deduce that the solution is indeed in $[C([0, T^*]; X(\mathbb{R}))]^M$.

This proves, by Lemma 2.3, the existence of a solution in $[C([0, T^*]; X(\mathbb{R}))]^M$, which satisfies the system (P_ε) - (ID_ε) in the sense of distributions. \square

3 Global existence of the solutions of the approximated system

In this Section, we will prove the global existence of solution for the system (P_ε) - (ID_ε) . Before going into the proof, we need the following lemma.

Lemma 3.1 (L^∞ bound)

Let $T > 0$. If $u^\varepsilon \in [C([0, T]; X(\mathbb{R}))]^M$ is a solution of system (P_ε) - (ID_ε) with initial data $u_0^\varepsilon \in X(\mathbb{R})$, then

$$\|u^\varepsilon\|_{[L^\infty([0, T] \times \mathbb{R})]^M} \leq \|u_0^\varepsilon\|_{[L^\infty(\mathbb{R})]^M}$$

The proof of this Lemma is a direct application of the Maximum Principle Theorem for parabolic equations (see Gilbarg-Trudinger [19, Th.3.1]).

Remark 3.2

Thanks to the previous Lemma, we notice that we can take the box U defined in (1.1) as the following

$$U = \Pi_{i=1}^M [-\|u_0^{\varepsilon, i}\|_{L^\infty(\mathbb{R})}, \|u_0^{\varepsilon, i}\|_{L^\infty(\mathbb{R})}].$$

For fixed ε , this definition guarantee that M_0 do not change in the course of time.

The result of this Section is the following.

Theorem 3.3 (Global existence)

Let $T > 0$ and $0 < \varepsilon \leq 1$. For initial data $u_0^\varepsilon \in [X(\mathbb{R})]^M$ satisfying (H1) and (H2). Then the system (P_ε) - (ID_ε) , admits a solution $u^\varepsilon \in [C([0, T]; X(\mathbb{R}))]^M$, with $u^\varepsilon(t, \cdot)$ satisfying (H1) and (H2) for all $t \in (0, T)$. Moreover, for all $t \in (0, T)$, we have the following inequalities:

$$\|u^{\varepsilon, i}(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \|u_0^{\varepsilon, i}\|_{L^\infty(\mathbb{R})}, \quad \text{for } i = 1, \dots, M, \quad (3.18)$$

Proof of Theorem 3.3:

We are going to prove that local in time solutions obtained by Theorem 2.2 can be extended to global solutions for the same system.

We argue by contradiction: assume that there exists a maximum time T_{max} such that, we have the existence of solutions of the system (P_ε) - (ID_ε) in the function space $[C([0, T_{max}); X(\mathbb{R}))]^M$.

For every small enough $\delta > 0$, we consider the system (P_ε) with the initial condition

$$u_0^{\varepsilon, \delta}(x) = u^\varepsilon(T_{max} - \delta, x).$$

From Theorem 2.2 to deduce that there exists a time $T^*(M_0, \varepsilon)$, independent of δ (see Remark 3.2), such that the system (P_ε) with initial data $u_0^{\varepsilon, \delta}$ has a solution $u^{\varepsilon, \delta}$ on the time interval $[0, T^*)$. Then for

$$T_0 = (T_{max} - \delta) + T^*,$$

we extend u^ε on the time interval $[0, T_0)$ as follows,

$$\tilde{u}^\varepsilon(t, x) = \begin{cases} u^\varepsilon(t, x), & \text{for } t \in [0, T_{max} - \delta], \\ u^{\varepsilon, \delta}(t, x), & \text{for } t \in [T_{max} - \delta, T_0) \end{cases}$$

and we can check that \tilde{u}^ε is a solution of (P_ε) - (ID_ε) on the time interval $[0, T_0)$. But from Lemma (3.1) we know that the time T^* is independent of δ (see Remark 3.2), which implies that $T_0 > T_{max}$ and so a contradiction.

The inequalities (3.18) is a consequence of Lemma 3.1. □

4 Properties of the solutions of the approximated system

In this section, we are going to prove that the solution of (P_ε) - (ID_ε) obtained by Theorem 2.2 is smooth and monotone.

Lemma 4.1 (Smoothness of the solution)

Let $T > 0$. For all initial data $u_0^\varepsilon \in [X(\mathbb{R})]^M$, where $\partial_x u_0^\varepsilon \in [W^{m,p}(\mathbb{R})]^M$ for all $m \in \mathbb{N}$, $1 \leq p \leq +\infty$.

If u^ε is a solution of the system (P_ε) - (ID_ε) , such that $u^\varepsilon \in [C([0, T]; X(\mathbb{R}))]^M$ and $\partial_x u^\varepsilon \in [L^\infty((0, T); L^1(\mathbb{R}))]^M$, then $u^\varepsilon \in [C^\infty([0, T) \times \mathbb{R})]^M$ and satisfies,

$$u^\varepsilon \in [W^{m,p}((0, T) \times \mathbb{R})]^M, \text{ for all } 1 < p \leq +\infty \text{ and } m \in \mathbb{N} \setminus \{0\}, \quad (4.19)$$

Proof of Lemma 4.1

Step 1 (Initialization of the Bootstrap):

For the sake of simplicity, we will set

$$F[u^\varepsilon] = -a(u^\varepsilon) \diamond \partial_x u^\varepsilon.$$

From the fact that $u^\varepsilon \in [C([0, T]; X(\mathbb{R}))]^M$ and $\partial_x u^\varepsilon \in [L^\infty((0, T); L^1(\mathbb{R}))]^M$, we deduce that $\partial_x u^\varepsilon, F[u^\varepsilon] \in [L^1((0, T) \times \mathbb{R})]^M \cap [L^8((0, T) \times \mathbb{R})]^M$, which proves by interpolation that

$$\partial_x u^\varepsilon, F[u^\varepsilon] \in [L^p((0, T) \times \mathbb{R})]^M \quad \text{for all } 1 \leq p \leq 8. \quad (4.20)$$

Because u^ε is a solution of (P_ε) , we see that

$$\partial_t u^\varepsilon - \varepsilon \partial_{xx} u^\varepsilon = F[u^\varepsilon], \quad (4.21)$$

$$\partial_{tx} u^\varepsilon - \varepsilon \partial_{xxx} u^\varepsilon = \partial_x F[u^\varepsilon]. \quad (4.22)$$

Applying the classical regularity theory of heat equations on (4.21), we deduce that:

$$\partial_t u^\varepsilon \quad \text{and} \quad \partial_{xx} u^\varepsilon \in [L^p((0, T) \times \mathbb{R})]^M, \quad \text{for all } 1 < p \leq 8. \quad (4.23)$$

For more details, see Ladyzenskaja [29, Theorem 9.1]. But we know that

$$\partial_x F[u^\varepsilon] = -a(u^\varepsilon) \diamond \partial_{xx} u^\varepsilon - Da(u^\varepsilon) \partial_x u^\varepsilon \diamond \partial_x u^\varepsilon \quad (4.24)$$

We notice that thanks to this better regularity on u^ε ((4.20) and (4.23), and by the Hölder inequality we can easily prove that

$$\partial_x F[u^\varepsilon] \in [L^p((0, T) \times \mathbb{R})]^M \quad \text{for all } 1 < p \leq 4.$$

Now, we apply again the classical regularity theory of heat equations on (4.22), to deduce that:

$$\partial_{tx} u^\varepsilon \quad \text{and} \quad \partial_{xxx} u^\varepsilon \in [L^p((0, T) \times \mathbb{R})]^M, \quad \text{for all } 1 < p \leq 4. \quad (4.25)$$

We know that

$$\partial_t F[u^\varepsilon] = -a(u^\varepsilon) \diamond \partial_{tx} u^\varepsilon - Da(u^\varepsilon) \partial_t u^\varepsilon \diamond \partial_x u^\varepsilon \quad (4.26)$$

Thanks this previous regularity on u^ε , we obtain by the Hölder inequality that

$$\partial_t F[u^\varepsilon] \in [L^p((0, T) \times \mathbb{R})]^M \quad \text{for all } 1 < p \leq 4.$$

Which gives that

$$\partial_x u^\varepsilon, F[u^\varepsilon] \in [W^{1,p}((0, T) \times \mathbb{R})]^M \quad \text{for all } 1 < p \leq 4,$$

and by the Sobolev embedding that $\partial_x u^\varepsilon \in [L^p((0, T) \times \mathbb{R})]^M$ for all $1 < p \leq \infty$.

Step 2 (Recurrence):

Now, we use the same steps, we can prove by recurrence that for all $m \in \mathbb{N}$ if,

$$(H_m) \quad \left\{ \begin{array}{l} \partial_x u^\varepsilon \in [L^\infty((0, T) \times \mathbb{R})]^M, \\ \partial_x u^\varepsilon, F[u^\varepsilon] \in [W^{m,p}((0, T) \times \mathbb{R})]^M \quad \text{for all } 1 < p \leq 4, \end{array} \right.$$

then

$$(H_m) \Rightarrow (H_{m+1}).$$

Indeed, as in (4.23) we can deduce here that

$$\partial_t u^\varepsilon \quad \text{and} \quad \partial_{xx} u^\varepsilon \in [W^{m,p}((0, T) \times \mathbb{R})]^M, \quad \text{for all } 1 < p \leq 4, \quad (4.27)$$

and From (4.24), because $\partial_x u^\varepsilon \in [L^\infty((0, T) \times \mathbb{R})]^M$, we can obtain here that

$$\partial_x F[u^\varepsilon] \in [W^{m,p}((0, T) \times \mathbb{R})]^M \quad \text{for all } 1 < p \leq 4.$$

Which proves that, as in (4.25) that

$$\partial_{tx} u^\varepsilon \quad \text{and} \quad \partial_{xxx} u^\varepsilon \in [W^{m,p}((0, T) \times \mathbb{R})]^M, \quad \text{for all } 1 < p \leq 4, \quad (4.28)$$

and From (4.26), we deduce that

$$\partial_t F[u^\varepsilon] \in [W^{m,p}((0, T) \times \mathbb{R})]^M \quad \text{for all } 1 < p \leq 4,$$

and then

$$\partial_x u^\varepsilon, F[u^\varepsilon] \in [W^{m+1,p}((0, T) \times \mathbb{R})]^M \quad \text{for all } 1 < p \leq 4,$$

Which proves by the Sobolev embedding the results. \square

Lemma 4.2 (Classical Maximum Principle)

Let $T > 0$. For all initial data $u_0^\varepsilon \in [X(\mathbb{R})]^M$, where $\partial_x u_0^\varepsilon \in [W^{m,p}(\mathbb{R})]^M$ for all $m \in \mathbb{N}$, $1 \leq p \leq +\infty$, and satisfying (H3).

If u^ε is a solution of the system (P_ε) - (ID_ε) , such that $u^\varepsilon \in [C([0, T]; X(\mathbb{R}))]^M$ and $\partial_x u^\varepsilon \in [L^\infty((0, T); L^1(\mathbb{R}))]^M$, then we have for $i = 1, \dots, M$, $\partial_x u^{\varepsilon,i} \geq 0$ on $(0, T) \times \mathbb{R}$.

Proof of Lemma 4.2

We first derive with respect to x the system (P_ε) - (ID_ε) , and get for $w^\varepsilon = (w^{\varepsilon,i})_{i=1,\dots,M} = \partial_x u^\varepsilon$

$$\partial_t w^\varepsilon - \varepsilon \partial_{xx} w^\varepsilon + a(u^\varepsilon) \diamond \partial_x w^\varepsilon + Da(u) w^\varepsilon \diamond w^\varepsilon = 0.$$

Since $u^\varepsilon \in [C^\infty([0, T] \times \mathbb{R})]^M$, we see, for $i = 1, \dots, M$, that $w^{\varepsilon,i}$ is smooth and satisfies $w^{\varepsilon,i}(0, x) = \partial_x u_0^{\varepsilon,i} \geq 0$. From the classical maximum principle we deduce that $w^{\varepsilon,i} \geq 0$ on $[0, T] \times \mathbb{R}$. \square

Remark 4.3 (L^1 uniform estimate on $\partial_x u^\varepsilon$)

Because $\partial_x u^{\varepsilon,i} \geq 0$, for $i = 1, \dots, M$, we deduce from Lemma 3.1 that:

$$\|\partial_x u^\varepsilon\|_{[L^\infty((0,T);L^1(\mathbb{R}))]^M} \leq 2 \|u^\varepsilon\|_{[L^\infty((0,T)\times\mathbb{R})]^M} \leq 2 \|u_0^\varepsilon\|_{[L^\infty(\mathbb{R})]^M}. \quad (4.29)$$

Corollary 4.4 (global existence of nondecreasing smooth solutions)

Let $T > 0$. The solution given in Theorem 2.2 can be chosen such that $u^\varepsilon = (u^{\varepsilon,i})_{i=1,\dots,M}$ smooth, satisfies (4.19) and for each $i = 1, \dots, M$, $\partial_x u^{\varepsilon,i} \geq 0$ on $(0, T) \times \mathbb{R}$.

The proof of Corollary 4.4 is a consequence of Theorem 2.2 and Lemmata 4.1, 4.2 and Remark 4.3.

5 ε -Uniform *a priori* estimates

In this Section, we show some ε -uniform estimates on the solutions of the system (P_ε) - (ID_ε) . These estimates will be used in Section 6 for the passage to the limit as ε tends to zero.

Lemma 5.1 (L^∞ bound on u^ε and L^1 bound on $\partial_x u^\varepsilon$)

Let $T > 0$, $0 < \varepsilon \leq 1$ and function $u_0 \in [L^\infty(\mathbb{R})]^M$ satisfying (H3). Then the solution of the system (P_ε) - (ID_ε) given in Theorem 3.3 with initial data $u_0^\varepsilon = u_0 * \eta_\varepsilon$, satisfies the following ε -uniform estimates:

$$(E1) \quad \|u^\varepsilon\|_{[L^\infty((0,T)\times\mathbb{R})]^M} \leq \|u_0\|_{[L^\infty(\mathbb{R})]^M},$$

$$(E2) \quad \|\partial_x u^\varepsilon\|_{[L^\infty((0,T),L^1(\mathbb{R}))]^M} \leq 2 \|u_0\|_{[L^\infty(\mathbb{R})]^M},$$

Proof of Lemma 5.1:

First, we remark that if $\partial_x u_0 \geq 0$, then $\partial_x u_0^\varepsilon = (\partial_x u_0) * \eta_\varepsilon(x) \geq 0$ (because η is positive).

The fact that $u_0 \in [L^\infty(\mathbb{R})]^M$ and $\partial_x u_0 \geq 0$, we obtain that $\partial_x u_0 \in [L^1(\mathbb{R})]^M$.

By classical properties of the mollifier $(\eta_\varepsilon)_\varepsilon$ we know that if $u_0 \in [L^\infty(\mathbb{R})]^M$ and $\partial_x u_0 \in [L^1(\mathbb{R})]^M$ we have $u_0^\varepsilon \in [X(\mathbb{R})]^M$ and $\partial_x u_0^\varepsilon \in [W^{m,p}(\mathbb{R})]^M$ for all $m \in \mathbb{N}$, $1 \leq p \leq +\infty$.

Now, we use Lemma 3.1 and Remark 4.3, we deduce by the classical properties of the mollifier (E1) and (E2).

Before going into the proof of the gradient entropy inequality defined in (5.30), we announce the main idea of this new gradient entropy estimate. Now, let us set for $w \geq 0$ the entropy function

$$\bar{f}(w) = w \ln w.$$

For any *non-negative* test function $\varphi \in C_c^1(\mathbb{R} \times [0, +\infty))$, let us define the following “*gradient entropy*” with $w^i := \partial_x u^i$:

$$\bar{N}(t) = \int_{\mathbb{R}} \varphi \left(\sum_{i=1, \dots, M} \bar{f}(w^i) \right) dx.$$

It is very natural to introduce such quantity $\bar{N}(t)$ which in the case $\varphi \equiv 1$, appears to be nothing else than the total entropy of the system of M type of particles of non-negative densities w^i . Then it is formally possible to deduce from (P) the equality in the following new *gradient entropy inequality* for all $t \geq 0$

$$\frac{d\bar{N}}{dt}(t) + \int_{\mathbb{R}} \varphi \left(\sum_{i,j=1, \dots, M} a_{i,j}^i w^i w^j \right) dx \leq R(t) \quad \text{for } t \geq 0, \quad (5.30)$$

with the rest

$$R(t) = \int_{\mathbb{R}} \left\{ (\partial_t \varphi) \left(\sum_{i=1, \dots, M} \bar{f}(w^i) \right) + (\partial_x \varphi) \left(\sum_{i=1, \dots, M} a^i \bar{f}(w^i) \right) \right\} dx,$$

where we only show the dependence on t in the integrals. We remark in particular that this rest is formally equal to zero if $\varphi \equiv 1$.

To guarantee the existence of continuous solutions, we assumed in (H2) a sign on the left hand side of inequality (5.30).

For we return this previous calculate more rigorous, we prove actually the following gradient entropy inequality

Proposition 5.2 (Gradient entropy inequality)

Let $T > 0$, $0 < \varepsilon \leq 1$ and function $u_0 \in [L^\infty(\mathbb{R})]^M$ satisfying (H3). We consider the solution u^ε of the system (P_ε) - (ID_ε) given in Theorem 3.3 with initial data $u_0^\varepsilon = u_0 * \eta_\varepsilon$. Then, there exists a constant $C(T, M, M_1, \|u_0\|_{[L^\infty(\mathbb{R})]^M}, \|\partial_x u_0\|_{[L \log L(\mathbb{R})]^M})$ such that

$$N(t) + \int_0^t \int_{\mathbb{R}} \sum_{i,j=1, \dots, M} a_{i,j}^i(u^\varepsilon) w^{\varepsilon,i} w^{\varepsilon,j} \leq C, \quad \text{with } N(t) = \int_{\mathbb{R}} \sum_{i=1, \dots, M} f(w^{\varepsilon,i}) dx. \quad (5.31)$$

where $w^\varepsilon = (w^{\varepsilon,i})_{i=1, \dots, M} = \partial_x u^\varepsilon$ and f is defined in (1.4).

For the proof of Proposition 5.2 we need the following Lemma:

Lemma 5.3 ($L \log L$ Estimate)

Let $(\eta_\varepsilon)_\varepsilon$ be a non-negative mollifier, f is the function defined in (1.4) and $h \in L^1(\mathbb{R})$ is a non-negative function. Then

i) $\int_{\mathbb{R}} f(h) < +\infty$ if and only if $h \in L \log L(\mathbb{R})$.

ii) If $h \in L \log L(\mathbb{R})$ the function $h_\varepsilon = h * \eta_\varepsilon \in L \log L(\mathbb{R})$ satisfies

$$\|h - h_\varepsilon\|_{L \log L(\mathbb{R})} \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0.$$

The proof of (i) is trivial, for the proof of (ii) see R. A. Adams [1, Th 8.20] for the proof of this Lemma.

Proof of Proposition 5.2:

Remark first that the quantity $N(t)$ is well-defined because $w^\varepsilon \in [L^\infty((0, T); L^1(\mathbb{R}))]^M \cap [L^\infty((0, T); L^8(\mathbb{R}))]^M$ (by Theorem 2.2 and Corollary 4.4) and we have the general inequality $\frac{-1}{e} \leq w \log w \leq w^2$ for all $w \geq 0$.

From Theorem 4.4 we know that $w^{\varepsilon, i}$ and smooth non-negative function. Now, we derive $N(t)$ with respect to t , this is well-defined because for $i = 1, \dots, M$, we have

$$\left| \int_{w^{\varepsilon, i} \geq \frac{1}{e}} \right| \leq e \|w^{\varepsilon, i}\|_{L^\infty((0, T); L^1(\mathbb{R}))} \text{ and for all } m \in \mathbb{N}, w^{\varepsilon, i} \in W^{m, \infty}((0, T) \times \mathbb{R}) \text{ (see (4.19)).}$$

Finally, we get that,

$$\begin{aligned} \frac{d}{dt} N(t) &= \int_{\mathbb{R}} \sum_{i=1, \dots, M} f'(w^{\varepsilon, i}) (\partial_t w^{\varepsilon, i}), \\ &= \int_{\mathbb{R}} \sum_{i=1, \dots, M} f'(w^{\varepsilon, i}) \partial_x (-a^i(u^\varepsilon) w^{\varepsilon, i} + \varepsilon \partial_x w^{\varepsilon, i}), \\ &= \overbrace{\int_{\mathbb{R}} \sum_{i=1, \dots, M} a^i(u^\varepsilon) w^{\varepsilon, i} f''(w^{\varepsilon, i}) \partial_x w^{\varepsilon, i}}^{J_1} - \varepsilon \overbrace{\int_{\mathbb{R}} \sum_{i=1, \dots, M} (\partial_x w^{\varepsilon, i})^2 f''(w^{\varepsilon, i})}^{J_2} \end{aligned}$$

But, it is easy to check that

$$f'(x) = \begin{cases} \ln(x) + 1 & \text{if } x \geq 1/e, \\ 0 & \text{if } 0 \leq x \leq 1/e, \end{cases} \quad \text{and} \quad f''(x) = \begin{cases} \frac{1}{x} & \text{if } x \geq 1/e, \\ 0 & \text{if } 0 \leq x \leq 1/e. \end{cases}$$

This proves that $J_2 \leq 0$. To control J_1 , we rewrite it under the following form

$$J_1 = \int_{\mathbb{R}} \sum_{i=1, \dots, M} a^i(u^\varepsilon) g'(w^{\varepsilon, i}) \partial_x w^{\varepsilon, i},$$

where

$$g(x) = \begin{cases} x - \frac{1}{e} & \text{if } x \geq 1/e, \\ 0 & \text{if } 0 \leq x \leq 1/e, \end{cases}$$

Then, we deduce that

$$\begin{aligned}
J_1 &= \int_{\mathbb{R}} \sum_{i=1, \dots, M} a^i(u^\varepsilon) \partial_x (g(w^{\varepsilon, i})) \\
&= - \int_{\mathbb{R}} \sum_{i, j=1, \dots, M} a_{,j}^i(u^\varepsilon) w^{\varepsilon, j} g(w^{\varepsilon, i}), \\
&= \underbrace{- \int_{\mathbb{R}} \sum_{i, j=1, \dots, M} a_{,j}^i(u^\varepsilon) w^{\varepsilon, j} w^{\varepsilon, i}}_{J_{11}} - \underbrace{\int_{\mathbb{R}} \sum_{i, j=1, \dots, M} a_{,j}^i(u^\varepsilon) w^{\varepsilon, j} (g(w^{\varepsilon, i}) - w^{\varepsilon, i})}_{J_{12}},
\end{aligned}$$

From (H2), we know that $J_{11} \leq 0$. We use the fact that $|g(x) - x| \leq \frac{1}{e}$ for all $x \geq 0$ and (H1), to deduce that

$$\begin{aligned}
|J_{12}| &\leq \frac{1}{e} M^2 M_1 \|w^{\varepsilon, i}\|_{[L^\infty((0, T), L^1(\mathbb{R}))]^M} \\
&\leq \frac{2}{e} M^2 M_1 \|u_0\|_{[L^\infty(\mathbb{R})]^M}
\end{aligned}$$

where we have use Lemma 5.1 (E2) in the last line. Finally, we deduce that, there exists a positive constant $C(\|u_0\|_{[L^\infty(\mathbb{R})]^M}, M_1, M)$ independent of ε such that

$$\begin{aligned}
\frac{d}{dt} N(t) &\leq J_{11} + J_{12} + J_2 \\
&\leq J_{11} + C.
\end{aligned}$$

Integrating in time we get by Lemma 5.3, there exists a another positive constant $C(T, M, M_1, \|u_0\|_{[L^\infty(\mathbb{R})]^M}, \|\partial_x u_0\|_{[L \log L(\mathbb{R})]^M})$ independent of ε such that

$$N(t) + \int_0^t \int_{\mathbb{R}} \sum_{i, j=1, \dots, M} a_{,j}^i(u^\varepsilon) w^{\varepsilon, j} w^{\varepsilon, i} \leq CT + N(0) \leq C.$$

□

Lemma 5.4 ($W^{-1,1}$ estimate on the time derivatives of the solutions)

Let $T > 0$, $0 < \varepsilon \leq 1$ and function $u_0 \in [L^\infty(\mathbb{R})]^M$ satisfying (H3). Then the solution of the system (P_ε) - (ID_ε) given in Theorem 3.3 with initial data $u_0^\varepsilon = u_0 * \eta_\varepsilon$, satisfies the following ε -uniform estimates:

$$\|\partial_t u^\varepsilon\|_{[L^2((0, T); W^{-1,1}(\mathbb{R}))]^M} \leq C \left(1 + \|u_0\|_{[L^\infty(\mathbb{R})]^M}^2 \right).$$

where $W^{-1,1}(\mathbb{R})$ is the dual of the space $W^{1,\infty}(\mathbb{R})$.

Proof of Lemma 5.4:

The idea to bound $\partial_t u^\varepsilon$ is simply to use the available bounds on the right hand side of the equation (P_ε) .

We will give a proof by duality. We multiply the equation (P_ε) by $\phi \in [L^2((0, T), W^{1,\infty}(\mathbb{R}))]^M$ and integrate on $(0, T) \times \mathbb{R}$, which gives

$$\int_{(0,T) \times \mathbb{R}} \phi \partial_t u^\varepsilon = \varepsilon \overbrace{\int_{(0,T) \times \mathbb{R}} \phi \partial_{xx}^2 u^\varepsilon}^{I_1} - \overbrace{\int_{(0,T) \times \mathbb{R}} \phi a(u^\varepsilon) \diamond \partial_x u^\varepsilon}^{I_2}.$$

We integrate by parts the term I_1 , and obtain that for $0 < \varepsilon \leq 1$:

$$\begin{aligned} |I_1| &\leq \left| \int_{(0,T) \times \mathbb{R}} \partial_x \phi \partial_x u^\varepsilon \right| \leq T \|\partial_x \phi\|_{[L^2((0,T), L^\infty(\mathbb{R}))]^M} \|\partial_x u^\varepsilon\|_{[L^2((0,T), L^1(\mathbb{R}))]^M}, \\ &\leq 2T^{\frac{3}{2}} \|\phi\|_{[L^2((0,T), W^{1,\infty}(\mathbb{R}))]^M} \|u_0\|_{[L^\infty(\mathbb{R})]^M}, \end{aligned} \quad (5.32)$$

here, we have used the inequality

$$\|\partial_x u^\varepsilon\|_{[L^2((0,T); L^1(\mathbb{R}))]^M} \leq 2T^{\frac{1}{2}} \|u_0\|_{[L^\infty(\mathbb{R})]^M}, \quad (5.33)$$

which follows from estimate (4.29) for bounded and nondecreasing function u^ε . Similarly, for the term I_2 , we have:

$$\begin{aligned} |I_2| &\leq M_0 \|u\|_{[L^\infty((0,T) \times \mathbb{R})]^M} \|\phi\|_{[L^2((0,T), L^\infty(\mathbb{R}))]^M} \|\partial_x u^\varepsilon\|_{[L^2((0,T), L^1(\mathbb{R}))]^M}, \\ &\leq 2T^{\frac{1}{2}} M_0 \|u_0\|_{[L^\infty(\mathbb{R})]^M}^2 \|\phi\|_{[L^2((0,T), W^{1,\infty}(\mathbb{R}))]^M}. \end{aligned} \quad (5.34)$$

Finally, collecting (5.32) and (5.34), we get that there exists a constant $C = C(T, M_0)$ independent of $0 < \varepsilon \leq 1$ such that:

$$\left| \int_{(0,T) \times \mathbb{R}} \phi \partial_t u^\varepsilon \right| \leq C \left(1 + \|u_0\|_{[L^\infty(\mathbb{R})]^M}^2 \right) \|\phi\|_{[L^2((0,T), W^{1,\infty}(\mathbb{R}))]^M}$$

which gives the announced result where we use that $L^2((0, T), W^{-1,1}(\mathbb{R}))$ is the dual of $L^2((0, T), W^{1,\infty}(\mathbb{R}))$ (see Cazenave and Haraux [9, Th 1.4.19, Page 17]). \square

Corollary 5.5 (ε -Uniform estimates)

Let $T > 0$, $0 < \varepsilon \leq 1$ and function $u_0 \in [L^\infty(\mathbb{R})]^M$ satisfying (H1) and (H2). Then the solution of the system (P_ε) - (ID_ε) given in Theorem 3.3 with initial data $u_0^\varepsilon = u_0 * \eta_\varepsilon$, satisfies the following ε -uniform estimates:

$$\|\partial_x u^\varepsilon\|_{[L^\infty((0,T); L \log L(\mathbb{R}))]^M} + \|u^\varepsilon\|_{[L^\infty((0,T) \times \mathbb{R})]^M} + \|\partial_t u^\varepsilon\|_{[L^2((0,T); W^{-1,1}(\mathbb{R}))]^M} \leq C.$$

where $C = C(T, M, M_0, M_1 \|u_0\|_{[L^\infty(\mathbb{R})]^M}, \|\partial_x u_0\|_{[L \log L(\mathbb{R})]^M})$.

We can easily prove this Corollary collecting Lemmata 5.1, 5.4 and 5.3 and Proposition 5.2.

6 Passage to the limit and the proof of Theorem 1.1

In this section, we prove that the system (P)-(ID) admits solutions u in the distributional sense. They are the limits of u^ε given by Theorem 3.3 when $\varepsilon \rightarrow 0$. To do this, we will justify the passage to the limit as ε tends to 0 in the system (P_ε) -(ID_ε) by using some compactness tools that are presented in a first Subsection.

6.1 Preliminary results

First, for all I open interval of \mathbb{R} , we denote by

$$L \log L(I) = \left\{ f \in L^1(I) \text{ such that } \int_I |f| \ln(1 + |f|) < +\infty \right\}.$$

Lemma 6.1 (Compact embedding)

Let I an open and bounded interval of \mathbb{R} . If we denote by

$$W^{1,L \log L}(I) = \{u \in L^1(I) \text{ such that } \partial_x u \in L \log L(I)\}.$$

Then the following injection:

$$W^{1,L \log L}(I) \hookrightarrow C(I),$$

is compact.

For the proof of this Lemma see R. A. Adams [1, Th 8.32].

Lemma 6.2 (Simon's Lemma)

Let X, B, Y be three Banach spaces, such that

$$X \hookrightarrow B \text{ with compact embedding and } B \hookrightarrow Y \text{ with continuous embedding.}$$

Let $T > 0$. If $(u^\varepsilon)_\varepsilon$ is a sequence such that,

$$\|u^\varepsilon\|_{L^\infty((0,T);X)} + \|u^\varepsilon\|_{L^\infty((0,T);B)} + \|\partial_t u^\varepsilon\|_{L^q((0,T);Y)} \leq C,$$

where $q > 1$ and C is a constant independent of ε , then $(u^\varepsilon)_\varepsilon$ is relatively compact in $C((0,T);B)$.

For the proof, see J. Simon [38, Corollary 4, Page 85].

In order to show the existence of solution system (P) in Subsection 6.2, we will apply this lemma to each scalar component in the particular case where $X = W^{1,\log}(I)$, $B = L^\infty(I)$ and $Y = W^{-1,1}(I) := (W^{1,\infty}(I))'$.

We denote by $K_{exp}(I)$ the class of all measurable function u , defined on I , for which,

$$\int_I (e^{|u|} - 1) < +\infty.$$

The space $EXP(I)$ is defined to be the linear hull of $K_{exp}(I)$. This space is supplemented with the following Luxemburg norm (see Adams [1, (13), Page 234]):

$$\|u\|_{EXP(I)} = \inf \left\{ \lambda > 0 : \int_I \left(e^{\frac{|u|}{\lambda}} - 1 \right) \leq 1 \right\},$$

Let us recall some useful properties of this space.

Lemma 6.3 (Weak star topology in $L \log L$)

Let $E_{exp}(I)$ be the closure in $EXP(I)$ of the space of functions bounded on I . Then $E_{exp}(I)$ is a separable Banach space which verifies,

i) $L \log L(I)$ is the dual space of $E_{exp}(I)$.

ii) $L^\infty(I) \hookrightarrow E_{exp}(I)$.

For the proof, see Adams [1, Th 8.16, 8.18, 8.20].

Lemma 6.4 (Generalized Hölder inequality, Adams [1, 8.11, Page 234])

Let $f \in EXP(I)$ and $g \in L \log L(I)$. Then $fg \in L^1(I)$, with

$$\|fg\|_{L^1(I)} \leq 2\|f\|_{EXP(I)}\|g\|_{L \log L(I)}.$$

The following Lemma, we allow to define later the restriction of a function $f \in W^{-1,1}(\mathbb{R})$ on all open interval I of \mathbb{R} .

Lemma 6.5 (Extension)

For all open interval I of \mathbb{R} , there exists a linear and continuous operator of extension $P : W^{1,\infty}(I) \rightarrow W^{1,\infty}(\mathbb{R})$ such that

i) $Pu|_I = u$ for $u \in W^{1,\infty}(I)$.

ii) $\|Pu\|_{W^{1,\infty}(\mathbb{R})} \leq \|u\|_{W^{1,\infty}(I)}$ for $u \in W^{1,\infty}(I)$.

for the proof of this Lemma see for instance Brezis [7, Th.8.5].

Thanks this Lemma, we can notice that, if $f \in W^{-1,1}(\mathbb{R})$, where $W^{-1,1}(\mathbb{R}) := (W^{1,\infty}(\mathbb{R}))'$, we can define, for all open interval I of \mathbb{R} , the function $f|_I$ as the following

$$\langle f|_I, h \rangle_{W^{-1,1}(I), W^{1,\infty}(I)} = \langle f, Ph \rangle_{W^{-1,1}(\mathbb{R}), W^{1,\infty}(\mathbb{R})}.$$

for all $h \in W^{1,\infty}(I)$.

6.2 Proof of Theorem 1.1

Step 1 (Existence):

First, by Corollary 5.5 we know that for any $T > 0$, the solutions u^ε of the system (P_ε) - (ID_ε) obtained with the help of Theorem 3.3, are ε -uniformly bounded in $[L^\infty((0, T) \times \mathbb{R})]^M$. Hence, as ε goes to zero, we can extract a subsequence still denoted by u^ε , that converges weakly- \star in $[L^\infty((0, T) \times \mathbb{R})]^M$ to some limit u . Then we want to show that u is a solution of the system (P)-(ID). Indeed, since the passage to the limit in the linear terms is trivial in $[\mathcal{D}'((0, T) \times \mathbb{R})]^M$, it suffices to pass to the limit in the non-linear term,

$$a(u^\varepsilon) \diamond \partial_x u^\varepsilon.$$

According to Corollary 5.5 we know that for all open and bounded interval I of \mathbb{R} there exists a constant C independent on ε such that:

$$\|u^\varepsilon\|_{[L^\infty((0, T); W^{1, L \log L}(I))]^M} + \|u^\varepsilon\|_{[L^\infty((0, T) \times I)]^M} + \|\partial_t u^\varepsilon\|_{[L^2((0, T); W^{-1, 1}(I))]^M} \leq C.$$

From the compactness of $W^{1, L \log L}(I) \hookrightarrow L^\infty(I)$ (see Lemma 6.3 (i)), we can apply Simon's Lemma (i.e. Lemma 6.2), with $X = [W^{1, L \log L}(I)]^M$, $B = [L^\infty(I)]^M$ and $Y = [W^{-1, 1}(I)]^M$, which shows that

$$u^\varepsilon \text{ is relatively compact in } [L^\infty((0, T) \times I)]^M \hookrightarrow [L^1((0, T); L^\infty(I))]^M. \quad (6.35)$$

Then from continuous injection of $L^\infty(I) \hookrightarrow E_{exp}(I)$ (see Lemma 6.3 (ii)), we deduce that,

$$u^\varepsilon \text{ is relatively compact in } [L^1((0, T); E_{exp}(\Omega))]^M. \quad (6.36)$$

On the other hand, by Corollary 5.5, we notice that $\partial_x u^\varepsilon$ is ε -uniformly bounded in $[L^\infty((0, T); L \log L(I))]^M$. Moreover, the space $[L^\infty((0, T); L \log L(I))]^M$ is the dual space of $[L^1((0, T); E_{exp}(I))]^M$, because $L \log L(I)$ is the dual space of $E_{exp}(I)$ (see Lemma 6.3 (ii) and Cazenave, Haraux [9, Th 1.4.19, Page 17]). Then, up to a subsequence

$$\partial_x u^\varepsilon \rightarrow \partial_x u \text{ weakly-}\star \text{ in } [L^\infty((0, T); L \log L(I))]^M. \quad (6.37)$$

Form (6.36) and (6.37), we see that we can pass to the limit in the non-linear term in the sense

$$[L^1((0, T); E_{exp}(I))]^M - strong \times [L^\infty((0, T); L \log L(I))]^M - weak - \star.$$

Because this is true for any bounded open interval I and for any $T > 0$, we deduce that,

$$a(u^\varepsilon) \diamond \partial_x u^\varepsilon \rightarrow a(u) \diamond \partial_x u \quad \text{in } \mathcal{D}'((0, T) \times \mathbb{R})$$

Consequently, we can pass to the limit in (P_ε) and get that,

$$\partial_t u + a(u) \diamond \partial_x u = 0 \quad \text{in } \mathcal{D}'((0, T) \times \mathbb{R}).$$

This solution u is also satisfy the following estimates (see for instance Brezis [7, Prop. 3.12]):

$$(E1') \quad \|\partial_x u\|_{[L^\infty((0, T); L \log L(\mathbb{R}))]^M} \leq \liminf \|\partial_x u^\varepsilon\|_{[L^\infty((0, T); L \log L(\mathbb{R}))]^M} \leq C,$$

$$(E2') \quad \|u\|_{[L^\infty((0, T) \times \mathbb{R})]^M} \leq \liminf \|u^\varepsilon\|_{[L^\infty((0, T) \times \mathbb{R})]^M} \leq \|u_0\|_{[L^\infty(\mathbb{R})]^M},$$

At this stage we remark that, thanks to these two estimates we obtain that $(a(u) \diamond \partial_x u) \in [L^\infty((0, T); L \log L(\mathbb{R}))]^M$, which gives, since $\partial_t u = -a(u) \diamond \partial_x u$, that $\partial_t u \in [L^\infty((0, T); L \log L(\mathbb{R}))]^M$, and then $u \in [C([0, T]; L \log L(\mathbb{R}))]^M$.

Step 2 (The initial conditions):

It remains to prove that the initial conditions (ID) coincides with $u(\cdot, 0)$. Indeed, by Corollary 5.5, we see that, for all open bounded interval I of \mathbb{R} , u^ε is ε -uniformly bounded in

$$[W^{1,2}((0, T); W^{-1,1}(I))]^M \hookrightarrow [C^{\frac{1}{2}}([0, T]; W^{-1,1}(I))]^M,$$

where $W^{-1,1}(I)$ is the dual of $W^{1,\infty}(I)$. It follows that, there exists a constant C independent on ε , such that, for all $t, s \in [0, T]$:

$$\|u^\varepsilon(t) - u^\varepsilon(s)\|_{[W^{-1,1}(I)]^M} \leq C|t - s|^{\frac{1}{2}}.$$

In particular if we set $s = 0$, we have:

$$\|u^\varepsilon(t) - u_0^\varepsilon\|_{[W^{-1,1}(I)]^M} \leq Ct^{\frac{1}{2}}. \quad (6.38)$$

Now we pass to the limit in (6.38). Indeed, the functions u^ε and u_0^ε are ε -uniformly bounded in $[W^{1,2}((0, T); W^{-1,1}(I))]^M$ and $[W^{-1,1}(I)]^M$ respectively. Moreover we know that $u^\varepsilon - u_0^\varepsilon$ converges weakly- \star in $[L^\infty((0, T) \times I)]^M$ to $u - u_0$.

Therefore, we can extract a subsequence still denoted by $u^\varepsilon - u_0^\varepsilon$, that weakly- \star converges in $[W^{1,2}((0, T); W^{-1,1}(I))]^M$ to $u - u_0$. In particular this subsequence converges, for all $t \in (0, T)$, weakly- \star in $[L^\infty((0, t); W^{-1,1}(I))]^M$, and consequently it verifies (see for instance Brezis [7, Prop. 3.12]),

$$\|u - u_0\|_{[L^\infty((0,t);W^{-1,1}(I))]^M} \leq \liminf \|u^\varepsilon - u_0^\varepsilon\|_{[L^\infty((0,t);W^{-1,1}(I))]^M} \leq Ct^{\frac{1}{2}}.$$

From (6.38) we deduce that

$$\|u(t) - u_0\|_{[W^{-1,1}(I)]^M} \leq Ct^{\frac{1}{2}},$$

which proves that $u(\cdot, 0) = u_0$ in $[\mathcal{D}'(\mathbb{R})]^M$.

Step 3 (Continuity of solution):

Now, we are going to prove the continuity estimate (1.5). For all $h > 0$ and $(t, x) \in (0, T) \times \mathbb{R}$, we have:

$$\begin{aligned} |u(t, x+h) - u(t, x)| &\leq \left| \int_x^{x+h} \partial_x u(t, y) dy \right| \\ &\leq 2 \|1\|_{EXP(x, x+h)} \|\partial_x u\|_{L \log L(x, x+h)}, \\ &\leq 2 \frac{1}{\ln(\frac{1}{h} + 1)} \|\partial_x u\|_{L^\infty((0, T); L \log L(\mathbb{R}))}, \\ &\leq C \frac{1}{\ln(\frac{1}{h} + 1)}, \end{aligned}$$

where we have used in the second line the generalized Hölder inequality (see Lemma 6.4) and in last line we have used that $\partial_x u \in L^\infty((0, T); L \log L(\mathbb{R}))$. Which proves finally the continuity in space. Now, we prove the continuity in time, for all $\delta > 0$ and $(t, x) \in (0, T) \times \mathbb{R}$, we have:

$$\begin{aligned} \delta |u(t + \delta, x) - u(t, x)| &= \int_x^{x+\delta} |u(t + \delta, x) - u(t, x)| dy, \\ &\leq \overbrace{\int_x^{x+\delta} |u(t + \delta, x) - u(t + \delta, y)| dy}^{K_1}, \\ &\quad + \overbrace{\int_x^{x+\delta} |u(t + \delta, y) - u(t, y)| dy}^{K_2}, \\ &\quad + \overbrace{\int_x^{x+\delta} |u(t, y) - u(t, x)| dy}^{K_3}. \end{aligned}$$

Similarly, as in the last estimate, we can show that:

$$\begin{aligned}
K_1 + K_3 &\leq \delta \int_x^{x+\delta} |\partial_x u(t + \delta, y)| dy, + \delta \int_x^{x+\delta} |\partial_x u(t, y)| dy, \\
&\leq 4\delta \|1\|_{EXP(x, x+\delta)} \|\partial_x u\|_{L^\infty((0, T); L \log L(\mathbb{R}))}, \\
&\leq C \frac{\delta}{\ln(\frac{1}{\delta} + 1)}.
\end{aligned}$$

Now, we use that u is a solution of (P), and we obtain that:

$$\begin{aligned}
K_2 &\leq \int_x^{x+\delta} \int_t^{t+\delta} |\partial_t u(s, y)| dy, \\
&\leq \int_t^{t+\delta} \int_x^{x+\delta} |a(u(s, y)) \diamond \partial_x u(s, y)| ds dy, \\
&\leq \delta M_0 \|u\|_{L^\infty((0, T) \times \mathbb{R})} \|1\|_{EXP(x, x+\delta)} \|\partial_x u\|_{L^\infty((0, T); L \log L(\mathbb{R}))}, \\
&\leq C \frac{\delta}{\ln(\frac{1}{\delta} + 1)},
\end{aligned}$$

where we have used in last line that $u \in L^\infty((0, T) \times \mathbb{R})$, collecting the estimates of K_1 , K_2 and K_3 , we prove that:

$$|u(t + \delta, x) - u(t, x)| \leq \frac{1}{\delta} (K_1 + K_2 + K_3) \leq C \frac{1}{\ln(\frac{1}{\delta} + 1)},$$

which proves finally the following:

$$|u(t + \delta, x + h) - u(t, x)| \leq C \left(\frac{1}{\ln(\frac{1}{\delta} + 1)} + \frac{1}{\ln(\frac{1}{h} + 1)} \right).$$

□

7 Some remarks on the uniqueness

In this Section we study the uniqueness of solution of the system (P)-(ID) with

$$a^i(u) = \sum_{j=1, \dots, M} A_{ij} u^j.$$

We show some uniqueness results for some particular matrices with $M \geq 2$.

For the proof of Theorem 1.5 in Subsection 7.2, we need to recall in the following Subsection the definition of viscosity solution and some well-known results in this framework.

7.1 Some useful results for viscosity solutions

The notion of viscosity solutions is quite recente. This concept has been introduced by Crandall and Lions [10, 11] in 1980, to solve the first-order Hamilton-Jacobi equations. The theory then extended to the second order equations by the work of Jensen [27] and Ishii [23]. For good introduction of this theory, we refer to Barles [5] and Bardi, Capuzzo-Dolcetta [3].

Now, we recall the definition of viscosity solution for the following problem for all $0 \leq \varepsilon \leq 1$:

$$\partial_t v + H(t, x, v, \partial_x v) - \varepsilon \partial_{xx} v = 0 \quad \text{with } x, v \in \mathbb{R}, t \in (0, T). \quad (7.39)$$

where $H : (0, T) \times \mathbb{R}^3 \mapsto \mathbb{R}$ is the Hamiltonian and is supposed continuous. We will set

$$USC((0, T) \times \mathbb{R}) = \{f \text{ such that } f \text{ is upper semicontinuous on } (0, T) \times \mathbb{R}\},$$

$$LSC((0, T) \times \mathbb{R}) = \{f \text{ such that } f \text{ is lower semicontinuous on } (0, T) \times \mathbb{R}\}.$$

Definition 7.1 (Viscosity subsolution, supersolution and solution)

A function $v \in USC((0, T) \times \mathbb{R})$ is a viscosity subsolution of (7.39) if it satisfies, for every $(t_0, x_0) \in (0, T) \times \mathbb{R}$ and for every test function $\phi \in C^2((0, T) \times \mathbb{R})$, that is tangent from above to v at (t_0, x_0) , the following holds:

$$\partial_t \phi + H(t_0, x_0, v, \partial_x \phi) - \varepsilon \partial_{xx} \phi \leq 0.$$

A function $v \in LSC((0, T) \times \mathbb{R})$ is a viscosity supersolution of (7.39) if it satisfies, for every $(t_0, x_0) \in (0, T) \times \mathbb{R}$ and for every test function $\phi \in C^2((0, T) \times \mathbb{R})$, that is tangent from below to v at (t_0, x_0) , the following holds:

$$\partial_t \phi + H(t_0, x_0, v, \partial_x \phi) - \varepsilon \partial_{xx} \phi \geq 0.$$

A function v is a viscosity solution of (7.39) if, and only if, it is a sub and a supersolution of (7.39).

Let us now recall some well-known results.

Remark 7.2 (Classical solution-viscosity solution)

If v is a C^2 solution of (7.39), then v is a viscosity solution of (7.39).

Lemma 7.3 (Stability result, see Barles [5, Th 2.3])

We suppose that, for $\varepsilon > 0$, v^ε is a viscosity solution of (7.39). If $v^\varepsilon \rightarrow v$ uniformly on every compact set then v is a viscosity solution of (7.39) with $\varepsilon = 0$.

Lemma 7.4 (Gronwall for viscosity solution)

Let v , a locally bounded $USC(0, T)$ function, which is a viscosity subsolution of the equation $\frac{d}{dt}v = \alpha v$ where $\alpha \geq 0$. Assume that $v(0) \leq v_0$ then $v \leq v_0 e^{\alpha T}$ in $(0, T)$.

The proof of this Lemma is a direct application of the comparison principle, (see Barles [5, Th 2.4]).

Remark 7.5

From Lemmata 7.2, 7.3 and from (6.35), we can notice that the solution u^i of our system (P) given in Theorem 1.1 is also a viscosity solution of (P) (where the w^j for $j \neq i$ are considered fixed to apply Definition 7.1).

7.2 Uniqueness results

In this Subsection we prove Theorem 1.5. Before going on, we recall in the following Remark a well-known uniqueness results and we recall in Theorem 7.7 the uniqueness results of $W^{1,\infty}$ solution of (P).

Remark 7.6 (Uniqueness for quasi-monotone Hamiltonians)

If the elements of the matrix A satisfy:

$$A_{ii} + \sum_{j \neq i, A_{ij} < 0} A_{ij} \geq 0 \quad \text{for all } i = 1, \dots, M.$$

and if $\partial_x u^i \geq 0$ for $i = 1, \dots, M$, then we can easily check that the Hamiltonian

$$H_i(u, \partial_x u^i) = \left(\sum_{j=1, \dots, M} A_{ij} u^j \right) \partial_x u^i,$$

is quasi-monotone in the sense of Ishii, Koike [25, (A.3)]. Then the result of Ishii, Koike [25, Th.4.7] shows that for any initial condition $u_0 \in [L^\infty(\mathbb{R})]^M$ satisfying (H1)-(H2), the system (P) satisfies the comparison principle which implies the uniqueness of the solution.

We have the following result which seems quite standard:

Theorem 7.7 (Uniqueness of the $W^{1,\infty}$ solution)

Let $u_0 \in [W^{1,\infty}(\mathbb{R})]^M$ and $T > 0$. Then system (P)-(ID) admits a unique solution in $[W^{1,\infty}([0, T) \times \mathbb{R})]^M$.

The proof of this Theorem is given in Appendix, because we have not found any proof of such a result in the literature.

Proof of Theorem 1.5:

Using Theorem 7.7 with $a^i(u) = \sum_{j=1, \dots, M} A_{ij} u^j$, it is enough to show that the system (P)-

(ID) admits a solution in $[W^{1,\infty}([0, T) \times \mathbb{R})]^M$. To do that, it is enough to prove that the solution u^ε of the approximated system obtained in Corollary 5.5 satisfies that $\partial_x u^\varepsilon$ is bounded in $[L^\infty([0, T) \times \mathbb{R})]^M$ uniformly in $0 < \varepsilon \leq 1$. Indeed, we then get the same

property for $\partial_x u$, where u is the limit of u^ε as $\varepsilon \rightarrow 0$. Moreover, from the equation (P) satisfied by u and the fact that

$$u \in [L^\infty((0, T) \times \mathbb{R})]^M \quad \text{and} \quad \partial_x u \in [L^\infty((0, T) \times \mathbb{R})]^M,$$

we deduce that $\partial_t u \in [L^\infty((0, T) \times \mathbb{R})]^M$ which shows that $u \in [W^{1,\infty}([0, T) \times \mathbb{R})]^M$.

To simplify, we denote

$$w^\varepsilon = \partial_x u^\varepsilon,$$

and we interest in the

$$\max_{x \in \mathbb{R}} w^{\varepsilon,i}(t, x) = m_i(t).$$

This maximum is reached at least at some point $x_i(t)$, because $w^{\varepsilon,i} \in C^\infty((0, T) \times \mathbb{R}) \cap W^{1,p}((0, T) \times \mathbb{R})$ for all $1 < p \leq +\infty$ (see Lemma 4.1, (4.19)).

In the following we prove in the two cases (i) and (ii) defined in Theorem 1.5 that m_i , for all $i = 1, \dots, M$, is bounded uniformly in ε . First, deriving with respect to x the equation (P_ε) satisfied by $u^\varepsilon \in [C^\infty((0, T) \times \mathbb{R})]^M$, we can see that w^ε satisfies the following equation

$$\partial_t w^{\varepsilon,i} - \varepsilon \partial_{xx} w^{\varepsilon,i} + \sum_{j=1, \dots, M} A_{ij} u^{\varepsilon,j} \partial_x w^{\varepsilon,i} + \sum_{j=1, \dots, M} A_{ij} w^{\varepsilon,j} w^{\varepsilon,i} = 0. \quad (7.40)$$

Now, we prove that m_i is a viscosity subsolution of the following equation,

$$\frac{d}{dt} m_i(t) + \sum_{j=1, \dots, M} A_{ij} w^{\varepsilon,j}(t, x_i(t)) w^{\varepsilon,i}(t, x_i(t)) \leq 0. \quad (7.41)$$

Indeed, let $\phi \in C^2(0, T)$ a test function, such that $\phi \geq m_i$ and $\phi(t_0) = m_i(t_0)$ for some $t_0 \in (0, T)$. From the definition of m_i , we can easily check that $\phi \geq w^{\varepsilon,i}(t, x)$ and $\phi(t_0) = w^{\varepsilon,i}(t_0, x_i(t_0))$. But, the fact that $w^{\varepsilon,i} \in C^\infty((0, T) \times \mathbb{R})$, by Remark 7.2 we know that $w^{\varepsilon,i}$ is a viscosity subsolution of (7.40). We apply Definition 7.1, and the fact that $\partial_x \phi = \partial_{xx} \phi = 0$, we get

$$\frac{d}{dt} \phi(t_0) + \sum_{j=1, \dots, M} A_{ij} w^{\varepsilon,j}(t_0, x_i(t_0)) w^{\varepsilon,i}(t_0, x_i(t_0)) \leq 0.$$

Which proves that m_i is a viscosity subsolution of (7.41).

Two cases may occur:

i) Here, we consider the case where $M \geq 2$ and $A_{ij} \geq 0$ for all $j \geq i$. We see the equation satisfied by m_1 , we deduce that satisfies (a viscosity subsolution)

$$\frac{d}{dt}m_1(t) \leq - \sum_{j=1,\dots,M} A_{1j}w^{\varepsilon,j}(t, x_1(t))w^{\varepsilon,1}(t, x_1(t)) \leq 0,$$

where we have used the fact that, for $j = 1, \dots, M$, $A_{1j} \geq 0$ and $w^{\varepsilon,j} \geq 0$. This proves by Lemma 7.4 (with $\alpha = 0$) that,

$$m_1(t) \leq m_1(0) = w^{\varepsilon,1}(t, x_1(t)) \leq \|\partial_x u_0^1\|_{L^\infty(\mathbb{R})}.$$

We reason by recurrence: we assume that $m_j \leq C$ for all $j \leq i$, where C is a positive constant independent of ε , and we prove that m_{i+1} is bounded uniformly in ε . Indeed, we know that

$$\begin{aligned} \frac{d}{dt}m_{i+1}(t) &\leq - \sum_{j=1,\dots,M} A_{i+1,j}w^{\varepsilon,j}(t, x_j(t))w^{\varepsilon,i+1}(t, x_{i+1}(t)), \\ &\leq - \sum_{j < i+1} A_{i+1,j}w^{\varepsilon,j}(t, x_j(t))w^{\varepsilon,i+1}(t, x_{i+1}(t)) \\ &\quad - \sum_{M \geq j \geq i+1} A_{i+1,j}w^{\varepsilon,j}(t, x_j(t))w^{\varepsilon,i+1}(t, x_{i+1}(t)), \end{aligned}$$

We use that $A_{i+1,j} \geq 0$, for $M \geq j \geq i+1$, we obtain that

$$\begin{aligned} \frac{d}{dt}m_{i+1}(t) &\leq - \sum_{j < i+1} A_{i+1,j}w^{\varepsilon,j}(t, x_j(t))w^{\varepsilon,i+1}(t, x_{i+1}(t)), \\ &\leq C \left(\sum_{j < i+1} |A_{i+1,j}| \right) m_{i+1}(t). \end{aligned}$$

This implies by Lemma 7.4, with $\alpha = C \left(\sum_{j < i+1} |A_{i+1,j}| \right)$, that

$$\begin{aligned} m_{i+1}(t) &\leq m_{i+1}(0)e^{\alpha T}, \\ &\leq \|\partial_x u_0^{i+1}\|_{L^\infty(\mathbb{R})}e^{\alpha T}. \end{aligned}$$

Which proves that for all $i = 1, \dots, M$, m_i is bounded uniformly in ε .

ii) Here, we consider the case where $M \geq 2$ and $A_{ij} \leq 0$ for all $i \neq j$. Taking the sum over the index i , from (7.41) we get that the quantity $m(t) = \sum_{i=1,\dots,M} m_i(t)$ satisfies (a viscosity subsolution see Bardi et al. [4])

$$\begin{aligned}
\frac{d}{dt}m(t) &\leq - \sum_{i,j=1,\dots,M} A_{ij}w^{\varepsilon,j}(t, x_i(t))w^{\varepsilon,i}(t, x_i(t)), \\
&\leq - \sum_{i,j=1,\dots,M} A_{ij}w^{\varepsilon,j}(t, x_j(t))w^{\varepsilon,i}(t, x_i(t)), \\
&\leq 0.
\end{aligned}$$

where we have used that the matrix A satisfies $(H2')$ and $w^{\varepsilon,i} \geq 0$, for $i = 1, \dots, M$. Using Lemma 7.4 with $\alpha = 0$, we get

$$\begin{aligned}
m(t) &\leq m(0) = \sum_{i=1,\dots,M} \partial_x u_0^{\varepsilon,i}, \\
&\leq \sup_{y \in \mathbb{R}} \sum_{i=1,\dots,M} \partial_x u_0^i(y).
\end{aligned}$$

which proves (1.7). □

8 Application on the dynamics of dislocations densities

In this Section, we present a model describing the dynamics of dislocations densities. We refer to [22] for a physical presentation of dislocations which are (moving) defects in crystals. Even if the problem is naturally a three-dimensional problem, we will first assume that the geometry of the problem is invariant by translations in the x_3 -direction. This reduces the problem to the study of dislocations densities defined on the plane (x_1, x_2) and propagation in a given direction \vec{b} belonging to the plane (x_1, x_2) (which is called the ‘‘Burger’s vector’’).

In this setting we consider a finite number of slip directions $\vec{b} \in \mathbb{R}^2$ and to each \vec{b} we will associate a dislocation density. For a detailed physical presentation of a model with multi-slip directions, we refer to Yefimov, Van der Giessen [41] and Yefimov [40, ch. 5.] and to Groma, Balogh [21] for the case of a model with a single slip direction. See also Cannone et al. [8] for a mathematical analysis of the Groma, Balogh model. In Subsection ??, we present the 2D-model with multi-slip directions.

In the particular geometry where the dislocations densities only depend on the variable $x = x_1 + x_2$, this two-dimensional model reduces to one-dimensional model which presented in In Subsection 8.2. See El Hajj [15] and El Hajj, Forcadel [16] for a study in the special case of a single slip direction. Finally in Subsection 8.3, we explain how to recover equation (P) as a model for dislocation dynamics with $a^i(u) = \sum_{j=1,\dots,M} A_{ij}u^j$ for some particular non-negative and symmetric matrix A .

8.1 The 2D-model

We now present in details the two-dimensional model. We denote by \mathbf{X} the vector $\mathbf{X} = (x_1, x_2)$. We consider a crystal filling the whole space \mathbb{R}^2 and its displacement $v = (v_1, v_2) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, where we have not yet introduced the time dependence for the moment.

We define the total strain by

$$\varepsilon(v) = \frac{1}{2}(\nabla v + {}^t\nabla v),$$

where ∇v is the gradient with $(\nabla v)_{ij} = \frac{\partial v_i}{\partial x_j}$, $i, j \in \{1, 2\}$.

Now, we assume that the dislocations densities under consideration are associated to edge dislocations. This means that we consider M slip directions where each direction is characterized by a Burgers vector $\vec{b}^k = (b_1^k, b_2^k) \in \mathbb{R}^2$, for $k = 1, \dots, M$. This leads to M type of dislocations which propagate in the plan (x_1, x_2) following the direction of \vec{b}^k , for $k = 1, \dots, M$.

The total strain can be splitted in two parts:

$$\varepsilon(v) = \varepsilon^e + \varepsilon^p.$$

Here, ε^e is the elastic strain and ε^p the plastic strain defined by

$$\varepsilon^p = \sum_{k=1, \dots, M} \varepsilon^{0,k} u^k, \quad (8.42)$$

where, for each $k = 1, \dots, M$, the scalar function u^k is the plastic displacement associated to the k -th slip system whose matrix $\varepsilon^{0,k}$ is defined by

$$\varepsilon^{0,k} = \frac{1}{2} \left(\vec{b}^k \otimes \vec{n}^k + \vec{n}^k \otimes \vec{b}^k \right), \quad (8.43)$$

where \vec{n}^k is unit vector orthogonal to \vec{b}^k and $(\vec{b}^k \otimes \vec{n}^k)_{ij} = b_i^k n_j^k$.

To simplify the presentation, we assume the simplest possible periodicity property of the unknowns.

Assumption (H):

i) The function v is \mathbb{Z}^2 -periodic with $\int_{(0,1)^2} v \, d\mathbf{X} = 0$.

ii) For each $k = 1, \dots, M$, there exists $L^k \in \mathbb{R}^2$ such that $u^k - L^k \cdot \mathbf{X}$ is a \mathbb{Z}^2 -periodic.

iii) The integer M is even with $M = 2N$ and $L^{k+N} = L^k$, and that

$$\begin{aligned} L^{k+N} &= L^k, \quad \vec{b}^{k+N} = -\vec{b}^k, \quad \vec{n}^{k+N} = \vec{n}^k, \\ \varepsilon^{0,k+N} &= -\varepsilon^{0,k}. \end{aligned}$$

iv) We denote by $\vec{\tau}^k = (\tau_1^k, \tau_2^k)$ a vector parallel to \vec{b}^k such that $\vec{\tau}^{k+N} = \vec{\tau}^k$. We require that L^k is chosen such $\vec{\tau}^k \cdot L^k \geq 0$.

The plastic displacement u^k is related to the dislocation density associated to the Burgers vector \vec{b}^k . We have

$$k\text{-th dislocation density} = \vec{\tau}^k \cdot \nabla u^k \geq 0. \quad (8.44)$$

The stress is then given by

$$\sigma = \Lambda : \varepsilon^e, \quad (8.45)$$

i.e. the coefficients of the matrix σ are:

$$\sigma_{ij} = \sum_{k,l=1,2} \Lambda_{ijkl} \varepsilon_{kl}^e \quad \text{for } i, j = 1, 2,$$

where $\Lambda = (\Lambda_{ijkl})_{i,j,k,l=1,2}$, are the constant elastic coefficients of the material, satisfying for $m > 0$:

$$\sum_{ijkl=1,2} \Lambda_{i,j,k,l} \varepsilon_{ij} \varepsilon_{kl} \geq m \sum_{i,j=1,2} \varepsilon_{ij}^2 \quad (8.46)$$

for all symmetric matrices $\varepsilon = (\varepsilon_{ij})_{ij}$, *i.e.* such that $\varepsilon_{ij} = \varepsilon_{ji}$.

Finally, for $k = 1, \dots, M$, the functions u^k and v are then assumed to depend on $(t, \mathbf{X}) \in (0, T) \times \mathbb{R}^2$ and to be solutions of the coupled system (see Yefimov [40, ch. 5.] and Yefimov, Van der Giessen [41]):

$$\left\{ \begin{array}{ll} \operatorname{div} \sigma = 0 & \text{on } (0, T) \times \mathbb{R}^2, \\ \sigma = \Lambda : (\varepsilon(v) - \varepsilon^p) & \text{on } (0, T) \times \mathbb{R}^2, \\ \varepsilon(v) = \frac{1}{2} (\nabla v + {}^t \nabla v) & \text{on } (0, T) \times \mathbb{R}^2, \\ \varepsilon^p = \sum_{k=1, \dots, M} \varepsilon^{0,k} u^k & \text{on } (0, T) \times \mathbb{R}^2, \\ \partial_t u^k = (\sigma : \varepsilon^{0,k}) \vec{\tau}^k \cdot \nabla u^k & \text{on } (0, T) \times \mathbb{R}^2, \quad \text{for } k = 1, \dots, M, \end{array} \right. \quad (8.47)$$

i.e. in coordinates

$$\left\{ \begin{array}{l} \sum_{j=1,2} \frac{\partial \sigma_{ij}}{\partial x_j} = 0 \quad \text{on } (0, T) \times \mathbb{R}^2, \quad \text{for } i = 1, 2, \\ \sigma_{ij} = \sum_{k,l=1,2} \Lambda_{ijkl} (\varepsilon_{kl}(v) - \varepsilon_{kl}^p) \quad \text{on } (0, T) \times \mathbb{R}^2, \\ \varepsilon_{ij}(v) = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \quad \text{on } (0, T) \times \mathbb{R}^2, \\ \varepsilon_{ij}^p = \sum_{k=1, \dots, M} \varepsilon_{ij}^{0,k} u^k \quad \text{on } (0, T) \times \mathbb{R}^2, \end{array} \right. \quad \text{for } i, j = 1, 2 \quad (8.48)$$

$$\left. \begin{array}{l} \partial_t u^k = \left(\sum_{i,j \in \{1,2\}} \sigma_{ij} \varepsilon_{ij}^{0,k} \right) \vec{\tau}^k \cdot \nabla u^k \quad \text{on } (0, T) \times \mathbb{R}^2, \quad \text{for } k = 1, \dots, M, \end{array} \right.$$

where the unknowns of the system are u^k and the displacement $v = (v_1, v_2)$ and with $\varepsilon^{0,k}$ defined in (8.43). Here the first equation of (8.47) is the equation of elasticity, while the last equation of (8.47) is the transport equation satisfied by the plastic displacement whose velocity is given by the Peach-Koehler force $\sigma : \varepsilon^{0,k}$. Remark that this implies in particular that each dislocation density satisfies a conservation law (see the equation obtained by derivation, using (8.44)). Remark also that our equations are compatible with our periodicity assumptions (H), (i)-(ii).

8.2 Derivation of the 1D-model

In this Subsection we are interested in a particular geometry where the dislocations densities depend only on the variable $x = x_1 + x_2$. This will lead to 1D-model. More precisely, we make the following:

Assumption (H'):

i) The functions $v(t, \mathbf{X})$ and $u^k(t, \mathbf{X}) - L^k \cdot \mathbf{X}$ depend on the variable $x = x_1 + x_2$.

ii) $\tau_1^k + \tau_2^k = 1$, for $k = 1, \dots, M$.

iii) $L_1^k = L_2^k$ for $k = 1, \dots, M$.

For this particular one-dimensional geometry, we denote by an abuse of notation the function $v = v(t, x)$ which is 1-periodic in x . If we set $l^k = \frac{L_1^k + L_2^k}{2}$, we have

$$L^k \cdot \mathbf{X} = l^k \cdot x + \left(\frac{L_1^k - L_2^k}{2} \right) (x_1 - x_2).$$

By assumption (H') , (iii) , we see (again by an abuse of notation) that $u = (u^k(t, x))_{k=1, \dots, M}$ is such that for $k = 1, \dots, M$, $u^k(t, x) - l^k \cdot x$ is 1-periodic in x .

Now, we can integrate the equations of elasticity, *i.e.* the first equation of (8.47). Using the \mathbb{Z}^2 -periodicity of the unknowns (see assumption (H) , (i) - (ii)), and the fact that $\varepsilon^{0, k+N} = -\varepsilon^{0, k}$ (see assumption (H) , (iii)), we can easily conclude that the strain

$$\varepsilon^e \text{ as a linear function of } (u^j - u^{j+N})_{j=1, \dots, N} \quad \text{and of } \left(\int_0^1 (u^j - u^{j+N}) dx \right)_{j=1, \dots, N}. \quad (8.49)$$

This leads to the following Lemma

Lemma 8.1 (Stress for the 1D-model)

Under assumptions (H) , (i) - (ii) - (iii) and (H') , (i) - (iii) and (8.46), we have

$$-\sigma : \varepsilon^{0, i} = \sum_{j=1, \dots, M} A_{ij} u^j + \sum_{j=1, \dots, M} Q_{ij} \int_0^1 u^j dx, \quad \text{for } i = 1, \dots, N. \quad (8.50)$$

where for $i, j = 1, \dots, N$

$$\begin{cases} A_{i,j} = A_{j,i} & \text{and} & A_{i+N,j} = -A_{i,j} = A_{i,j+N}, \\ Q_{i,j} = Q_{j,i} & \text{and} & Q_{i+N,j} = -Q_{i,j} = Q_{i,j+N}. \end{cases} \quad (8.51)$$

Moreover the matrix A is non-negative.

The proof of Lemma 8.1 will be given at the end of this Subsection.

Finally using Lemma 8.1, we can eliminate the stress and reduce the problem to a one-dimensional system of M transport equations only depending on the function u^i , for $i = 1, \dots, M$. Naturally, from (8.50) and (H') , (ii) this 1D-model has the following form

$$\partial_t u^i + \left(\sum_{j=1, \dots, M} A_{ij} u^j + \sum_{j=1, \dots, M} Q_{ij} \int_0^1 u^j dx \right) \partial_x u^i = 0, \quad \text{on } (0, T) \times \mathbb{R}, \quad \text{for } i = 1, \dots, M, \quad (8.52)$$

with from (8.44)

$$\partial_x u^i \geq 0 \quad \text{for } i = 1, \dots, M. \quad (8.53)$$

Now, we give the proof of Lemma 8.1.

Proof of Lemma 8.1:

For the 2D-model, let us consider the elastic energy on the periodic cell (using the fact that ε^e is \mathbb{Z}^2 -periodic)

$$E^{el} = \frac{1}{2} \int_{(0,1)^2} \Lambda : \varepsilon^e : \varepsilon^e d\mathbf{X}.$$

By definition of σ and ε^e , we have for $i = 1, \dots, M$

$$\sigma : \varepsilon^{0,i} = -\nabla_{u^i} E^{el}. \quad (8.54)$$

On the other hand usind (H') , (i) - (iii) , (with $x = x_1 + x_2$) we can check that we can rewrite the elastic energy as

$$E^{el} = \frac{1}{2} \int_0^1 \Lambda : \varepsilon^e : \varepsilon^e dx.$$

Replacing ε^e by its expression (8.49), we get:

$$\begin{aligned} E^{el} &= \frac{1}{2} \int_0^1 \sum_{i,j=1,\dots,N} A_{ij} (u^j - u^{j+N})(u^i - u^{i+N}) dx \\ &\quad + \frac{1}{2} \sum_{i,j=1,\dots,N} Q_{ij} \left(\int_0^1 (u^j - u^{j+N}) dx \right) \left(\int_0^1 (u^i - u^{i+N}) dx \right), \end{aligned}$$

for some symmetric matrices $A_{i,j} = A_{j,i}$, $Q_{i,j} = Q_{j,i}$. In particular, joint to (8.54) this gives exactly (8.50) with (8.51).

Let us now consider the functions $w^i = u^i - u^{i+N}$ such that

$$\int_0^1 w^i dx = 0 \quad \text{for } i=1,\dots,N, \quad (8.55)$$

From (8.46) that we deduce that

$$0 \leq E^{el} = \frac{1}{2} \int_0^1 \sum_{i,j=1,\dots,N} A_{ij} w^i w^j dx.$$

More precisely, for all $i = 1, \dots, N$ and for all $\bar{w}^i \in \mathbb{R}$, we set

$$w^i = \begin{cases} \bar{w}^i & \text{on } [0, \frac{1}{2}], \\ -\bar{w}^i & \text{on } [\frac{1}{2}, 1], \end{cases}$$

which satisfies (8.55). Finally, we obtain that

$$0 \leq E^{el} = \frac{1}{2} \int_0^1 \sum_{i,j=1,\dots,N} A_{ij} \bar{w}^i \bar{w}^j dx.$$

Because this is true for every \bar{w}^i , we deduce that A a non-negative matrix. \square

8.3 Heuristic derivation of the non-periodic model

Starting from the model (8.52)-(8.53) where for $i = 1, \dots, M$, $u^i(t, x) - l^i \cdot x$ is 1-periodic in x , we now want to rescale the unknowns to make the periodicity disappear. More precisely, we have the following Lemma:

Lemma 8.2 (Non-periodic model)

Let u be a solution of (8.52)-(8.53) assuming Lemma 8.1 and $u^i(t, x) - l^i \cdot x$ is 1-periodic in x . Let

$$u_\delta^j(t, x) = u^j(\delta t, \delta x), \quad \text{for a small } \delta > 0 \text{ and for } j = 1, \dots, M,$$

such that, for all $j = 1, \dots, M$

$$u_\delta^j(0, \cdot) \rightarrow \bar{u}^j(0, \cdot), \quad \text{as } \delta \rightarrow 0, \quad \text{and } \bar{u}^j(0, \pm\infty) = \bar{u}^{j+N}(0, \pm\infty) \quad (8.56)$$

Then $\bar{u} = (\bar{u}^j)_{j=1, \dots, M}$ formally is a solution of

$$\partial_t \bar{u}^i + \left(\sum_{j=1, \dots, M} A_{ij} \bar{u}^j \right) \partial_x \bar{u}^i = 0, \quad \text{on } (0, T) \times \mathbb{R}, \quad (8.57)$$

with the matrix A is non-negative and $\partial_x \bar{u}^i \geq 0$ for $i = 1, \dots, M$.

We remark that the limit problem (8.57) is of type (P) with (H1') and (H2').

Now, we give a formal proof of Lemma 8.2.

Formal proof of Lemma 8.2:

Here, we know that $u_\delta^i - \delta l^i \cdot x$ is $\frac{1}{\delta}$ -periodic in x , and satisfies for $i = 1, \dots, M$

$$\partial_t u_\delta^i + \left(\sum_{j=1, \dots, M} A_{ij} u_\delta^j + \delta \sum_{j=1, \dots, M} Q_{ij} \int_0^{\frac{1}{\delta}} u_\delta^j dx \right) \partial_x u_\delta^i = 0, \quad \text{on } (0, T) \times \mathbb{R}, \quad (8.58)$$

To simplify, assume that the initial data $u_\delta(0, \cdot)$ converge to a function $\bar{u}(0, \cdot)$ such that $\partial_x u_\delta(0, \cdot)$ has a support in $(-R, R)$, uniformly in δ , where R a positive constant. We expect heuristically that the velocity in (8.58) remains uniformly bounded as $\delta \rightarrow 0$.

Therefore, using the finite propagation speed, we see that, there exists a constant C independent in δ , such that $\partial_x u_\delta(t, \cdot)$ has a support in $(-R - Ct, R + Ct)$ uniformly in δ . Moreover, from (8.56) and the fact that

$$\sum_{j=1, \dots, M} Q_{ij} \int_0^{\frac{1}{\delta}} u_\delta^j dx = \sum_{j=1, \dots, N} Q_{ij} \int_0^{\frac{1}{\delta}} (u^j - u^{j+N}) dx,$$

we deduce that

$$\sum_{j=1,\dots,M} Q_{ij} \int_0^{\frac{1}{\delta}} u_{\delta}^j dx,$$

remains bounded uniformly in δ . Then formally the non-local term vanishes and we get for $i = 1, \dots, M$

$$\sum_{j=1,\dots,M} A_{ij} u_{\delta}^j + \delta \sum_{j=1,\dots,M} Q_{ij} \int_0^{\frac{1}{\delta}} u_{\delta}^j dx \rightarrow \sum_{j=1,\dots,M} A_{ij} \bar{u}^j, \quad \text{as } \delta \rightarrow 0,$$

which proves that \bar{u} is solution of (8.57), with the matrix A is non-negative. \square

9 Appendix: proof of Theorem 7.7

Let $u_1 = (u_1^i)_i$ and $u_2 = (u_2^i)_i$, for $i = 1, \dots, M$, be two solutions of the system (P) in $[W^{1,\infty}((0, T) \times \mathbb{R})]^M$, such that $u_1^i(0, \cdot) = u_2^i(0, \cdot)$.

Then by definition u_1^i and u_2^i satisfy respectively the following system, for $i = 1, \dots, M$:

$$\partial_t u_1^i = -a^i(u_1) \partial_x u_1^i,$$

$$\partial_t u_2^i = -a^i(u_2) \partial_x u_2^i,$$

Subtracting the two equations we get:

$$\partial_t (u_1^i - u_2^i) = - (a^i(u_1) - a^i(u_2)) \partial_x u_1^i - a^i(u_2) \partial_x (u_1^i - u_2^i).$$

Multiplying this system by $(u_1^i - u_2^i) (\psi)^2$ where $\psi(x) = e^{-|x|}$, and integrating in space, we deduce that:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|(u_1^i - u_2^i) \psi\|_{L^2(\mathbb{R})}^2 &= - \int_{\mathbb{R}} (a^i(u_1) - a^i(u_2)) (u_1^i - u_2^i) \psi^2 \partial_x u_1^i \\ &\quad - \int_{\mathbb{R}} a^i(u_2) \psi^2 (u_1^i - u_2^i) \partial_x (u_1^i - u_2^i). \end{aligned}$$

Taking the sum over i , we get:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\sum_{i=1,\dots,M} \|(u_1^i - u_2^i) \psi\|_{L^2(\mathbb{R})}^2 \right) &= \overbrace{- \int_{\mathbb{R}} \sum_{i=1,\dots,M} (a^i(u_1) - a^i(u_2)) (u_1^i - u_2^i) \psi^2 \partial_x u_1^i}^{I_1} \\ &\quad \overbrace{- \frac{1}{2} \int_{\mathbb{R}} \sum_{i=1,\dots,M} a^i(u_2) \psi^2 \partial_x (u_1^i - u_2^i)^2}^{I_2}. \end{aligned}$$

Integrating I_2 by part, we obtain:

$$I_2 = \overbrace{\frac{1}{2} \int_{\mathbb{R}} \sum_{i,j=1,\dots,M} a_{i,j}^i(u_2) (\partial_x u_2^j) \psi^2 (u_1^i - u_2^i)^2}_{I_{21}} + \overbrace{\frac{1}{2} \int_{\mathbb{R}} \sum_{i=1,\dots,M} a^i(u_2) (u_1^i - u_2^i)^2 \partial_x (\psi^2)}_{I_{22}}.$$

Next, using the fact that u_2^i is bounded in $W^{1,\infty}((0, T) \times \mathbb{R})$, for $i = 1, \dots, M$, we deduce that:

$$\begin{aligned} |I_{21}| &\leq \frac{1}{2} M M_1 \|u_2\|_{[W^\infty((0,T)\times\mathbb{R})]^M} \left(\sum_{i=1,\dots,M} \|(u_1^i - u_2^i) \psi\|_{L^2(\mathbb{R})}^2 \right), \\ &\leq C \left(\sum_{i=1,\dots,M} \|(u_1^i - u_2^i) \psi\|_{L^2(\mathbb{R})}^2 \right). \end{aligned} \tag{9.59}$$

Since $\partial_x (\psi(x))^2 = -2 \text{sign}(x) (\psi(x))^2$ and u_2^i is bounded in $W^{1,\infty}((0, T) \times \mathbb{R})$, for $i = 1, \dots, M$, we obtain:

$$\begin{aligned} |I_{22}| &\leq \frac{1}{2} M_0 \left(\sum_{i=1,\dots,M} \|(u_1^i - u_2^i) \psi\|_{L^2(\mathbb{R})}^2 \right) \\ &\leq C \left(\sum_{i=1,\dots,M} \|(u_1^i - u_2^i) \psi\|_{L^2(\mathbb{R})}^2 \right) \end{aligned} \tag{9.60}$$

Now, using the fact that u_1^i is bounded in $W^{1,\infty}((0, T) \times \mathbb{R})$, for $i = 1, \dots, M$, and the inequality $|ab| \leq \frac{1}{2}(a^2 + b^2)$, we get:

$$\begin{aligned} |I_1| &\leq \frac{1}{2} M_1 (M + 1) \|u_1\|_{[W^\infty((0,T)\times\mathbb{R})]^M} \int_{\mathbb{R}} \sum_{i=1,\dots,M} |u_1^i - u_2^i|^2 \psi^2, \\ &\leq \frac{1}{2} M_1 (M + 1) \|u_1\|_{[W^\infty((0,T)\times\mathbb{R})]^M} \left(\sum_{i=1,\dots,M} \|(u_1^i - u_2^i) \psi\|_{L^2(\mathbb{R})}^2 \right), \\ &\leq C \left(\sum_{i=1,\dots,M} \|(u_1^i - u_2^i) \psi\|_{L^2(\mathbb{R})}^2 \right). \end{aligned} \tag{9.61}$$

Finally, (9.61), (9.59) and (9.60), imply:

$$\frac{d}{dt} \left(\sum_{i=1, \dots, M} \|(u_1^i - u_2^i)\psi\|_{L^2(\mathbb{R})}^2 \right) \leq 2(|I_1| + |I_{21}| + |I_{22}|) \leq C \left(\sum_{i=1, \dots, M} \|(u_1^i - u_2^i)\psi\|_{L^2(\mathbb{R})}^2 \right).$$

Now, we apply the Gronwall Lemma and we use that $u_1^i(0, \cdot) = u_2^i(0, \cdot)$, to deduce that:

$$\sum_{i=1, \dots, M} \|(u_1^i - u_2^i)\psi\|_{L^\infty((0, T); L^2(\mathbb{R}))}^2 \leq \sum_{i=1, \dots, M} \|(u_1^i(0, \cdot) - u_2^i(0, \cdot))\psi\|_{L^2(\mathbb{R})}^2 e^{CT} = 0,$$

i.e., $u_1 = u_2$ a.e in $(0, T) \times \mathbb{R}$. □

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