Numerical Approximation in Nonlinear Elasticity

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GOAL : Compute Sructures in Large Deformation (nonlinear PDE, potentially unstable, strong heterogeneities, complex geometries, expensive, but relevant in many industrial or biological situations)

- Variational Formulations.
- Finite Element Approximations
- Adaptive Mesh Refinement
- Algebraic Solution
- Conclusions

1 Variational Formulation

Present configuration Ω^φ is unknown when large displacements are present. For solution, transport back to a known configuration Ω where constitutive laws are easy to write. Define :

$$\begin{aligned}
x^\varphi &= x + u(x) \\
F(x) &= \frac{\partial x^\varphi}{\partial x}(x) = Id + \nabla u(x) \\
&= \text{deformation gradient,} \\
T(x) &= T^\varphi(x^\varphi(x))F(x)^{-T}det(F(x)) \\
&= \text{first Piola Kirchhoff stress tensor satisfying} \\
&\quad\ T(x)\cdot nda = T^\varphi \cdot n^\varphi da^\varphi \\
v(x) &= v^\varphi(x^\varphi(x)), \\
f(x) &= f^\varphi(x^\varphi(x))det(F(x)) \\
g(x) &= g^\varphi \frac{da^\varphi}{da} \\
&= \text{density of loads in reference config.}
\end{aligned}$$

By transporting equation of equilibrium on Ω, get standard form

$$0 = \int_{\Omega} T : \nabla v dx - \int_{\Omega} f \cdot v dx - \int_{\Gamma} g \cdot v da, \forall v : \bar{\Omega} \to \mathbb{E}.$$

but acting on first Piola Kirchhoff stress tensor T.

Constitutive Laws

- GOAL : obtain stress from deformation history of constitutive material.
- ELASTICITY : simplified case where stress only depend on deformation

$$F(x) = \frac{\partial x^{\varphi}}{\partial x}(x) = Id + \nabla u(x).$$

- FUNDAMENTAL ASSUMPTION :

  No energy dissipation during a cycle and energy insensitive to any rigid body motion (change of observer) ($F \longmapsto QF$), implying $\hat{W}(F) = \tilde{W}(F^T F) := \tilde{W}(C)$.
- FINAL FORM :

$$T = 2F \frac{\partial \tilde{W}(x, F^T F)}{\partial C}.$$

- Automatically ensure symmetry of Cauchy stress tensor $det(F(x))T^{\varphi} = 2F\dfrac{\partial \tilde{W}(x, F^T F)}{\partial C}F^T$.
- Classical choices are

$$\hat{W}(F) = \frac{\lambda}{2}(Tr\frac{(C - Id)}{2})^2 + \mu Tr(\frac{(C - Id)}{2})^2,$$

only isotropic choice quadratic in C

$$\hat{W}(F) = C_1(I_1 - 3) + C_2(I_2 - 3) + a(J^2 - 1)$$
$$-(C_1 + 2C_2 + a)lnJ^2,$$

reversible in C, polyconvex, coercive, local invertibility through lnJ.

- a locally incompressible material amounts to set

$$\tilde{W}(x, C) = +\infty \text{ if } detC = J^2 \neq 1.$$

- Extension to viscoelasticity or to multiscale mechanics by introducing local differential equations at each material point.
- **Problem:** which form, which coefficients ?

Variational Formulations

By elimination of stresses by constitutive laws, equilibrium equations reduce to variational form (compressible case, displacement u as unknown)

$$\begin{aligned}\int_\Omega 2F \cdot \frac{\partial \hat{W}}{\partial C}(x, Id + \nabla u) \,:\, \nabla v \, dx \\ = \int_\Omega f \cdot v \, dx + \int_{\Gamma_1} g \cdot v \, da \ , \\ \forall \ v \in \mathrm{V} \ , \ (u - u_0) \in \mathrm{V} \ .\end{aligned}$$

To get bounded energy integrals in C, choose space V of unknown displacements and test functions as

$$\mathrm{V} = \{v \in W^{1,s}(\Omega; \mathbb{E}), \quad v|_{\Gamma_o} = 0\}.$$

Mixed formulations

Numerical difficulties when dealing with most materials which are very stiff in compression (almost incompressible)

$$W = W((I+\nabla u)^T(I+\nabla u), \det(I+\nabla u)) = W(C, J),$$

$$W_{,JJ} \approx 10^6 W_{,CC} >> W_{,CC}$$

To get approximation uniformly insensitive to compression stiffness, treat $p = W_{,J} = \dfrac{\partial W}{\partial J}$ as an independent variable. Inverting relation $p = W_{,J}(J-1)$, get *symmetric mixed problem* acting on displacement u and pressure field p

$$\int_\Omega 2F(W_{,C}(x, \nabla u) + p\frac{\partial J^2}{\partial C}) \cdot \nabla v =$$
$$\int_\Omega f \cdot v \, dx + \int_{\Gamma_1} g \cdot v \, da, \forall v \in \mathrm{V},$$

$$\int_\Omega (J^2 - 1 - W^{-1}_{,J^2}(x, p))\hat{p} = 0, \forall \hat{p}.$$

Comments

- strong form of equation in simplest case

$$((\delta_{ij} + u_{i,j})(2W_{,I_1}\delta_{jk} + pJC^{-1}_{jk}))_{,k} = f_i,$$
$$J^2 - 1 = W^{-1}_{,J^2}(p).$$

  is second order, quasilinear, nonlinear (terms in $C^{-1} = (I+\nabla u)^{-1}(I+\nabla u)^{-T}$ and $J = det(I + \nabla u)$)

- intrinsically mixed (special role of stiffness to volume change)

- no real existence theory in mixed form (only as minimisation problem)

- no guarantee on global injectivity (only local through energy terms like $detJ$).

2 Finite element approximation

Replace space V of displacements by isoparametric finite element (*same flexibility on shape as on deformation*), such as

$$\mathrm{V}_h = \{ \; v_h : \bar{\Omega} \to \mathbb{E} \;, \; v_h \, \text{continuous}, \;\; v_h = 0 \;\text{ on}\, \Gamma_0,$$

$$v_{h_{|\Omega_l}} = v_l \circ \varphi_l^{-1} \;, \; v_h \in [Q_2(\hat{\Omega})]^N \;, \; \forall l = 1, \ldots, N_h \}. \tag{1}$$

If needed, introduce finite element space for pressure

$$P_h = \{ p_h : \bar{\Omega} \to \mathbb{E}, p_{h_{|\Omega_l}} = p_l \in P_1 \}.$$

Mixed finite element problem becomes

Find displacement and pressure $(u_h, p_h) \in \mathrm{V}_h \times P_h$

$$\int_\Omega 2F \left(W_{,C}(x, C, J(p_h)) + p_h \frac{\partial \det\,(C)}{\partial C} \right) \cdot \nabla v_h$$
$$= \int_\Omega f \cdot v_h \, dx + \int_{\Gamma_1} g \cdot v_h \, da, \forall v_h \in \mathrm{V}_h,$$

$$\int_\Omega (\det\,(C) - 1 - W_{,J^2}^{-1}(x, C, p_h)) \hat{p}_h = 0, \forall \hat{p}_h \in P_h.$$

After linearisation around a given state, reduces to a standard mixed symmetric formulation

$$\begin{pmatrix} A & B^T \\ B & -C \end{pmatrix} \begin{pmatrix} \delta U \\ \delta P \end{pmatrix} = \begin{pmatrix} R_U \\ R_P \end{pmatrix}$$

under the notation

$$A_{ij} = \int_\Omega \delta C(\nabla \hat{u}_i)$$
$$\cdot \left(W_{,CC} - W_{,CJ^2} \cdot W^{-1}_{,J^2J^2} \cdot W_{,J^2C} + p_h \frac{\partial^2 \mathrm{det} C}{\partial C^2} \right)$$
$$\cdot \delta C(\nabla \hat{u}_j)$$
$$+ \quad \int_\Omega \nabla \hat{u}_i \cdot (F^{-1}T) \cdot \nabla \hat{u}_j$$

$$B_{\alpha i} = \int_\Omega \left(W_{,CJ^2} \cdot (W_{,J^2J^2})^{-1} \cdot \hat{p}_\alpha + \hat{p}_\alpha \frac{\partial \mathrm{det}\ (C)}{\partial C} \right)$$
$$\cdot \delta C(\nabla \hat{u}_i)$$

$$C_{\alpha\beta} = \int_\Omega (W_{,J^2J^2})^{-1} \hat{p}_\alpha \hat{p}_\beta$$

(small, positive, symmetric).

Can be rewritten as (much simpler for numerical analysis)

$$-(BA^{-1}B^T + C)\delta P = R_P - BA^{-1}R_U,$$
$$A\delta U = R_U - B^T\delta P.$$

Theorem 2.1 *Consider the mixed problem*

$$\begin{aligned} \mathcal{A}v + \mathcal{B}^T p &= L_v \in \mathrm{V}' \\ \mathcal{B}v &= L_p \in \mathbb{P}', \end{aligned}$$

The operator $\mathcal{B}$ is onto iif

$$\inf_{q\in\mathbb{P},\|q\|=1}\{\sup_{v\in\mathrm{V},\|v\|=1}\{\langle\mathcal{B}v, q\rangle\}\} = \beta > 0.$$

Moreover, the mixed problem is well-posed iif $\mathcal{B}$ is onto and $\mathcal{A}$ is invertible.

Here, at the incompressible limit the above inf-sup condition writes

$$\inf_{q_h\in \mathbb{P}_h}\left\{\sup_{v_h\in \mathrm{V}_h}\left\{\frac{\int_\Omega q_h F_h \frac{\partial \det C_h}{\partial C} : \nabla v_h}{\|q_h\|\|v_h\|}\right\}\right\} = C_0(h) > 0.$$

By introducing $\varphi_h(x) = u_h(x) + x$, and using

$$\frac{\partial \det}{\partial F}(F) : G = (\mathrm{cof}\ F) : G = Tr(GF^{-1})\det\ (F)$$

the condition reduces to classical condition on deformed configuration :

$$\inf_{q_h\in \mathbb{P}_h}\left\{\sup_{v_h\in \mathrm{V}_h}\left\{\frac{\int_{\varphi_h(\Omega)}(q_h\circ\varphi_h^{-1})\mathrm{div}\ (v_h\circ\varphi_h^{-1})}{\|q_h\|\|v_h\|}\right\}\right\} > 0.$$

REMARKS

- Inf-sup condition and coercivity must be uniform in h to achieve uniform stability in energy norm (data or error of small energy leading to a solution or a perturbation of small energy).
- If no inf-sup condition, need to stabilize discrete problem by adding local residuals of equilibirum equations in the kinematic equation.
- Extension to compressible case ($C \neq 0$) obtained by spectral equivalence of

$$\begin{pmatrix} A & B \\ B^t & -C \end{pmatrix}, \qquad \begin{pmatrix} A & 0 \\ 0 & -B^T A^{-1} B - C \end{pmatrix}$$

and

$$\begin{pmatrix} A & 0 \\ 0 & -B^T A^{-1} B \end{pmatrix}.$$

Nonlinear Analysis

If polyconvexity and coercivity of energy $\hat{W}$, the compressible finite element problem is well posed and satisfies

Theorem 2.2 *For any isolated minimum u of the total energy , there exists a sequence of finite element solutions $\{u_h\}_h$ strongly converging towards u*

$$\lim_{h\to 0}(\|u-u_h\|_{1,s} \\ +\|cof(Id+\nabla u)-cof(Id+\nabla u_h)\|_{0,q})=0.$$

Same theorem applies in incompressible case, under the previous nonlinear inf-sup condition relating the solutions u_h, and the spaces V_h and $\mathbb{P}_h$.

Practical choices of Mixed Finite Elements

- *Accuracy for displacement and pressure.* Problems are second order, solutions are locally H^2, rarely more. Good accuracy achieved by continuous elements (Q^k, R^k or P^k).
- *Kinematics.* Strong gradients in stresses are frequent. Requires piecewise linear deformations (quadratic displacements) as test functions (shear locking).
- *Stability of discrete problem.* To achieve stability of solution with respect to data in energy norm (leading to best approximation error), one needs
  1. coercivity of operator A on discrete set of admissible unknowns.
  2. stability of $BA^{-1}B^T$, that is stability of constraints B.

Stability is obtained either by surjectivity (inf-sup condition) or by stabilisation, with addition of local residuals, that is complex algebraic terms at each element level, and an interelement (Laplacian) coupling by penalisation of pressure jumps

$$\sum_{i \neq j} \int_{\partial T_i \cap \partial T_j} (p_{T_i} - p_{T_j})^2.$$

Very complex. Practical choice is therefore second order elements which do not need stabilisation if piecewise linear discontinuous pressures (but importance of face nodes).

Satisfying the Inf-Sup Condition

Theorem 1. (Stenberg, scale invariance of inf-sup). *The inf-sup condition is satisfied iff the finite element mesh can be split in overlapping macroelements* $\Omega = \cup_m M_m$, $M_m = \cup_i T_i$ *(patches) with*

- *each internal face is an internal face of at least one and of at most L macroelements,*
- *each macroelement M is equivalent (homeomorphic) to a given reference macroelement $\hat{M}$, satisfying*

$$\begin{aligned} P_N &:= \{p_h \in P_h(M), \textstyle\int_M p_h \; div \; v_h = 0, \\ &\qquad \forall v_h \in V_h(M) \cap H_0^1(M)\} \\ &= P_0(M). \end{aligned}$$

Theorem 2. *The inf-sup condition holds for disc. linear pressures if the velocity space contains element bubble b_i for each element T_i and face bubbles b_{ij} for each interface $\partial T_i \cap \partial T_j$.*

Proof. Choose macroelements M with two elements T_i and T_j sharing one face. To characterize all functions $p_h \in P_h(M)$ such that

$$\int_M p_h \text{ div } v_h = 0, \forall v_h \in V_h(M) \cap H_0^1(M)$$

Choose $v_h = b_i(x)\vec{k}, \forall \vec{k} \in R^3$.

$$0 = \int_M p_h \text{ div } v_h = -\int_{T_i} (\nabla p_h \cdot \vec{k}) b_i(x), \forall \vec{k} \in R^3,$$

Thus ∇p_h is locally zero, and p_h is constant per element. Choosing then $v_h = b_{ij}(x) n_{ij}$, and integrating, get

$$0 = \int_M p_h \text{ div } v_h = -\int_{\partial T_i \cap \partial T_j} (p_i - p_j) n_{ij} \cdot b_{ij}(x) n_{ij},$$

implying $p_i = p_j = p_0$.

A list of good elements with discontinuous piecewise linear pressures

1. $Q2 \times P1_{\text{disc}}$. Hexaedra of order 2 with 27 nodes in velocity.
2. $(R2 \oplus b_i) \times P1_{\text{disc}}$. Second order prismes of order 2 with 19 velocity nodes (18 usual + internal node).

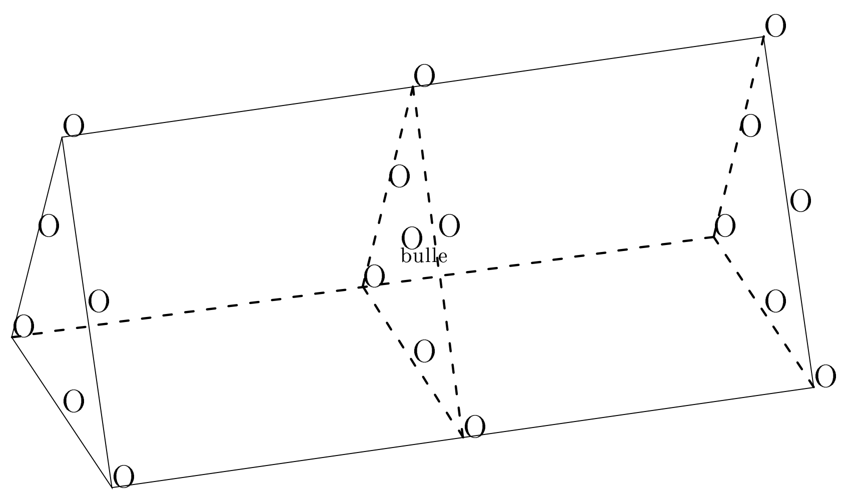

Figure 1: Geometric definition of primes

3. $(Pk \oplus b_{ij} \oplus b_i) \times P1_{\text{disc}}$Tetraedra of order 1 or 2, with velocity nodes added on mid faces and mid element.

The case of second order tetrahedra with

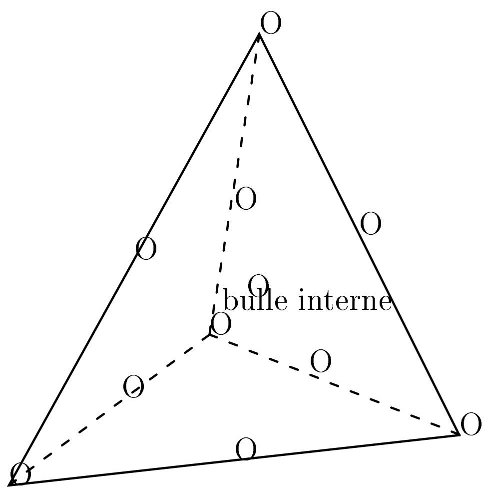

Figure 2: Geometric definition of tetraedras

internal velocity node is uncertain in 3D.

3 Adaptive mesh refinement

How many elements do you choose ? For this, use aposteriori error estimates, based on weighted residuals.

$$
\begin{aligned}
< Res, \hat{v} > \; = \; & \sum_{T} \int_{T} (divT(I + \nabla u_h) + f)\hat{v} \\
& + \sum_{e} \int_{e} [T(I + \nabla u_h) \cdot n_e]\hat{v}.
\end{aligned}
$$

Error estimate

$$
\begin{aligned}
\|e\|_a^2 &= \|A^{-1}Res\|_a^2 = \sup_{v\in V} \frac{< Res, v >^2}{\|v\|_a^2} \\
&\approx \sup_{v\in V_{fin}} \frac{< Res, v >^2}{\|v\|_a^2} \qquad \text{saturation lemma} \\
&= a(v_R, v_R) \text{dual energy norm} \\
&\approx \sum_T a(P_T v_R, P_T v_R) + \sum_e a(P_e v_R, P_e v_R)) \\
&\quad \text{partition lemma} \\
&\approx \sum_T a(A_T^{-1} I_T^t Res, A_T^{-1} I_T^t Res) \\
&\qquad + \sum_e a(A_e^{-1} I_e^t Res, A_e^{-1} I_e^t Res) \\
&= \sum_T \eta_{R,T}^2
\end{aligned}
$$

Here v_R solves global dual problem in error, and its projection $P_i v_R$ is defined by solving local element or face problem $P_i v_R = A_i^{-1} I_i^t Res$.

Incompressible extension : add energy error term on volume change

$$\|e_P\|_a^2 = \sum_T \int_T E_T(\det\ (C_) - 1 - W_{,J^2}^{-1}(C_h, p_h))^2$$

Above formula is SCALE INVARIANT, and proved to be asymptotically correct for heterogeneous isotropic materials INDEPENDENTLY OF COEFFICIENTS JUMPS.

Limitations

No anisotropic effects.

For better estimates, use more advanced tools combining (Ainsworth-Oden, Babuska, Ladeveze-Pelle) :

- residual balancing (solving one local problem per node);
- calculation in dual energy norm (i.e. solving one local energy problem per element);
- possible pollution coefficients through dual problem (Rannacher, Babuska).

MESH REFINEMENT

DATA : error estimate gives indication of optimal local refinement factor r_T by minimizing number of elements of order p

$$NE = \sum_T (r_T)^{dim}$$

under the constraint of given error

$$\epsilon^2 = \sum_T \int_T ||u - u_h||^2_{e,T} \approx \sum_T \eta^2_{R,T} r_T^{-2p}$$

yielding (Ladeveze, Pelle)

$$r_T = C \eta_{R,T}^{2/(2p+dim)}.$$

PROBLEM : industrial meshes are overly constrained (material discontinuities, geometric singularities).

CONSEQUENCE : Global adaptive remeshing with imposed refinement might be impossible.

SOLUTION : local refinement of preexisting mesh, independently from one " material " region to another.

COST: apparition of nonconforming interfaces F_{ij} to be treated as contact finite elements on which to impose either strong (slave node) or weak (mortar) continuity.

In slave nodes, get pointwise interface continuity on shared nodes

$$u(P_k^i) = u(P_k^j)$$

plus local slave constraint continuity (imposed locally inside Ω_i by introducing a corresponding contact elements)

$$u(P_l^i) = \sum_k u(P_k^i)\phi_k^T(P_l^i).$$

Mortar
PRINCIPLE :

- on each region interface Γ_{ij}, define a glue space V_{ijh}, usually as the trace of the coarser grid TrV_{jh}
- impose weak continuity by introduction of lagrange multipliers (traction forces) $\lambda_h \in V_{ijh}$

$$\int_{\Gamma_{ij}} (v_i - v_j)\mu_h = 0, \forall \mu_h \in V_{ijh}$$

leading to new mixed problem (pressures omitted here)

$$\text{Find } (u_{ih}) \in \prod_i \mathbb{H}_h(\Omega_i), \text{ and } (\lambda_{ij}) \in \prod_{i<j} \mathrm{V}_{ijh}$$

$$\int_{\Omega_i} T(\nabla u_{ih}) : \nabla v_{ih}$$
$$+ \sum_{i<j} \int_{\Gamma_{ij}} \lambda_{ij} v_{ih} - \sum_{i>j} \int_{\Gamma_{ji}} \lambda_{ji} v_{ih} = L(v_{ih}),$$
$$\forall v_{ih} \in \mathbb{H}_h(\Omega_i), \forall i > 0,$$

$$\int_{\Gamma_{ij}} (u_{ih} - u_{jh})\mu_h = 0, \forall \mu_h \in \mathrm{V}_{ijh}, \forall i < j.$$

Numerical analysis : framework

Norm

Right energy norm on interface should be $H^{\frac{1}{2}}(\Gamma_{ij})'$ for multipliers. Completely impractical. Good finite element norm is (Barbara Wohlmuth)

$$\|\mu_h\|^2_{h^{-1/2}(\Gamma_{ij})} = \sum_{F \subset \Gamma_{ij}} h_F \|\mu_h\|^2_{L^2(F)}.$$

Assumptions on V_{ijh}. Not too small, not too large

1. Each traction force $\mu = T \cdot n$ must be approached at oder k (typically $k = 2$) in mortar space

$$\inf_{\lambda_{ij} \in \mathrm{V}_{ijh}} \|\mu - \lambda_{ij}\|_{h^{-1/2}(\Gamma_{ij})} \leq C h^k \|\mu\|_{H^{k-1/2}(\Gamma_{ij})}. \tag{2}$$

2. On each interface Γ_{ij}, verify

$$\inf_{\lambda_{ij} \in \mathrm{V}_{ijh}} \sup_{v_h} \frac{\int_{\Gamma_{ij}} \lambda_{ij} v_h}{\|\lambda_{ij}\|_{h^{-1/2}(\Gamma_{ij})} \|v_h\|_{H^1(\Omega_k)}} \geq \beta.$$

Numerical analysis : Result

Theorem 3.1 *Under these assumptions, the discrete solutions satisfy (in linearized case)*

$$\|u - u_h\|_H \leq Ch^k(\|u\|^2_{H^{k+1}(\Omega)} + (\sum_{i<j} \|\sigma.n\|^2_{H^{k-1/2}(\Gamma_{ij})})^{1/2}),$$

$$\left(\sum_{i<j} \|\sigma.n - \lambda_{ij}\|^2_{h^{-1/2}(\Gamma_{ij})}\right) \leq C(\|u - u_h\|^2_H + \sum_{i<j} \inf_{\mu_{ijh} \in \mathrm{V}_{ijh}} \|\sigma.n - \mu_{ijh}\|^2_{h^{-1/2}(\Gamma_{ij})}).$$

IMPLEMENTATION : copy nodal values of mortar Trv_j on side Ω_i. Get as before interface continuity

$$\tilde{v}_i(P_k^i) = v_j(P_k^j)$$

plus local mortar constraint (imposed locally inside Ω_i by introducing a corresponding contact element)

$$\int_{\Gamma_{ij}} (v_i - \tilde{v}_i)\mu_h = 0, \forall \mu_h \in V_{ijh}.$$

The system to solve

After expansion in a finite element basis $\{\varphi_i\}_{i=1,N_h}$ of V_h and $\{\psi_i\}_{i=1,M_h}$ of $\mathbb{P}_h$, get the algebraic system

Find $\{U, P\} \in \mathbb{R}^{N_h} \times \mathbb{R}^{M_h}$ such that

$$\mathcal{F}(U, P, f, g) \;=\; 0 \qquad \text{in} \;\; \mathbb{R}^{N_h} \times \mathbb{R}^{M_h} \;.$$

Very large ($10^6 unknowns$), very nonlinear, nonsymmetric if friction or dissipation, very stiff (stiffness varies from 1 to 10^6, with second (or fourth) order diferential operators), multiple length scales (from 1 to 10^3).

Needs robustness.

The basic Newton's method

Solve iteratively (replacement of curve by tangent)

$$\mathcal{F}(U^n, P^n, f, g) + \frac{D\mathcal{F}}{D(U,P)}(U^{n+1} - U^n, P^{n+1} - P^n) = 0.$$

Fast convergence if properly initialized.

Theorem 4.1 *Let (U, P) be a solution of $\mathcal{F}(U, P) = 0$, with $\dfrac{D\mathcal{F}(U, P, f, g)}{D(U, P)}$ invertible and locally Lipschitz continuous. Then, if (U^0, P^0) is sufficiently close to (U, P), we have*

$$\|(U^{n+1}, P^{n+1}) - (U, P)\| \leq \beta \|(U^n, P^n) - (U, P)\|^2.$$

Arc-length continuation

For limit points, or for proper initialization, replace calculation of a point by calculation of a map

$$s \longmapsto \{U(s), P(s), \lambda(s)\}$$

defined on the solution curve

$$\mathcal{F}(U, P, \lambda f, \lambda g) \ = \ 0 \ ,$$

and satisfying the additional differential equation

$$\|\frac{dU}{ds}\|^2 + \|\frac{dP}{ds}\|^2 + |\frac{d\lambda}{ds}|^2 \ = \ 1 \ .$$

Here, λ is an auxiliary unknown load scaling factor.

Differential curve to be computed by predictor corrector based on forward Euler prediction.

5 Conclusion and state of the art

1. Standard approach uses compressible or nearly incompressible materials (penalization), first order isoparametric elements, and a basic Newton's algorithm.
2. Sophisticated methods involve incompressible models, second-order elements, adaptive mesh refinement and arclength continuation.
3. Main danger is to be too poor in kinematics.

4. problem in cost reduction (domain decomposition, parallel frontal solvers, localized nonconforming refinement)
5. problem in identification of constitutive laws (a lot of insight from microscopic understanding)
6. contact with friction is not yet properly solved
7. nonlinear shell models for steel rubbers mixtures are still under development.

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