Competitive Variance Reduction for Monte Carlo Computation

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1 Introduction

The RM algorithm is a part of the large family of non deterministic methods used to find the extrema of unknown functions provided the evaluation of these functions are based on random measurements. The general form of the RM algorithm is given by :

$$X_{n+1} = X_n - \gamma_n f(X_n) + \gamma_n \epsilon_{n+1} \tag{1}$$

or

$$X_{n+1} = X_n - \gamma_n Y_{n+1} \tag{2}$$

where the ϵ_n are random errors made when evaluating the function f. In financial case Y_n is given by :

$$Y_{n+1} = (X_n - Z_{n+1}) G^2(Z_{n+1}) e^{-X_n \cdot Z_{n+1} + \frac{1}{2}\|X_n\|^2} \tag{3}$$

where G is slitghly a payoff function and $(Z_n)_n$ is a sequence of independant identically distributed standard gaussian vectors.

2 Mathematical Context

2.1 Financial background

The dynamic of the asset S_t is

$$dS_t = S_t(rdt + \sigma dW_t)$$

whose solution is given by

$$S_t = S_0 \exp\left((r - \frac{\sigma^2}{2})t + \sigma W_t \right)$$

Then S can excatly be simulated at dates $0 < T_1 < \cdots < T_m = T$ by

$$S_{T_i} = S_{T_{i-1}} \exp\left((r - \frac{\sigma^2}{2})(T_i - T_{i-1}) + \sigma \sqrt{T_i - T_{i-1}} Z_i \right) , \tag{4}$$

where $Z_1, \dots , Z_m$ are indepedent gaussian variables with mean 0 and variance 1.

So if we want to price an option whose payoff is given by $f(S_{T_1}, \ldots, S_{T_m})$ we have to evaluate

$$V_0 = \mathbf{E}[e^{-rT} f(S_{T_1}, \ldots, S_{T_m})]. \tag{5}$$

Clearly V_0 can be expressed as

$$V_0 = \mathbf{E}[G(Z_1, \ldots, Z_m)] \ , \tag{6}$$

2.2 Importance sampling

Follwing [3], we change the law of $Z = (Z_1, \ldots Z_m)$ adding a drift vector $\mu = (\mu_1, \ldots \mu_m)$. An elementary version of Girsanov theorem leads to the following representation of V_0

$$V_0 = \mathbf{E}(\alpha(\mu)) \ , \tag{7}$$

where

$$\alpha(\mu) = G(Z + \mu) e^{-\mu \cdot Z - \frac{1}{2}\mu \cdot \mu} \ . \tag{8}$$

The authors in [3] give approximative procedure to minimize the variance of $\alpha(\mu)$ or equivalently $\mathbf{E}(\alpha^2(\mu))$. Here, we present Robbins-Monro algorithm which enable us to find $\mu^* \in \mathbf{R}^m$ such that

$$H(\mu^*) = \min_{\mu \in \mathbf{R}^m} H(\mu), \tag{9}$$

where

$$H(\mu) = \mathbf{E}(\alpha^2(\mu)) \ . \tag{10}$$

One have :

Proposition 2.1 *If* $\mathbf{E}(G^2(Z)) < \infty$, *then* H *is indefinitely differentiable in* $\mathbf{R}^m$ *and there exist a unique* $\mu^* \in \mathbf{R}^m$ *such that*

$$H(\mu^*) = \min_{\mu \in \mathbf{R}^m} H(\mu). \tag{11}$$

Now we can develop the version of the RM algorithm which deals with our problem.

3 RM Algorithm

3.1 General form

Let (γ_n) be a sequence of positive numbers such that

$$\sum \gamma_n = +\infty \quad \text{and} \quad \sum \gamma_n^2 < +\infty. \tag{12}$$

As said previously, the general form of the RM algorithm is given by

$$X_{n+1} = X_n - \gamma_n \cdot Y_{n+1} \tag{13}$$

with

$$\mathbf{E}[Y_{n+1}/\mathcal{F}_n] = \nabla H(X_n) \ , \tag{14}$$

where $\mathcal{F}_n = \sigma\{X_k,\ Y_k,\ k \leq n\}$ is the σ-algebra generated by X_k, Y_k for $k \leq n$.

In our case, the sequence (Y_n) is define by

$$Y_{n+1} = (X_n - Z_{n+1})G^2(Z_{n+1})e^{-X_n\cdot Z_{n+1}+\frac{1}{2}X_n\cdot X_n} \ , \tag{15}$$

where $(Z_n)_{n\geq 0}$ is a sequence of vectors $i.i.d$ with law $\mathcal{N}(0, I_d)$. clearly Z_{n+1} is independent of $\mathcal{F}_n$. There are some important classical results on this algorithm (see for exemple [2] or [1] for details).

Remark 3.1 One of the most important hypothesis that allowed the algorithm to converge is :

$$\mathbf{E}[\|Y_{n+1}\|^2/\mathcal{F}_n] \leq K(1+\|X_n\|^2) \quad a.s., \tag{16}$$

(see [2]) or

$$\sum_n \gamma_n^2 \mathbf{E}[\|Y_{n+1}\|^2/\mathcal{F}_n] < +\infty \quad a.s. \tag{17}$$

Unfortunately the exponential part in the expression of our Y_n makes impossible the proof of this kind of hypothesis. that is why we try to use the truncated version of the RM algorithm introduce in [5] by H.F. Chen and Y.M. Zhu .

3.2 Randomly varying bounds truncation

Let $v(x) = \|x - \mu^*\|^2$. To describe the new algorithm, first take two arbitrary points $x_1^* \neq x_2^*$ in $\mathbf{R}^m$ and $M > 0$ such that

$$\max(v(x_1^*), v(x_2^*)) < \min(M, \inf(v(x); \|x\| > M)).$$

Let $(u_n)_n$ be an arbitrary increasing sequence of positive numbers tending to infinity, with $u_0 > M + 8$.
Define, for $n = 0, 1, \ldots,$

$$X_{n+1} = \begin{cases} X_n - \gamma_n Y_{n+1} & \text{si} \quad \|X_n - \gamma_n Y_{n+1}\| \leq u_{\sigma(n)}, \\ x_n^* & \text{sinon} \end{cases} \tag{18}$$

where

$$\sigma(n) = \sum_{k=0}^{n-1} \mathbf{1}_{\|X_n - \gamma_n Y_{n+1}\| > u_{\sigma(n)}}, \quad \sigma(0) = 0,$$

$$x_n^* = \begin{cases} x_1^* & \text{si } \sigma(n) \text{ est pair }, \\ x_2^* & \text{sinon} \quad . \end{cases}$$

The main result of [4] states that

Theorem 1 *Let* $\epsilon_{n+1} = Y_{n+1} - \nabla H(X_n)$ *satisfies* $\overline{\lim}\, \gamma_n \| \sum_{i=0}^{n-1} \epsilon_{i+1} \| = 0$ *a.s.* . *Then the algorithm above converges to* μ^* *a.s.* *and the number of truncations* σ_n *is finite.*

See [4] for the proof.

Remark 3.2 We prooved that if the payoff function G is bounded, then it is possible to choose the sequence $(u_n)_n$ such that the truncated algorithm converges to μ^*.

4 Numerical tests

We first note that our method is very simple to implement and is always applicable since the asset S_t can be simulate . The bound condition on the payoff function is not strong : there are many products whose payoff satisfies this condition (put, digital, ...). To price a Call product, one can first price the corresponding Put and use Call-Put parity to recorver the Call price. Tables4.1 and 4.2 contain some interesting numerical results which confirm that our method is effective. The choice of points $x_i^*, \quad i = 1, 2$ (see (18)) has no sgnificant influence on the limit of the algorithm. However when the state dimension m is large, it seems better to take $x_i^* \approx \epsilon \frac{1}{m}, \quad i = 1, 2$ whith $\epsilon = 1$ if we deal whith a Call option and $\epsilon = -1$ otherwise.

TABLE4.1

Estimated Variance Reduction Ratios for the European Call[e]

Parameters			Importance Sampling			
Volatility	Strike	Mu	Std Ratio	Price	MCC Price	BS Price
0.3	45	1.004	3.23	11.22	11.32	11.26
	50	1.118	3.29	8.34	8.43	8.37
	55	1.139	3.32	6.04	6.14	6.07
0.1	40	0.364	5.18	13.800	13.815	13.807
	50	0.610	2.51	5.14	5.17	5.15
	60	0.171	1.23	0.612	0.614	0.601

[e]All results are based on a total of 15,000 gaussian paths. All results use

$S_0 = 50, \quad r = 0.1, \quad T = 1.0, \quad \alpha = 0.05, \quad \beta = 1.0, \quad x_1^* = 0.625, \quad x_2^* = 1.0725 \quad \text{and} \quad M = 100.0.$

TABLE4.2

Estimated Variance Reduction Ratios for an European Basket Put

Parameters		Importance Sampling		
Volatility	Strike	Std Ratio	VarRM	VarC
0.3	45	1.73	39.1	116.5
	50	2.07	37.1	158.6
	55	1.87	58.7	204.9

All results are based on a total of 25,000 gaussian paths. All results use

$m = 5, \quad S_0 = 50, \quad r = 0.1, \quad T = 1.0, \quad \alpha = 0.005, \quad \beta = 1.0, \quad x_1^* = -0.205, \quad x_2^* = -0.195 \quad \text{and} \quad M = 100.0.$

5 Conclusion

The method we present is

- simple
- easy to implement,
- not very expensive,
- effective.

Our next goal is to combine this procedure with a stratification method along an 'optimal direction' which can be found by the same algorithm.

References

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[4] H.-F. Chen, L. Guo, and A.-J. Gao. Convergence and robustness of the robbins-monro algorithm truncated at randomly varying bounds. *Stochastic Processes and their Applications*, 27:217–231, 1988.

[5] H.F. Chen and Y.M. Zhu :. Stochastic approximation procedure with randomly varying truncations. *Scientia Sinica (series A)*, 29.