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Méthode de décomposition de domaine avec maillages non conformes pour l'électromagnétisme

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Outline

- Domain decomposition method with non–matching grids:
  - A Model problem and its variational formulation;
  - Definition of the mortar method;
  - Convergence results;
- Applications
  - magnetostatics;
  - 3D electric engines.

$$\mathbf{curl}\ \mathbf{E} = -\partial_t \mathbf{B} \qquad \mathrm{div}\,(\varepsilon \mathbf{E}) = \rho$$

$$\mathbf{curl}\ \mu^{-1}\mathbf{B} = \varepsilon \partial_t \mathbf{E} + \mathbf{J}_s + \sigma \mathbf{E} \qquad \mathrm{div}\,(\mathbf{B}) = 0$$

$$\mathbf{E} \wedge \mathbf{n}_{|\partial\Omega} = 0 \quad \mathbf{B} \cdot \mathbf{n}_{|\partial\Omega} = 0$$

$$\mathbf{E}(\cdot, 0) = \mathbf{E_0} \quad \mathbf{B}(\cdot, 0) = \mathbf{B_0}.$$

- Good parallel/domain decomposition methods are of KEY IMPORTANCE in electromagnetic computation.
  - huge calculation: 3D vectors in the space, sometimes $6D$ vectors in the space.
  - Often the coefficients involved (ε, μ, σ) are discontinuous at the interface between different materials.

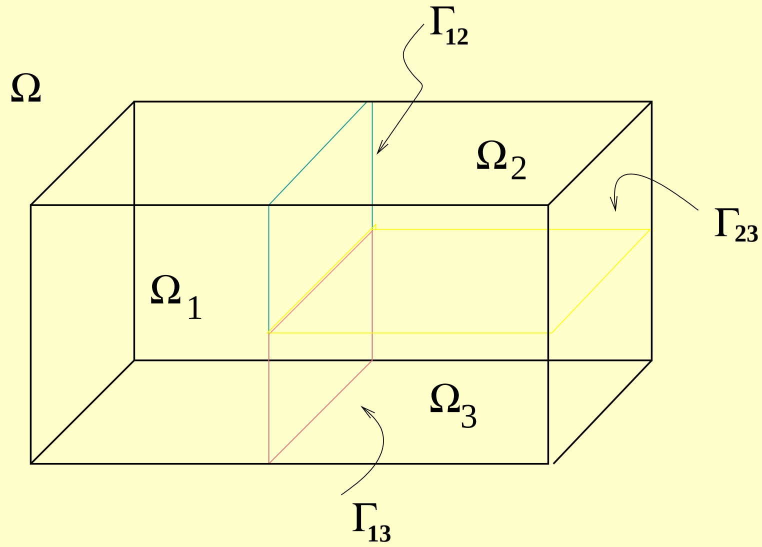
Maxwell's equations, model problem

$$\mathbf{curl}\ \mathbf{E} = -\partial_t \mathbf{B}$$

$$\mathbf{curl}\ \mu^{-1}\mathbf{B} = \varepsilon \partial_t \mathbf{E} + \mathbf{J}_s + \sigma \mathbf{E} \quad \Longrightarrow \quad \mathbf{u} + \mathbf{curl}\,\mathbf{curl}\,\mathbf{u} = \mathbf{f}$$

$$\mathbf{E} \wedge \mathbf{n}_{|\partial\Omega} = 0 \quad \mathbf{B} \cdot \mathbf{n}_{|\partial\Omega} = 0 \qquad \mathbf{u} \wedge \mathbf{n} = 0 \quad \text{at } \partial\Omega$$

$$\mathbf{E}(\cdot, 0) = \mathbf{E_0} \quad \mathbf{B}(\cdot, 0) = \mathbf{B_0}.$$



3D continuous/discrete problem

$$H_0(\mathbf{curl},\Omega) = \{\mathbf{u} \in L^2(\Omega)^3 \ : \quad \mathbf{curl}\,\mathbf{u} \in L^2(\Omega)^3\ ,\ \mathbf{u}\wedge\mathbf{n} = 0 \text{ at } \partial\Omega\}$$
$$= \{(\mathbf{u}_i)_i \in L^2(\Omega_i)^3,\ \mathbf{curl}\,\mathbf{u}_i \in L^2(\Omega_i)^3\ ,\ \mathbf{u}\wedge\mathbf{n} = 0 \text{ at } \partial\Omega$$
$$\mathbf{u}_i\wedge\mathbf{n} = \mathbf{u}_j\wedge\mathbf{n} \text{ at } \Gamma_{ij}\}$$

Variational formulation:

$$\mathbf{u} \in H_0(\mathbf{curl},\Omega) \quad \forall \mathbf{v} \in H_0(\mathbf{curl},\Omega)$$

$$\int_\Omega \mathbf{u}\cdot\mathbf{v} + \int_\Omega \mathbf{curl}\,\mathbf{u}\cdot\mathbf{curl}\,\mathbf{v} = \int_\Omega \mathbf{f}\cdot\mathbf{v}$$

Discretization:

- Independent grids on each subdomain , $\mathcal{T}_{i,h}$;

  Edge Elements in each subdomain;

- Easy and optimal coupling among subdomains;

  $$\left(\llbracket \mathbf{u}_h\wedge\mathbf{n}\rrbracket, \boldsymbol{\varphi}_h\right)_{\Gamma_{ij}} = 0 \quad \boldsymbol{\varphi}_h \in ??$$

  Mortar method: the constraint is in the discrete space

Σ = Skeleton, $\Sigma = \overline{\bigcup_{\ell=1}^{L} \Gamma_\ell}$, non-overlapping partition

$(\mathcal{T}_{i,h}, X_{i,h})$, conforming $\mathbf{H}(\mathbf{curl}, \Omega_i)$ finite elements , $i = 1, .., N$

$$X_h = \bigotimes_i X_{i,h} + \text{boundary condition at } \partial\Omega$$

$(\mathcal{T}^{\Gamma}_{\ell,h}, M_{\ell,h})$ finite elements on Γ_ℓ, $\ell = 1, .., L$

$$M_h = \bigotimes_\ell M_{\ell,h}$$

We set $\mathbf{u}_h = (\mathbf{u}_h^1, .., \mathbf{u}_h^N)$

$$a(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^{M} \int_{\Omega_i} \left(\mathbf{u}_h^i \cdot \mathbf{v}_h^i + \mathbf{curl}\,\mathbf{u}_h^i \cdot \mathbf{curl}\,\mathbf{v}_h^i \right) d\Omega \qquad \forall \mathbf{u}, \mathbf{v} \in X_h$$

$$b(\mathbf{u}, \boldsymbol{\lambda}) = \sum_{\ell=1}^{L} \int_{\Gamma_\ell} [\![\mathbf{u}_h \wedge \mathbf{n}]\!] \cdot \boldsymbol{\lambda}_h \, d\Gamma \qquad \forall \mathbf{u} \in X_h, \boldsymbol{\lambda}_h \in M_h$$

Discrete space: $\mathcal{X}_h = \{ \mathbf{v}_h \in X_h \quad \text{s.t.} \quad b(\mathbf{v}_h, \boldsymbol{\varphi}_h) = 0 \quad \forall \boldsymbol{\varphi}_h \in M_h \}$

Discrete problem: Find $\mathbf{u}_h \in \mathcal{X}_h$ such that

$$a(\mathbf{u}_h, \mathbf{v}_h) = \int_\Omega \mathbf{f} \mathbf{v}_h \quad \forall \mathbf{v}_h \in \mathcal{X}_h$$

Edge Elements

Families of $\mathbf{H}(\mathbf{curl},\Omega)$ conforming finite elements, **Tetrahedra**

Nédélec ('80): $(\mathcal{E}\mathcal{E}_1^0(\Omega_i))$ $\hat{\mathbf{v}} : \hat{T} \to \mathbb{R}^3$, $\hat{\mathbf{v}} = \mathbf{a} + \mathbf{b} \wedge \mathbf{x}$; $\quad \mathbf{a}, \mathbf{b} \in \mathbb{R}^3$

$$\int_e \hat{\mathbf{v}} \cdot \hat{\tau}_{\hat{T}}\, \varphi\, ds \quad \forall \varphi \in P_0(e) \;\; e = e_i,\; i = 1,.,6 \qquad \text{(6 D.o.F.)}$$

Nédélec ('86): $(\mathcal{E}\mathcal{E}_1(\Omega_i))$ $\hat{\mathbf{v}} : \hat{T} \to \mathbb{R}^3$, $\hat{\mathbf{v}} \in \mathbb{P}_1(\hat{T})^3$

$$\int_e \hat{\mathbf{v}} \cdot \hat{\tau}_{\hat{T}}\, \varphi\, ds \quad \forall \varphi \in P_1(e) \;\; e = e_i,\; i = 1,.,6 \qquad \text{(12 D.o.F.)}$$

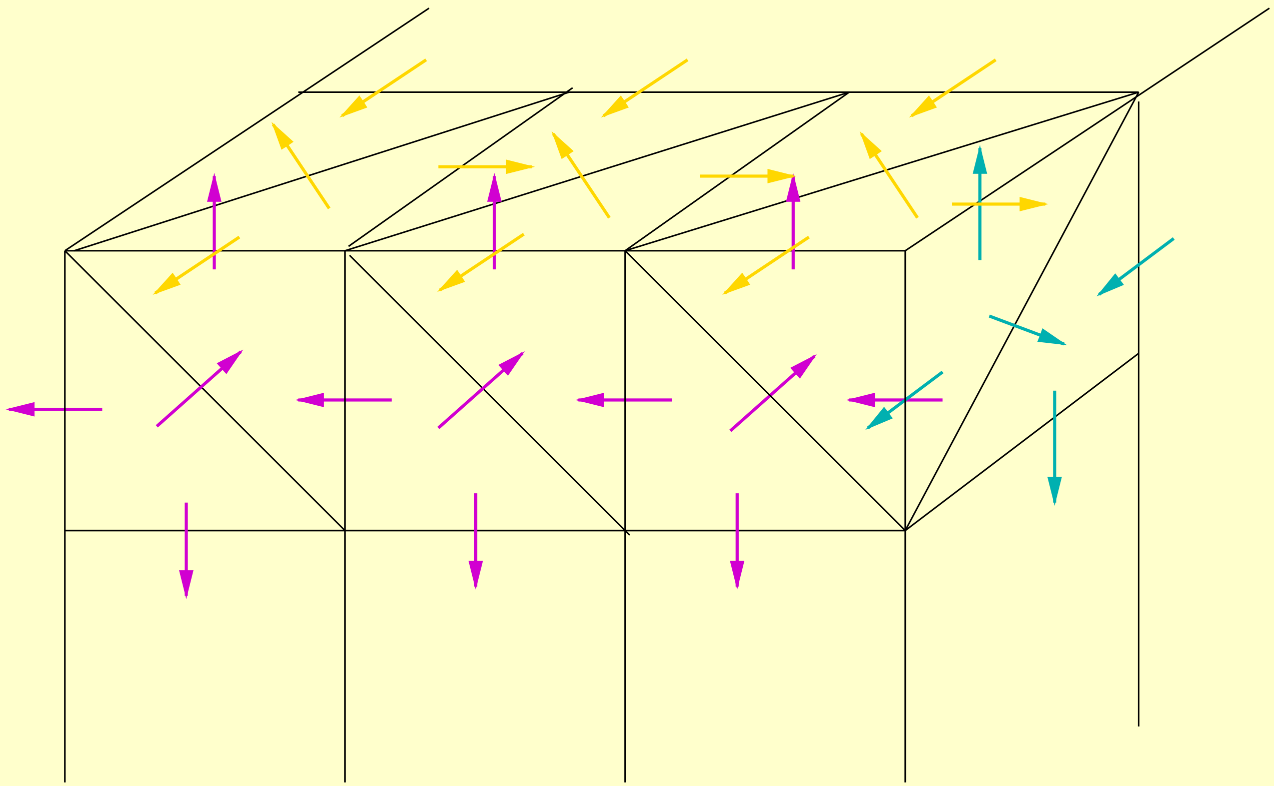
Tangetial Traces: $\boldsymbol{\lambda} = \mathbf{u} \wedge \mathbf{n}$ $\qquad$ ($\mathbf{n}$ outward normal)

$$\mathcal{E}\mathcal{E}_1^0(\Omega_i) \wedge \mathbf{n}_i \Rightarrow \quad \mathcal{R}\mathcal{T}_1(\partial\Omega_i)$$

$$\mathcal{E}\mathcal{E}_1(\Omega_i) \wedge \mathbf{n}_i \Rightarrow \quad \mathcal{B}\mathcal{D}\mathcal{M}_1(\partial\Omega_i)$$

$\partial\Omega_i$ $\quad$ non-smooth surfaces

$H(div)$-conforming, normal components.



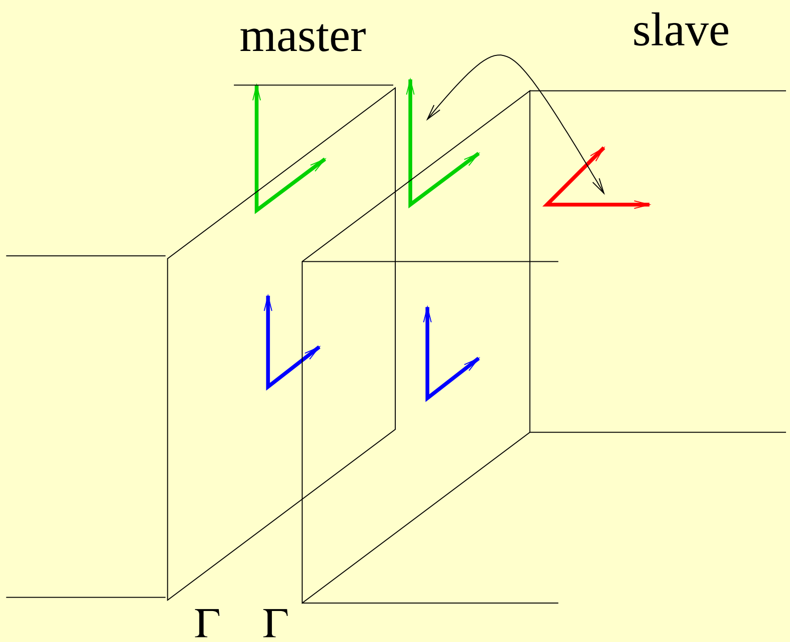
Choice of discrete spaces

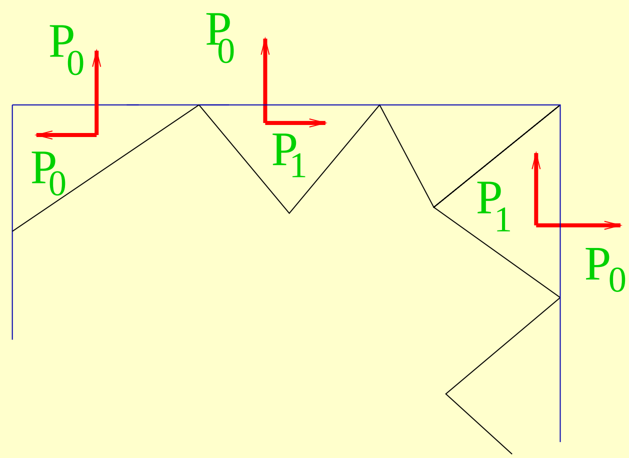
$$X_{i,h} \approx \mathcal{E}\mathcal{E}_1, \quad T_{\ell,h} := \{\mathbf{v}_{i(\ell),h} \wedge \mathbf{n}, \mathbf{v}_{i(\ell),h} \in X_{i(\ell),h}\}$$

$$T^0_{\ell,h} = T_{\ell,h} \cap H_0(\mathrm{div}\,, \Gamma_\ell),$$

$M_{\ell,h} \subset T_{\ell,h}$ such that $dim\big(M_h\big) = dim\big(T^0_{\ell,h}\big)$.

$M_{\ell,h} = T_{\ell,h} \;\Rightarrow$ no existence of solutions.





Constraint space formulation

$\mathcal{X}_h = \{\mathbf{v}_h \in X_h \quad \text{s.t.} \quad b(\mathbf{v}_h, \boldsymbol{\varphi}) = 0 \quad \forall \boldsymbol{\varphi} \in M_h\}$

Find $\mathbf{u}_h \in \mathcal{X}_h$ such that $a(\mathbf{u}_h, \mathbf{v}_h) = \int_\Omega \mathbf{f}\mathbf{v}_h \ \ \forall \mathbf{v}_h \in \mathcal{X}_h$

We have a non–conforming approximation on Ω:

Lemma: (Berger-Scott-Strang) If $\mathbf{curl}\,\mathbf{u} \in H^1(\Omega)$, then:

$$||\mathbf{u}-\mathbf{u}_h||_{\star,\mathbf{curl}} \lesssim \inf_{\mathbf{v}_h \in X_h} ||\mathbf{u}-\mathbf{v}_h||_{\star,\mathbf{curl}} + \sup_{\mathbf{v}_h \in X_h} \frac{\displaystyle\sum_{\ell=1}^{L} {}_{-1/2,\Gamma_\ell} < [\![\mathbf{u}_h \wedge \mathbf{n}_\ell]\!]_\ell, (\mathbf{curl}\,\mathbf{u})_T >_{1/2,\Gamma_\ell}}{||\mathbf{v}_h||_{\star,\mathbf{curl}}}$$

$(\mathbf{curl}\,\mathbf{u})_T = \mathbf{n} \wedge (\mathbf{curl}\,\mathbf{u} \wedge \mathbf{n})$

$|| \cdot ||^2_{\star,\mathbf{curl}} = \sum_{i=1}^{M} || \cdot ||^2_{\mathbf{curl},\Omega_i}$ (broken norms)

Error Estimates When $\max_{1\le i\le M} h_i \lesssim \min_{1\le i\le M} h_i$ (mesh sizes)

$$||\mathbf{u}-\mathbf{u}_h||_{\star,\mathbf{curl}} \lesssim h\sqrt{|\ln h|}\ ||\mathbf{u}||_{\star,2,\mathbf{curl},\Omega}$$

Ingredients

$\Pi_h : L^2(\Gamma) \to T^1_{h,0}\ \int_\Gamma(\boldsymbol{\lambda}-\Pi_h\boldsymbol{\lambda})\varphi_h = 0\ \forall\varphi_h \in M_h$

$||\Pi_h\boldsymbol{\lambda}||_{L^2(\Gamma)} \lesssim ||\boldsymbol{\lambda}||_{L^2(\Gamma)}$

$\mathcal{I}^i_h : H^1(\mathbf{curl},\Omega_i) \to \mathcal{E}\mathcal{E}_1(\Omega_i)$ interpolation operator

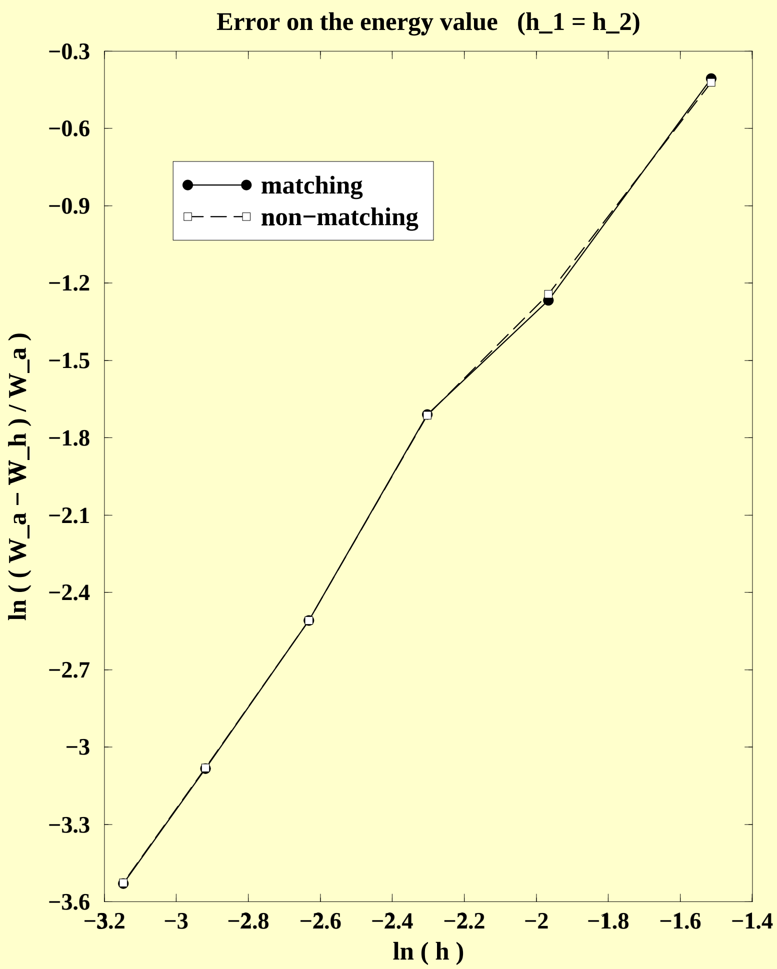
$||\mathbf{u}_i-\mathcal{I}^i_h\mathbf{u}_i||_{\mathbf{curl}} + h_i^{-1}||\mathbf{u}_i-\mathcal{I}^i_h\mathbf{u}_i||_{L^2} \lesssim h_i^s|\mathbf{u}_i|_{s+1,\Omega_i}$

$||\mathbf{v}_{i,h}\wedge\mathbf{n}||_{0,\Gamma} \lesssim h_i^{-1/2}||\mathbf{v}_{i,h}\wedge\mathbf{n}||_{(H^{1/2}_{00}(\Gamma))'}$ inverse inequality

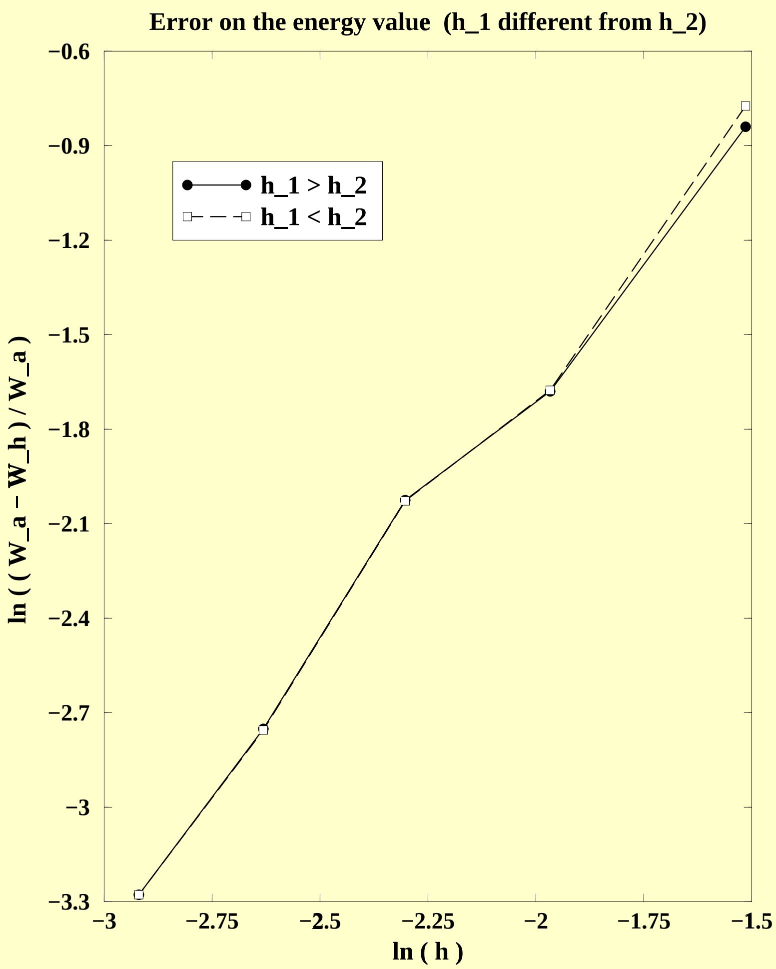
$|(\mathbf{curl}\,\mathbf{u})_T|_{s,B\Gamma} \lesssim h_1^{1/2}\sqrt{|\ln h_1|}\ ||(\mathbf{curl}\,\mathbf{u})_T||_{s+1/2,\Gamma}$ Sobolev inequality

- The regularity of $\mathbf{u}$ is requested only in a neighborhood of Γ.

same mesh size



different mesh size



Magnetostatics

$$\mathbf{curl}(\nu\ \mathbf{curl}\ \mathbf{u}) = \mathbf{J} \quad \text{in } \Omega \qquad \mathbf{u} \times \mathbf{n} = \mathbf{0} \quad \text{on } \partial\Omega.$$

here $\mathbf{u}\ : \mathbf{curl}\,\mathbf{u} = \mathbf{B}$ and $\mathbf{u}$ solution implies $\mathbf{u} + \nabla\phi$ is solution.

Variational formulation $\mathbf{T}\ : \mathbf{curl}\,\mathbf{T} = \mathbf{J}$, and

Given $\mathbf{T} \in L^2(\Omega)^3$, find $\mathbf{u} \in H_0(\mathbf{curl}, \Omega)$ such that

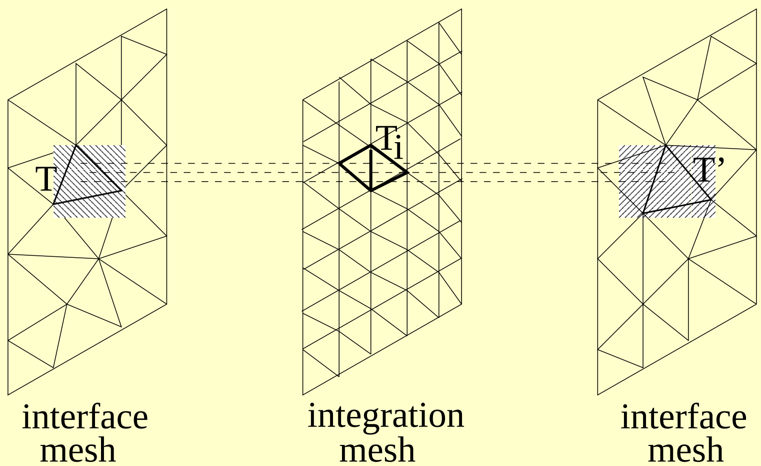
$$\int_\Omega \nu\ \mathbf{curl}\ \mathbf{u} \cdot \mathbf{curl}\ \mathbf{v}\, d\Omega = \int_\Omega \mathbf{T} \cdot \mathbf{curl}\ \mathbf{v}\, d\Omega\ ,\ \forall \mathbf{v} \in H_0(\mathbf{curl}, \Omega)$$

Discrete Variational formulation

Given $\mathbf{T} \in L^2(\Omega)^3$, find $\mathbf{u}_h \in \mathcal{X}_h$ such that

$$\int_\Omega \nu\ \mathbf{curl}\ \mathbf{u}_h \cdot \mathbf{curl}\ \mathbf{v}_h\, d\Omega = \int_\Omega \mathbf{T} \cdot \mathbf{curl}\ \mathbf{v}_h\, d\Omega\ ,\ \forall \mathbf{v}_h \in \mathcal{X}_h$$

Features of the discrete problem:

- We need to construct a basis for $\mathcal{X}_h$ ... their are supported on 3/4 tetrahedra;
- 

- The final system has the form:

$$\mathcal{M}\mathbf{u}_h = \mathcal{T}_h$$

where the matrix $\mathcal{M}$ has a non-trivial kernel. The system has a solution since $\mathcal{T}_h \in range\{\mathcal{M}\}$. Conjugate gradient method!

Table 1: Conforming discretizations

meshes	dof	CG residual/ 10^{-7}	CG iterations	energy/10^6
I - I	68	4.7063	20	1.5419
II - II	296	9.7081	30	1.9261
III - III	712	9.9794	43	2.0271
IV - IV	2652	9.9279	52	2.1267
V - V	2604	8.9139	67	2.1267
VI - VI	6416	9.4174	90	2.1622
VII - VII	12820	9.4468	113	2.1787

Analytic value of the energy: 2208 MJ

Table 2: Non-conforming discretizations

meshes	dof	CG residual/10^{-7}	CG iterations	energy/10^6
I - I rotated	68	7.3256	53	1.5521
III - I	358	7.8839	35	1.7466
I - III	422	9.2364	141	1.7764
I - III rotated	422	9.6712	127	1.7779
II - II rotated	296	9.7265	75	1.9195
II - IV	1539	9.7055	143	2.0216
II - IV rotated	1539	9.6157	142	2.0264
III - III rotated	712	9.7021	92	2.0277
IV - IV rotated	2652	9.6514	108	2.1318
V - V rotated	2604	9.9828	116	2.1267
VI - VI rotated	6416	9.9618	135	2.1621
VI - VII (*)	9722	9.8627	193	2.1703
VII - VI (*)	9514	9.7641	186	2.1703
VII - VII rotated	12820	9.7970	194	2.1786

Analytic value of the energy: 2208 MJ

$$\sigma \frac{\partial \mathbf{u}_1}{\partial t} + \mathbf{curl}\,(\nu\,\mathbf{curl}\,\mathbf{u}_1) = \mathbf{J}_s \qquad \Omega_1(0)$$

$$\mathbf{curl}\,(\nu\,\mathbf{curl}\,\mathbf{u}_2) = \mathbf{J}_s \qquad \Omega_2$$

$$\mathrm{div}(\mathcal{R}_t \mathbf{u}) = 0 \qquad \mathcal{I}\times]0,T[$$

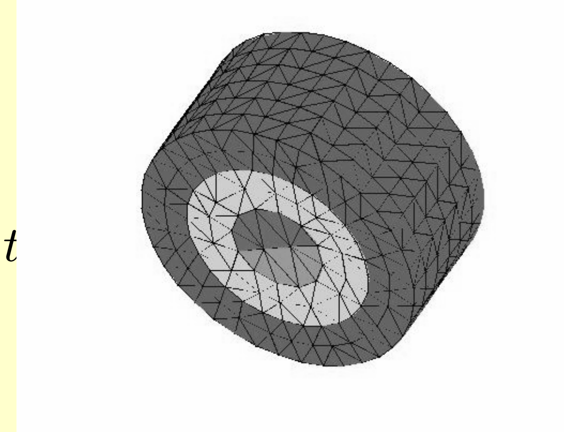
$$\mathbf{T}_m = \int_{\Omega_1} \mathbf{r} \times \left[\left(-\sigma \frac{\partial \mathbf{u}_1}{\partial t} + \mathbf{J}_s\right) \times \mathbf{curl}\,\mathbf{u}_1\right] d\Omega$$

$$J\,\frac{d\omega}{dt} + k\,\omega = \mathrm{T}_m \quad , \quad \omega = \frac{d\theta}{dt}$$

$$(\mathcal{R}_t(\mathbf{u}_1))_{\tau,\Gamma}(\mathbf{x},t) = (\mathbf{u}_2)_{\tau,\Gamma}(\mathbf{x},t)$$

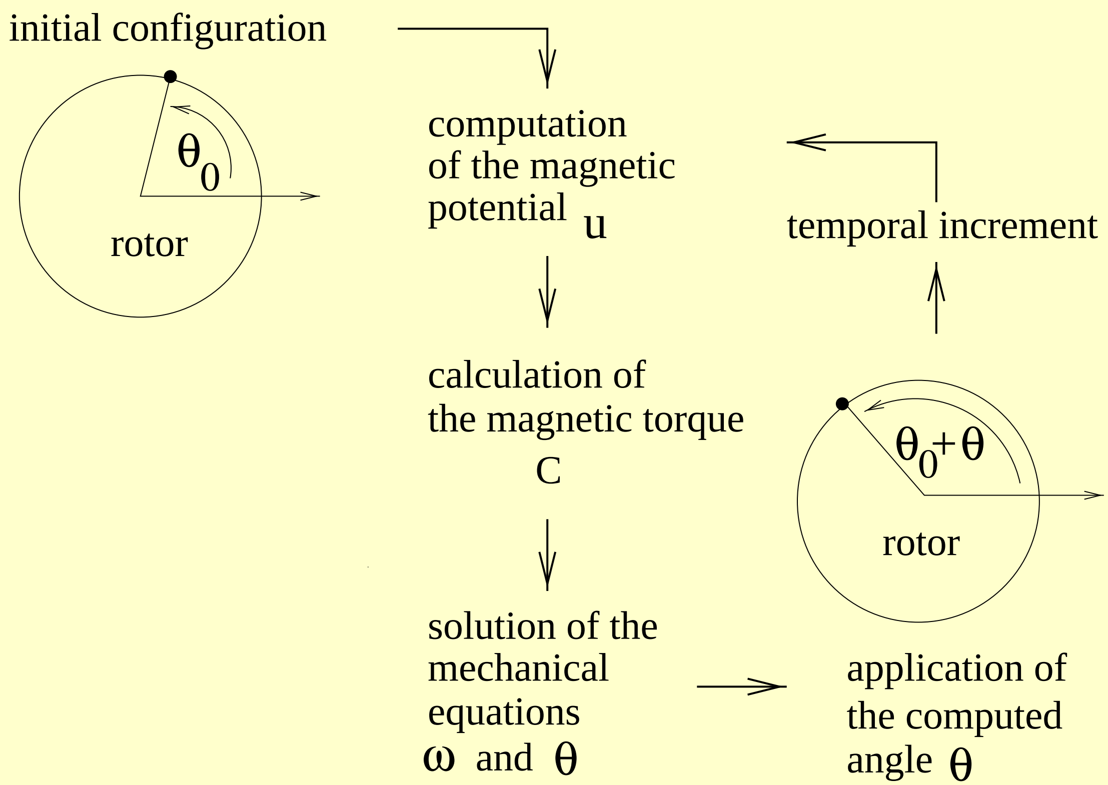
$$(\mathcal{R}_t(\nu\,\mathbf{curl}\,\mathbf{u}_1))_{\tau,\Gamma}(\mathbf{x},t) = (\nu\,\mathbf{curl}\,\mathbf{u}_2)_{\tau,\Gamma}(\mathbf{x},t$$

where $R_t\mathbf{u} = R_t(\mathbf{u}(R_{-t}\mathbf{x},t))$.



Theorems: Existence, uniqueness for J suff. large, discretization by mortar edge element + Implicit/explicit Euler in time.

Time discretisation and coupling



Coupling flowchart

Time discretisation by a simple implicit/explicit Euler.

Download at `http://www.ian.pv.cnr.it/~annalisa`